

Approximate Stationary Solution for Beam-Beam Interaction Models with Parametric Poisson White Noise

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Abstract: In this paper, a stochastic averaging method is derived for a class of non-linear stochastic systems under parametrical Poisson white noise excitation, which may be used to model the beam-beam interaction models in particle accelerators. The averaged Generalized Fokker-Planck equation is derived and the approximate stationary solution of the averaged Generalized Fokker-Planck equation is solved by using perturbation method. The present method applied in this paper can reduce the dimensions of stochastic ODE from $2n$ to n , which simplify the complex stochastic ODE, and then the analytical stationary solutions can be obtained. An example is employed to demonstrate the procedure of our proposed method. The analytical solution of approximate stationary probability density function is obtained, and the theoretical results are verified through numerical simulations. Finally, the stability of the amplitude process is investigated.

Keywords: Poisson white noise, Stochastic averaging, Generalized Fokker-Planck equation, Stationary probability density, Stochastic stability

1 Introduction

Stochastic dynamical systems have been an attracting subject in the past two decades, since more and more uncertainty factors are being introduced into the considerations of conventional systems. For example, mechanical and electrical systems are normally modeled as second order differential equations, however unmodeled dynamics and structural behavior contribute to noise for such system [Rajan and Raha (2008)]. For certain kind of devices taking the noise into account for numerical simulation is essential in order to get correct estimates of design behavior with respect to manufacturing variations, thus affecting yield. Also, in solid mechanics,

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some of the most important parameters may be uncertain mechanical properties such as modulus of elasticity, density and heat transfer coefficients. The uncertain mechanical properties can be randomly generated using the Monte Carlo simulation with various coefficients of variations (COVs) and normal distribution. The stochastic analysis of dynamic problems can be helpful for predicting all possible responses caused by uncertainties of mechanical properties. Hoseini, Shahabian Sladek and Sladek (2011) proposed a Stochastic Meshless local Petrov-Galerkin method (MLPG) to solve the stochastic boundary value problem, where the efficient MLPG [Atluri and Zhu (1998); Atluri (2004)] is employed, accompanied by the Monte-Carlo simulation method. Silva, Deus, Mantovani, and Beck (2010) used the Galerkin method to analyze the stochastic beam bending on winkler foundations. Kaminski (2011) carried out the probabilistic analysis of transient problems by the least squares stochastic perturbation-based finite element method. In the fluid mechanics, Kami and Ossowski (2011) applied the stochastic finite volume method in Navier-Stokes problems.

In our study, a typical beam-beam interaction model is considered. Recently, beam-beam interaction models widely appear in many fields of applied science and practical problems [Month and Herrera (1979); Bountis and Mahmoud (1987); Mahmoud (1992)]. Therefore it has attracted the attention of many scholars. For example, Mahmond (1995) studied it under additive broad-band random noise using stochastic averaging. Later, Xu and Xu (2004) generalized stochastic averaging method to complex beam-beam interaction models with broad-band random excitation. Xu, Xu, Mahmoud and Lei (2005) also applied the method of multiple scales to discuss that beam-beam interaction models in the presence of narrow-band excitation, and Zhang, Xu and Xu (2009) explored the case of colored noise excitation by using the method of stochastic averaging combined with the perturbation technique.

The stochastic averaging method was first proposed by Statonovich (1963) and it has been proven that it was a powerful approximate technique for analyzing Gaussian white noise and (or) Gaussian colored noise excitation stochastic differential equation [Roberts and Spanos (1986); Zhu (1988); Roy (1994); Liu and Liew (2005)]. Recently, it was applied to study the response of quasi-linear systems to Poisson white noise excitation successfully [Zeng and Zhu (2010a)]. Later, this method was extended to n -dimensional non-linear dynamical systems subject to filtered Poisson white noise excitation [Zeng and Zhu (2010b)]. Zeng and Zhu (2011) studied the response of single-degree-of-freedom (SDOF) strongly nonlinear oscillators under Poisson white noise excitation by using the generalized harmonic functions procedure and combine perturbation method.

In this paper, we apply stochastic averaging method and perturbation method to research a class of non-linear stochastic systems with the approximate stationary

probability density function under parametrical Poisson white noise excitation. We also consider the stochastic stability of stochastic differential equation for the amplitude process.

2 Averaged Stochastic Method

In this paper, we consider Beam-Beam interaction models with parametric Poisson white noise of the form

$$\ddot{x} + \omega_0^2 x + \varepsilon^2 \gamma g(\dot{x}) + \varepsilon^2 f(x) p(\omega_0 t) = \varepsilon h(x, \dot{x}) W_p(t) \quad (1)$$

where ω_0 is nature frequency, ε is a small positive parameter, $p(\omega_0 t)$ is a periodic function, $f(x)$, $g(\dot{x})$, $h(x, \dot{x})$ are generally nonlinear functions, $W_p(t)$ is a Poisson white noise which can be considered as formal derivatives of the homogeneous compound Poisson processes:

$$W_p(t) = \frac{dC(t)}{dt} = \sum_{k=1}^{N(t)} Y_k \delta(t - t_k) \quad (2)$$

Where $N(t)$ is the total number of pulses that arrive in the time interval $(-\infty, t]$ which with intensity $\lambda > 0$, Y_k are independent and identically distributed (i.i.d.) random variables with zero mean. Thus,

$$E \left[(dC(t))^k \right] = \lambda E \left[Y^k \right] dt \quad (3)$$

Applying the stochastic averaging method, the joint response process (x, \dot{x}) can be transformed into the amplitude $A(t)$ and phase $\Theta(t)$ processes, according to the follow relationships:

$$x(t) = A(t) \cos \Phi(t), \quad \dot{x}(t) = -A(t) \omega_0 \sin \Phi(t), \quad \text{where } \Phi(t) = \omega_0 t + \Theta(t).$$

Therefore, Eq. (1) can be changed as following

$$\begin{aligned} \dot{A} &= \varepsilon^2 m_1(A, \Phi, \omega_0 t) dt + \varepsilon n_1(A, \Phi) W_p(t) \\ \dot{\Theta} &= \varepsilon^2 m_2(A, \Phi, \omega_0 t) dt + \varepsilon n_2(A, \Phi) W_p(t) \end{aligned} \quad (4)$$

Where

$$m_1(A, \Phi, \omega_0 t) = \frac{\gamma}{\omega_0} g(A, \Phi) \sin \Phi + \frac{1}{\omega_0} f(A, \Phi) p(\omega_0 t) \sin \Phi \quad (5)$$

$$m_2(A, \Phi, \omega_0 t) = \frac{\gamma}{\omega_0} g(A, \Phi) \sin \Phi + \frac{1}{\omega_0} f(A, \Phi) p(\omega_0 t) \sin \Phi \quad (6)$$

$$m_2(A, \Phi, \omega_0 t) = \frac{\gamma}{A\omega_0} g(A, \Phi) \cos \Phi + \frac{1}{A\omega_0} f(A, \Phi) p(\omega_0 t) \cos \Phi \quad (7)$$

$$n_1(A, \Phi) = -\frac{1}{\omega_0} h(A, \Phi) \sin \Phi \quad n_2(A, \Phi) = -\frac{1}{A\omega_0} h(A, \Phi) \cos \Phi \quad (8)$$

Equation (4) can be considered as Stratonovich stochastic differential equation and then transformed into a stochastic differential equation by Paola and Falsone (1993a, 1993b). The result is

$$dA = \varepsilon^2 m_1(A, \Phi, \omega_0 t) dt + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} G_1^{(k)}(A, \Phi) (dC(t))^k$$

$$d\Theta = \varepsilon^2 m_2(A, \Phi, \omega_0 t) dt + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} G_2^{(k)}(A, \Phi) (dC(t))^k \quad (9)$$

in which

$$G_i^{(k)}(A, \Phi) = n_1(A, \Phi) \frac{\partial G_i^{(k-1)}(A, \Phi)}{\partial A} + n_2(A, \Phi) \frac{\partial G_i^{(k-1)}(A, \Phi)}{\partial \Theta} \quad (10)$$

$$G_i^{(1)}(A, \Phi) = n_i(A, \Phi), \quad i = 1, 2; \quad k = 1, 2, \dots$$

The generalized Fokker-Planck equation associated with Eq. (9) is

$$\frac{\partial \rho(a, \theta, t)}{\partial t} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial a^i} M_{1i}(a, \vartheta, \omega_0 t) \rho(a, \theta, t) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial \theta^j} M_{2j}(a, \vartheta, \omega_0 t)$$

$$\times \rho(a, \theta, t) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^{m+n}}{\partial a^m \partial \theta^n} N_{mn}(a, \vartheta) \rho(a, \theta, t) \quad (11)$$

Where

$$M_{i1}(a, \vartheta, \omega_0 t) = \varepsilon^2 m_i(a, \vartheta, \omega_0 t) + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} G_i^{(j)}(a, \vartheta) \lambda E \left[Y^j \right]$$

$$M_{ij}(a, \vartheta) = \sum_{k_1=1}^{\infty} \dots \sum_{k_i=1}^{\infty} \frac{\varepsilon^{k_1+\dots+k_i}}{k_1! \dots k_i!} G_i^{(k_1)}(a, \vartheta) \dots G_i^{(k_i)}(a, \vartheta) \lambda E \left[Y^{k_1+\dots+k_i} \right] \quad (12)$$

$$i = 1, 2; \quad j = 2, 3, \dots$$

$$N_{mn}(a, \vartheta) = \sum_{k_1=1}^{\infty} \dots \sum_{k_m=1}^{\infty} \sum_{s_1=1}^{\infty} \dots \sum_{s_n=1}^{\infty} \frac{\varepsilon^{k_1+\dots+k_m+s_1+\dots+s_n}}{k_1! \dots k_m! s_1! \dots s_n!} G_1^{(k_1)}(a, \vartheta) \dots G_1^{(k_m)}(a, \vartheta)$$

$$\times G_2^{(s_1)}(a, \vartheta) \dots G_2^{(s_n)}(a, \vartheta) \lambda E \left[Y^{k_1+\dots+k_m+s_1+\dots+s_n} \right] \quad m, n = 1, 2, 3, \dots$$

(13)

Averaging generalized Fokker-Planck equation coefficients with respect to a yields the following,

$$\begin{aligned}\bar{M}_{ij}(a) &= \frac{1}{2\pi} \int_0^{2\pi} M_{ij}(a, \vartheta, \omega_0 t) d\vartheta \\ \bar{N}_{ij}(a) &= \frac{1}{2\pi} \int_0^{2\pi} N_{ij}(a, \vartheta, \omega_0 t) d\vartheta, \quad i = 1, 2 \quad j = 1, 2, 3, \dots\end{aligned}\quad (14)$$

Therefore, the averaging generalized Fokker-Planck equation is obtained as follows:

$$\frac{\partial \rho(a, t)}{\partial t} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial a^i} \bar{M}_{1i}(a) \rho(a, t) \quad (15)$$

Where

$$\begin{aligned}\bar{M}_{11}(a) &= \varepsilon^2 \frac{1}{2\pi} \int_0^{2\pi} m_1(a, \vartheta, \omega_0 t) d\vartheta + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \lambda E [Y^k] \frac{1}{2\pi} \int_0^{2\pi} G_1^{(k)}(a, \vartheta) d\vartheta \\ \bar{M}_{1i}(a) &= \sum_{k_1=1}^{\infty} \dots \sum_{k_i=1}^{\infty} \frac{\varepsilon^{k_1+\dots+k_i}}{2\pi k_1! \dots k_i!} \lambda E [Y^{k_1+\dots+k_i}] \int_0^{2\pi} G_1^{(k_1)}(a, \vartheta) \dots G_1^{(k_i)}(a, \vartheta) d\vartheta,\end{aligned}\quad (16)$$

Base on the power exponent of ε , Eq. (15) becomes:

$$\begin{aligned}\frac{\partial \rho(a, t)}{\partial t} &= - \frac{\partial}{\partial a} \bar{Q}_{11}(a) \rho(a, t) + \frac{1}{2!} \frac{\partial^2}{\partial a^2} \bar{Q}_{12}(a) \rho(a, t) - \frac{1}{3!} \frac{\partial^3}{\partial a^3} \bar{Q}_{13}(a) \rho(a, t) \\ &\quad + \frac{1}{4!} \frac{\partial^4}{\partial a^4} \bar{Q}_{14}(a) \rho(a, t) + O(\varepsilon^5)\end{aligned}\quad (17)$$

where

$$\begin{aligned}\bar{Q}_{11}(a) &= \varepsilon^2 \left(\overline{m_1(a, \vartheta)} + \frac{1}{2!} \lambda E [Y^2] \overline{G_1^{(2)}(a, \vartheta)} \right) + \varepsilon^3 \frac{1}{3!} \lambda E [Y^3] \overline{G_1^{(3)}(a, \vartheta)} \\ &\quad + \varepsilon^4 \frac{1}{4!} \lambda E [Y^4] \overline{G_1^{(4)}(a, \vartheta)} \\ \bar{Q}_{12}(a) &= \varepsilon^2 \lambda E [Y^2] \overline{G_1^{(1)}(a, \vartheta) G_1^{(1)}(a, \vartheta)} + \varepsilon^3 \lambda E [Y^3] \overline{G_1^{(1)}(a, \vartheta) G_1^{(2)}(a, \vartheta)} \\ &\quad + \varepsilon^4 \lambda E [Y^4] \left(\overline{\frac{1}{3} G_1^{(1)}(a, \vartheta) G_1^{(3)}(a, \vartheta)} + \frac{1}{4} \left[\overline{G_1^{(2)}(a, \vartheta)} \right]^2 \right)\end{aligned}$$

$$\begin{aligned}\bar{Q}_{13}(a) &= \varepsilon^3 \lambda E [Y^3] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^3 + \varepsilon^4 \frac{3}{2} \lambda E [Y^4] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^2 \overline{G_1^{(2)}(a, \vartheta)} \\ \bar{Q}_{14}(a) &= \varepsilon^4 \lambda E [Y^4] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^4\end{aligned}\quad (18)$$

The stationary solution form of the nonlinear system can be written by using perturbation technique in Lin and Cai (1995) as

$$\rho(a) = \rho_0(a) + \varepsilon \rho_1(a) + \varepsilon^2 \rho_2(a) \quad (19)$$

Substituting Eq. (19) into (18) leads to the following system of equations:

$$\begin{cases} L_0 \rho_0(a) = 0 \\ L_0 \rho_1(a) = -L_1 \rho_0(a) \\ L_0 \rho_2(a) = -L_1 \rho_1(a) - L_2 \rho_0(a) \\ \dots \dots \dots \end{cases} \quad (20)$$

where

$$\begin{aligned}L_0 &= -\frac{\partial}{\partial a} \left(\frac{\overline{m_1(a, \vartheta)}}{m_1(a, \vartheta)} + \frac{\lambda E [Y^2] \overline{G_1^{(2)}(a, \vartheta)}}{2!} \right) \\ &\quad + \frac{\partial^2}{\partial a^2} \left(\frac{\lambda E [Y^2] \overline{G_1^{(1)}(a, \vartheta)} \overline{G_1^{(1)}(a, \vartheta)}}{2!} \right)\end{aligned}\quad (21)$$

$$\begin{aligned}L_1 &= -\frac{\partial}{\partial a} \left(\frac{\lambda E [Y^3] \overline{G_1^{(3)}(a, \vartheta)}}{3!} \right) + \frac{\partial^2}{\partial a^2} \left(\frac{\lambda E [Y^3] \overline{G_1^{(1)}(a, \vartheta)} \overline{G_1^{(2)}(a, \vartheta)}}{2!} \right) \\ &\quad - \frac{\partial^3}{\partial a^3} \left(\frac{\lambda E [Y^3] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^3}{3!} \right)\end{aligned}\quad (22)$$

$$\begin{aligned}L_2 &= -\frac{\partial}{\partial a} \left(\frac{\lambda E [Y^4] \overline{G_1^{(4)}(a, \vartheta)}}{4!} \right) + \frac{\partial^2}{\partial a^2} \left(\frac{\lambda E [Y^4] \overline{G_1^{(4)}(a, \vartheta)}}{2! \times 4!} \right) \\ &\quad - \frac{\partial^3}{\partial a^3} \left(\frac{3 \lambda E [Y^4] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^2 \overline{G_1^{(2)}(a, \vartheta)}}{2 \times 3!} \right) \\ &\quad + \frac{\partial^4}{\partial a^4} \left(\frac{\lambda E [Y^4] \left[\overline{G_1^{(1)}(a, \vartheta)} \right]^4}{4!} \right)\end{aligned}\quad (23)$$

3 Example

3.1 Approximate stationary solution of an averaged generalized FPK equation

Suppose $g(\dot{x}) = \dot{x}^3$, $f(x) = x^3$, $p(\omega_0 t) = \cos \omega_0 t$ and $h(x, \dot{x}) = \omega_0 \sigma x$, Eq.(1) becomes

$$\ddot{x} + \omega_0^2 x + \varepsilon^2 \gamma \dot{x}^3 + \varepsilon^2 x^3 \cos \omega_0 t = \varepsilon \omega_0 \sigma x W_p(t) \quad (24)$$

From Eq. (5) to Eq. (10), we get

$$\frac{\partial \rho(a, t)}{\partial t} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial a^i} \bar{M}_{1i}(a) \rho(a, t) \quad (25)$$

where

$$\begin{aligned} m(a, \vartheta) &= -\gamma \omega_0^2 a^3 \sin^4 \vartheta + \frac{1}{\omega_0} a^3 \cos(\omega_0 t) \sin \vartheta \cos^3 \vartheta \\ n_1(a, \vartheta) &= G_1^{(1)}(a, \vartheta) = -\sigma a \cos \vartheta \sin \vartheta, \quad n_2(a, \vartheta) = G_2^{(1)}(a, \vartheta) = -\sigma \cos^2 \vartheta \\ G_1^{(k)}(a, \vartheta) &= n_1(a, \vartheta) \frac{\partial G_1^{(k-1)}(a, \vartheta)}{\partial a} + n_2(a, \vartheta) \frac{\partial G_1^{(k-1)}(a, \vartheta)}{\partial \theta} \\ \bar{M}_{11}(a) &= \varepsilon^2 \frac{1}{2\pi} \int_0^{2\pi} m(a, \vartheta) d\vartheta + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{2\pi k!} \lambda E[Y^k] \int_0^{2\pi} G_1^{(k)}(a, \vartheta) d\vartheta \\ \bar{M}_{1i}(a) &= \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \frac{\varepsilon^{k_1+\cdots+k_i}}{2\pi k_1! \cdots k_i!} \lambda E[Y^{k_1+\cdots+k_i}] \int_0^{2\pi} G_1^{(k_1)}(a, \vartheta) \cdots G_1^{(k_i)}(a, \vartheta) d\vartheta, \end{aligned} \quad (26)$$

Substituting Eq. (21) into (23), we have

$$L_0 = -\frac{\partial}{\partial a} \left(-\frac{3}{8} \gamma \omega_0^2 a^3 + \frac{3}{16} \lambda E[Y^2] \sigma^2 a^2 \right) + \frac{1}{2!} \frac{\partial^2}{\partial a^2} \left(\lambda E[Y^2] \frac{1}{8} \sigma^2 a^2 \right)$$

$$L_1 = 0$$

$$\begin{aligned} L_2 &= -\frac{\partial}{\partial a} \left(\frac{15}{1024} \lambda E[Y^4] \sigma^4 a \right) + \frac{\partial^2}{\partial a^2} \left(\frac{15}{1024} \lambda E[Y^4] \sigma^4 a^2 \right) \\ &\quad - \frac{\partial^3}{\partial a^3} \left(\frac{10}{1024} \lambda E[Y^4] \sigma^4 a^3 \right) + \frac{\partial^4}{\partial a^4} \left(\frac{1}{1024} \lambda E[Y^4] \sigma^4 a^4 \right) \end{aligned}$$

Solving Eq. (20), we obtain

$$\rho_0(a) = a \cdot \exp \left\{ -3\gamma \omega_0^2 a^2 / \sigma^2 \lambda E[Y^2] \right\}$$

$$\rho_1(a) = 0$$

$$\rho_2(a) = \left(-\frac{9\gamma^2\omega_0^4 E[Y^4]}{8\lambda^2\sigma^2(E[Y^2])^3}a^5 + \frac{9\gamma^3\omega_0^6 E[Y^4]}{16\lambda^3\sigma^4(E[Y^2])^4}a^7 \right) \cdot \exp\left\{-3\gamma\omega_0^2 a^2 / \sigma^2 \lambda E[Y^2]\right\}$$

So, we get the approximate stationary probability density function of order ε

$$\rho(a) = \rho_0(a) + \varepsilon\rho_1(a) + \varepsilon^2\rho_2(a) = c \cdot \left[a + \varepsilon^2 \cdot \left(-\frac{9\gamma^2\omega_0^4 E[Y^4]}{8\lambda^2\sigma^2(E[Y^2])^3}a^5 + \frac{9\gamma^3\omega_0^6 E[Y^4]}{16\lambda^3\sigma^4(E[Y^2])^4}a^7 \right) \right] \times \exp\left\{-3\gamma\omega_0^2 a^2 / \sigma^2 \lambda E[Y^2]\right\} \quad (27)$$

3.2 Numerical analysis

For illustrative purposes, we carry out the numerical simulation to verify the analysis results. In figures 1-3, we choose the parameters: $\omega_0=1$, $\gamma=2.5$, $\sigma=1.5$, $\varepsilon=0.1$, $\lambda=1$, random pulse amplitude was assumed to have a Gaussian distribution with $E(Y) = 0$ and $E(Y^2) = 2$.

As shown in fig. 1, the approximate stationary probability density $\rho(a)$ is very close to Monte Carlo simulation, and is better than the Gaussian approximate which means that the Poisson white noise is Gaussian white noise.

It is shown in Fig. 2 and Fig. 3 that the Mean-square response and fourth-order moment of the response of amplitude process $a(t)$. Approximate stationary moments of the response for different values of γ . The comparison shows the good accuracy of the method as the Monte Carlo simulation.

3.3 Stochastic stability for the amplitude process

Equation (25) may be changed into

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{\partial \rho(a, t)}{\partial t} - \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial a^i} M_{1i}(a, \vartheta) \rho(a, t) \right] d\vartheta = 0 \quad (28)$$

Therefore,

$$\frac{\partial \rho(a, t)}{\partial t} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial a^i} M_{1i}(a, \vartheta) \rho(a, t) \quad (29)$$

where

$$M_{11}(a, \vartheta) = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2 a^3 + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \lambda E[Y^k] G_1^{(k)}(a, \vartheta) \quad (30)$$

$$M_{1i}(a, \vartheta) = \sum_{k_1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \frac{\varepsilon^{k_1+\cdots+k_i}}{k_1! \cdots k_i!} G_1^{(k_1)}(a, \vartheta) \cdots G_1^{(k_i)}(a, \vartheta) \lambda E[Y^{k_1+\cdots+k_i}]$$

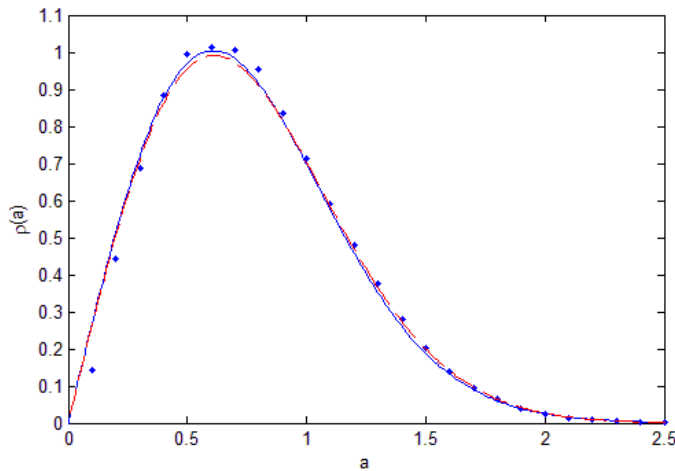


Figure 1: $\gamma=2.0$, Stationary probability density of amplitude process a . “—”: Approximate stationary probability density function. “---”: Gaussian approximate simulation result. “.”: Monte Carlo simulation result.

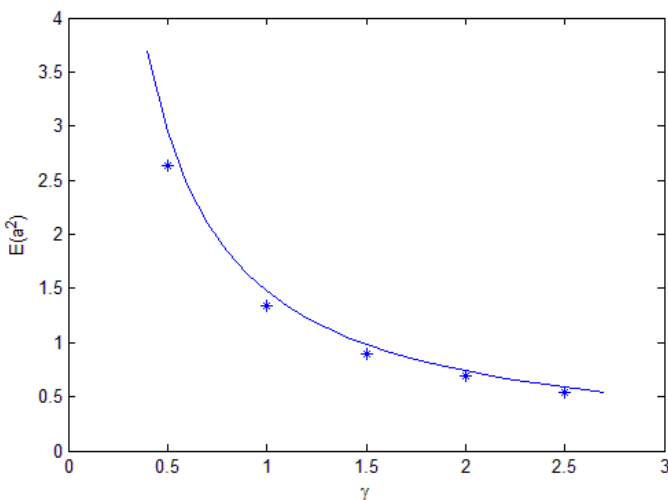


Figure 2: Mean-square response of amplitude process a . “—”: Approximate Mean-square response. “*”: Monte Carlo simulation Mean-square response.

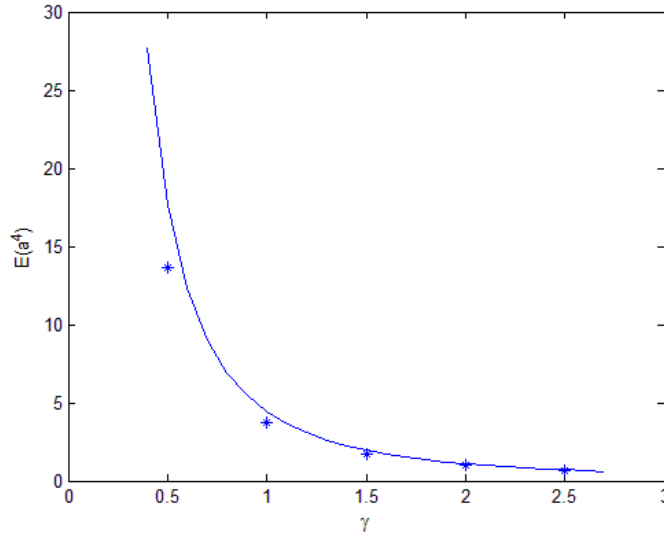


Figure 3: Fourth-order moment of response of amplitude process a . "—": Approximate fourth-order moment of response. "*": Monte Carlo simulation fourth-order moment of response progress.

The corresponding Itô stochastic differential equation is

$$da = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2a^3dt + \sum_{k=1}^{\infty}\varepsilon^k\frac{1}{k!}S_k(\vartheta)a(dC(t))^k \quad (31)$$

where

$$S_k(\vartheta)a = G_1^{(k)}(a, \vartheta)$$

The corresponding integral Eq. (19) of Eq. (31) can be written as follows:

$$da = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2a^3dt + \int_y \kappa(a, \vartheta, Y, t)M(dt, dY) \quad (32)$$

in which

$$\kappa(a, \vartheta, Y, t) = \sum_{k=1}^{\infty}\varepsilon^k\frac{1}{k!}S_k(\vartheta)aY^k$$

Consider the difference $\tilde{a} = a - a_s$, where a_s and a are stationary and transient solution respectively. So

$$d\tilde{a}(t) = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2(a^3(t) - a_s^3(t))dt + \int_y \left\{ \sum_{k=1}^{\infty}\varepsilon^k\frac{1}{k!}S_k(\vartheta)\tilde{a}(t)Y^k \right\} M(dt, dY) \quad (33)$$

Since a_s and a are stationary and transient solution of Eq. (32), we have

$$a^3 - a_s^3 = (a - a_s)(a^2 + aa_s + a_s^2) = 3a_s^2 \tilde{a}$$

The exponential growth rates of \tilde{a} are given by the Lyapunov exponent in Grigoriu (1996), Duan, Xu, Su and Zhou (2011) as:

$$\lambda_e = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\tilde{a}(t)\| \quad (34)$$

Applying the generalized formula (19) to the function $\phi(t) = \ln \|\tilde{a}(t)\|$, we have

$$d\phi(t) = -\frac{9}{8}\varepsilon^2\gamma\omega_0^2 a_s^2(t)dt + \int_y \ln \left\| 1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k \right\| M(dt, dY) \quad (35)$$

So that

$$\phi(t) = \phi(0) - \frac{9}{8}\varepsilon^2\gamma\omega_0^2 \int_0^t a_s^2(u)du + \int_0^t \int_y \ln \left\| 1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k \right\| M(du, dY) \quad (36)$$

Therefore, Lyapunov exponent is

$$\begin{aligned} \lambda_e &= \lim_{t \rightarrow \infty} \frac{1}{t} \left(\phi(0) - \frac{9}{8}\varepsilon^2\gamma\omega_0^2 \int_0^t a_s^2(u)du + \int_0^t \int_y \ln \left(1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k \right) M(du, dY) \right) \\ &= -\frac{9}{8}\varepsilon^2\gamma\omega_0^2 E(a_s^2(t)) + \lambda E \left(\ln \left(1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k \right) \right) \end{aligned} \quad (37)$$

(i) When $a_s(t)=0$ (trivial stationary solution), $E(a_s(t))=0$, the Lyapunov exponent can be written as:

$$\lambda_e = \lambda E \left(\ln \left(1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k \right) \right) \quad (38)$$

The trivial stationary solution $a_s(t) = 0$ is stable with probability one when $\lambda_e < 0$, that is

$$\sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k < 0, \text{ a.s.} \quad (39)$$

(ii) When $a_s \neq 0$ (that is: non-trivial stationary solution)

$$da_s(t) = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2 a_s^3(t)dt + \int_y \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) a_s(t) Y^k M(dt, dY) \quad (40)$$

So,

$$\frac{dE(\ln \|a_s(t)\|)}{dt} = -\frac{3}{8}\varepsilon^2\gamma\omega_0^2 E(a_s^2(t)) + \lambda E\left(\ln\left\|1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k\right\|\right) \quad (41)$$

Because of the non-trivial stationary solution $a_s \neq 0$, therefore

$$E(a_s^2) = \frac{8\lambda}{3\varepsilon^2\gamma\omega_0^2} E\left(\ln\left\|1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k\right\|\right) \quad (42)$$

Applying Eq. (37)

$$\begin{aligned} \lambda_e &= -\frac{9}{8}\varepsilon^2\gamma\omega_0^2 E(a_s^2) + \lambda E\left(\ln\left\|1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k\right\|\right) \\ &= -2\lambda E\left(\ln\left\|1 + \sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k\right\|\right) \end{aligned} \quad (43)$$

The non-trivial stationary solution $a_s(t) \neq 0$ is stable with probability one when $\lambda_e < 0$, that is

$$\sum_{k=1}^{\infty} \varepsilon^k \frac{1}{k!} S_k(\vartheta) Y^k > 0, \text{ a.s.} \quad (44)$$

4 Conclusions

In this paper, beam-beam interaction models with parametric Poisson white noise are considered. The stochastic averaging method in conjunction with the perturbation method is applied to derive an approximate stationary solution of average Fokker-Plank equation. An example is used to get an approximate stationary probability density. The theory analytical results are verified using Monte Carlo simulation. The stability of the amplitude process is discussed base on the Lyapunov exponent. Results indicate that the response stability is related to the distribution of jump of Poisson white noise.

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