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Numerical Approximate Solutions of Nonlinear Fredholm Integral Equations of Second Kind Using B-spline Wavelets and Variational Iteration Method

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Abstract: In this paper, nonlinear integral equations have been solved numerically by using B-spline wavelet method and Variational Iteration Method (VIM). Compactly supported semi-orthogonal linear B-spline scaling and wavelet functions together with their dual functions are applied to approximate the solutions of nonlinear Fredholm integral equations of second kind. Comparisons are made between the variational Iteration Method (VIM) and linear B-spline wavelet method. Several examples are presented to compare the accuracy of linear B-spline wavelet method and Variational Iteration Method (VIM) with their exact solutions.

Keywords: Nonlinear Fredholm integral equation, Linear B-spline wavelets, Semi-orthogonal, Scaling function, Variational Iteration Method (VIM).

1 Introduction

Integral equation has been one of the essential tools for various area of applied mathematics. Integral equations occur naturally in many fields of science and engineering [Wazwaz (2011)]. Wavelet analysis has been applied in a wide range of engineering disciplines; particularly wavelets are very successfully used in signal analysis, time-frequency analysis and fast algorithms for easy implementation [Chui (1997)].

In this paper, we consider the second kind nonlinear Fredholm integral equation of the following form

$$u(x) = f(x) + \int_0^1 K(x,t)F(t,u(t))dt, \ 0 \leq x \leq 1. \tag{1}$$

where $K(x,t)$is the kernel of the integral equation, $f(x)$ and $K(x,t)$ are known functions and $u(x)$ is the unknown function that is to be determined.

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A computational approach to solve integral equation is an essential work in scientific research. Some methods for solving second kind Fredholm integral equation are available in open literature. The Petrove-Galerkin method and the iterated Petrove-Galerkin method [Chen and Xu (1998); Kaneko, Noren and Novaprateep (2003)] have applied to solve nonlinear integral equations. A variation of the Nystrom method for nonlinear integral equations of second kind was presented by Lardy (1981). The learned researchers Maleknejad (2011) proposed a numerical method for solving nonlinear Fredholm integral equations of the second kind using sinc-collocation method.

A novel meshless technique termed the Random Integral Quadrature (RIQ) method has been developed by Zou, H. and Li, H. (2010). By applying this method, the governing equations in the integral form are discretized directly with the field nodes distributed randomly or uniformly, which is achieved by discretizing the integral governing equations with the generalized integral quadrature (GIQ) technique over a set of background virtual nodes, and then interpolating the function values at the virtual nodes over a set of field nodes with Local Kriging method, where the field nodes are distributed either randomly or uniformly.

The Fictitious Time Integration Method (FTIM) previously developed by Liu and Atluri (2009) has been employed to solve a system of ill-posed linear algebraic equations, which may result from the discretization of a first-kind linear Fredholm integral equation.

Gauss-Legendre Nyström method [Kelmanson and Tenwick (2010)] has been applied for determining approximate solutions of Fredholm integral equations of the second kind on finite intervals. The authors' recent continuous-kernel approach is generalized in order to accommodate kernels that are either singular or of limited continuous differentiability at a finite number of points within the interval of integration.

Quadratic integral equations are a class of nonlinear integral equations having many important uses in engineering and sciences. Adomian decomposition method has been applied to solve the quadratic integral equations of Volterra type [Fu, Wang and Duan (2013)].

In the present paper, we apply compactly supported linear semi-orthogonal B-spline wavelets to solve the second kind nonlinear Fredholm integral equation. The obtained numerical solutions are then compared with the results obtained by Variational Iteration Method [He (1999); He (2006); He (2007)]. The Variational Iteration Method (VIM) has proposed originally by He [He (1999)]. This method has been applied by many renowned researchers as a powerful mathematical tool for solving various kinds of linear and nonlinear problems [Biazar and Ghazvini

(2010); Biazar and Ghazvini (2007); Abbasbandy and Shirzadi (2008); Dehghan and Shakeri (2008); Odibat and Momani (2006); Wazwaz (2008)].

Unlike the traditional numerical methods, VIM needs no discretization, linearization, transformation or perturbation. The method has been widely applied to solve nonlinear problems, more and more merits have been discovered and some modifications are suggested to overcome the demerits arising in the solution procedure [Soltani and Shirzadi (2010)].

In this paper, the B-spline wavelet method converts the integral equation (1) into a set of algebraic equations by expanding the unknown function as linear B-spline wavelets with unknown coefficients. The properties of these wavelets are then utilized to evaluate the unknown coefficients. Moreover, the VIM has been also successfully applied to find the numerically approximate solution of nonlinear Fredholm integral equation. This method is based on the incorporation of a general Lagrange multiplier in the construction of correction functional for the integral equation.

2 Linear B-Spline scaling and wavelet functions on the interval [0,1]

Semi-orthogonal wavelets using B-spline specially constructed for the bounded interval and this wavelet can be represented in a closed form. This provides a compact support. Semi-orthogonal wavelets form the basis in the space $L^2(R)$.

Using this basis, an arbitrary function in $L^2(R)$ can be expressed as the wavelet series [Chui (1992)]. For the finite interval [0, 1], the wavelet series cannot be completely presented by using this basis. This is because supports of some basis are truncated at the left or right end points of the interval. Hence a special basis has to be introduced into the wavelet expansion on the finite interval. These functions are referred to as the boundary scalar functions and boundary wavelet functions.

Let m and n be two positive integers and

$$a = x_{-m+1} = \dots = x_0 < x_1 < \dots < x_n = x_{n+1} = \dots = x_{n+m-1} = b, \tag{2}$$

be an equally spaced knots sequence. The functions

$$B_{m,j,X}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j,X}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j+1,X}(x), \quad j = -m+1, \dots, n-1. \tag{3}$$

and

$$B_{1,j,X}(x) = \begin{cases} 1, \ x \in [x_j, x_{j+1}), \\ 0, \text{ otherwise}, \end{cases} \tag{4}$$

are called cardinal B-spline functions of order $m \geq 2$ for the knot sequence $X = \{x_i\}_{i=-m+1}^{n+m-1}$, and $SuppB_{m,j,X}(x) = [x_j, x_{j+m}] \cap [a,b]$.
By considering the interval $[a,b] = [0,1]$, at any level $j \in \mathrm{Z}^+$, the discretization step is 2^{-j}, and this generates $n = 2^j$ number of segments in $[0,1]$ with knot sequence

$$X^{(j)} = \begin{cases} x_{-m+1}^{(j)} = \ldots = x_0^{(j)} = 0, & \\ x_k^{(j)} = \frac{k}{2^j}, & k = 1, \ldots, n-1, \\ x_n^{(j)} = \ldots = x_{n+m-1}^{(j)} = 1. & \end{cases} \tag{5}$$

Let j_0 be the level for which $2^{j_0} \geq 2m-1$; for each level, $j \geq j_0$ the scaling functions of order m can be defined as follows in [Maleknejad and Sahlan (2010)]:

$$\varphi_{m,j,i}(x) = \begin{cases} B_{m,j_0,i}(2^{j-j_0}x) & i = -m+1, \ldots, -1, \\ B_{m,j_0,2^j-m-i}(1-2^{j-j_0}x) & i = 2^j-m+1, \ldots, 2^j-1, \\ B_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0}i) & i = 0, \ldots, 2^j-m. \end{cases} \tag{6}$$

And the two scale relation for the m-order semi-orthogonal compactly supported B-wavelet functions are defined as follows:

$$\psi_{m,j,i-m} = \sum_{k=i}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \; i = 1, \ldots, m-1 \tag{7}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{2i+2m-2} q_{i,k} B_{m,j,k-m}, \; i = m, \ldots, n-m+1 \tag{8}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{n+i+m-1} q_{i,k} B_{m,j,k-m}, \; i = n-m+2, \ldots, n \tag{9}$$

where $q_{i,k} = q_{k-2i}$.
Hence there are $2(m-1)$ boundary wavelets and $(n-2m+2)$ inner wavelets in the bounded interval $[a,b]$. Finally, by considering the level j with $j \geq j_0$, the B-wavelet functions in [0,1] can be expressed as follows:

$$\psi_{m,j,i}(x) = \begin{cases} \psi_{m,j_0,i}(2^{j-j_0}x) & i = -m+1, \ldots, -1, \\ \psi_{m,2^j-2m+1-i,i}(1-2^{j-j_0}x) & i = 2^j-2m+2, \ldots, 2^j-m, \\ \psi_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0}i) & i = 0, \ldots, 2^j-2m+1. \end{cases} \tag{10}$$

The scaling functions $\varphi_{m,j,i}(x)$ occupy m segments and the wavelet functions $\psi_{m,j,i}(x)$ occupy $2m-1$ segments.
When the semi-orthogonal wavelets are constructed from B-spline of order m, the lowest octave level $j = j_0$ is determined in [Goswami, Chan and Chui (1995)] by

$$2^{j_0} \geq 2m-1, \tag{11}$$

so as to have a minimum of one complete wavelet on the interval [0, 1].

3 Function approximation

A function $f(x)$ defined over interval [0, 1] may be approximated by B-spline wavelets as

$$f(x) = \sum_{k=-1}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=-1}^{2^j-2} d_{j,k}\psi_{j,k}(x). \tag{12}$$

In particular, for $j_0 = 2$, if the infinite series in equation (12) is truncated at M then eq. (12) can be written as [Maleknejad and Sahlan (2010); Maleknejad, Nosrati and Najafi (2012)]

$$f(x) \approx \sum_{k=-1}^{3} c_k\varphi_{2,k}(x) + \sum_{j=2}^{M}\sum_{k=-1}^{2^j-2} d_{j,k}\psi_{j,k}(x) = C^T\Psi(x) \tag{13}$$

where $\varphi_{2,k}$ and $\psi_{j,k}$ are scaling and wavelet functions, respectively, and C and Ψ are $(2^{M+1}+1)\times 1$ vectors given by

$$C = [c_{-1}, c_0, ..., c_3, d_{2,-1}, ..., d_{2,2}, d_{3,-1}, ..., d_{3,6}, ..., d_{M,-1}, ..., d_{M,2^M-2}]^T, \tag{14}$$

$$\Psi = [\varphi_{2,-1}, \varphi_{2,0}, ..., \varphi_{2,3}, \psi_{2,-1}, ..., \psi_{2,2}, \psi_{3,-1}, ..., \psi_{3,6}, ..., \psi_{M,-1}, ..., \psi_{M,2^M-2}]^T, \tag{15}$$

with

$$\begin{aligned} c_k &= \int_0^1 f(x)\tilde{\varphi}_{2,k}(x)dx,\ k = -1, 0, ..., 3, \\ d_{j,k} &= \int_0^1 f(x)\tilde{\psi}_{j,k}(x)dx,\ j = 2, 3, 4, ..., M,\ k = -1, 0, 1, ..., 2^j - 2, \end{aligned} \tag{16}$$

where $\tilde{\varphi}_{2,k}(x)$ and $\tilde{\psi}_{j,k}(x)$ are dual functions of $\varphi_{2,k}$, and $\psi_{j,k}$ respectively. These can be obtained by linear combinations of $\varphi_{2,k}, k = -1, ..., 3$, and $\psi_{j,k}, j = 2, ..., M$, $k = -1, ..., 2^j - 2$, as follows. Let

$$\Phi = [\varphi_{2,-1}(x), \varphi_{2,0}(x), \varphi_{2,1}(x), \varphi_{2,2}(x), \varphi_{2,3}(x)]^T \quad , \tag{17}$$

$$\bar{\Psi} = [\psi_{2,-1}(x), \psi_{2,0}(x), ..., \psi_{M,2^M-2}(x)]^T \quad . \tag{18}$$

Using eq. (6) and eq. (17) we get

$$\int_0^1 \Phi\Phi^T dx = P_1 = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix} \quad , \tag{19}$$

and from eq (10) and eq. (18) we have

$$\int_0^1 \bar{\Psi}\bar{\Psi}^{\mathrm{T}} dx = P_2 = \begin{bmatrix} N_{4\times 4} & & & & \\ & \frac{1}{2}N_{8\times 8} & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & & \frac{1}{2^{M-2}}N_{2^M\times 2^M} \end{bmatrix} \tag{20}$$

where P_1 and P_2 are 5×5 and $(2^{M+1}-4)\times(2^{M+1}-4)$ matrices, respectively, and N is a five-diagonal matrix given by

$$N = \begin{bmatrix} \frac{2}{27} & \frac{1}{96} & -\frac{1}{864} & 0 & 0 & . & . & . & 0 \\ \frac{1}{96} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & 0 & . & . & . & 0 \\ -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} & -\frac{1}{864} & . & . & . & 0 \\ . & . & . & . & . & . & & & . \\ . & & . & . & . & . & . & & . \\ . & & & . & . & . & . & . & . \\ 0 & . & . & . & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} \\ 0 & . & . & . & 0 & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} \\ 0 & . & . & . & 0 & 0 & -\frac{1}{864} & \frac{1}{96} & \frac{2}{27} \end{bmatrix} \tag{21}$$

Suppose $\tilde{\Phi}$ and $\tilde{\bar{\Psi}}$ are the dual functions of Φ and $\bar{\Psi}$, respectively, given by

$$\begin{aligned} \tilde{\Phi} &= [\tilde{\varphi}_{2,-1}(x), \tilde{\varphi}_{2,0}(x), \tilde{\varphi}_{2,1}(x), \tilde{\varphi}_{2,2}(x), \tilde{\varphi}_{2,3}(x)]^T , \\ \tilde{\bar{\Psi}} &= [\tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), ..., \tilde{\psi}_{M,2^M-2}(x)]^T . \end{aligned} \tag{22}$$

And combining the above two, we can get

$$\tilde{\Psi} = [\tilde{\varphi}_{2,-1}(x), \tilde{\varphi}_{2,0}(x), \tilde{\varphi}_{2,1}(x), \tilde{\varphi}_{2,2}(x), \tilde{\varphi}_{2,3}(x), \tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), ..., \tilde{\psi}_{M,2^M-2}(x)]^T \tag{23}$$

Using eqs. (17), (18) and (22) we have

$$\int_0^1 \tilde{\Phi}\,\Phi^T dx = I_1, \quad \int_0^1 \tilde{\bar{\Psi}}\,\bar{\Psi}^T dx = I_2, \tag{24}$$

where I_1 and I_2 are 5×5 and $(2^{(M+1)}-4)\times(2^{(M+1)}-4)$ identity matrices, respectively. Then eqs. (19), (20), and (24) yield

$$\tilde{\Phi} = P_1^{-1}\Phi, \quad \tilde{\bar{\Psi}} = P_2^{-1}\bar{\Psi}. \tag{25}$$

4 Application of B-spline wavelet method to nonlinear Fredholm integral equation of second kind

In this section, we have solved nonlinear Fredholm integral equation of second kind of the form given in eq. (1) by using B-spline wavelets. First, we assume

$$y(x) = F(x, u(x)), \quad 0 \le x \le 1. \tag{26}$$

Now from eq. (13), we can approximate the functions $u(x)$ and $y(x)$ as

$$u(x) = A^T \Psi(x), \quad \text{and} \quad y(x) = B^T \Psi(x), \tag{27}$$

where A and B are $(2^{M+1}+1) \times 1$ column vectors similar to C defined in eq. (14). Again by using dual of the wavelet functions, we can approximate the functions $f(x)$ and $K(x,t)$ as follows

$$f(x) = D^T \tilde{\Psi}(x), \quad \text{and} \quad K(x,t) = \tilde{\Psi}^T(t)\Theta\tilde{\Psi}(x), \tag{28}$$

where

$$\Theta_{(i,j)} = \int_0^1 \left[\int_0^1 K(x,t)\Psi_i(t)dt \right] \Psi_j(x)dx.$$

From eqs. (26)- (28), we get

$$\begin{aligned} \textstyle\int_0^1 K(x,t)F(t,u(t))dt &= \textstyle\int_0^1 B^T \Psi(t)\tilde{\Psi}^T(t)\Theta\tilde{\Psi}(x)dt \\ &= B^T \left[\textstyle\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt \right] \Theta\tilde{\Psi}(x) \\ &= B^T \Theta\tilde{\Psi}(x), \text{ since } \textstyle\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt = I \end{aligned} \tag{29}$$

Applying eqs. (26)- (29) in eq. (1), we get

$$A^T \Psi(x) = D^T \tilde{\Psi}(x) + B^T \Theta\tilde{\Psi}(x) \tag{30}$$

Multiplying eq. (30) by $\Psi^T(x)$ both sides from the right and integrating from 0 to 1, we have

$$\begin{aligned} &A^T P = D^T + B^T \Theta \\ &A^T P - D^T - B^T \Theta = 0 \end{aligned} \tag{31}$$

where P is a $(2^{M+1}+1) \times (2^{M+1}+1)$ square matrix given by
$P = \int_0^1 \Psi(x)\Psi^T(x)dx = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}$, and $\int_0^1 \tilde{\Psi}(x)\Psi^T(x)dx = I$.

Eq. (31) gives a system of $(2^{M+1}+1)$ algebraic equations with $2(2^{M+1}+1)$ unknowns for A and B vectors given in eq. (27).

To find the solution $u(x)$ in eq. (27), we first utilize the following equation

$$F(x, A^T\Psi(x)) = B^T\Psi(x), \tag{32}$$

with the collocation points $x_i = \frac{i-1}{2^{M+1}}$, where $i = 1, 2, ..., 2^{M+1}+1$.

Eq. (32) gives a system of $(2^{M+1}+1)$ algebraic equations with $2(2^{M+1}+1)$ unknowns, for A and B vectors given in eq. (27).

Combining eqs. (31) and (32), we have a total of $2(2^{M+1}+1)$ system of algebraic equations with $2(2^{M+1}+1)$ unknowns for A and B. Solving those equations for the unknown coefficients in the vectors A and B, we can obtain the solution $u(x) = A^T\Psi(x)$.

5 Variational Iteration Method (VIM)

Let us consider, a nonlinear Fredholm integral equation of second kind given in eq. (1). For solving eq. (1) by variational iteration method, first we have to take the partial derivative of eq. (1) with respect to x.

$$u'(x) = f'(x) + \int_0^1 K'(x,t)F(t,u(t))dt, \tag{33}$$

We apply variational iteration method for the eq. (33). According to this method, correction functional can be defined as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - f'(\xi) - \int_a^b K'(\xi,t)F(t,\tilde{u}_n(t))dt\right)d\xi, \tag{34}$$

where $\lambda(\xi)$ is a general Lagrange multiplier which can be identified optimally by the variational theory, the subscript n denotes the nth order approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. The successive approximations $u_n(x), n \geq 1$ for the solution $u(x)$ can be readily obtained after determining the Lagrange multiplier and selecting an appropriate initial function $u_0(x)$. Consequently the approximate solution may be obtained by using $u(x) = \lim\limits_{n\to\infty} u_n(x)$.

To make the above correction functional stationary, we have

$$\begin{aligned}\delta u_{n+1}(x) &= \delta u_n(x) + \delta\int_0^x \lambda(\xi)\left(u_n'(\xi) - f'(\xi) - \int_a^b K'(\xi,t)F(t,\tilde{u}_n(t))dt\right)d\xi \\ &= \delta u_n(x) + \int_0^x \lambda(\xi)\delta(u_n'(\xi))d\xi \\ &= \delta u_n(x) + \lambda\delta u_n|_{\xi=x} - \int_0^x \lambda'(\xi)\delta u_n(\xi)d\xi\end{aligned} \tag{35}$$

Under stationary condition $\delta u_{n+1} = 0$, implies the following Euler Lagrange equation

$$\lambda'(\xi) = 0, \tag{36}$$

with the following natural boundary condition

$$1 + \lambda(\xi)|_{\xi=x} = 0 \quad . \tag{37}$$

Solving the eq. (36) along with boundary condition (37), we get the general Lagrange multiplier $\lambda = -1$.

Substituting the identified Lagrange multiplier into eq. (34), results in the following iterative scheme

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - f'(\xi) - \int_a^b K'(\xi,t)F(t,\tilde{u}_n(t))dt \right) d\xi,\ n \geq 0 \tag{38}$$

By starting with initial approximate function $u_0(x) = f(x)$ (say), we can determine the approximate solution $u(x)$ of the eq. (1).

6 Illustrative examples

Example 1. Consider the equation

$$u(x) = -\frac{x}{9} - \frac{x^2}{8} + x^3 + \int_0^1 (x^2t + xt^2)u^2(t)dt, \quad 0 \leq x \leq 1,$$

with the exact solution $u(x) = x^3$. The approximate solution is obtained by the method of B-spline wavelets explained in section 4 for $M = 2$ and $M = 4$ and also by VIM explained in section 5. The following table 1 cites the numerical solutions obtained by B-spline method and VIM accomplished with corresponding exact solutions and table 2 cites the absolute errors obtained by B-spline method and VIM. Figure 1-2 and Figure 3-4 present the comparison graphically between the numerical solutions obtained by B-spline wavelet method with exact solutions and VIM solutions respectively.

Example 2.

Consider the equation

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xtu^2(t)dt, 0 \leq x \leq 1,$$

with the exact solution $u(x) = x$. The approximate solution is obtained by the method of B-spline wavelets explained in section 4 for $M = 2$ and $M = 4$ and

Table 1: Comparison of numerical solutions obtained by B-spline method and VIM with exact solution

x	*Linear B-spline wavelet method*		*Variational Iteration Method (VIM)*	*Exact solution*
	$M=2$	$M=4$		
0	-0.000249587	-3.61701e-6	0	0
0.1	0.00166976	0.00105608	0.000965618	0.001
0.2	0.0105762	0.00816602	0.00792412	0.008
0.3	0.0313409	0.0272625	0.0268755	0.027
0.4	0.068719	0.0642781	0.0638198	0.064
0.5	0.12735	0.125145	0.124757	0.125
0.6	0.223711	0.216493	0.215687	0.216
0.7	0.355417	0.343778	0.34261	0.343
0.8	0.527106	0.512931	0.511526	0.512
0.9	0.743534	0.729887	0.728434	0.729
1.0	1.00957	1.00058	0.999336	1

Table 2: Absolute errors obtained by B-spline method and VIM

x \ Absolute error	*Linear B-Spline wavelet method*		*Variational Iteration Method (VIM)*
	$M=2$	$M=4$	
0	0.0002495	3.61701e-6	0
0.1	0.0006695	0.0000560	0.0000343
0.2	0.0025762	0.0001660	0.0000758
0.3	0.0043409	0.0002625	0.0001245
0.4	0.0047190	0.0002781	0.0001802
0.5	0.0023496	0.0001454	0.0002430
0.6	0.0077114	0.0004932	0.0003130
0.7	0.0124172	0.0007776	0.0003901
0.8	0.0151063	0.0009313	0.0004743
0.9	0.0145338	0.0008869	0.0005656
1.0	0.0095707	0.0005807	0.0006641

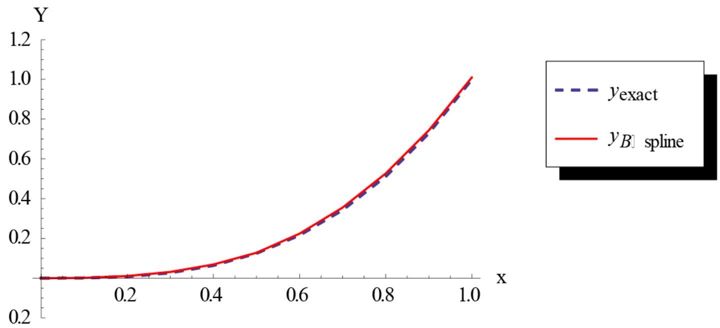

Figure 1: Comparison of numerical solution obtain by B-spline $(M = 2)$ with exact solution

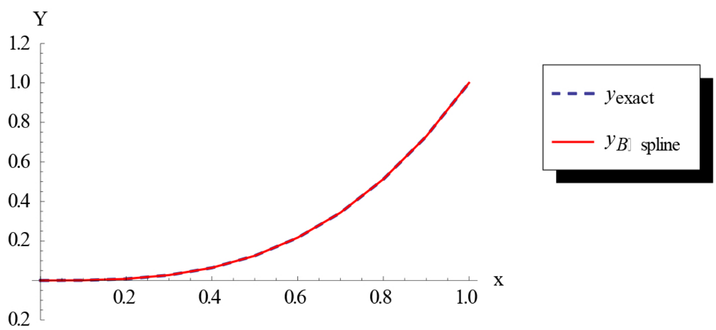

Figure 2: Comparison of numerical solution obtain by B-spline $(M = 4)$ with exact solution

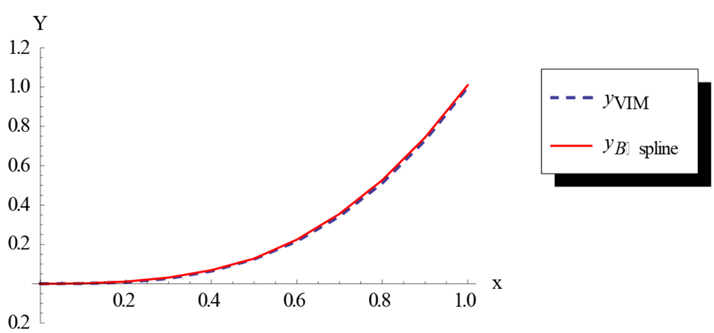

Figure 3: Comparison of numerical solution obtain by B-spline $(M = 2)$ with VIM solution

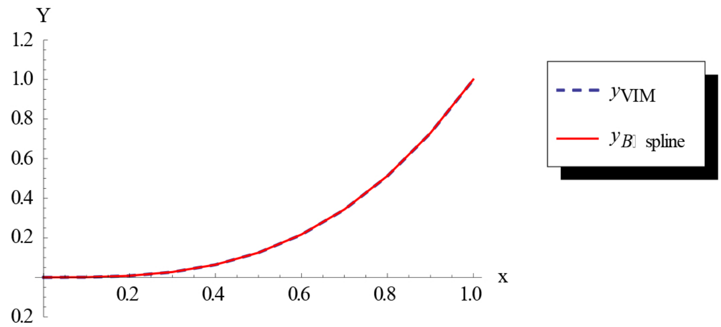

Figure 4: Comparison of numerical solution obtain by B-spline $(M = 4)$ with VIM solution

also by VIM explained in section 5. The following table 3 cites the numerical solutions obtained by B-spline method and VIM accomplished with corresponding exact solutions and table 4 cites the absolute errors obtained by B-spline method and VIM. Figure 5-6 and Figure 7-8 present the comparison graphically between the numerical solutions obtained by B-spline wavelet method with exact solutions and VIM solutions respectively.

Table 3: Comparison of numerical solutions obtained by B-spline method and VIM with exact solution

x	*Linear B-spline wavelet method*		*Variational Iteration Method (VIM)*	*Exact solution*
	$M = 2$	$M = 4$		
0	0	0	0	0
0.1	0.100087	0.100005	0.1	0.1
0.2	0.200174	0.200011	0.2	0.2
0.3	0.300261	0.300016	0.3	0.3
0.4	0.400348	0.400022	0.4	0.4
0.5	0.500435	0.500027	0.5	0.5
0.6	0.600522	0.600033	0.6	0.6
0.7	0.700609	0.700038	0.7	0.7
0.8	0.800696	0.800043	0.8	0.8
0.9	0.900783	0.900049	0.9	0.9
1.0	1.00087	1.00005	1	1

Table 4: Absolute errors obtained by B-spline method and VIM

x \ Absolute error	*Linear B-Spline wavelet method*		*Variational Iteration Method(VIM)*
	$M=2$	$M=4$	
0	0	0	0
0.1	0.0000869	5.42599e-6	1.09593e-8
0.2	0.0001739	0.0000108	2.19186e-8
0.3	0.0002609	0.0000162	3.28778e-8
0.4	0.0003478	0.0000217	4.38371e-8
0.5	0.0004348	0.0000271	5.47964e-8
0.6	0.0005218	0.0000325	6.57557e-8
0.7	0.0006087	0.0000379	7.6715e-8
0.8	0.0006957	0.0000434	8.76743e-8
0.9	0.0007827	0.0000488	9.86335e-8
1.0	0.0008696	0.0000542	1.09593e-7

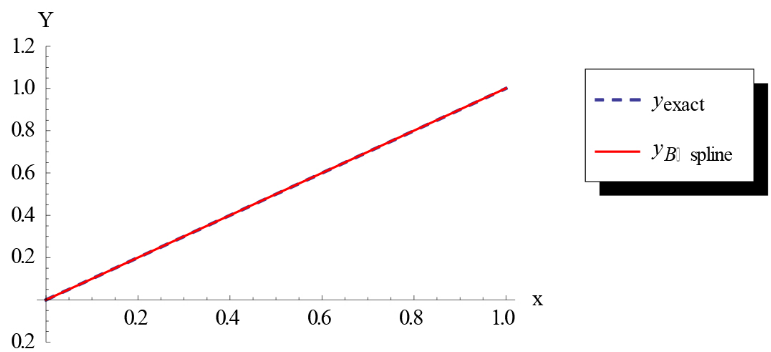

Figure 5: Comparison of numerical solution obtain by B-spline ($M=2$) with exact solution

Example 3.

Consider the equation

$$u(x) = \cos(x) - \frac{1}{2}x\sin(2) + \int_0^1 x(u^2(t) - \sin^2(t))dt, \quad 0 \leq x \leq 1,$$

with the exact solution $u(x) = \cos(x)$. The approximate solution is obtained by the method of B-spline wavelets explained in section 4 for $M=2$ and $M=4$ and also by VIM explained in section 5. The following table 5 cites the numerical solutions

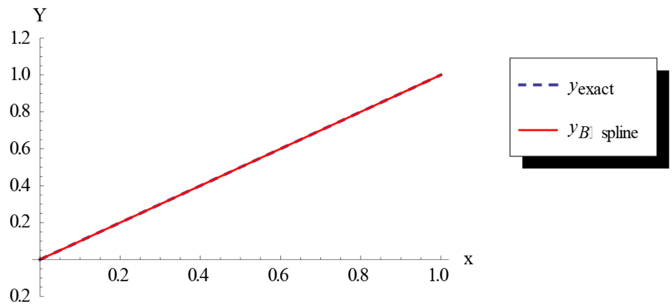

Figure 6: Comparison of numerical solution obtain by B-spline $(M = 4)$ with exact solution

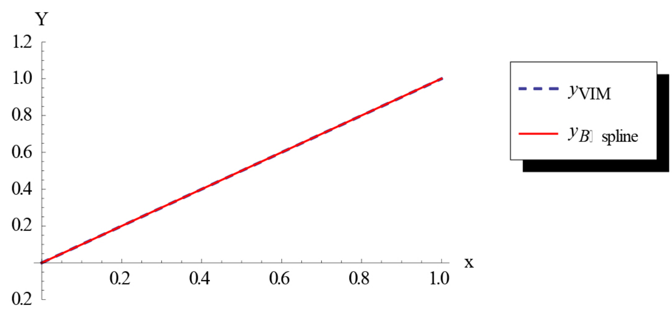

Figure 7: Comparison of numerical solution obtain by B-spline $(M = 2)$ with VIM solution

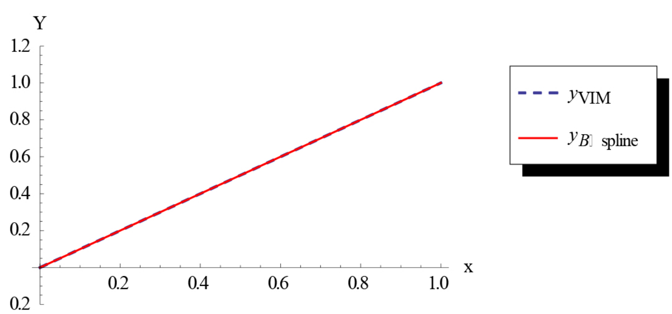

Figure 8: Comparison of numerical solution obtain by B-spline $(M = 4)$ with VIM solution

obtained by B-spline method and VIM accomplished with corresponding exact solutions and table 6 cites the absolute errors obtained by B-spline method and VIM. Figure 9-10 and Figure 11-12 present the comparison graphically between the numerical solutions obtained by B-spline wavelet method with exact solutions and VIM solutions respectively.

Table 5: Comparison of numerical solutions obtained by B-spline method and VIM with exact solution

x	*Linear B-spline wavelet method*		*Variational Iteration Method (VIM)*	*Exact solution*
	$M = 2$	$M = 4$		
0	1.0013	1.00008	1	1
0.1	0.994854	0.994995	0.994897	0.995004
0.2	0.979103	0.980006	0.979853	0.980067
0.3	0.954196	0.955265	0.955016	0.955336
0.4	0.920326	0.921014	0.920633	0.921061
0.5	0.877732	0.877591	0.877048	0.877583
0.6	0.824168	0.825263	0.824694	0.825336
0.7	0.763003	0.764727	0.764093	0.764842
0.8	0.694726	0.696581	0.695851	0.696707
0.9	0.619877	0.621499	0.620647	0.62161
1.0	0.53905	0.540222	0.539233	0.540302

Example 4.

Consider the equation

$$u(x) = -\sin(x) - x^3\left(-\tfrac{367}{4096}\cos(4)\sin(4) + \tfrac{11357}{98304} - \tfrac{2095}{32768}\cos^2(4)\right) + \int_0^1 x^3 t^5 u^2(t)dt,$$

$$0 \le x \le 1,$$

with the exact solution $u(x) = \sin(-4x)$. The approximate solution is obtained by the method of B-spline wavelets explained in section 4 for $M = 2$ and $M = 4$ and also by VIM explained in section 5. The following table 7 cites the numerical solutions obtained by B-spline method and VIM accomplished with corresponding exact solutions and table 8 cites the absolute errors obtained by B-spline method and VIM. Figure 13-14 and Figure 15-16 present the comparison graphically between the numerical solutions obtained by B-spline wavelet method with exact solutions and VIM solutions respectively.

Table 6: Absolute errors obtained by B-spline method and VIM

x \ Absolute error	*Linear B-Spline wavelet method* $M=2$	*Linear B-Spline wavelet method* $M=4$	*Variational Iteration Method (VIM)*
0	0.0013027	0.0000813	0
0.1	0.0001501	9.23203e-6	0.0001069
0.2	0.0009632	0.0000600	0.0002139
0.3	0.0011401	0.0000718	0.0003209
0.4	0.0007350	0.0000472	0.0004279
0.5	0.0001495	8.82354e-6	0.0005348
0.6	0.0011679	0.0000721	0.0006418
0.7	0.0018390	0.0001148	0.0007488
0.8	0.0019810	0.0001252	0.0008558
0.9	0.0017320	0.0001110	0.0009628
1.0	0.0012520	0.0000807	0.0010697

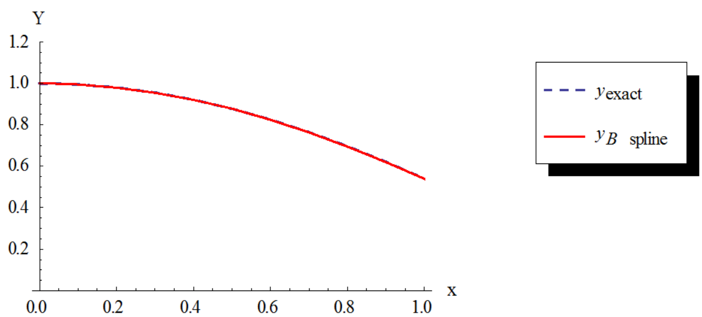

Figure 9: Comparison of numerical solution obtain by B-spline ($M=2$) with exact solution

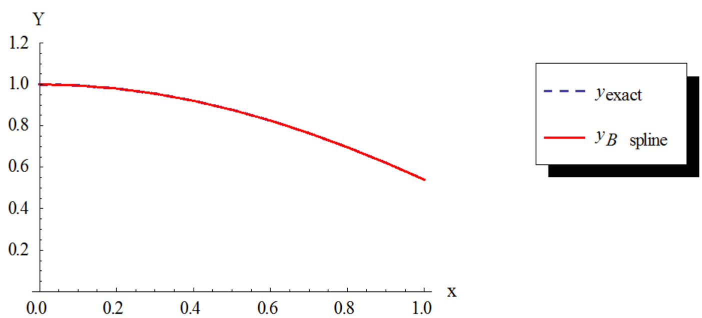

Figure 10: Comparison of numerical solution obtain by B-spline ($M=4$) with exact solution

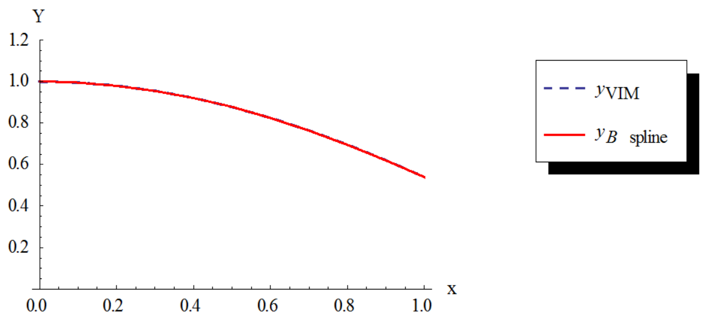

Figure 11: Comparison of numerical solution obtain by B-spline ($M = 2$) with VIM solution

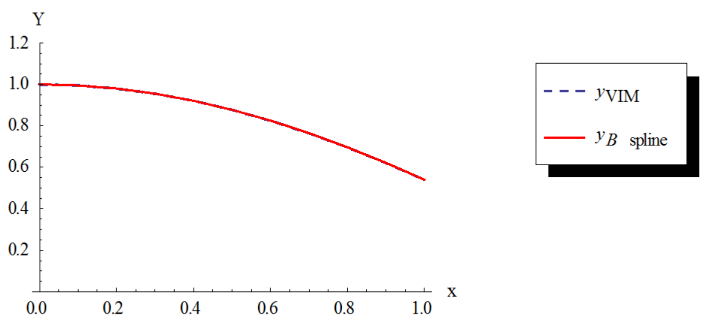

Figure 12: Comparison of numerical solution obtain by B-spline ($M = 2$) with VIM solution

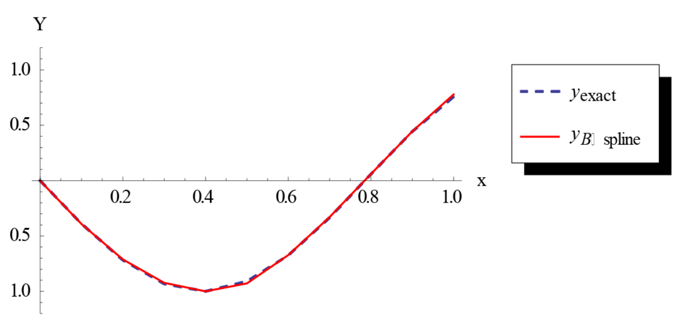

Figure 13: Comparison of numerical solution obtain by B-spline ($M = 2$) with exact solution

Table 7: Comparison of numerical solutions obtained by B-spline method and VIM with exact solution

x	*Linear B-spline wavelet method*		*Variational Iteration Method (VIM)*	*Exact solution*
	$M = 2$	$M = 4$		
0	-0.00247969	-0.0000376591	0	0
0.1	-0.391555	-0.389409	-0.389418	-0.389418
0.2	-0.711074	-0.716931	-0.717356	-0.717356
0.3	-0.922784	-0.9315	-0.93204	-0.932039
0.4	-1.00002	-0.999602	-0.999576	-0.999574
0.5	-0.927696	-0.910438	-0.909302	-0.909297
0.6	-0.673206	-0.675445	-0.675471	-0.675463
0.7	-0.328869	-0.334689	-0.335	-0.334988
0.8	0.0599757	0.0585377	0.058356	0.0583741
0.9	0.4454819	0.44283	0.442495	0.44252
1.0	0.77698	0.758118	0.756767	0.756809

Table 8: Absolute errors obtained by B-spline method and VIM

Absolute error / x	*Linear B-Spline wavelet method*		*Variational Iteration Method (VIM)*
	$M = 2$	$M = 4$	
0	0.0024796	0.0000376	0
0.1	0.0021369	9.45907e-6	3.53614e-8
0.2	0.0062817	0.0004255	2.82891e-7
0.3	0.0092549	0.0005389	9.54757e-7
0.4	0.0004443	0.0000283	2.26313e-6
0.5	0.0183982	0.0011404	4.42017e-6
0.6	0.0022571	0.0000183	7.63806e-6
0.7	0.0061188	0.0002991	0.0000121
0.8	0.0016015	0.0001635	0.0000181
0.9	0.0028986	0.0003096	0.0000257
1.0	0.0201778	0.0013154	0.0000353

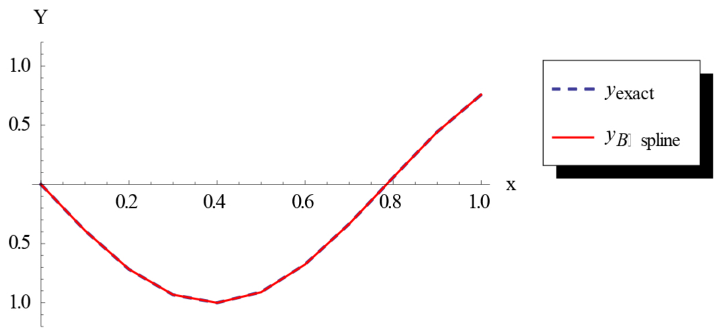

Figure 14: Comparison of numerical solution obtain by B-spline $(M = 4)$ with exact solution

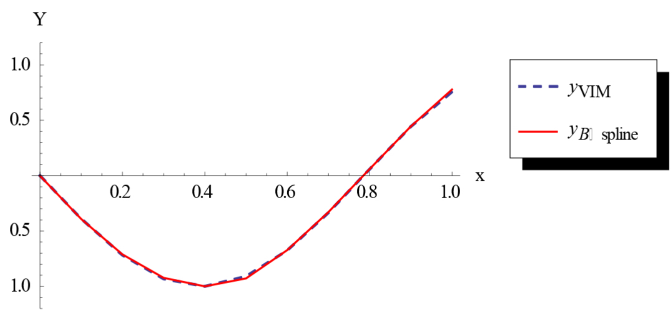

Figure 15: Comparison of numerical solution obtain by B-spline $(M = 2)$ with VIM solution

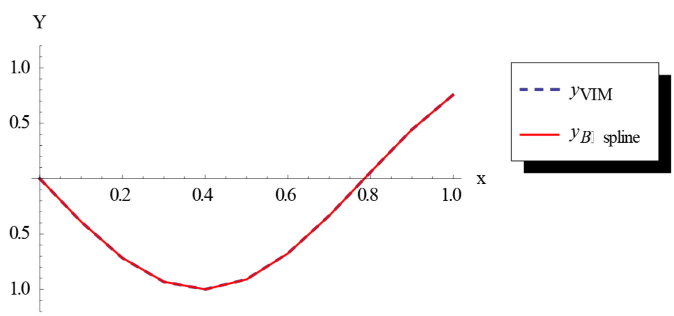

Figure 16: Comparison of numerical solution obtain by B-spline $(M = 4)$ with VIM solution

7 Conclusion

In this present work, semi-orthogonal compactly supported linear B-spline wavelets have been applied to find the numerical solution of nonlinear Fredholm integral equation of second kind. Using this procedure, the integral equation has been reduced to solve a system of nonlinear algebraic equations. The aim of this work was to derive the approximate solution of nonlinear Fredholm integral equation. We have achieved this goal by applying Variational Iteration Method and the approximate solutions are then compared with B-spline wavelet method solution. The obtained results are found to be in good agreement with the B-spline wavelet solution. The absolute errors can be reduced if we take B-splines of higher order. Comparisons with B-spline wavelet method reveal that the VIM is very effective and convenient. In addition, no linearization, discretization or perturbation is required by VIM.

References

Abbasbandy, S.; Shirzadi, A. (2008): The variational iteration method for a class of eight-order boundary-value differential equations. *Zeitschrift fuer Naturforschung 63a*, pp. 745-751.

Biazar, J.; Ghazvini, H. (2007): He's Variational Iteration Method for solving hyperbolic differential equations. *Int. J. Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 3, pp. 311-314.

Biazar, J.; Ghazvini, H. (2008): An analytical approximation to the solution of a wave equation by a variational iteration method. *Applied Mathematics Letters*, vol. 21, no. 8, pp. 780-785.

Chen, Z.; Xu, Y. (1998): The Petrov-Galerkin and Iterative Petrov-Galerkin method for second kind integral equations. *SIAM J. Numer. Anal.*, vol. 35, no. 11, pp. 406-434.

Chui, C. K. (1992): *An introduction to Wavelets, Wavelet Analysis and Its Applications, vol. 1.* Academic press, Massachusetts.

Chui, C. K. (1997): Wavelets, A mathematical tool for signal analysis; in SIAM monographs on mathematical modeling and computation. SIAM, Pennsylvania.

Dehghan, M.; Shakeri, F. (2008): Application of He's Variational iteration method for solving the Cauchy reaction diffusion problem. *Journal of Computational and Applied Mathematics*, vol. 214, no. 2, pp. 435-446.

Fu, S. Z.; Wang, Z.; Duan, J. S. (2013): Solution of Quadratic Integral Equations by the Adomian Decomposition Method. *Comput. Model. Eng. Sci. (CMES)*, vol. 92, no. 4, pp. 369-385.

Goswami, J. C.; Chan, A. K.; Chui, C. K. (1995): On solving first-kind integral equtions using wavelets on a bounded interval. *IEEE Trans. Antennas propagate.*, vol. 43, pp. 614-622.

He, J. H. (1999): Variational iteration method- a kind of nonlinear analytical technique: some examples. *Int. J. Non-Linear mechanics*, vol. 34, pp. 699-708.

He, J. H. (2006): Some asymptotic methods for strongly nonlinear equations. *Int. J. Modern Physics B*, vol. 20, no. 10, pp. 1141-1199.

He, J. H. (2007): Variational Iteration Method- Some recent results and new interpretations. *Journal of Computational and Applied Mathematics*, vol. 207, pp. 3-17.

Kaneko, H.; Noren, R. D.; Novaprateep, B. (2003): Wavelet applications to the Petrov-Galerkin method for Hammerstein equations. *Appl. Numer. Math.*, vol. 45, pp. 255-273.

Kelmanson, M. A.; Tenwick, M. C. (2010): Error Reduction in Gauss-Jacobi-Nyström Quadrature for Fredholm Integral Equations of the Second Kind. *Comput. Model. Eng. Sci. (CMES)*, vol. 55, no. 2, pp. 191-210.

Lardy, L. J. (1981): A Variation of Nystroms method for Hammerstein equations *J. Integral equation*, vol. 3, pp. 43-60.

Liu, C. S.; Atluri, S. N. (2009): A fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentiation of noisy data, and its relation to the filter theory. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 41, no. 3, pp. 243-261.

Maleknejad, K.; Nedaiasl, K. (2011): Application of sinc-collocation method for solving a class of nonlinear Fredholm integral equations. *Comput. Math. Appl.*, vol. 62, pp. 3292-3303.

Maleknejad, K.; Nosrati, M.; Najafi, E. (2012): Wavelet Galerkin Method for Solving Singular Integral Equations. *Comput. And Appl. Math.*, vol. 31, no. 2, pp. 373-390.

Maleknejad, K.; Sahlan, M. N. (2010): The Method of Moments for Solution of Second Kind Fredholm Integral Equations Based on B-Spline Wavelets. *Int. J. Comp. Math.*, vol. 87, no. 7, pp. 1602-1616.

Odibat, Z. M.; Momani, S. (2006): Application of variational iteration method to nonlinear differential equations of fractional order. *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, pp. 27-34.

Soltani, L. A.; Shirzadi, A. (2010): A new modification of the variational iteration method. *Computers and Mathematics with Applications*, vol. 59, pp. 2528-2535.

Wazwaz, A. M. (2008): A study on linear and nonlinear Schrodinger equations by the variational iteration method. *Chaos, Solitons and Fractals*, vol. 37, pp. 1136-1142.

Wazwaz, A. M. (2011): Linear and nonlinear integral equations: methods and applications Higher education, springer.

Zou, H.; Li, H. (2010): A Novel Meshless Method for Solving the Second Kind of Fredholm Integral Equations. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 67, no. 1, pp. 55-77.