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Wavelet solution of a class of two-dimensional nonlinear boundary value problems

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Abstract: By combining techniques of boundary extension and Coiflet-type wavelet expansion, an approximation scheme for a function defined on a two-dimensional bounded space is proposed. In this wavelet approximation, each expansion coefficient can be directly obtained by a single-point sampling of the function. And the boundary values and derivatives of the bounded function can be embedded in the modified wavelet basis. Based on this approximation scheme, a modified wavelet Galerkin method is developed for solving two-dimensional nonlinear boundary value problems, in which the interpolating property makes the solution of such strong nonlinear problems very effective and accurate. As an example, we have applied the proposed method to the solution of two-dimensional Bratu-like equations. Results demonstrate an excellent numerical accuracy, a 2.5-order convergence rate for the present wavelet method, and a very good capability in dealing with the nonlinear boundary value problems with multiple solution branches. Interestingly, unlike most existing methods, numerical errors of the present solutions are not sensitive to the nonlinear intensity of the two-dimensional equations.

Keywords: modified wavelet Galerkin method, two-dimensional, strong nonlinearity, boundary value problems, Bratu-like equation

1 Introduction

Most applications in science and engineering will inevitably subject to solutions of nonlinear differential equations, which is still a very difficult task either by theoretical or numerical means [He (2006); Bebernes and Eberly (1989); Abbott (1978); Frank-Kamenetskii (1969); Li and Liao (2005); Wazwaz (2005); Zhou, Wang and Zheng (1998); Liu, Zhou, Wang and Wang (2013); Tsai, Liu and Yeih (2010); Yi

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and Chen (2012)]. Here, we consider a general class of two-dimensional nonlinear differential equations, which can include for instance the large-deformation bending equation of plates and two-dimensional steady-state Navier-Stokes equation of fluids etc., as follows:

$$\mathbf{L}^0 u(x,y) + \mathbf{L}^1 \mathbf{N}[u(x,y)] = 0, \quad 0 < x,y < 1,$$
$$\partial^{i_s^{0,1}} u/\partial x^{i_s^{0,1}}|_{x=0,1} = 0, \quad \partial^{j_s^{0,1}} u/\partial y^{j_s^{0,1}}|_{y=0,1} = 0 \tag{1}$$

in which $\mathbf{L}^0$ and $\mathbf{L}^1$ denote differential operators, $\mathbf{N}$ is a nonlinear functional of the unknown function $u(x,y)$, $i_s^{0,1}$, $j_s^{0,1}$ and s are non-negative integers.

In our recent work [Liu, Zhou, Wang and Wang (2013)], we have developed a modified wavelet Galerkin method for the solution of strong nonlinear boundary value problems in one-dimensional form (1), which overcomes some drawbacks of existing conventional wavelet Galerkin methods [Liu, Qin, Liu and Cen (2010); Liang, Guo and Gong (2009); Restrepo and Leaf (1995); Schneider and Vasilyev (2010)]. And by using the one-dimensional Bratu equation as an example, we have demonstrated that the wavelet algorithm [Liu, Zhou, Wang and Wang (2013)] has a convergence rate of order 4, and shows a much better accuracy than many other numerical methods, such as the Non-polynomial spline method [Jalilian (2010)], the B-spline method [Caglar, Caglar, Özer, Valaristos and Anagnostopoulos (2010)], the Lie-group shooting method [Abbasbandy, Hashemi and Liu (2011)], the differential transformation method [Hassan and Ertürk (2007)], the Laplace transform decomposition method [Khuri (2004)] and the decomposition method [Deeba, Khuri and Xie (2000)]. Most importantly, the computational accuracy of the wavelet method [Liu, Zhou, Wang and Wang (2013)] is almost independent of the nonlinear intensity of the equation, in contrary to most other methods [Jalilian (2010); Caglar, Caglar, Özer, Valaristos and Anagnostopoulos (2010); Abbasbandy, Hashemi and Liu (2011); Hassan and Ertürk (2007); Khuri (2004); Deeba, Khuri and Xie (2000)], whose numerical accuracy usually decays very fast along with the nonlinear intensity. In addition, such a wavelet method [Liu, Zhou, Wang and Wang (2013)] also has been demonstrated the good capability in dealing with the nonlinear boundary value problems with multiple solution branches.

In the present study, as an extension of such a work [Liu, Zhou, Wang and Wang (2013)] on the one-dimensional nonlinear differential equations, we propose a new wavelet approximation scheme for the two-dimensionally bounded functions based on techniques of boundary extension and Coiflet-type wavelet expansion. Similar to the method for one-dimensional problems, boundary extension treatment in the present two-dimensional scheme can also eliminate the undesired oscillating error near boundary points due to function value jump [Liu, Zhou, Wang and Wang

(2013); Zhou, Wang, Wang and Liu (2011); Zhou, Wang and Zheng (1999)]. In addition, the two-dimensional wavelet approximation also has the interpolating property, which can efficiently avoid numerical calculation of the integral of multifold-products of scaling functions and their derivatives. Most importantly, the present two-dimensional approximation scheme can explicitly express each nonlinear term of unknown functions in the equation as a linear summation of single-point samplings of unknown function composite, which guarantees successful employment of the Galerkin-type methods [Zhou and Wang (1999)].

In the following, we introduce in detail the proposed method and demonstrate how one can use such a method to solve the two-dimensional nonlinear differential equations (1). Especially, we illustrate the efficiency and accuracy of the proposed wavelet algorithm by numerically solving the following two-dimensional Bratu-like equation [Boyd (1986); Chang and Chien (2003); Odejide and Aregbesola (2006); Mohsen, Sedeek and Mohamed (2008)]

$$\partial^2 u(x,y)/\partial x^2 + \partial^2 u(x,y)/\partial y^2 + \lambda e^{u(x,y)} = f(x,y), \quad 0 < x,y < 1, \quad u(x,y)|_{x,y=0,1} = 0 \tag{2}$$

where parameter $\lambda > 0$. When $f(x,\ y)$=0, Eq. (2) reduces to the classical two-dimensional Bratu equation [Boyd (1986); Chang and Chien (2003); Odejide and Aregbesola (2006); Mohsen, Sedeek and Mohamed (2008)], which represents a class of nonlinear eigenvalue problems, will be investigated in this study.

2 Wavelet approximation of an interval-bounded L^2-function

Following the theory of multiresolution analysis, a set of scaling bases for two-dimensional space can be directly extended by the tensor products of one-dimensional wavelet bases [Meyer (1992)]. Based on our previous work [Liu, Zhou, Wang and Wang (2013)], for a function $f(x,y) \in L^2[0,1]^2$, we have

$$f(x,y) \approx P^j f(x,y) = \sum_{k=0}^{2^j}\sum_{l=0}^{2^j} f(k/2^j, l/2^j)\varphi_{j,k}^x(x)\varphi_{j,l}^y(y), \quad x,y \in [0,\ 1] \tag{3}$$

in which

$$\varphi_{j,k}^{x(y)}(x) = \begin{cases} \sum\limits_{i=-9}^{-1} T_{0,k}^{x(y)}(\frac{i}{2^j})\phi(2^j x - i + 7) + \phi(2^j x - k + 7) & k \in [0,\ 3] \\ \phi(2^j x - k + 7) & k \in [4,\ 2^j - 4] \\ \sum\limits_{i=2^j+1}^{2^j+6} T_{1,2^j-k}^{x(y)}(\frac{i}{2^j})\phi(2^j x - i + 7) + \phi(2^j x - k + 7) & k \in [2^j - 3,\ 2^j] \end{cases} . \tag{4}$$

In Eq. (4), $\phi(x)$ is the generalized Coiflet-type orthogonal scaling function developed by Wang [Wang (2001)], and

$$T_{0,k}^{x(y)}(x) = \sum_{i=0}^{3} \frac{p_{0,i,k}^{x(y)}}{i!} x^i, \quad T_{1,k}^{x(y)} = \sum_{i=0}^{3} \frac{p_{1,i,k}^{x(y)}}{i!} (x-1)^i \tag{5}$$

where $\mathbf{P_0} = \{2^{-ij} p_{0,i,k}^{x(y)}\}$, $\mathbf{P}_1 = \{2^{-ij} p_{1,i,k}^{x(y)}\}$, $i,k = 0,1,2,3$ are defined by [Liu, Zhou, Wang and Wang (2013); Wang (2001)]

$$\mathbf{P_0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11/6 & -3 & 3/2 & -1/3 \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{bmatrix}. \tag{6}$$

The accuracy of the wavelet approximation (3) of the function, $f(x,y) \in L^2[0,1]^2$, depends on the number of vanishing moment N=6 of the wavelet function associated with the scaling function $\phi(x)$ in Eq. (4), and the decomposition level j, as has been determined by [Liu, Zhou, Wang and Wang (2013); Wang (2001); Resnikoff and Wells (1998)]

$$||f(x) - P^j f(x)||_{L^2[0,\ 1]^2} \le C_1 2^{-jN} \tag{7}$$

For the approximation of various derivatives of the function $f(x,\ y)$, we similarly have

$$||\frac{\partial^{n+m} f(x,y)}{\partial x^n \partial y^m} - \frac{\partial^{n+m} P^j f(x,y)}{\partial x^n \partial y^m}||_{L^2[0,\ 1]^2} \le C_2 2^{-j(N-l)} \tag{8}$$

in which constants C_1 and C_2 depend on the smoothness and boundary extension property of $f(x,\ y)$, n and m are non-negative integers satisfying $n,\ m < N$, and l=max$\{n,\ m\}$.

We note that, the wavelet approximation (3) can easily satisfy homogeneous boundary conditions by setting coefficients of corresponding rows of the matrices in Eq. (6) as zeros [Liu, Zhou, Wang and Wang (2013); Wang (2001)]. For example, $\partial^n u/\partial x^n|_{x=0} = 0$ can be realized by setting $p_{0,n,k}^x = 0$ $(k = 0,1,2,3)$ and keeping all other elements $p_{0,m,k}^x$, $m \neq n$ unchanged.

Similar to the one dimensional case [Liu, Zhou, Wang and Wang (2013)], the single-point reconstruction formula of function $f(x,\ y)$ in Eq. (3) has a very interesting characteristic: for any nonlinear operator $\mathbf{N}$ satisfying $\mathbf{N}[f(x,y)] \in L^2[0,1]^2$, by treating $\mathbf{N}[f(x,y)]$ as a new function and applying Eq. (3), we have

$$\mathbf{N}[f(x,y)] \approx \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} \mathbf{N}[f(k/2^j, l/2^j)] \varphi_{j,k}^x(x) \varphi_{j,l}^y(y), \quad x,y \in [0,\ 1]. \tag{9}$$

We should note that the way of approximation like Eq. (9) is not valid for Fourier or Fourier-like bases, making such form of series expansion very useful when using Galerkin type method to solve nonlinear differential equations.

3 Solutions of two-dimensional nonlinear boundary value problems

In the following, we propose a modified Galerkin method based on Eq. (3) to solve the two-dimensional nonlinear boundary value problems as shown in Eq. (1).

First, the coefficients $p^{x(y)}_{0,i,k}$ and $p^{x(y)}_{1,i,k}$ in Eq. (5) are given by Eq. (6), except that$p^{x}_{0,i^0_s,k}=0$, $p^{x}_{1,i^1_s,k}=0$, $p^{y}_{0,j^0_s,k}=0$ and $p^{y}_{1,j^1_s,k}=0$, k=0, 1, 2, 3, are assignedto meet the boundary conditions in Eq. (1). Then the modified scaling basis $\varphi^{x(y)}_{j,k}$ in Eq. (4) will be specified accordingly, which is re-denoted as $h^{x(y)}_{j,k}$. Thus the unknown function $u(x,y)$ can be approximated by Eq. (3) as

$$u(x,y)\approx\sum_{k=0}^{2^j}\sum_{l=0}^{2^j}u(k/2^j,l/2^j)h^x_{j,k}(x)h^y_{j,l}(y). \tag{10}$$

Similar to Eq. (9), the nonlinear term $\mathbf{N}[u(x,y)]$ can be expressed as

$$\mathbf{N}[u(x,y)]\approx\sum_{k=0}^{2^j}\sum_{l=0}^{2^j}\mathbf{N}[u(k/2^j,l/2^j)]\varphi^x_{j,k}(x)\varphi^y_{j,l}(y) \tag{11}$$

in which the modified scaling basis $\varphi^{x(y)}_{j,k}(x)$ is denoted in Eq. (4) with coefficients $p^{x(y)}_{0,i,k}$ and $p^{x(y)}_{1,i,k}$ specified in Eq. (6). Substituting Eqs. (10) and (11) into Eq. (1), gives

$$\sum_{k=0}^{2^j}\sum_{l=0}^{2^j}\{u(k/2^j,l/2^j)\mathbf{L}^0[h^x_{j,k}(x)h^y_{j,l}(y)]+\mathbf{N}[u(k/2^j,l/2^j)]\mathbf{L}^1[\varphi^x_{j,k}(x)\varphi^y_{j,l}(y)]\}\approx 0. \tag{12}$$

Multiplying both sides of Eq. (12) by $h^x_{j,n}(x)h^y_{j,m}(y)$, $n,m=0,\ 1,\ 2,\ ...,\ 2^j$, respectively and perform integration over the interval [0, 1], yields

$$\mathbf{AU}+\mathbf{BD}\approx\mathbf{0} \tag{13}$$

where

matrix $\mathbf{A}=\{a_{op}=\int_0^1\int_0^1\mathbf{L}^0[h^x_{j,k}(x)h^y_{j,l}(y)]h^x_{j,n}(x)h^y_{j,m}(y)dxdy\}$,

matrix $\mathbf{B}=\{b_{op}=\int_0^1\int_0^1\mathbf{L}^1[\varphi^x_{j,k}(x)\varphi^y_{j,l}(y)]h^x_{j,n}(x)h^y_{j,m}(y)dxdy$, vectors $\mathbf{U} = \{u_p = u(k/2^j, l/2^j)\}$ and $\mathbf{D} = \{d_p = \mathbf{N}[u(k/2^j, l/2^j)]\}$, in which, the subscripts $o = (2^j+1)n+m$, $p = (2^j+1)k+l$ and k, $l, n, m =$ 0, 1, 2, ..., 2^j. We note that the generalized connection coefficients a_{op} and b_{op} can be obtained exactly by using the procedure suggested by Wang [Wang (2001)], and the expression of the modified scaling basis have been given by Eq. (4).

By using the iterative method, such as the classical Newton iteration method, the dynamic Newton-like method [Ku and Yeih] and the residual-norm based algorithms [Liu and Atluri], to solve the system of nonlinear algebraic equations (13), we can obtain the nodal values of unknown function $u(k/2^j, l/2^j), k, l = 0, 1, 2, ..., 2^j$, which can be used to reconstruct $u(x,y)$ in terms of Eq. (10).

4 Numerical examples

In this section, we will demonstrate the efficiency and accuracy of the proposed method by numerically solving the two-dimensional Bratu-like equations (2).

In order to fulfill the boundary conditions in Eq. (2), we set $p_{0,0,k}^{x(y)} = 0$, $p_{1,0,k}^{x(y)} = 0$ ($k = 0, 1, 2, 3$), and keep all other coefficients $p_{0,i,k}^{x(y)}$, $p_{1,i,k}^{x(y)}$, ($i = 1, 2, 3$) in Eq. (6) unchanged. Thus the approximation expression, Eq. (10), to the unknown function $u(x,y)$ and function $f(x, y)$ can be respectively rewritten as

$$u(x,y) \approx \sum_{k=1}^{2^j-1} \sum_{l=1}^{2^j-1} u(k/2^j, l/2^j) h_{j,k}^x(x) h_{j,l}^y(y), \tag{14}$$

$$f(x,y) \approx \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} f(k/2^j, l/2^j) \varphi_{j,k}^x(x) \varphi_{j,l}^y(y). \tag{15}$$

Considering $\mathbf{N}[u(x,y)] = \lambda e^{u(x,y)}$, similar to the derivations of Eqs (11), (12) and (13), we have

$$e^{u(x,y)} \approx \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} e^{u(k/2^j, l/2^j)} \varphi_{j,k}^x(x) \varphi_{j,l}^y(y), \tag{16}$$

$$\begin{aligned} &\sum_{k=1}^{2^j-1} \sum_{l=1}^{2^j-1} u(k/2^j, l/2^j)\left[\frac{d^2 h_{j,k}^x(x)}{dx^2} h_{j,l}^y(y) + h_{j,k}^x(x) \frac{d^2 h_{j,l}^y(y)}{dy^2}\right] \\ &+ \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} \left[\lambda e^{u(k/2^j, l/2^j)} - f\left(\frac{k}{2^j}, \frac{l}{2^j}\right)\right] \varphi_{j,k}^x(x) \varphi_{j,l}^y(y) \approx 0 \end{aligned} \tag{17}$$

and

$$\mathbf{AU} + \lambda \mathbf{BD} + \lambda \mathbf{C} \approx \mathbf{Q}. \tag{18}$$

where

matrices $\mathbf{A} = \{a_{op} = \bar{a}_{nk}\tilde{a}_{ml} + \tilde{a}_{nk}\bar{a}_{ml}\}$, $\mathbf{B} = \{b_{op} = \tilde{b}_{nk}\tilde{b}_{ml}\}$,

vectors $\mathbf{U} = \{u_p = u(k/2^j, l/2^j\}$,

$$\mathbf{C} = \{c_o = (\tilde{b}_{n0} + \tilde{b}_{n2^j})\sum_{l=0}^{2^j}\tilde{b}_{ml} + (\tilde{b}_{m0} + \tilde{b}_{m2^j})\sum_{k=1}^{2^j-1}\tilde{b}_{nk}\},$$

vectors $\mathbf{D} = \{d_p = e^{u(k/2^j, l/2^j)}\}$, and $\mathbf{Q} = \{q_o = \sum_{k=0}^{2^j}\sum_{l=0}^{2^j} f(k/2^j, l/2^j)\tilde{b}_{nk}\tilde{b}_{ml}\}$.

Here $\bar{a}_{lk} = \int_0^1 d^2h^x_{j,k}(x)/dx^2 h^x_{j,l}(x)dx$, $\tilde{a}_{lk} = \int_0^1 h^x_{j,k}(x)h^x_{j,l}(x)dx$, $\tilde{b}_{lk} = \int_0^1 \varphi^x_{j,k}(x)h^x_{j,l}(x)dx$, and the subscripts $o = (2^j - 1)(n-1) + m$, $p = (2^j - 1)(k-1) + l$ and k, l, n, m=1, 2, …, 2^j-1.

To obtain the nodal values of unknown function $u(k/2^j, l/2^j)$, $k, l = 1, 2, ..., 2^j - 1$, the classical Newton iteration method is employed to numerically solve the nonlinear algebraic equations (18), where initial values $u(k/2^j, l/2^j) = 0$, $k, l = 1, 2, ..., 2^j - 1$ are adopted.

As a numerical test, we assign $f(x,y) = \lambda e^{\sin(\pi x)\sin(\pi y)} - 2\pi^2\sin(\pi x)\sin(\pi y)$ to Eq. (2), then an exact solution of Eq. (2) can be easily obtained, which is $u(x,y) = \sin(\pi x)\sin(\pi y)$. Figs. 1 and 2 show the errors of numerical solutions under resolution level j =4, 5 and 6 corresponding to the number of grid points n=(2^j-1)×(2^j-1)=15×15, 31×31 and 63×63, for λ =1, 4 and 7, respectively. It can be seen from these figures that under the same resolution level j and parameter λ, the numerical accuracy of this modified wavelet Galerkin method for solving two-dimensional nonlinear boundary value problems is almost the same as that in one-dimensional case [Liu, Zhou, Wang and Wang (2013)]. And from Fig. 2, we find out that the absolute values of the absolute errors decrease by $n^{-2.5}C_0$ as the number of grid points n increases where C_0 is a positive constant, implying that the convergence rate of the present modified Galerkin method is in order 2.5. Most interestingly, the absolute values of these relative errors in Figs. 1 and 2 are almost unchanged for different values of λ, which may have implied a very weak dependence of the errors of the present numerical solutions on the nonlinear strength of the equation. This fact can be further illustrated by Fig. 3, where exact and numerically obtained values of $u(1/2,\ 1/2)$ as a function of parameter λ have been compared. It can be seen from Fig. 3 that, when λ increases from 0.1 to 7, the relative errors increases no more than one order of magnitude for each number of grid points n, demonstrating the increase of errors along with λ is much slower than that of all other methods [Jalilian (2010); Caglar, Caglar, Özer, Valaristos and Anagnostopoulos (2010); Abbasbandy, Hashemi and Liu (2011); Hassan and Ertürk (2007); Khuri (2004); Deeba, Khuri and Xie (2000)]. As a special example when setting $f(x,\ y)$=0 in Eq. (2), we use the present method under resolution level j=4 to solve the resulting classical two-

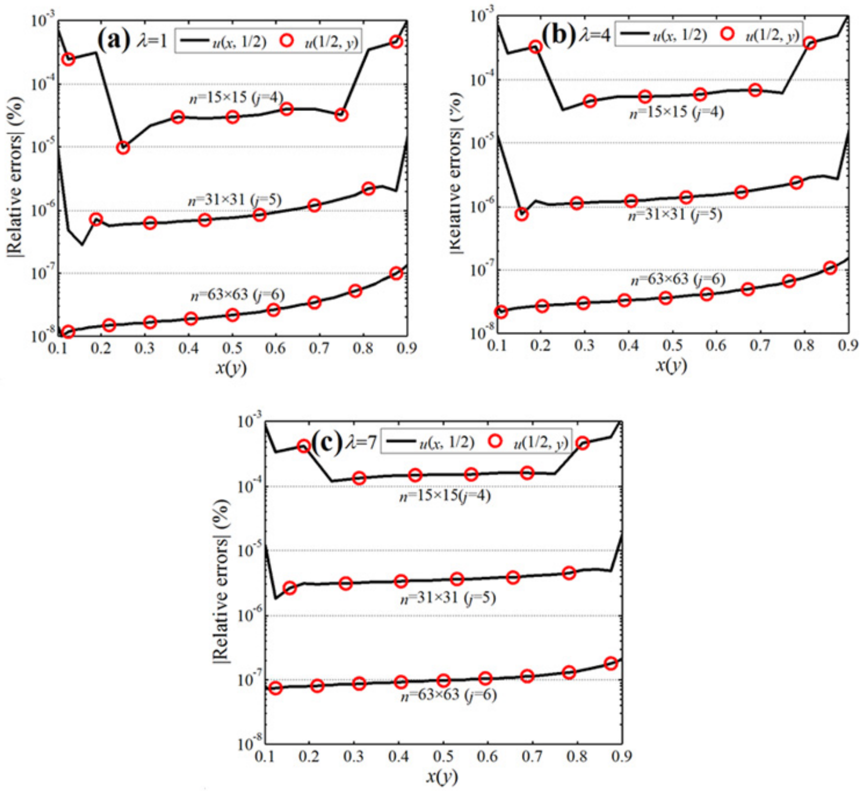

Figure 1: Distribution along x and y axes of the relative errors of numerical solutions of Eq. (2) under various number of grid points n (corresponding to resolution level j, through relation n=(2^j-1)×(2^j-1)) for parameter λ: (a) λ=1, (b) λ=4, and (c) λ=7.

dimensional Bratu equation which has zero, one or two solutions associated with $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$, respectively [Boyd (1986); Chang and Chien (2003); Odejide and Aregbesola (2006); Mohsen, Sedeek and Mohamed (2008)]. We find out that the critical value λ_c lies between 6.80 and 6.81, which is in good agreement with a convincing value $\lambda_c = 6.808124423$ [Chang and Chien (2003); Mohsen, Sedeek and Mohamed (2008); Doedel and Sharifi (2000); Fedoseyev, Friedman and Kansa (2000)], and is much better than the prediction on λ_c is located between 7.12222 and 6.939571823 by respectively using the finite difference method and classical weighted residual method [Odejide and Aregbesola (2006)]. And Table 1 shows the two solution branches of u(1/2, 1/2), which are close to those obtained by the weighted residual method [Odejide and Aregbesola (2006)]. Following the previous discussion on errors as shown in Fig. 3, the absolute values of relative errors of the present numerical solutions $u(1/2,\ 1/2)$ may be less than 10^{-6}. In addition, it indicates that the present modified wavelet Galerkin method is capable of finding multiple solutions for two-dimensional boundary value problems with strong nonlinearity.

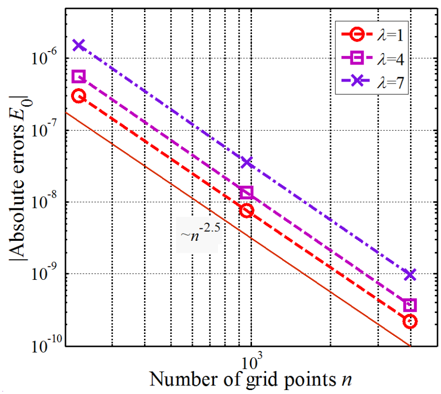

Figure 2: Absolute errors $E_0(1/2,\ 1/2)$ of numerical solutions of Eq. (2) under λ=1, 4 and 7, as a function of the number of grid points n, corresponding to resolution level j, through relation n=$(2^j$-1$)\times(2^j$-1$)$.

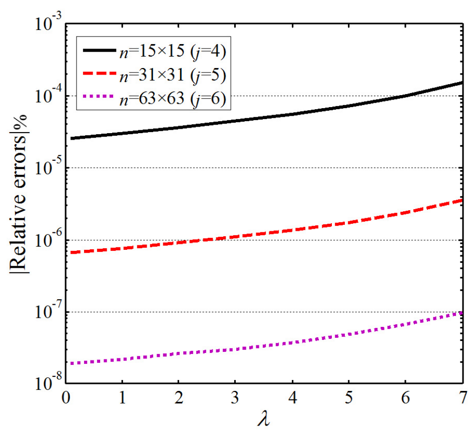

Figure 3: Absolute values of the relative errors of numerical results of $u(1/2,\ 1/2)$ as a function of parameter λ under number of grid points n =15$\times$15, 31$\times$31 and 63$\times$63, corresponding to resolution level j=4, 5 and 6.

Table 1: Numerical solutions $u(1/2,\ 1/2)$ of the two-dimensional Bratu equation under various λ.

λ	$u_{min}(1/2,\ 1/2)$	$u_{max}(1/2,\ 1/2)$	λ	$u_{min}(1/2,\ 1/2)$	$u_{max}(1/2,\ 1/2)$
0.1	7.408093229E-03	1.138218777E+01	4.0	3.955259988E-01	3.454936050E+00
1.0	7.810101876E-02	6.548494384E+00	5.0	5.569597638E-01	2.845945007E+00
2.0	1.668957564E-01	5.072459550E+00	6.0	7.971089810E-01	2.239910842E+00
3.0	2.703663546E-01	4.156756192E+00	6.8	1.323450119E+00	1.462270493E+00

5 Conclusion

In this paper, we proposed an approximation scheme for aL^2-function defined on a two-dimensional bounded space by combining techniques of boundary extension and Coiflet-type wavelet expansion. Using such approximation scheme, we developed a modified wavelet Galerkin method for the solution of two-dimensional boundary value problems with strong nonlinearity. By using the two-dimensional Bratu-like equation as a numerical example, we find that this wavelet algorithm is capable of treating nonlinear boundary value problems with multiple solution branches. Further detailed numerical analysis shows that convergence rate of the proposed method can reach order 2.5, and, interestingly, the computational accuracy of the present method is almost independent of the nonlinear intensity of the equation. For most other existing methods, numerical accuracy usually decays fast along with the nonlinear intensity.

As our new wavelet approximation scheme on the nonlinear terms in a equation is a complete expansion in a Riesz basis of a closed linear subspace of $L^2[0,\ 1]^2$, we expect that the subsequently proposed Galerkin method has the so-called closure property. This unique property makes the proposed method be able to successfully deal with strong nonlinear problems, shedding new light on the accurate quantitative analysis in general nonlinear science.

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