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A set-based method for structural eigenvalue analysis using Kriging model and PSO algorithm

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Abstract: The set-based structural eigenvalue problem is defined, by expressing the uncertainties of the structural parameters in terms of various convex sets. A new method based on Kriging model and Particle Swarm Optimization (PSO) is proposed for solving this problem. The introduction of the Kriging model into this approach can effectively reduce the computational burden especially for large-scale structures. The solutions of the non-linear and non-monotonic problems are more accurate than those obtained by other methods in the literature with the PSO algorithm. The experimental points for Kriging model are sampled according to Latin hypercube sampling method. Two approaches of imposing the constraint of the hyper-ellipsoid are presented for global optimization. One is by adding penalty terms to the original objective function; the other one is use objective function with interval spherical coordinates by coordinate transformation. An engineering example revealed the feasibility and accuracy of the proposed method.

Keywords: structural eigenvalue, natural frequency, convex model, Kriging model, Particle Swarm Optimization

1 Introduction

The finite element method (FEM) has become one of the most popular methods for solving ordinary differential equations and partial differential equations, because its simplicity, efficiency, and established convergence, etc. It has been widely applied in various fields of engineering and sciences, such as solid/fluid mechanics, heat transfer, electromagnetics, acoustics, see [Atluri (2005)]. The traditional finite element methods are based on deterministic model, which can be called deterministic finite element methods. In recent years, it is realized that many uncertain factors exist in practical structures, such as environmental loads, material parame-

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ters, geometries and boundary conditions. The uncertainties should be considered in finite element analysis if they are expected to have great influences on structural behaviors.

As the first uncertain finite element method, the Probabilistic Finite Element Method (PFEM) has been studied systematically by many researchers. However, PFEM has significant limitations when very limited statistical information is available, i.e. the accurate Probability Density Function (PDF) of uncertainties cannot be obtained. [Elishakoff (2000)] implemented a universal survey and systematically summarized the limitations of the probabilistic methods in engineering problems. Affected by the random factors, complex problems that require large computational resources become more complex and mostly cannot be effectively solved.

The interval or convex models are new mathematical tools for uncertain problems. The convex models do not depend heavily on statistical information, and are therefore very suitable for problems with small sample-sizes. Many studies about the applications of the non-probabilistic convex models have been conducted in recent years, see [Ben-Haim and Elishakoff (1990); Qiu (2005); Li, Luo and Sun (2011); Gao, Song and Tin-Loi (2009); Qiu, Chen, Su and Elishakoff (1995)]. They brought up a new research direction called Interval Finite Element Method (IFEM) or Non-probabilistic Finite Element Method (NFEM), and many studies in this area is summarized by [Moens and Vandepitte (2005,2006); Moens and Hanss (2011)]. The "Set-based Finite Element Method" is used in this paper to represent the general cases of various convex sets, of which the IFEM is merely a special case.

The non-probabilistic analysis of structural eigenvalues have been studied in [Yang, Chen and Lian (2001); Chen, Lian and Yang (2003); Sim, Qiu and Wang (2007); Muhanna, Kreinovich and Solin (2006)], mainly based on interval models. The purpose of these studies is to seek the upper and lower bounds of the structural eigenvalues, from which the studies can be divided into two categories [Qiu (2005)]: one is the standard interval eigenvalue problem (the matrix vertex solving theorem, the positive semi-definite solution theorem and the parameter decomposition based solving theorem, etc.), the other one is the generalized interval eigenvalue problem (parameter vertex method and the method based on the perturbation of interval parameters, etc.).

In this paper, a method based on Kriging model and Particle Swarm Optimization (PSO) algorithm is presented to study the bounds of structural eigenvalues caused by various uncertaities. South Africa geologists Krige firstly proposed this model in [Krige (1951)], which had been applied mainly in Geology for determination of mineral reserves distribution, before being introduced into the field of design optimization. [Irfan (2005); Zhang and Li (2006)] have used Kriging model in struc-

tural reliability analysis, in order to construct the alternative model of structural Limit State Function (LSF). One of the prominent advantages of Kriging model is its excellent local fitting performance. In this study, Kriging model is used to construct the response function of structural natural frequency. The PSO algorithm [Shi and Eberhart (1998); Eberhart and Shi (2000)] which has excellent performance for global optimization is then used to seek the extreme values of natural frequency. An engineering example is given, which has revealed the feasibility and accuracy of the proposed methods.

The rest of paper is organized as follows: in section 2, the concepts regarding convex models are briefly reviewed in section 2; the generalized interval eigenvalue problem is defined in section 3; in section 4, the method prosed in this study with Kriging model and Particle Swarm Optimization is presented with detail; in section 5, we demonstrated the feasibility and accuracy of this method by numerical examples; in section 6, we complete this paper with some concluding remarks.

2 Convex models

In the 1960s, a branch of mathematics called Convex Analysis appeared due to the various needs of mathematical programming, optimal control, calculus of variations, numerical approximation, mathematical economics, and etc. The basic researches are about convex sets and convex functions. The former one will be introduced here.

Assume $\boldsymbol{R}$ denotes the space of real number, $\boldsymbol{V}$ is a linear space in $\boldsymbol{R}$, $\boldsymbol{R}^n$ denotes n-dimensional Euclidean space, and $\|x\|$denotes the Euclidean norm in $\boldsymbol{R}^n$.

Definition 1: Assume $\boldsymbol{C}$ belongs to $\boldsymbol{R}^n$. If the relation

$$(1-t)\boldsymbol{x}+t\boldsymbol{y} \in \boldsymbol{C} \tag{1}$$

is always true as long as $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $\boldsymbol{C}$ and t is between 0 and 1, $\boldsymbol{C}$ is a convex set model.

One of the properties of the convex set model is that, if two different points are included in it, the line segment between them must be included too.

Definition 2: Assume $\boldsymbol{x}_i$ belongs to $\boldsymbol{V}$, λ_i is greater than or equal to 0 and the sum of $\lambda_i$is equal to 1. The point

$$\boldsymbol{x} = \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \tag{2}$$

is called the convex combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$

Property 1: Assume all $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, ... , $\boldsymbol{C}_n$ are convex sets and α_i (i=1, 2, ..., n) belong to $\boldsymbol{R}$. Then,

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{C}_i = \left\{ \boldsymbol{x} : \boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i, \boldsymbol{x}_i \in \boldsymbol{C}_i \right\} \tag{3}$$

must be a convex set.

Property 2: Assume $\boldsymbol{C}_j$ ($j \in \boldsymbol{J}$) are convex sets and $\boldsymbol{J}$ is an indicator set. Then,

$$\boldsymbol{C} = \underset{j \in \boldsymbol{J}}{\cap} \boldsymbol{C}_j \tag{4}$$

must be a convex set.

Property 3: Assume $\boldsymbol{W}$ is a linear space and $\boldsymbol{T}$ is a linear mapping from $\boldsymbol{V}$ to $\boldsymbol{W}$.

1. If $\boldsymbol{A}$ is a convex set in $\boldsymbol{V}$, $\boldsymbol{TA}$ must be a convex set in $\boldsymbol{W}$
2. If $\boldsymbol{B}$ is a convex set in $\boldsymbol{W}$, $\boldsymbol{T}^{-1}\boldsymbol{B}$ must be a convex set in $\boldsymbol{V}$, where $\boldsymbol{T}^{-1}$ is the inverse mapping of $\boldsymbol{T}$.

Definition 3: Assume $\boldsymbol{A}$ belongs to $\boldsymbol{V}$. The intersection of all the convex sets that belong to $\boldsymbol{V}$ and contain $\boldsymbol{A}$ is called the convex hull of $\boldsymbol{A}$ and denoted by co($\boldsymbol{A}$).

Theorem 1: If $\boldsymbol{A}$ belongs to $\boldsymbol{V}$, the convex hull of $\boldsymbol{A}$ is composed of all the finite convex combinations of the elements in $\boldsymbol{A}$

Definition 4: If $\boldsymbol{A}$ is a finite set of points $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m\}$in $\boldsymbol{V}$. The convex hull of $\boldsymbol{A}$ is

$$\text{co}\,(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m) = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\} \tag{5}$$

which is called convex polyhedron. The points $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m$ are called the generators of the convex polyhedron.

3 The generalized interval eigenvalue problem

The eigenvalue problem of undamped structure can be expressed as

$$\boldsymbol{K}(\boldsymbol{\alpha})\boldsymbol{u} = \lambda \boldsymbol{M}(\boldsymbol{\alpha})\boldsymbol{u} \tag{6}$$

where α is the vector of the structural parameters, $\boldsymbol{K}(\boldsymbol{\alpha})$ and $\boldsymbol{M}(\boldsymbol{\alpha})$ are the stiffness matrix and the mass matrix, respectively, λ is the structural eigenvalue which is equal to the square of natural frequency, $\boldsymbol{u}$ is the corresponding eigenvector.

Assume the parameter vector α is an interval vector expressed as

$$\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \bar{\boldsymbol{\alpha}}] \tag{7}$$

where $\underline{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\alpha}}$ are the lower and upper boundaries of the parameter vector, respectively. Then both the stiffness matrix and the mass matrix contain interval parameters. Equation (6) can be written as

$$\boldsymbol{K}\left(\boldsymbol{\alpha}^{\mathrm{I}}\right)\boldsymbol{u} = \lambda \boldsymbol{M}\left(\boldsymbol{\alpha}^{\mathrm{I}}\right)\boldsymbol{u} \tag{8}$$

where $\boldsymbol{K}(\boldsymbol{\alpha}^I)$ and $\boldsymbol{M}(\boldsymbol{\alpha}^I)$ are the collections of the stiffness matrix and the mass matrix, which can be expressed by $\{\boldsymbol{K} : \boldsymbol{K} = \boldsymbol{K}(\boldsymbol{\alpha}), \underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \bar{\boldsymbol{\alpha}}\}$ and $\{\boldsymbol{M} : \boldsymbol{M} = \boldsymbol{M}(\boldsymbol{\alpha}), \underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \bar{\boldsymbol{\alpha}}\}$, respectively.

The corresponding collection of the eigenvalue vectors is

$$\boldsymbol{\Gamma} = \{\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \boldsymbol{R}^n, \boldsymbol{K}(\boldsymbol{\alpha})\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{M}(\boldsymbol{\alpha})\boldsymbol{u},\ \boldsymbol{u} \neq 0, \underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \bar{\boldsymbol{\alpha}}\} \tag{9}$$

The above collection is generally a complex non-convex set, but what are of more interest are the upper and lower boundaries of this area expressed by the following interval forms

$$\boldsymbol{\lambda}^I = \left[\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}\right] = \left\{\lambda_i^I, i = 1, .., n\right\}, \lambda_i^I = \left[\underline{\lambda_i}, \overline{\lambda_i}\right] \tag{10}$$

where $\underline{\lambda_i}$ and $\overline{\lambda_i}$ can be expressed as

$$\underline{\lambda_i} = \min_{\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I} \lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}), \boldsymbol{M}(\boldsymbol{\alpha})>) \tag{11}$$

$$\overline{\lambda_i} = \max_{\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I} \lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}), \boldsymbol{M}(\boldsymbol{\alpha})>) \tag{12}$$

where $\lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}), \boldsymbol{M}(\boldsymbol{\alpha})>)$ can be further expressed as [Qiu (2005)]

$$\lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}), \boldsymbol{M}(\boldsymbol{\alpha})>) = \min_{\boldsymbol{\Phi}_i \subset \boldsymbol{R}^n} \max_{\substack{\boldsymbol{u} \in \boldsymbol{\Phi}_i \\ \boldsymbol{u} \neq 0}} \frac{\boldsymbol{u}^T \boldsymbol{K} \boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{M} \boldsymbol{u}} \tag{13}$$

where $\boldsymbol{\Phi}_i$ is the arbitrary i-dimensional subspace in Euclidean space $\boldsymbol{R}^n$.

Up to now, the non-probabilistic structural eigenvalue analyses are mainly based on the interval model. Eigenvalue or natural frequency analyses using other types of convex set models needs further studies.

4 Structural eigenvalue analysis using Kriging and PSO algorithms

Optimization is considered to be one of the most suitable methods for set-based problems. The feasibility of various optimization methods depend mainly on the following two aspects [Moens and Hanss (2011)]:

(1) The level of the confidence about which the results can represent the global optimums.

(2) The computation cost for reaching the confidence level.

The first aspect depends on the stability of the optimization algorithm and the capability of it to explore new space, especially for complex nonlinear objective functions. The second aspect is mainly associated with two problems: (a) The volume of the optimization space grows exponentially with increasing number of uncertain parameters; (b) Most of the engineering problems are faced with large-scale finite element model. Typically, problem (b) is crucial. Therefore, in order to reduce the computational burden, an alternative model of the objective function can be sought for. A new set-based method which combines the Kriging model and the PSO algorithm will be developed to analyze the set-based structural eigenvalues.

4.1 Kriging model

Generally, Kriging model contains two parts: polynomials and random distribution as follows

$$y(\boldsymbol{x}) = F(\boldsymbol{\beta}, \boldsymbol{x}) + z(\boldsymbol{x}) = f^T(\boldsymbol{x})\boldsymbol{\beta} + \boldsymbol{z}(\boldsymbol{x}) \tag{14}$$

where β is the vector of regression coefficient, $f(\boldsymbol{x})$ is the polynomial basis function which provides the global simulation, $z(\boldsymbol{x})$ is a random process which provides the local simulation and follows normal distribution $N(0,\sigma^2)$. The covariance of $z(\boldsymbol{x})$ is non-zero and the covariance matrix of $z(\boldsymbol{x})$ is

$$Cov[Z(\boldsymbol{x}_i), Z(\boldsymbol{x}_j)] = \sigma^2 R[R(\boldsymbol{x}_i, \boldsymbol{x}_j)] \tag{15}$$

where $R(\boldsymbol{x}_i,\boldsymbol{x}_j)$ is the correlation equation about any two sample points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, which determines the accuracy of the algorithm. Sacks, Koehler and Owen have presented some types of correlation equations. The most widely used is the Gaussian correlation equation as the following form

$$R(\boldsymbol{x}_i, \boldsymbol{x}_j) = \exp\left(-\sum_{k=1}^{n} \theta_k \left| \boldsymbol{x}_k^i - \boldsymbol{x}_k^j \right|^2 \right) \tag{16}$$

where n denotes the dimension of the correlation function and θ_k is anisotropic parameter. Then, the correlation matrix can be obtained based on the experimental

data, and has the following form

$$\boldsymbol{R} = \begin{pmatrix} R(\boldsymbol{x}_1,\boldsymbol{x}_1) & \cdots & R(\boldsymbol{x}_1,\boldsymbol{x}_N) \\ \vdots & \ddots & \vdots \\ R(\boldsymbol{x}_N,\boldsymbol{x}_1) & \cdots & R(\boldsymbol{x}_N,\boldsymbol{x}_N) \end{pmatrix} \tag{17}$$

where N is the total number of the experimental samples. According to the weighted least squares method, the parameters of the polynomial can be obtained

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^T \boldsymbol{R}^{-1} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{R}^{-1} \boldsymbol{Y} \tag{18}$$

where $\boldsymbol{X}$ is the coefficient matrix and $\boldsymbol{Y}$ is the corresponding vector of response values expressed as

$$\boldsymbol{X} = \begin{pmatrix} f_1(\boldsymbol{x}_1) & \cdots & f_p(\boldsymbol{x}_1) \\ \vdots & \ddots & \vdots \\ f_1(\boldsymbol{x}_N) & \cdots & f_p(\boldsymbol{x}_N) \end{pmatrix} \tag{19}$$

$$\boldsymbol{Y} = [y(\boldsymbol{x}_1)\, y(\boldsymbol{x}_2) \cdots y(\boldsymbol{x}_N)]^T \tag{20}$$

The estimated value of the variance is

$$\hat{\sigma}_z^2 = \frac{1}{N}\left(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\right)^T R^{-1} \left(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\right) \tag{21}$$

Because $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}_z$ are associated with θ, the optimal value of θ must be firstly obtained based on the experimental data. According to the Maximum Likelihood Estimation Method, the following optimization problem can be obtained

$$\max_{\theta > \theta_0} -\frac{\left[N \ln\left(\sigma^2\right) + \ln\left(\det \boldsymbol{R}\right)\right]}{2} \tag{22}$$

Finally, the predictive value at $\boldsymbol{x}$ is

$$\hat{y}(\boldsymbol{x}) = \boldsymbol{f}^T(\boldsymbol{x})\,\hat{\boldsymbol{\beta}} + \boldsymbol{r}^T(\boldsymbol{x})\,\boldsymbol{R}^{-1}\left(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\right) \tag{23}$$

where $r(\boldsymbol{x})$ is the correlation matrix between $\boldsymbol{x}$ and the experimental data, which is expressed as

$$\boldsymbol{r}^T(\boldsymbol{x}) = [R(\boldsymbol{x},\boldsymbol{x}_1)\, R(\boldsymbol{x},\boldsymbol{x}_2) \cdots R(\boldsymbol{x},\boldsymbol{x}_N)] \tag{24}$$

The second item in the right of eq. (23) can ensure that the fitted surface gets through the experimental points.

4.2 PSO algorithm

Particle Swarm Optimization (PSO) method appeared in 1990s, which is a Swarm Intelligence (SI) algorithm that simulates bird predation. The particles in the optimization space are abstracted as birds. These particles constitute a population and constantly change their position and velocity by sharing information in order to seek the global optimums.

Let the position and velocity of the particle i be denoted by $\boldsymbol{x}_i$=$(x_{i1},x_{i2},\ldots,x_{in})^T$ and $\boldsymbol{v}_i$=$(v_{i1},v_{i2},\ldots,v_{in})^T$, respectively. Every particle knows its own current location $\boldsymbol{x}_i$, its history optimal location $\boldsymbol{x}_b$and the population's history optimal location $\boldsymbol{x}_{gb}$.

Every particle changes the velocity and the location by the following two equations

$$v_{ik}(t+1)=\omega v_{ik}(t)+c_1 rand_1(0,1)\left[x_{bk}(t)-x_{ik}(t)\right]+c_2 rand_2(0,1)\left[x_{gk}(t)-x_{ik}(t)\right] \tag{25}$$

$$x_{ik}(t+1)=x_{ik}(t)+v_{ik}(t+1) \tag{26}$$

where

i is the serial number of the particle,

k is the dimension number of location and velocity,

t is the number of iterations,

$\text{rand}_1(0,1)$ and $\text{rand}_2(0,1)$ are random numbers between 0 and 1,

ω is the inertia weight,

c_1 and c_2 are acceleration factors.

The location and velocity components x_{ik} andv_{ik} are bounded by the intervals $[x_{k,min}, x_{k,max}]$ and $[-v_{k,max},v_{k,max}]$, respectively.

When ω is relatively large, the algorithm has stronger global search capability, otherwise, when ω is relatively small, the algorithm has stronger local search capability [Shi and Eberhart (1998)]. Using dynamic inertia weight can improve the performance of the algorithm, for this method focuses on global exploration at early stages and local development at later stages. The linear decreasing inertia weight is used in this study, which can be expressed as

$$\omega(t)=\omega_{\max}-(\omega_{\max}-\omega_{\min})\cdot(t-1)/(T_{\max}-1) \tag{27}$$

where $T_{\max}$ denotes the iteration times for reaching $\omega_{\min}$.

There are two criterions for iteration termination: (1) The maximum number of iterations is done; (2) $\boldsymbol{x}_{gb}$ remains stable during m steps and the error is less than the required accuracy.

4.3 The program for the natural frequency analysis

4.3.1 Latin hypercube sampling and the evaluation of Kriging model

Before building the Kriging model, we need to select the experimental sample points firstly. Latin hypercube sampling, due to [McKay, Conover, Beckman (1979)], is a strategy for generating random sample points ensuring that all portions of the vector space is represented.

Consider the case where we wish to sample m points in the n-dimensional vector space. The Latin hypercube sampling strategy is as follows:

(1) Divide the interval of each dimension into $m$non-overlapping intervals having equal probability.

(2) Sample randomly from a uniform distribution a point in each interval in each dimension.

(3) Pair randomly the points from each dimension.

The experimental points (20 in total) in a 2-dimensional space are shown in Fig. 1.

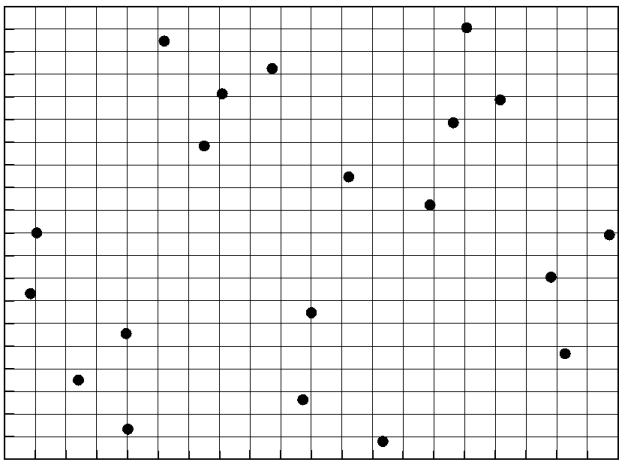


Figure 1: The Latin sampling points (20 in total) in 2-dimensional space

The Kriging model should be evaluated before further application of it. Thus, we need to sample some test points besides the experimental points. Two commonly used evaluation indexes are as follows:

(1) Empirical Integrated Squared Error

$$EISE = \frac{1}{m}\sum_{i=1}^{m}\left[y(\boldsymbol{x}_i) - \hat{y}(\boldsymbol{x}_i)\right]^2 \tag{28}$$

where m denotes the total number of the test points, $y(\boldsymbol{x}_i)$ is the actual response value and $\hat{y}(\boldsymbol{x}_i)$ is the predictive value.

(2) Average relative error

$$err = \frac{1}{m}\sum_{i=1}^{m}\frac{|y(\boldsymbol{x}_i) - \hat{y}(\boldsymbol{x}_i)|}{|y(\boldsymbol{x}_i) + \varepsilon|}, \varepsilon = \begin{cases} 0 & |y(\boldsymbol{x}_i)| \neq 0 \\ 0.01 & |y(\boldsymbol{x}_i)| = 0 \end{cases} \tag{29}$$

4.3.2 The natural frequency analysis method

(I) For structures with interval parameters

When the uncertainties of the parameters are quantified by intervals, the problem of eigenvalue analysis has been defined as eq. (11) and eq. (12).

The strategy for natural frequency analysis of structure with interval parameters is as follows:

(a) Sample the experimental points in the variable space according to Latin hypercube sampling method.

(b) Get the values of natural frequency corresponding to the experimental points through modal analysis.

(c) Choose the type of correlation function and provide the boundaries of the correlation parameters such as θ. Define the type of simulation (isotropic or anisotropic) and establish the Kriging model.

(d) Sample a certain number of test points and evaluate the Kriging model according to eq. (28) or eq. (29). Adjust the optimization parameters until the accuracy meets the requirements.

(e) Set the parameters about PSO such as population size ps, inertia weight ω, acceleration factors c_1 and c_2, the search scopes and the termination criterion.

(f) Apply the Kriging model as the objective function and get the upper and lower boundaries of the structural natural frequency by using PSO method.

(II) For structures with hyper-ellipsoidal uncertain parameters

Assume the uncertainty of the structural parameters can be expressed as

$$\boldsymbol{\alpha} \in \boldsymbol{E}(\boldsymbol{\alpha}, \theta) = \left\{\boldsymbol{\alpha} : (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^T \boldsymbol{W} (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0) \leq \phi^2\right\} \tag{30}$$

where $\boldsymbol{W}$ is a positive definite matrix, ϕ is a positive real number and $\boldsymbol{\alpha}_0$ is the nominal vector of $\boldsymbol{\alpha}$. The upper and lower boundaries of the i^{th} eigenvalue can be

expressed by

$$\underline{\lambda_i} = \min_{\boldsymbol{\alpha}\in\boldsymbol{E}(\boldsymbol{\alpha},\theta)} \lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}),\boldsymbol{M}(\boldsymbol{\alpha})>) \tag{31}$$

$$\overline{\lambda_i} = \max_{\boldsymbol{\alpha}\in\boldsymbol{E}(\boldsymbol{\alpha},\theta)} \lambda_i(<\boldsymbol{K}(\boldsymbol{\alpha}),\boldsymbol{M}(\boldsymbol{\alpha})>) \tag{32}$$

The Kriging model can be constructed in the smallest external hyper-cuboid that encloses the hyper-ellipsoid. However, the application of PSO needs the constraint condition of the hyper-ellipsoid. Two approaches for imposing the constraint are presented as follows:

1) Add penalty terms to the original problems and establish the following optimization problem

$$\begin{cases} \underline{\lambda_i} = \min\limits_{\boldsymbol{\alpha}\in\diamondsuit(E(\boldsymbol{\alpha},\theta))} \{\lambda_i(\boldsymbol{\alpha}) + C_1 \cdot sgn[T(\boldsymbol{\alpha})]\} \\ \overline{\lambda_i} = \max\limits_{\boldsymbol{\alpha}\in\diamondsuit(E(\boldsymbol{\alpha},\theta))} \{\lambda_i(\boldsymbol{\alpha}) - C_2 \cdot sgn[T(\boldsymbol{\alpha})]\} \end{cases} \tag{33}$$

where $\diamondsuit(E(\boldsymbol{\alpha},\theta))$ is a hyper-cuboid containing the ellipsoidal model, C_1 and C_2 are penalty factors taking positive real numbers, sgn$(\cdot)$ is sign function. $T(\boldsymbol{\alpha})$is expressed as follows

$$T(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)^T \boldsymbol{W}(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0) - \phi^2 \tag{34}$$

Thus, the original problem is converted to an unconstrained optimization problem.

2) Convert the orthogonal coordinates to the spherical coordinates

We can make eigenvalue decomposition for the positive definite matrix $\boldsymbol{W}$ as

$$\boldsymbol{W} = \boldsymbol{Q}^T\boldsymbol{D}\boldsymbol{Q}, \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \tag{35}$$

where $\boldsymbol{D}$ is a diagonal matrix and $\boldsymbol{I}$ is a unit matrix. Let us introduce the following vector

$$\boldsymbol{u\prime} = \left(1/\phi\right)\boldsymbol{D}^{1/2}\boldsymbol{Q}\boldsymbol{\alpha} \tag{36}$$

Consequently, the original hyper-ellipsoidal model can be converted to the following forms

$$\boldsymbol{u\prime} \in \left\{\boldsymbol{u\prime} : (\boldsymbol{u\prime}-\boldsymbol{u\prime}_0)^T(\boldsymbol{u\prime}-\boldsymbol{u\prime}_0) \le 1\right\} \tag{37}$$

$$\Delta\boldsymbol{u\prime} \in \left\{\Delta\boldsymbol{u\prime} : \Delta\boldsymbol{u\prime}^T\Delta\boldsymbol{u\prime} \le 1\right\} \tag{38}$$

Through the above transformation, the original hyper-ellipsoidal model has been converted to a unit hyper-sphere. From Eq. (36), we can get

$$\boldsymbol{\alpha} = \phi \boldsymbol{Q}^T \boldsymbol{D}^{-1/2} \boldsymbol{u}\prime = \phi \boldsymbol{Q}^T \boldsymbol{D}^{-1/2} (\Delta \boldsymbol{u}\prime + \boldsymbol{u}\prime_0) \tag{39}$$

Assume the dimension of $\Delta \boldsymbol{u}$ is $n$and the spherical coordinates of the unit hyper-sphere is $(r,\theta_1,\theta_2,\ldots,\theta_{n-1})$. Then all the components are interval parameters: $r \in[0,1]$, $\theta_1 \sim\theta_{n-2} \in[0,\pi]$, $\theta_{n-1} \in[0,2\pi]$. The spherical coordinates and the orthogonal coordinates have the following relations

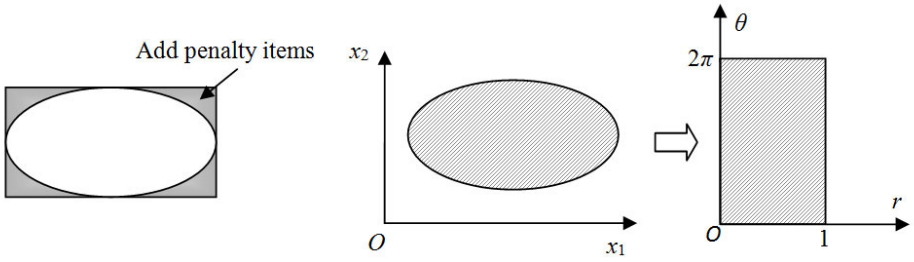
$$\begin{cases} \Delta u_1 = r\cos\theta_1 \\ \Delta u_2 = r\sin\theta_1\cos\theta_2 \\ \vdots \\ \Delta u_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ \Delta u_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{cases} \tag{40}$$

Thus, the objective function with interval spherical coordinates can be obtained by the above coordinate conversion, and the following optimization problem can be established

$$\begin{cases} \underline{\lambda_i} = \min\limits_{\boldsymbol{\delta}\in\boldsymbol{\delta}^{\mathrm{I}}} \lambda_i\prime(\boldsymbol{\delta}) \\ \overline{\lambda_i} = \max\limits_{\boldsymbol{\delta}\in\boldsymbol{\delta}^{\mathrm{I}}} \lambda_i\prime(\boldsymbol{\delta}) \end{cases} \tag{41}$$

where δ is the vector of spherical coordinates of the unit hyper-sphere.

The principles of the above two methods are shown in Fig. 2.



(a) Add penalty items (b) Conversion of coordinates

Figure 2: Two methods for imposing the hyper-ellipsoidal constraint

The strategy for natural frequency analysis of structure with hyper-ellipsoidal uncertain parameters is as follows:

(a) Sampling the experimental points within the external hyper-cuboid that enclosing the hyper-ellipsoid according to Latin sampling methods.

(b) Take the experimental sample data as the inputs of the deterministic Finite Element Analysis, and the corresponding $1^{st} \sim k^{th}$ order natural frequencies can be obtained.

(c) Set the type of the correlation equation in Kriging model and provide the boundaries of the correlation parameters such as θ. Define the type of simulation (isotropic or anisotropic) and establish the Kriging models for the $1^{st} \sim k^{th}$ order natural frequencies.

(d) Sample a certain number of test sample points, and evaluate the Kriging models according to eq. (28) and eq. (29). Adjust the optimization parameters until the obtained model meets the accuracy requirement.

(e) Add penalty items to the original objective function, and build the optimization problem as eq. (33). If choose the way of coordinate conversion, then go to step (f).

(f) Get the objective functions with spherical coordinates through coordinate conversion. Establish the optimization problem as eq. (41).

(g) Provide the parameters about PSO such as population size ps, inertia weight ω, acceleration factors c_1 and c_2, the search scopes and termination criterion.

(h) Get the upper and lower boundaries of the $1^{st} \sim k^{th}$ order natural frequencies based on PSO methods and the objective functions with penalty items or spherical coordinates.

5 Numerical examples

Take a gas turbine blade shown in Fig. 3 as the object of discussion. The parameters of the blade are listed in Tab. 1.

Table 1: The structural parameters

elastic modulus $E(\times 10^{11}\text{Pa})$	density $\rho(\times 10^{3}\text{kg/m}^{3})$	angular velocity $\omega(\times 10^{2}\text{rad/s})$	Poisson ratio$v$
2.17($\pm$5%)	8.489($\pm$5%)	3.14($\pm$5%)	0.32(deterministic)

20 experimental points and 10 test points are extracted using Latin sampling method. The Gaussian correlation function is chosen for Kriging model. The adjustment procedure of correlation parameter θ is listed in Tab. 2 for the first order natural

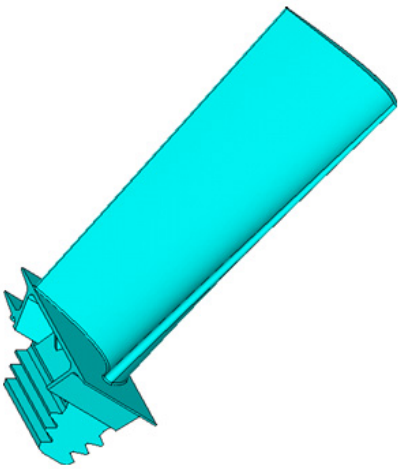


Figure 3: The geometric model of the turbine blade

frequency.

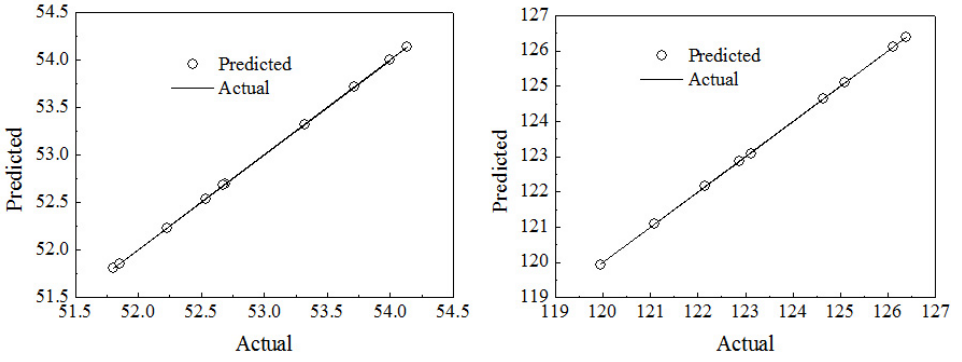
Table 2: The adjustment of the Kriging model for the 1^{st} natural frequency

Lower	Upper	initial value	θ_1	θ_2	θ_3	EISE($\times 10^{-4}$)	err(‰)
0.1,0.1,0.1	20,20,20	10,10,10	0.899	0.143	0.100	182.33	1.88
0.1,0.1,0.1	10,10,10	5,5,5	0.450	0.100	0.100	68.131	1.21
0.02,0.01,0.01	10,5,5	5,2,2	0.450	0.030	0.010	10.350	0.39
0.01,0.01,0.01	5,5,5	2,2,2	0.180	0.020	0.010	3.6490	0.28
0.01,0.01,0.005	5,5,3	2,2,1	0.180	0.019	0.005	2.7062	0.24
0.01,0.01,0.0001	5,5,3	2,2,1	0.0196	0.010	0.000107	0.0518	0.028

It can be found from Tab. 2 that choosing the anisotropic parameters is better than choosing the isotropic parameters. The finally obtained Kriging model can meet the accuracy requirement after parameter adjustment.

The comparison of the predicted and actual values at the test points are shown in Fig. 4.

Fig. 5 and Fig. 6 show the optimization procedure for the 1^{st} and 2^{nd} natural frequencies. The termination criterion is that the population's optimal value remains stable during 50 steps. The optimal values are reached after only about 20 steps,



(a) The first order (b) The second order

Figure 4: The comparison of the predicted and actual values of the 1^{st} and 2^{nd} natural frequencies

and the high efficiency of the PSO algorithm is revealed.

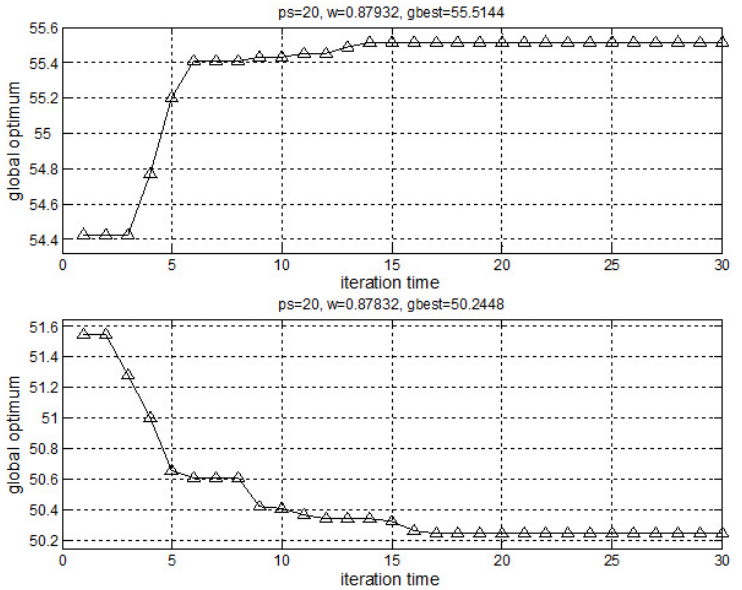


Figure 5: The optimization process for the 1^{st} natural frequency

The intervals and the uncertainty degrees of the $1^{st} \sim 5^{th}$ order natural frequencies are listed in Tab. 3.

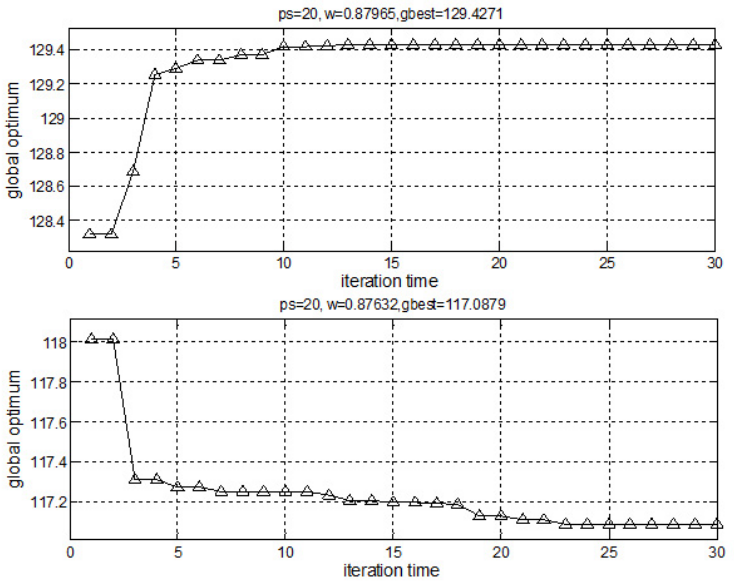


Figure 6: The optimization process for the 2^{st} natural frequency

Assume the uncertainty degree of the structural parameter changes, then the boundaries of the natural frequencies are calculated which are listed in Tab. 4.

It can be seen from Tab. 4 that the intervals of the natural frequencies become wider when the uncertainty degree of the structural parameters increases.

If the uncertainties of the structural parameters are quantified by the hyper-ellipsoidal model, the Kriging model obtained above can also be used, but the hyper-ellipsoidal constraint must be imposed for PSO.

The boundaries of the $1^{st} \sim 5^{th}$ order natural frequencies based on the internal

Table 3: The uncertainty degrees and the intervals of $1^{st} \sim 5^{th}$ order natural frequencies

	f_1/Hz	f_2/Hz	f_3/Hz	f_4/Hz	f_5/Hz
nominal/Hz	52.880	123.26	172.87	247.02	472.13
interval/Hz	[50.245, 55.514]	[117.09, 129.43]	[164.27, 181.47]	[234.86, 259.19]	[448.48, 495.79]
uncertainty/%	4.98	5.01	4.98	4.93	5.01

Table 4: The natural frequency boundaries based on different uncertainty degrees of structural parameter

β	f_1/Hz	f_2/Hz	f_3/Hz	f_4/Hz	f_5/Hz
0.01	[52.293,53.348]	[121.84,124.30]	[170.91,174.36]	[244.23,249.15]	[467.13,476.57]
0.02	[51.771,53.883]	[120.62,125.55]	[169.21,176.11]	[241.79,251.65]	[462.48,481.37]
0.03	[51.255,54.423]	[119.42,126.81]	[167.54,177.88]	[239.38,254.18]	[457.85,486.20]
0.04	[50.745,54.967]	[118.24,128.11]	[165.89,179.67]	[237.05,256.71]	[453.20,491.02]

hyper-ellipsoid are listed in Tab. 5.

Table 5: The $1^{st} \sim 5^{th}$ natural frequency intervals based on the internal ellipsoidal model

	f_1/Hz	f_2/Hz	f_3/Hz	f_4/Hz	f_5/Hz
nominal/Hz	52.849	123.131	172.743	246.846	472.105
interval/Hz	[51.17, 54.53]	[118.87, 127.39]	[166.74, 178.75]	[238.30, 255.39]	[455.58, 488.63]
uncertainty/%	3.18	3.46	3.48	3.46	3.50

Interval model is simple and intuitive, which is easy to operate. However, the hyper-ellipsoidal model can more flexibly and objectively reflect the uncertainties of the structure. The above analysis showed the feasibility of the proposed method based on Kriging model and PSO algorithm.

6 Conclusions

In this paper, a new non-probabilistic method for structural eigenvalue analysis is developed, which is featured with the Kriging model and the PSO algorithm. To reduce the computational cost and deal with the small sample-size problem, the convex model is introduced to this study to model the structural uncertainties. Thus, how to obtain the accurate bounds of the structural eigenvalues is the problem of interest in this study. The Kriging method which has excellent performance for local fitting, is very suitable for complex non-linear problems and is used here to construct the approximate models of the structural eigenvalues. The PSO algorithm which has excellent performance for global optimization is used here to seek the bounds of the structural eigenvalues. Two methods for imposing the hyper-ellipsoidal constraints are also presented. The proposed new method successfully solved the set-based eigenvalue problems for complex and random structures. It is

simple, highly efficient and accurate, and can be a useful complement when it is difficult to get the accurate probability densities of the uncertain structural parameters.

We would also like to point out that, although this study is focused on structural eigenvalue analysis, the essential methodology can be easily extended to solve other practical engineering problems with uncertainties, such as cracks, life-time prediction, buckling and postbuckling, structural reliability, etc. And the structural model is not limited to FEM, the method proposed in this study can also be easily combined with BEM [Brebbia, Telles and Wrobel (1984)], and novel methods like the MLPG method [Atluri (2003)], etc. These can be possible interesting studies in future.

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References

Atluri, S. N. (2005): *Methods of computer modeling in engineering & the sciences*. Palmdale: Tech Science Press.

Atluri, S. N. (2003): *The meshless method (MLPG) for domain & BIE discretizations* . Forsyth: Tech Science Press

Brebbia, C. A., Telles, J. C. F., Wrobel, L. C. (1984). *Boundary element techniques: theory and applications in engineering*. Berlin and New York, Springer-Verlag.

Ben-Haim, Y., Elishakoff, I.(1990): *Convex models of uncertainty in applied mechanics. Amsterdam*: Elsevier Science Publishers.

Chen, S., Lian, H., Yang, X.(2003): Interval eigenvalue analysis for structures with interval parameters. *Finite Elements in Analysis and Design*, vol.39, no.5∼6, pp.419-431.

Elishakoff, I.(2000): Possible limitations of probabilistic methods in engineering. *Appl. Mech. Rev.*, vol.53, no.2, pp.19-36.

Eberhart, R.C., Shi, Y.(2000): Comparing inertia weights and constriction factors in particle swarm optimization. *Proceedings of IEEE congress evolutionary*

computation, San Diego, pp.84-88.

Gao, W., Song, C., Tin-Loi, F. (2009): Probabilistic interval response and reliability analysis of structures with a mixture of random and interval properties. *CMES: Computer Modeling in Engineering and Sciences*, vol.46, no.2, pp.151-189.

Krige, D. G.(1951): *A statistical approach to some mine valuations and allied problems at the Witwatersrand.* Witwatersrand: University of Witwatersrand, 1951.

Kaymaz., I. (2006): Application of kriging method to structural reliability problems. *Structural Safety*, vol.27, pp.133-151.

Li, F., Luo, F., Sun, G.(2011): Reliability-based multiobjective design optimization under interval uncertainty. *CMES: Computer Modeling in Engineering and Sciences*, vol.74, no.1, pp.39-64.

Muhanna, R, Kreinovich, V, Solin, P. (2006): Interval finite element methods: new directions. *Proceedings of the NSF workshop on reliable engineering computing*, 2006, Georgia, USA.

Moens, D., Hanss, M.(2011): Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances. *Finite Elements in Analysis and Design*, vol.47, pp.4-16.

McKay, M. D, Conover, W. J., Beckman, R. J.(1979): A comparison of Three Methods for Selecting Values of Input Variables in the Analysis of Output from a Computer Code. *Technometrics,* vol.42, no.1, pp.55-61.

Moens, D., Vandepitte, D.(2005): A survey of non-probabilistic uncertainty treatment in finite element analysis. *Comput. Methods Appl. Mech. Engrg.*, vol.194, pp.1527-1555.

Moens, D., Vandepitte, D.(2006): Recent advances in non-probabilistic approaches for non-deterministic dynamic finite element analysis. *Arch. Comput. Mech. Engng.,* vol.13, no.3, pp.389-464.

Qiu, Z. P. (2005): *Convex Method Based on Non-probabilistic Set-theory and its Application (in Chinese).* Beijing: National Defense Industry Press.

Qiu, Z. P., Chen S., Elishakoff, I.(1995): Natural frequencies of structures with uncertain- but-non-random parameters. *Journal of Optimization Theory and Applications*, vol.6, pp.669-683.

Qiu, Z, Chen. S., Elishakoff, I.(1996): Bounds of eigenvalues for structures with an interval description of uncertain-but-non-random parameters. *Chaos, Solitons and Fractals*, vol.7, pp.425-434.

Shi, Y., Eberhart, R. (1998): Parameter selection in particle swarm optimization. *Evolutionary Programming VII:Proceedings of the Seventh Annual Conference on Evolutionary Programming*, New York, pp.591-600.

Sim, J., Qiu, Z., Wang, X.(2007): Modal analysis of structures with uncertain-but-bounded parameters via interval analysis. *Journal of Sound and Vibration*, vol.303, pp.29-45.

Yang, X., Chen. S., Lian, H.(2001): Bounds of complex eigenvalues of structures with interval parameters. *Engineering Structures*, vol.23, no.5, pp.557-563.

Zhang, Q., Li, X.(2006): Analysis of structural reliability based on Kriging model. Chinese Journal of *Computational Mechanics*, vol.23, no.2, pp.175-179.