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Numerical solution of fractional partial differential equations using Haar wavelets

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Abstract: In this paper, we present a computational method for solving a class of fractional partial differential equations which is based on Haar wavelets operational matrix of fractional order integration. We derive the Haar wavelets operational matrix of fractional order integration. Haar wavelets method is used because its computation is sample as it converts the original problem into Sylvester equation. Finally, some examples are included to show the implementation and accuracy of the approach.

Keywords: Haar wavelets, Operational matrix, Fractional partial differential equations, Sylvester equation, Numerical solution.

1 Introduction

In the last decades, fractional derivative and fractional differential equations have found their applications in several different disciplines. Many practical problems can be elegantly modeled with the help of the fractional derivative [Sun, Chen and Wei (2011); Sun, Chen and Li (2010); Chen (2007)]. For example, the fluid dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [He (1999)], and nonlinear oscillation of earthquake can be modeled with fractional derivatives [He (1998)]. According to the increasing applications, a lot of attention has been given to numerical and exact solution of fractional differential equations. The analytical solutions of fractional differential equations are still in a preliminary stage. However, it is difficult to obtain their exact solutions. In recent years, both mathematicians and physicists have engaged in discussing the numerical methods for solving fractional differential equations. The most commonly used ones are Adomian Decomposition Method [EI-Kalla (2008); Hosseini (2006)], Generalized Differential Transform Method [Momani and Odibat (2007); Odibat and Momani (2008)], Varia-

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tional Iteration Method [Odibat (2010)], Finite Difference Method [Sun, Chen and Li (2012); Meerschaert, Scheffler and Tadjeran (2006)], Homtopy Analysis Method [Hashim and Abdulaziz (2009)], and Wavelet Method [Chen and Wu (2010); Jafari and Yousefi (2011); Chen, Yiand Yu (2012)].

In this paper, we consider a class of fractional partial differential equations

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}=f(x,t) \tag{1}$$

subject to

$$\left.\frac{\partial u}{\partial x}\right|_{t=0}=\delta_1(x),\ \left.\frac{\partial u}{\partial t}\right|_{x=0}=\delta_2(t) \tag{2}$$

$$u(0,t)=\theta_1(t),u(x,0)=\theta_2(x) \tag{3}$$

where $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ and $\frac{\partial^\beta u(x,t)}{\partial t^\beta}$ are fractional derivative of Caputo sense, $f,\delta_1,\delta_2,\theta_1,\theta_2$ are the known continuous functions, $u(x,t)$is the unknown function, $0<\alpha,\beta\le 1$.

There have been several methods for solving the fractional partial differential equations. [Podlubny (1999)] used the Laplace transform method to solve the fractional partial differential equations with constant coefficients. [Zhang (2009)] discussed a practical implicit method to solve a class of initial boundary value space-time fractional convection-diffusion equations with variable coefficients. [Odibat and Momani (2008)] applied generalized differential transform method to solve the numerical solution of linear partial differential equations of fractional order.

Wavelets theory is a new and emerging area in mathematical research, it is very successfully used in signal analysis for waveform representation and segmentations, time frequency analysis. In this paper, our purpose is to proposed Haar wavelets operational matrix method to solve a class of fractional partial differential equations.

2 Definition of fractional derivative and integral

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this work [Podlubny (1999)].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha\ge 0$of a function is defined as

$$J^\alpha v(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}v(t)dt, \quad \alpha>0,\ x>0 \tag{4}$$

$$J^0 v(x)=v(x) \tag{5}$$

The properties of the operator J^{α} are given as follows
i) $J^{\alpha}J^{\beta}v(x) = J^{\alpha+\beta}v(x)$,
ii) $J^{\alpha}J^{\beta}v(x) = J^{\beta}J^{\alpha}v(x)$,
iii) $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

Definition 2.2 The fractional derivative of $v(x)$ in the Caputo sense is defined as

$$D_*^{\alpha}v(x) = \begin{cases} \frac{d^r v(x)}{dx^r}, & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)}\int_0^x \frac{v^{(r)}(t)}{(x-t)^{\alpha-r+1}}dt, & 0 \le r-1 < \alpha < r. \end{cases} \tag{6}$$

The Caputo fractional derivative of order α is also defined as $D_*^{\alpha}v(x) = J^{r-\alpha}D^r v(x)$, where D^r is the usual integer differential operator of order r.

3 Haar wavelets and function approximation

For $x \in [0,1]$, the orthogonal set of Haar wavelets functions are defined by [Chen and Hsiao (1997)]:

$$h_0(x) = \frac{1}{\sqrt{m}} \tag{7}$$

$$h_i(x) = \frac{1}{\sqrt{m}}\begin{cases} 2^{j/2}, & \frac{k-1}{2^j} \le x < \frac{k-1/2}{2^j} \\ -2^{j/2}, & \frac{k-1/2}{2^j} \le x < \frac{k}{2^j} \\ 0, & otherwise \end{cases} \tag{8}$$

where $i = 0,1,2,\ldots,m-1$, $m = 2^{p+1}$ and p is a positive integer. j and k represent the integer decomposition of the index i, i.e. $i = 2^j + k - 1$.
Any function $v(x) \in L^2([0,1))$ can be expanded into Haar wavelets by

$$v(x) = \sum_{i=0}^{\infty} c_i h_i(x) \tag{9}$$

where $c_i = \int_0^1 v(x)h_i(x)dx$ are wavelet coefficients.
If $v(x)$ is approximated as piecewise constant during each subinterval, Eq. (9) will be terminated at finite terms

$$v(x) \cong \sum_{i=0}^{m-1} c_i h_i(x) = c^T H_m(x) \tag{10}$$

where $c = [c_0, c_1, \ldots, c_{m-1}]^T$, $H_m(x) = [h_0(x), h_1(x), \ldots, h_{m-1}]^T$, m is a power of 2.

The matrix form of Eq.(10) is

$$v = c^T H \tag{11}$$

where the row vector $v$is the discrete form of the function $v(x)$. $H$is Haar wavelets matrix of order $m = 2^{p+1}$, $p = 0, 1, 2, \ldots$, i.e.

$$H = \begin{bmatrix} h_0(t_0) & h_0(t_1) & \cdots & h_0(t_{m-1}) \\ h_1(t_0) & h_1(t_1) & \cdots & h_1(t_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_0) & h_{m-1}(t_1) & \cdots & h_{m-1}(t_{m-1}) \end{bmatrix}.$$

For arbitrary function $u(x,t) \in L^2([0,1) \times [0,1))$, it can be also expanded into Haar series by [Wu (2009)]

$$u(x,t) \cong \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} u_{ij} h_i(x) h_j(t) \tag{12}$$

where $u_{ij} = \langle h_i(x), \langle u(x,t), h_j(t) \rangle \rangle$ are wavelet coefficients,
$\langle h_i(x), h_j(x) \rangle = \int_0^1 h_i(x) h_j(x) dx.$
Eq.(12) will be written as

$$u(x,t) \cong H_m^T(x) U H_m(t) \tag{13}$$

In this paper, we apply wavelet collocation method to determine the coefficients u_{ij}. These collocation points are shown in the following

$$x_l = t_l = (l - 1/2)/m, \quad l = 1, 2, \ldots, m. \tag{14}$$

Discreting Eq.(13) by the step Eq.(14), we can obtain the matrix form of Eq.(13)

$$C = H^T U H \tag{15}$$

where $U = [u_{ij}]_{m \times m}andC = [u(x_i, t_j)]_{m \times m}$.
From the definition of Haar wavelets functions, we may know easily that $H$is a orthogonal matrix.

4 Convergence of the Haar wavelets bases

In this part, we assume that $\frac{\partial u(x,t)}{\partial x}$is continuous and bounded on $(0,1) \times (0,1)$, then

$$\exists M > 0, \quad \forall x, t \in (0,1) \times (0,1), \quad \left| \frac{\partial u(x,t)}{\partial x} \right| \leq M \tag{16}$$

Suppose $u_m(x,t)$is the following approximation of $u(x,t)$

$$u_m(x,t)=\sum_{n=0}^{m-1}\sum_{l=0}^{m-1}u_{nl}h_n(x)h_l(t) \tag{17}$$

Then we have

$$u(x,t)-u_m(x,t)=\sum_{n=m}^{\infty}\sum_{l=m}^{\infty}u_{nl}h_n(x)h_l(t)=\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty}u_{nl}h_n(x)h_l(t) \tag{18}$$

The orthonormality of the sequence $\{h_i(x)\}$on $[0,1)$implies that

$$\int_0^1 h_n(x)h_{n'}(x)dx=\begin{cases}1/m, & n=n'\\ 0, & n\neq n'\end{cases} \tag{19}$$

Therefore

$$\begin{aligned}&\|u(x,t)-u_m(x,t)\|_E^2=\int_0^1\int_0^1[u(x,t)-u_m(x,t)]^2dxdt\\&=\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty}\sum_{n'=2^{p+1}}^{\infty}\sum_{l'=2^{p+1}}^{\infty}u_{nl}u_{n'l'}\left(\int_0^1 h_n(x)h_{n'}(x)dx\right)\left(\int_0^1 h_n(t)h_{n'}(t)dt\right)\\&=\frac{1}{m^2}\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty}u_{nl}^2\end{aligned} \tag{20}$$

where $u_{nl}=\langle h_n(x),\langle u(x,t),h_l(t)\rangle\rangle$.
According to Eq.(7) and Eq.(8), we have

$$\begin{aligned}\langle u(x,t),h_l(t)\rangle&=\int_0^1 u(x,t)h_l(t)dt\\&=\frac{2^{j/2}}{\sqrt{m}}\left(\int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}}u(x,t)dt-\int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}}u(x,t)dt\right)\end{aligned} \tag{21}$$

Using mean value theorem of integrals:

$$\exists t_1,t_2:\quad (k-1)\cdot 2^{-j}\leq t_1<(k-\frac{1}{2})\cdot 2^{-j},\quad (k-\frac{1}{2})\cdot 2^{-j}\leq t_2<k\cdot 2^{-j}$$

such that

$$\begin{aligned}&\langle u(x,t),h_l(t)\rangle\\&=\frac{2^{j/2}}{\sqrt{m}}\left\{[(k-\tfrac{1}{2})2^{-j}-(k-1)2^{-j}]u(x,t_1)-[k2^{-j}-(k-\tfrac{1}{2})2^{-j}]u(x,t_2)\right\}\\&=\frac{2^{-j/2-1}}{\sqrt{m}}(u(x,t_1)-u(x,t_2))\end{aligned} \tag{22}$$

hence

$$
\begin{aligned}
u_{nl} &= \left\langle h_n(x), \frac{2^{-j/2-1}}{\sqrt{m}}(u(x,t_1)-u(x,t_2)) \right\rangle \\
&= \frac{2^{-j/2-1}}{\sqrt{m}} \int_0^1 h_n(x)(u(x,t_1)-u(x,t_2))\,dx \\
&= \frac{2^{-j/2-1}}{\sqrt{m}} \left(\int_0^1 h_n u(x,t_1)\,dx - \int_0^1 h_n(x) u(x,t_2)\,dx \right) \\
&= \frac{1}{2m} \left(\int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}} u(x,t_1)\,dx - \int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}} u(x,t_1)\,dx \right. \\
&\quad \left. - \int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}} u(x,t_2)\,dx + \int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}} u(x,t_2)\,dx \right)
\end{aligned}
$$

Using mean value theorem of integrals again:

$$
\begin{aligned}
\exists x_1,x_2,x_3,x_4: \quad & (k-1)\cdot 2^{-j} \le x_1,x_3 < (k-\frac{1}{2})\cdot 2^{-j}, \\
& (k-\frac{1}{2})\cdot 2^{-j} \le x_2,x_4 < k\cdot 2^{-j}
\end{aligned}
$$

such that

$$
\begin{aligned}
u_{nl} &= \tfrac{1}{2m} \left\{ [(k-\tfrac{1}{2})2^{-j} - (k-1)2^{-j}]u(x_1,t_1) - [k2^{-j} - (k-\tfrac{1}{2})2^{-j}]u(x_2,t_1) \right. \\
&\quad \left. -[(k-\tfrac{1}{2})2^{-j} - (k-1)2^{-j}]u(x_3,t_2) + [k2^{-j} - (k-\tfrac{1}{2})2^{-j}]u(x_4,t_2) \right\} \\
&= \tfrac{1}{2^{j+2}m} [(u(x_1,t_1)-u(x_2,t_1)) - (u(x_3,t_2)-u(x_4,t_2))]
\end{aligned} \tag{23}
$$

hence

$$
u_{nl}^2 = \frac{1}{2^{2j+4}m^2} [(u(x_1,t_1)-u(x_2,t_1)) - (u(x_3,t_2)-u(x_4,t_2))]^2 \tag{24}
$$

Using mean value theorem of derivatives:

$$
\exists \xi_1,\xi_2: \quad x_1 \le \xi_1 < x_2,\ x_3 \le \xi_2 < x_4
$$

such that

$$
\begin{aligned}
u_{nl}^2 &= \tfrac{1}{2^{2j+4}m^2} \left[(x_2-x_1)\tfrac{\partial u(\xi_1,t_1)}{\partial x} - (x_4-x_3)\tfrac{\partial u(\xi_2,t_2)}{\partial x} \right]^2 \\
&\le \tfrac{1}{2^{2j+4}m^2} \left\{ (x_2-x_1)^2 \left[\tfrac{\partial u(\xi_1,t_1)}{\partial x}\right]^2 + (x_4-x_3)^2 \left[\tfrac{\partial u(\xi_2,t_2)}{\partial x}\right]^2 \right. \\
&\qquad \left. +2(x_2-x_1)(x_4-x_3) \left|\tfrac{\partial u(\xi_1,t_1)}{\partial x}\right| \left|\tfrac{\partial u(\xi_2,t_2)}{\partial x}\right| \right\}
\end{aligned} \tag{25}
$$

Combining Eq.(16) and Eq.(25) , we obtain

$$u_{nl}^2 \le \frac{4M^2}{2^{4j+4}m^2} = \frac{M^2}{2^{4j+2}m^2} \tag{26}$$

Substituting Eq.(26) into Eq.(20), then we have

$$\begin{aligned}\|u(x,t)-u_m(x,t)\|_E^2 &= \tfrac{1}{m^2}\sum_{n=2^{p+1}}^{\infty}\sum_{l=2^{p+1}}^{\infty}u_{nl}^2 = \tfrac{1}{m^2}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1}u_{nl}^2\right)\\ &\le \tfrac{1}{m^2}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1}\tfrac{M^2}{2^{4j+2}m^2}\right)\\ &= \tfrac{M^2}{m^4}\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{l=2^j}^{2^{j+1}-1}\tfrac{1}{2^{4j+2}}\right)\\ &= \tfrac{M^2}{3m^4}\tfrac{1}{2^{2(p+1)}} = \tfrac{M^2}{3}\tfrac{1}{m^6}\end{aligned} \tag{27}$$

Therefore

$$\|u(x,t)-u_m(x,t)\|_E^{\le} \frac{M}{\sqrt{3}}\frac{1}{m^3} \tag{28}$$

From the Eq.(28), we can find that $\|u(x,t)-u_m(x,t)\|_E \to 0$when $m \to \infty$. The larger the value of m, the more accurate the numerical solution.

5 Haar wavelets operational matrix of fractional order integration

Now, we derive the Haar wavelets operational matrix of fractional order integration. For this purpose, we may use the definition of Riemann-Liouville fractional integral operator J^α.

The Haar wavelets operational matrix of fractional order integration P^αcan be deduced by

$$\begin{aligned}P^\alpha H_m(x) &= J^\alpha H_m(x)\\ &= [J^\alpha h_0(x), J^\alpha h_1(x), \ldots, J^\alpha h_{m-1}(x)]^T\\ &= [\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_0(t)dt, \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_1(t)dt,\\ &\ldots, \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_{m-1}(t)dt]^T\\ &= [Ph_0(x), Ph_1(x), \ldots, Ph_{m-1}(x)]^T\end{aligned}$$

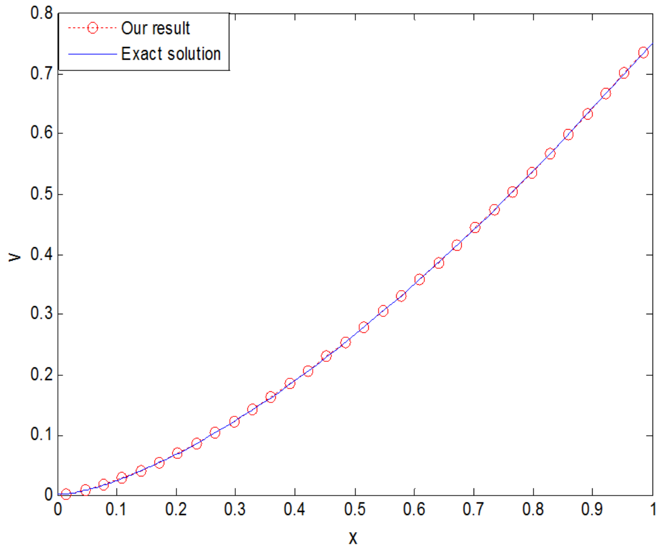

Figure 1: 0.5-order integration of the function $v(x) = x$

where

$$Ph_0(x) = \frac{1}{\sqrt{m}} \frac{x^{\alpha}}{\Gamma(\alpha+1)} \qquad x \in [0,1) \tag{29}$$

$$Ph_i(x) = \frac{1}{\sqrt{m}} \begin{cases} 0, & 0 \le x < \frac{k-1}{2^j} \\ 2^{j/2}\varphi_1(x), & \frac{k-1}{2^j} \le x < \frac{k-1/2}{2^j} \\ 2^{j/2}\varphi_2(x), & \frac{k-1/2}{2^j} \le x < \frac{k}{2^j} \\ 2^{j/2}\varphi_3(x), & \frac{k}{2^j} \le x < 1 \end{cases} \tag{30}$$

where

$$\varphi_1(x) = \frac{1}{\Gamma(\alpha+1)} \left(x - \frac{k-1}{2^j}\right)^{\alpha};$$

$$\varphi_2(x) = \frac{1}{\Gamma(\alpha+1)} \left(x - \frac{k-1}{2^j}\right)^{\alpha} - \frac{2}{\Gamma(\alpha+1)} \left(x - \frac{k-1/2}{2^j}\right)^{\alpha};$$

$$\varphi_3(x) = \frac{1}{\Gamma(\alpha+1)} \left(x - \frac{k-1}{2^j}\right)^{\alpha} - \frac{2}{\Gamma(\alpha+1)} \left(x - \frac{k-1/2}{2^j}\right)^{\alpha} + \frac{1}{\Gamma(\alpha+1)} \left(x - \frac{k}{2^j}\right)^{\alpha}.$$

The derived Haar wavelets operational matrix of fractional integration is $P^{\alpha} = (P^{\alpha}H)\cdot H^{T}$. For instance, if $\alpha = 0.5,\ m = 8$, we have

$$P^{1/2} = \begin{bmatrix} 0.7549 & -0.2180 & -0.1072 & -0.0579 & -0.0516 & -0.0289 & -0.0223 & -0.0189 \\ 0.2180 & 0.3190 & -0.1072 & 0.1565 & -0.0516 & -0.0289 & 0.0809 & 0.0389 \\ 0.0579 & 0.1565 & 0.2337 & -0.0312 & -0.0730 & 0.1052 & -0.0229 & -0.0044 \\ 0.1072 & -0.1072 & 0 & 0.2337 & 0 & 0 & -0.0730 & 0.1052 \\ 0.0189 & 0.0389 & 0.1052 & -0.0044 & 0.1788 & -0.0189 & -0.0025 & -0.0009 \\ 0.0223 & 0.0809 & -0.0730 & -0.0229 & 0 & 0.1788 & -0.0189 & -0.0025 \\ 0.0289 & -0.0289 & 0 & 0.1052 & 0 & 0 & 0.1788 & -0.0189 \\ 0.0516 & -0.0516 & 0 & -0.0730 & 0 & 0 & 0 & 0.1788 \end{bmatrix}$$

The fractional order integration of the function x is selected to verify the correctness of matrix P^{α}. The fractional order integration of the function $v(x) = x$ is obtained as follows

$$J^{\alpha}v(x) = \frac{\Gamma(2)}{\Gamma(\alpha+2)}x^{\alpha+1} \tag{31}$$

When $\alpha = 0.5,\ m = 32$, the comparison result for fractional integration is shown in Fig. 1.

6 Applications and results

In this section, we will use the Haar wavelets operational matrix of fractional order integration to solve the fractional partial differential equation Eq.(1). To demonstrate the effectiveness of this method, we consider four numerical examples.

6.1 Example 1

Consider the following nonhomogeneous partial differential equation

$$\frac{\partial^{1/4}u}{\partial x^{1/4}} + \frac{\partial^{1/4}u}{\partial t^{1/4}} = f(x,t), \qquad 0 \le x,t \le 1 \tag{32}$$

such that $\frac{\partial u}{\partial x}\Big|_{t=0} = \frac{\partial u}{\partial t}\Big|_{x=0} = u(0,t) = u(x,0) = 0$, $f(x,t) = \frac{4(x^{3/4}t+xt^{3/4})}{3\Gamma(3/4)}$, the exact solution is xt.
Let $\frac{\partial^2 u}{\partial x \partial t} \cong H_m^T(x)UH_m(t)$, then

$$\begin{aligned} \frac{\partial u}{\partial x} &= \int_0^t \frac{\partial^2 u}{\partial x \partial t}dt + \frac{\partial u}{\partial x}\Big|_{t=0} \cong \int_0^t [H_m^T(x)UH_m(t)]dt + \frac{\partial u}{\partial x}\Big|_{t=0} \\ &= H_m^T(x)UP^1H_m(t) \end{aligned} \tag{33}$$

$$\frac{\partial u}{\partial t} = \int_0^x \frac{\partial^2 u}{\partial x \partial t} dx + \left.\frac{\partial u}{\partial t}\right|_{x=0} \cong \int_0^x [H_m^T(x) U H_m(t)] dx + \left.\frac{\partial u}{\partial t}\right|_{x=0} \\ = H_m^T(x)[P^1]^T U H_m(t) \tag{34}$$

Therefore

$$u(x,t) \cong H_m^T(x)[P^1]^T U P^1 H(t) + u(0,t) = H_m^T(x)[P^1]^T U P^1 H(t) \tag{35}$$

Then we have

$$\frac{\partial^{1/4} u}{\partial x^{1/4}} = J^{3/4}\left(\frac{\partial u}{\partial x}\right) \cong J^{3/4}\left(H_m^T(x) U P^1 H_m(t)\right) = H_m^T(x)[P^{3/4}]^T U P^1 H_m(t) \tag{36}$$

$$\frac{\partial^{1/4} u}{\partial t^{1/4}} = J^{3/4}\left(\frac{\partial u}{\partial t}\right) \cong J^{3/4}\left(H_m^T(x)[P^1]^T U H_m(t)\right) = H_m^T(x)[P^1]^T U P^{3/4} H_m(t) \tag{37}$$

Similarly, $f(x,t)$may be expanded by the Haar wavelets functions as follows

$$f(x,t) \cong H_m^T(x) F H_m(t) \tag{38}$$

where $F = [f_{ij}]_{m\times m}$.
Substituting Eq.(36), Eq.(37) and Eq.(38) into Eq.(32), we have

$$H_m^T(x)[P^{3/4}]^T U P^1 H_m(t) + H_m^T(x)[P^1]^T U P^{3/4} H_m(t) = H_m^T(x) F H_m(t) \tag{39}$$

Dispersing Eq.(39) by the points (x_i, t_j), $i = 1,2,\cdots,m$and$j = 1,2,\cdots,m$, we can obtain

$$H_m^T[P^{3/4}]^T U P^1 H_m + H_m^T[P^1]^T U P^{3/4} H_m = H_m^T F H_m \tag{40}$$

Eq.(40) can be also written as

$$[P^{-1}]^T[P^{3/4}]^T U + U P^{3/4} P^{-1} = [P^{-1}]^T F P^{-1} \tag{41}$$

Eq.(41) is a Sylvester equation which is solved easily by using Matlab software. Solving it, we can get U. Then using Eq.(35), we obtain the approximation $u(x,t)$. The numerical results for $m = 8$, m=16,m=32are shown in Fig. 2, Fig. 3, Fig. 4. The exact solution is shown in Fig. 5. From the Fig. 2-5, we can see clearly that numerical solutions are in very good agreement with exact solution.

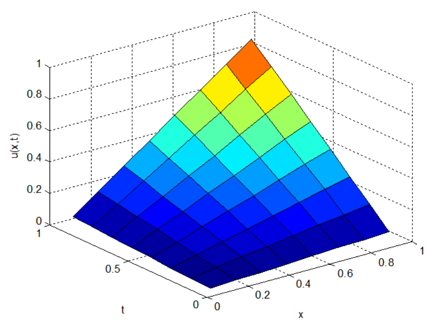

Figure 2: Numerical solution of m=8

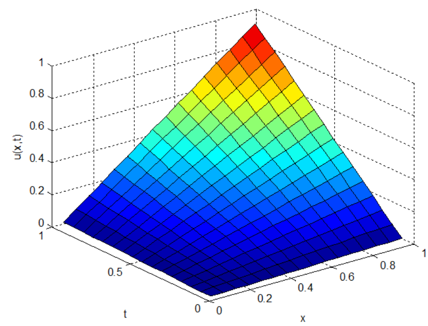

Figure 3: Numerical solution of m=16

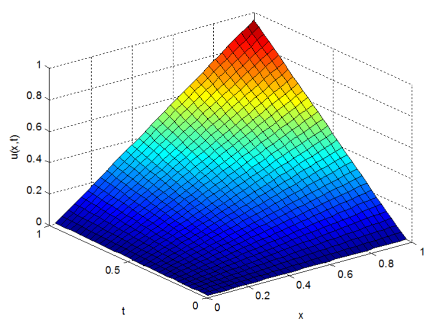

Figure 4: Numerical solution of m=32

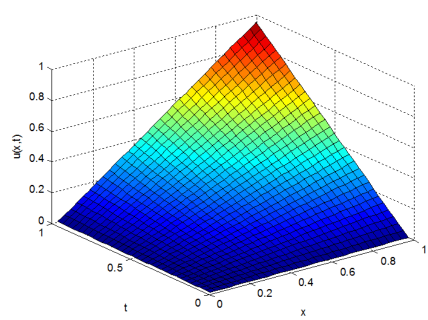

Figure 5: Exact solution for Example 1

6.2 Example 2

Consider the following fractional partial differential equation

$$\frac{\partial^{1/3}u}{\partial x^{1/3}}+\frac{\partial^{1/2}u}{\partial t^{1/2}}=f(x,t), \qquad 0\le x,t\le 1 \tag{42}$$

subject to $\left.\frac{\partial u}{\partial x}\right|_{t=0}=2x,\ \left.\frac{\partial u}{\partial t}\right|_{x=0}=2t,\ u(0,t)=t^2,\ u(x,0)=x^2,\ f(x,t)=\frac{\Gamma(3)x^{5/3}}{\Gamma(8/3)}+\frac{\Gamma(3)t^{3/2}}{\Gamma(5/2)}$.

The exact solution of the equation is x^2+t^2.

Let $\frac{\partial^2 u}{\partial x\partial t}\cong H_m^T(x)UH_m(t)$, then

$$\begin{aligned}\frac{\partial u}{\partial x}&=\int_0^t\frac{\partial^2 u}{\partial x\partial t}dt+\left.\frac{\partial u}{\partial x}\right|_{t=0}\cong\int_0^t[H_m^T(x)UH_m(t)]dt+\left.\frac{\partial u}{\partial x}\right|_{t=0}\\&=H_m^T(x)UP^1H_m(t)+2x\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial u}{\partial t}&=\int_0^x\frac{\partial^2 u}{\partial x\partial t}dx+\left.\frac{\partial u}{\partial t}\right|_{x=0}\cong\int_0^x[H_m^T(x)UH_m(t)]dx+\left.\frac{\partial u}{\partial t}\right|_{x=0}\\&=H_m^T(x)[P^1]^TUH_m(t)+2t\end{aligned} \tag{44}$$

Hence

$$u(x,t)\cong H_m^T(x)[P^1]^TUP^1H(t)+x^2+u(0,t)=H_m^T(x)[P^1]^TUP^1H(t)+x^2+t^2 \tag{45}$$

Then we have

$$\begin{aligned}\frac{\partial^{1/3}u}{\partial x^{1/3}}&=J^{2/3}\left(\frac{\partial u}{\partial x}\right)\cong J^{2/3}\left(H_m^T(x)UP^1H_m(t)+2x\right)\\&=H_m^T(x)[P^{2/3}]^TUP^1H_m(t)+\frac{2\Gamma(2)}{\Gamma(8/3)}x^{5/3}\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{\partial^{1/2}u}{\partial t^{1/2}}&=J^{1/2}\left(\frac{\partial u}{\partial t}\right)\cong J^{1/2}\left(H_m^T(x)[P^1]^TUH_m(t)+2t\right)\\&=H_m^T(x)[P^1]^TUP^{1/2}H_m(t)+\frac{2\Gamma(2)}{\Gamma(5/2)}t^{3/2}\end{aligned} \tag{47}$$

Substituting Eq.(46) and Eq.(47) into Eq.(42), we have

$$H_m^T(x)[P^{2/3}]^TUP^1H_m(t)+H_m^T(x)[P^1]^TUP^{1/2}H_m(t)=0 \tag{48}$$

According to Eq.(48), we may find that $U=0$is the solution of Eq.(48). Substituting $U=0$into Eq.(45), we get $u(x,t)=x^2+t^2$ which is the exact solution of the initial fractional partial differential equation.

6.3 Example 3

Consider this equation

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}=f(x,t), \qquad 0\le x,t\le 1 \tag{49}$$

such that $\left.\frac{\partial u}{\partial x}\right|_{t=0}=2x,\ \left.\frac{\partial u}{\partial t}\right|_{x=0}=2t,\ u(0,t)=t^2+1,\ u(x,0)=x^2+1,$
and$f(x,t)=\frac{\Gamma(3)x^{2-\alpha}(t^2+1)}{\Gamma(3-\alpha)}+\frac{\Gamma(3)(x^2+1)t^{2-\beta}}{\Gamma(3-\beta)}$, the exact solution is $(x^2+1)(t^2+1)$.
Let $\frac{\partial^2 u}{\partial x\partial t}\cong H_m^T(x)UH_m(t)$, then we have

$$\begin{aligned}\frac{\partial u}{\partial x}&=\int_0^t\frac{\partial^2 u}{\partial x\partial t}dt+\left.\frac{\partial u}{\partial x}\right|_{t=0}\cong\int_0^t[H_m^T(x)UH_m(t)]dt+\left.\frac{\partial u}{\partial x}\right|_{t=0}\\&=H_m^T(x)UP^1H_m(t)+2x\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial u}{\partial t}&=\int_0^x\frac{\partial^2 u}{\partial x\partial t}dx+\left.\frac{\partial u}{\partial t}\right|_{x=0}\cong\int_0^x[H_m^T(x)UH_m(t)]dx+\left.\frac{\partial u}{\partial t}\right|_{x=0}\\&=H_m^T(x)[P^1]^TUH_m(t)+2t\end{aligned} \tag{51}$$

Therefore

$$\begin{aligned}u(x,t)&\cong H_m^T(x)[P^1]^TUP^1H(t)+x^2+u(0,t)\\&=H_m^T(x)[P^1]^TUP^1H(t)+x^2+t^2+1\end{aligned} \tag{52}$$

Then we can get

$$\begin{aligned}\frac{\partial^\alpha u}{\partial x^\alpha}&=J^{1-\alpha}\left(\frac{\partial u}{\partial x}\right)\cong J^{1-\alpha}\left(H_m^T(x)UP^1H_m(t)+2x\right)\\&=H_m^T(x)[P^{1-\alpha}]^TUP^1H_m(t)+\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha}\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{\partial^\beta u}{\partial t^\beta}&=J^{1-\beta}\left(\frac{\partial u}{\partial t}\right)\cong J^{1-\beta}\left(H_m^T(x)[P^1]^TUH_m(t)+2t\right)\\&=H_m^T(x)[P^1]^TUP^{1-\beta}H_m(t)+\frac{\Gamma(3)}{\Gamma(3-\beta)}t^{2-\beta}\end{aligned} \tag{54}$$

Substituting Eq.(53), Eq.(54) into Eq.(49), we have

$$H_m^T(x)[P^{1-\alpha}]^TUP^1H_m(t)+H_m^T(x)[P^1]^TUP^{1-\beta}H_m(t)=g(x,t) \tag{55}$$

where $g(x,t)=\frac{\Gamma(3)x^{2-\alpha}t^2}{\Gamma(3-\alpha)}+\frac{\Gamma(3)x^2t^{2-\beta}}{\Gamma(3-\beta)}$. Similarly, $g(x,t)$can be expressed as follows

$$g(x,t)\cong H_m^T(x)GH_m(t) \tag{56}$$

where $G = [g_{ij}]_{m\times m}$. Dispersing Eq.(55) and Eq.(56) by the points (x_i, t_j), $i = 1,2,\cdots,m$and$j = 1,2,\cdots,m$, we can obtain

$$H_m^T[P^{1-\alpha}]^T UP^1 H_m + H_m^T[P^1]^T UP^{1-\beta} H_m = H_m^T G H_m \tag{57}$$

Namely

$$[P^{-1}]^T[P^{1-\alpha}]^T U + UP^{1-\beta}P^{-1} = [P^{-1}]^T GP^{-1} \tag{58}$$

Eq.(58) is a Sylvester equation. We can obtain $U$by solving it . Then using Eq.(52), we get the numerical solution of $u(x,t)$.

Table 1: Numerical solution of α=3/4,β=2/3

(x,t)	$m=8$	$m=16$	$m=32$	$m=64$
(0,0)	4.308840e-006	2.727204e-007	1.725163e-008	1.090640e-009
(1/8,1/8)	7.640897e-005	8.671441e-006	3.222372e-006	1.100921e-006
(2/8,2/8)	1.351753e-004	5.072255e-005	1.723830e-005	5.859165e-006
(3/8,3/8)	4.827074e-004	1.354669e-004	4.566683e-005	1.565391e-005
(4/8,4/8)	7.985028e-004	2.699575e-004	9.138683e-005	3.147262e-005
(5/8,5/8)	1.433897e-003	4.606640e-004	1.567464e-004	5.411168e-005
(6/8,6/8)	2.126908e-003	7.134967e-004	2.437371e-004	8.425278e-005
(7/8,7/8)	3.104180e-003	1.033717e-003	3.541178e-004	1.224983e-004

Taking $\alpha = 1/2$, $\beta = 1/3$, we may achieve the absolute errors for different m. The absolute errors are shown in Tab. 1. From the Tab. 1, we can see clearly that the absolute errors become more and more small when $m$increases. The numerical results and the exact result for$x = 0.25$, $m = 64$ are shown in Fig. 6. From the Fig. 6, we find easily that the numerical solutions are in good agreement with the exact solution.

6.4 Example 4

Consider the below fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\beta u}{\partial t^\beta} = \cos x + \cos t, \qquad 0 \le x,t \le 1 \tag{59}$$

subject to $\left.\frac{\partial u}{\partial x}\right|_{t=0} = \cos x$, $\left.\frac{\partial u}{\partial t}\right|_{x=0} = \cos t$, $u(0,t) = \sin t$, $u(x,0) = \sin x$.

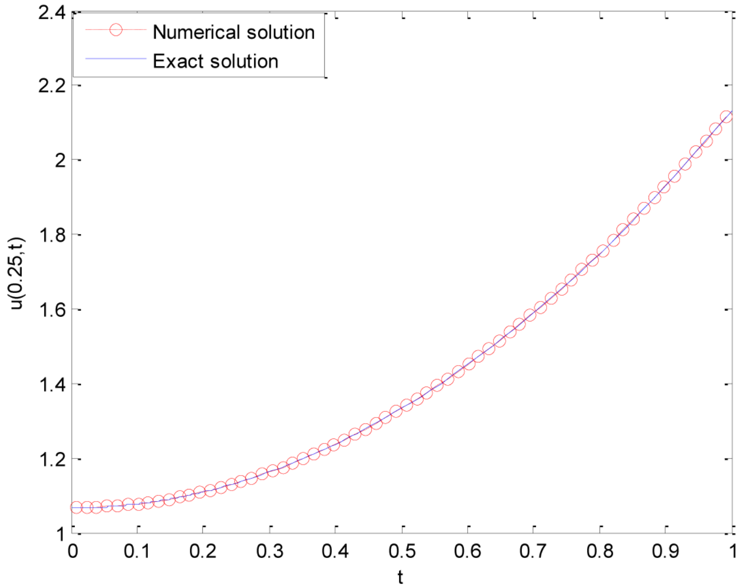

Figure 6: : Numerical solution of α=3/5,β=1/3

Let $\frac{\partial^2 u}{\partial x \partial t} \cong H_m^T(x)UH_m(t)$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \int_0^t \frac{\partial^2 u}{\partial x \partial t}dt + \left.\frac{\partial u}{\partial x}\right|_{t=0} \cong \int_0^t [H_m^T(x)UH_m(t)]dt + \left.\frac{\partial u}{\partial x}\right|_{t=0} \\ &= H_m^T(x)UP^1H_m(t) + \cos x\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &= \int_0^x \frac{\partial^2 u}{\partial x \partial t}dx + \left.\frac{\partial u}{\partial t}\right|_{x=0} \cong \int_0^x [H_m^T(x)UH_m(t)]dx + \left.\frac{\partial u}{\partial t}\right|_{x=0} \\ &= H_m^T(x)[P^1]^T UH_m(t) + \cos t\end{aligned} \tag{61}$$

Hence we have

$$\begin{aligned}u(x,t) &\cong H_m^T(x)[P^1]^T UP^1H(t) + \sin x + u(0,t) \\ &= H_m^T(x)[P^1]^T UP^1H(t) + \sin x + \sin t\end{aligned} \tag{62}$$

Substituting Eq.(60) and Eq.(61) into Eq.(59) when α=β=1

$$H_m^T(x)UP^1H_m(t) + H_m^T(x)[P^1]^T UH_m(t) = 0 \tag{63}$$

$U = 0$is the exact solution of Eq.(63). We can get $u(x,t) \cong \sin x + \sin t$by using Eq.(62). When α=β=1, the exact solution of initial partial differential equation is $\sin x + \sin t$.

When $\alpha, \beta \neq 1$, we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial x^\alpha} &= J^{1-\alpha}\left(\frac{\partial u}{\partial x}\right) \cong J^{1-\alpha}\left(H_m^T(x)UP^1H_m(t)+\cos x\right)\\ &= H_m^T(x)[P^{1-\alpha}]^T UP^1 H_m(t) + J^{1-\alpha}(\cos x)\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\partial^\beta u}{\partial t^\beta} &= J^{1-\beta}\left(\frac{\partial u}{\partial t}\right) \cong J^{1-\beta}\left(H_m^T(x)[P^1]^T UH_m(t)+\cos t\right)\\ &= H_m^T(x)[P^1]^T UP^{1-\beta} H_m(t) + J^{1-\beta}(\cos t)\end{aligned} \tag{65}$$

Substituting Eq.(64), Eq.(65) into Eq.(59), we have

$$H_m^T(x)[P^{1-\alpha}]^T UP^1 H_m(t) + H_m^T(x)[P^1]^T UP^{1-\beta} H_m(t) = g(x,t) \tag{66}$$

where $g(x,t) = \cos x - J^{1-\alpha}(\cos x) + \cos t - J^{1-\beta}(\cos t)$.
Let $\cos x \cong u_1^T H_m(x), \ \ \cos t \cong u_2^T H_m(t)$,
where $u_1 = [c_0, c_1, \ldots, c_{m-1}]^T, \ \ u_2 = [c'_0, c'_1, \ldots, c'_{m-1}]^T$.
Then $g(x,t)$will be

$$g(x,t) = \cos x - u_1^T P^{1-\alpha} H_m(x) + \cos t - u_1^T P^{1-\beta} H_m(t) \tag{67}$$

Similarly, $g(x,t)$can be also expressed as follows

$$g(x,t) \cong H_m^T(x) G H_m(t) \tag{68}$$

where $G = [g_{ij}]_{m\times m}$.
Dispersing Eq.(66) and Eq.(68) by the points (x_i, t_j), $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, m$, we have

$$H_m^T [P^{1-\alpha}]^T UP^1 H_m + H_m^T [P^1]^T UP^{1-\beta} H_m = H_m^T G H_m \tag{69}$$

Thus

$$[P^{-1}]^T [P^{1-\alpha}]^T U + UP^{1-\beta} P^{-1} = [P^{-1}]^T G P^{-1} \tag{70}$$

Eq.(70) is a Sylvester equation. We can get $U$by solving Eq.(70) . Then applying Eq.(62), we obtain the approximation of $u(x,t)$. Fig. 7 and Fig. 8 show the numerical solutions for different values of α, β. Here, we may take $m = 32$. Compared with the generalized differential transform method in [Odibat and Momani (2008)], taking advantage of above technique greatly reduces computation. What's more, the method in this paper is easy implementation.

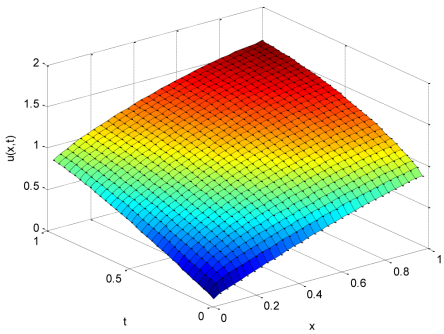

Figure 7: Numerical solution of $\alpha = 3/4, \beta = 2/3$

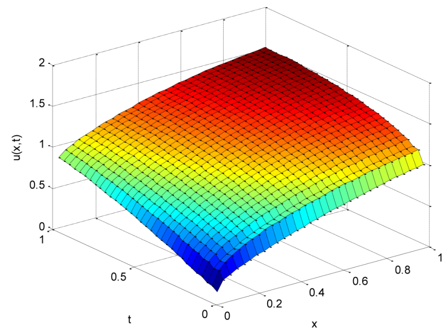

Figure 8: Numerical solution of $\alpha = 3/5, \beta = 1/3$

7 Conclusion

A numerical method for the fractional partial differential equations based on Haar wavelets operational matrix of fractional integration has been proposed. A general procedure of forming the matrix $P^{\alpha}$is summarized. This matrix is used to obtain the numerical solutions of a class of fractional partial differential equations. The convergence analysis of the Haar wavelet bases is given in section 4. The initial fractional partial differential equations have been transformed into Sylvester equation. Some numerical examples are provided to verify the validity of the method and the correctness of the theoretical analysis.

Compared with the Haar wavelets operational matrix method in the Ref.[Yi and Chen (2012)], we can also obtain the same Haar wavelets operational matrix of fractional integration. However, we needn't calculate the Haar wavelets operational matrix of fractional differentiation to solve the fractional partial differential equations. Therefore, our method may greatly reduce the computation and achieve the numerical solutions with good coincidence.

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