# Richardson Extrapolation Method for Singularly Perturbed Convection-Diffusion Problems on Adaptively Generated Mesh

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**Abstract:** Adaptive mesh generation has become a valuable tool for the improvements of accuracy and efficiency of numerical solutions over fixed number of meshes. This paper gives an interpretation of the concept of equidistribution for singularly perturbed problems to obtain higher-order accuracy. We have used the post-processing Richardson extrapolation technique to improve the accuracy of the parameter uniform computed solution, obtained on a mesh which is adaptively generated by equidistributing a monitor function. Numerical examples demonstrate the high quality behavior of the computed solution.

**Keywords:** Singularly perturbed problem, Layer-adapted mesh, Mesh equidistribution, Uniform convergence, Higher-order convergence, Extrapolation technique.

## 1 Introduction

The solution of singularly perturbed problem typically exhibits sharp layers of different widths at the boundary as well as at the interior part of the domain. This problem arises in the modeling of convection dominated flow problems in fluid dynamics for e.g., linearized Navier-Stoke equation with high Reynolds number. Because of the presence of these layers, standard numerical methods fail to give accurate results. To obtain a reliable numerical approximation, one well-known technique is to use, locally refined meshes that are fine in narrow layer regions and coarse outside. Several research have been done to get uniform convergence where the mesh is chosen *apriori*. Here our aim is to construct higher-order accurate solution on adaptively generated fully nonuniform meshes by the moving mesh method which can automatically detect accurate locations and widths of the layers.

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In the present paper, we consider the following singularly perturbed convection-diffusion problem:

$$\begin{cases} \mathfrak{L}u(x) \equiv -\varepsilon u''(x) - (a(x)u(x))' = f(x), & x \in \Omega = (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{1}$$

where $\varepsilon$ $(0 < \varepsilon \ll 1)$ is the parameter. It will be assumed that $a(x)$ and $f(x)$ are in $\mathcal{C}^2(\Omega)$. Under these assumptions, the problem (1) admits a unique solution $u(x) \in \mathcal{C}^4(\Omega) \cap \mathcal{C}(\overline{\Omega})$ where $\overline{\Omega} = [0,1]$. In general, the solution $u(x)$ of (1) exhibits a boundary layer at $x = 0$, if $a(x)$ has a positive lower bound.

In the last few decades, the construction of $\varepsilon$-uniformly convergent schemes for singularly perturbed problems are attracted by several researchers. The *apriori* chosen piecewise-uniform Shishkin mesh and Bakhvalov mesh are investigated by several authors (see Miller, O'Riordan, and Shishkin (1996); Roos, Stynes, and Tobiska (2008)) for singularly perturbed convection-diffusion problems. But, most of them are almost first-order accurate. In this context, Natividad and Stynes (2003) considered a higher-order convergent technique using Richardson extrapolation method which improves the order of convergence $O(N^{-1} \ln N)$ to $O(N^{-2} \ln^2 N)$ on piecewise uniform Shishkin mesh. The extension of this technique for system of differential equations and partial differential equations can be seen in Deb and Natesan (2008); Mukherjee and Natesan (2011) for Shishkin mesh. On this mesh, Das and Natesan (2013) proposed a hybrid scheme which provides $O(N^{-2} \ln^2 N)$ solution for a system of Robin type singularly perturbed reaction-diffusion problems. In the present paper, our aim is to achieve the higher-order convergent solution on the adaptively generated mesh for a singularly perturbed convection-diffusion problem of the form (1).

A commonly used technique in adaptive mesh generation is based on the idea of equidistribution. A mesh $\Omega^N \equiv \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ is said to be equidistributed, if

$$\int_{x_{j-1}}^{x_j} M\big(s, u(s)\big) ds = \int_{x_j}^{x_{j+1}} M\big(s, u(s)\big) ds, \quad j = 1, \ldots, N-1, \tag{2}$$

where $M\big(s, u(s)\big) > 0$ is called the monitor function. This monitor function is normally some measure of computational error or solution variation, specially where the solution changes rapidly. Equivalently, (2) can be expressed as

$$\int_{x_{j-1}}^{x_j} M\big(s, u(s)\big) ds = \frac{1}{N} \int_0^1 M\big(s, u(s)\big) ds, \quad j = 1, \ldots, N. \tag{3}$$

It is common to use monitor functions which are bounded away from zero by a constant to prevent mesh starvation outside the layer. In practice, the monitor func-

tions are based on the simple functions which involves the derivatives of the unknown solution. In this regard, one can refer Das and Natesan (2012) where a monitor function, based on the curvature of the solution is used to achieve higher-order accurate solution for singularly perturbed Robin type reaction-diffusion problems. The extension of this monitor function for parabolic problems can be seen in Gowrisankar and Natesan (2012). The monitor function considered in this article is originally due to Beckett and Mackenzie (2000). A first-order convergence is observed by them for singularly perturbed convection-diffusion problems. Here, our aim is to construct a higher-order convergent solution on equidistributed mesh using Richardson extrapolation method.

In this paper, the following two monitor functions will be used for the error analysis

$$M\big(x, u(x)\big) = 1 + |u''(x)|^{1/2}, \text{ and } M\big(x, w(x)\big) = 1 + |w''(x)|^{1/2}. \tag{4}$$

Here $u(x)$ is the solution of (1) and $w(x)$ is the singular component of the solution $u(x)$.

The outline of this paper is as follows: In Section 2, the derivative bounds of the analytical solution $u(x)$ of (1) is introduced. Its decomposition into the smooth and singular components and their derivatives bounds are established in this section. The stability of the continuous solution is also provided here. A finite difference discretization of the continuous problem (1) and the stability of the discrete solution is introduced in Section 3. Section 4 is devoted to study the detailed error analysis. Also, two monitor functions are generated from the error analysis, which will lead to the first-order parameter-uniform convergence on the equidistributed mesh. Richardson extrapolation technique is used at the end of this section, where the error is improved to second-order accuracy by equidistributing the proposed monitor functions. Finally, Section 5 provides numerical experiments to support the theoretical findings using an adaptive algorithm.

Throughout this paper, $C$ denotes a generic positive constant independent of $\varepsilon$, $x_i$-the discretized grid and $N$-number of mesh intervals, and can take different values at different places, even in the same argument. A subscripted $C$ (*for e.g., $C_1$*) is a constant that is independent of $\varepsilon$ and of the nodal point $x_i$, but whose value is fixed. To simplify the notation, we set $g_i = g(x_i)$ for any function $g$, while the superscript $G_i^N$ or/ $G_i$ denotes an approximation of $g$ at $x_i$. We denote the discrete maximum norm as $||\cdot||_\infty$ where $||\phi||_\infty = \max_{\xi \in \Omega} |\phi(\xi)|$ for a function $\phi$ defined on some domain $\Omega$. When the domain is obvious, or of no particular significance, we simply write $||\cdot||$.

## 2   Solution Decomposition and Derivative Bounds

This section presents the standard *apriori* bounds of the analytical solution of (1) and its derivatives. For the analysis presented here, we assume that $a(x) = a$, a constant in (1) such that $a \gg \varepsilon > 0$. The next lemma provides *apriori* bounds of the solution and its derivatives.

**Lemma 2.1** *The solution $u(x)$ of (1) and its derivatives satisfy the following bounds for any prescribed r,*

$$|u^{(k)}(x)| \leq C\big(1 + \varepsilon^{-k}\exp(-ax/\varepsilon)\big), \quad \text{for } k = 0, \cdots, r.$$

**Proof.** The proof of this lemma can be seen in Kellogg and Tsan (1978).     ∎

To establish the parameter-uniform properties of the numerical methods, we shall decompose the analytic solution $u$ into the smooth component $v$ and the singular component $w$ such that $u = v + w$. The following lemma provides an insight about the derivative bounds of the smooth component $v$ and the singular component $w$.

**Lemma 2.2** *The smooth component $v(x)$ of the solution $u(x)$ of (1) satisfying*

$$\begin{cases} \mathfrak{L}v(x) = f(x), & x \in \Omega, \\ v(0) = v_\varepsilon, & v(1) = 0, \quad \text{with a suitable} \quad v_\varepsilon, \end{cases}$$

*admits the following bound*

$$|v^{(k)}(x)| \leq C, \quad \text{for } k = 0, \cdots, r, \text{ and any prescribed } r,$$

*while the singular component $w(x)$ satisfying*

$$\begin{cases} \mathfrak{L}w(x) = 0, & x \in \Omega, \\ w(0) = -v(0), & w(1) = 0, \end{cases}$$

*is of the form $w(x) = A + B\exp(-ax/\varepsilon)$, where*

$$A = \frac{v(0)\exp(-a/\varepsilon)}{1 - \exp(-a/\varepsilon)}, \quad B = \frac{-v(0)}{1 - \exp(-a/\varepsilon)}.$$

**Proof.** To obtain the derivative bounds for the smooth component $v(x)$, consider the decomposition

$$v(x) = \sum_{i=0}^{r+1} \varepsilon^i v_i(x), \text{ with } v(1) = 0.$$

Now, comparing the powers of $\varepsilon$, we get

$$-av'_0(x) = f(x), \quad v_0(1) = 0,$$

$$-av'_i(x) = v''_{i-1}(x), \quad v_i(1) = 0, \quad i = 1, \cdots, r,$$

$$\mathfrak{L}v_{r+1}(x) = v''_r(x), \quad v_{r+1}(1) = 0, \quad v_{r+1}(0) = u(0) = 0.$$

Since the last equation is similar to (1), Lemma 2.1 can be used to bound the term $v_{r+1}(x)$. Combining the derivative bounds of each component, we obtain the required bound. A direct calculation from

$$\mathfrak{L}w(x) = 0, \quad w(0) = -v(0), \quad w(1) = 0,$$

leads to

$$w(x) = A + B\exp(-ax/\varepsilon), \quad \text{with} \quad A = \frac{v(0)\exp(-a/\varepsilon)}{1 - \exp(-a/\varepsilon)}, \quad B = \frac{-v(0)}{1 - \exp(-a/\varepsilon)},$$

which proves the lemma. ∎

The continuous operator $\mathfrak{L}$ defined in (1) enjoys the following stability property

$$||v||_{\infty,\overline{\Omega}} \leq 2a^{-1}||\mathfrak{L}v||_{*,\overline{\Omega}}, \quad \text{for all } v \text{ with} \quad v(0) = v(1). \tag{5}$$

This result is obtained by Andreev (2001), using the Green's function associated with the operator $\mathfrak{L}$.

## 3 Discretization of the Continuous Problem

In this section, we explicitly describe a upwind finite difference discretization for the problem (1). For a given a discrete function $\{v_i\}_{i=0}^{N}$ on $\Omega^N \equiv \{0 = x_0 < x_1 < \cdots < x_N = 1\}$, define the forward and backward operators

$$D^+v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad \text{and} \quad D^-v_i = \frac{v_i - v_{i-1}}{h_i},$$

respectively, where $h_i = x_i - x_{i-1}$. The boundary-value problem (1) is discretized by the following difference scheme:

$$\begin{cases} [\mathfrak{L}^N U]_i \equiv -\varepsilon[D^+D^-U]_i - [D^+(aU)]_i = f_i, & \text{for} \quad i = 1, \cdots, N-1, \\ U_0 = U_N = 0. \end{cases} \tag{6}$$

The discrete operator $\mathfrak{L}^N$ satisfies the following stability property

$$||U||_{\infty,\Omega^N} \leq 2a^{-1}||\mathfrak{L}^N U||_{*,\Omega^N}, \quad \text{for all} \quad U \in R_0^{N+1}, \tag{7}$$

(see Andreev (2001)), where

$$||U||_{*,\Omega^N} = ||U||_{-1,\infty,\Omega^N} = \max_{i=0,\cdots,N-1} \left| \sum_{p=i}^{N-1} h_{p+1}U_p \right|. \tag{8}$$

The next lemma is very useful to have an upper bound of the mesh spacings throughout the domain.

**Lemma 3.1** *The mesh widths generated by equidistribution of the monitor functions (4) satisfy*

$$h_i \leq CN^{-1}, \text{ for } i = 1, \cdots, N.$$

**Proof.**  It is clear that the monitor function (4) satisfies $M(x, w(x)) \geq 1$. Again the derivative estimates from Lemma 2.2 imply $\int_0^1 M(x, w(x))dx \leq C_1$ for some constant $C_1$. Hence, the equidistribution principle (3) leads to

$$h_i \leq \int_{x_{i-1}}^{x_i} M(x, w(x))dx = \frac{1}{N} \int_0^1 M(x, w(x))dx \leq C_1 N^{-1} \Rightarrow h_i \leq C_1 N^{-1}.$$

This implies that

$$h_i \leq CN^{-1}.$$

By adopting a similar technique, it is easy to check that $h_i \leq CN^{-1}$ for the monitor function $M(x, u(x))$.                                                                                         ∎

## 4   Error Analysis

In this section, the standard error analysis involving the monitor functions given in (4) is presented. These monitor functions are generated from the error expansion. First, we provide a technical lemma, which is used for the proof of $\varepsilon$-uniform error estimate.

**Lemma 4.1** *Consider the following continuous problem*

$$\mathfrak{L}\phi = \Phi', \quad x \in \Omega, \qquad \phi(0) = 0, \quad \phi(1) = 0. \tag{9}$$

*Here $\Phi(x)$ is a piecewise continuously differentiable function where the differentiation can be understood in the distributional sense. If $\phi(x) \in \mathcal{C}^2(\Omega)$, then it satisfies the following equality*

$$\sum_{p=i}^{N-1} h_{p+1}[\mathfrak{L}^N \phi]_p = \varepsilon([D^-\phi]_i - \phi'_{i-0}) - \varepsilon([D^-\phi]_N - \phi'_{N-0}) + \Phi_{N-0} - \Phi_{i-0}.$$

**Proof.** This result can be obtained by integrating the continuous problem (9) and the discrete problem (6) over the interval $(x_i, 1)$ and combining them (see Linß (2004)). ∎

From this lemma, we can obtain the truncation error estimate of the numerical solution $U$ of (6). This error becomes

$$\sum_{p=i}^{N-1} h_{p+1}[\mathfrak{L}^N(u-U)]_p = \varepsilon([D^-u]_i - u'_{i-0}) \quad - \quad \varepsilon([D^-u]_N - u'_{N-0}) + \tag{10}$$

$$\int_{x_i}^{1} f(x)dx - \sum_{p=i}^{N-1} h_{p+1}f_p.$$

Thus, using the discrete norm defined in (8), we have

$$||\mathfrak{L}^N(u-U)||_{*,\Omega^N} \leq 2\varepsilon \max_{i=1,\cdots,N} |[D^-u]_i - u'_{i-0}| + \max_{i=0,\cdots,N-1} \left| \int_{x_i}^{1} f(x)dx - \sum_{p=i}^{N-1} h_{p+1}f_p \right|. \tag{11}$$

One can observe that the terms appearing in the right-hand side of the truncation error bound will provide at most first-order convergence. In fact, with the help of discrete stability estimate (7), it is proved in Linß (2001) that

$$||u-U||_{*,\Omega^N} \leq C \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + |w''(x)|^{1/2} \right) dx.$$

Here, we have used Lemma 2.2. From this expression, it is clear that the equidistribution principle (2) of the monitor function $M(x, w(x))$ leads to a first-order convergent solution of (6). This monitor function is originally due to Beckett and Mackenzie (2000) where a technique used in Pereyra and Sewell (1975) followed for the convergence analysis.

Now onwards, for the sake of convenience, the left-hand derivative of $u(x)$ at $x_i$ will be denoted by $u'_i$ instead of $u'_{i-0}$. To improve the accuracy of the numerical solution, we have to eliminate the first-order dominating terms from the expression (11). This can be achieved by introducing the extrapolation technique. Assume that $\chi(x)$, the leading term of the error expansion is the solution of the following problem

$$\mathfrak{L}\chi = \Lambda', \quad \chi(0) = \chi(1) = 0, \quad \text{with} \quad \Lambda(x) = \frac{\varepsilon h(x)u''(x)}{2} - \int_{x}^{1} h(t)f'(t)dt, \tag{12}$$

where

$$h(x) = x - x_{p-1}, \quad \text{on} \quad x \in (x_{p-1}, x_p).$$

Then, using Lemma 4.1, we have

$$\sum_{p=i}^{N-1} h_{p+1}[\mathfrak{L}^N(u-\chi - U)]_p = \varepsilon([D^-u]_i - u'_i + \frac{h_i}{2}u''_i) - \varepsilon([D^-u]_N - u'_N$$

$$+\frac{h_N}{2}u''_N) + \int_{x_i}^1 \left(f(x) - h(x)f'(x)\right)dx - \sum_{p=i}^{N-1} h_{p+1}f_p +$$

$$\varepsilon([D^-\chi]_N - \chi'_N) - \varepsilon([D^-\chi]_i - \chi'_i). \tag{13}$$

From the above equality, it can be noted that the expressions involving $u(x)$ and $f(x)$ appearing in the right-hand side of (13) are of second-order. Third expression involving $\chi(x)$ is also of second-order, as the leading error $\chi(x)$ itself is first-order.

Now, we shall analyze each term of the error expression appearing in (13) separately. This will give us an insight to the place, where the monitor functions and the given boundary-value problem (BVP) (1) are used. Observe that, the given equation (1) implies that $|u'| \leq C(1 + \varepsilon|u''|)$. Again, differentiating equation (1) we get $\varepsilon u''' = -f' - a''u - 2a'u' - au''$, which implies that $|\varepsilon u'''| \leq C(1 + |u''|)$.

Now, consider the first term of the right-hand side expression in (13). Noting the above observations, Taylor series expansion with integral form of remainder yields

$$\varepsilon\left([D^-u]_i - u'_i + \frac{h_i}{2}u''_i\right) = \frac{\varepsilon}{2h_i}\left|\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2 u'''(x)dx\right|$$

$$\leq \frac{C}{2h_i}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2[1+|u''(x)|]dx$$

$$\leq C\max_{[x_{i-1},x_i]} h_i^2[1+|u''(x)|]$$

$$\leq C\max_{[x_{i-1},x_i]} h_i^2[1+|w''(x)|]$$

$$\leq C\max_{[x_{i-1},x_i]} h_i^2[1+|w''(x)|^{1/2}]^2, \tag{14}$$

where Lemma 2.2 is used. Again Taylor series expansion yields

$$f(s) = f_{i-1} + (s-x_{i-1})f'(s) - \int_{x_{i-1}}^s (x-x_{i-1})f''(x)dx.$$

Integrating the above expression over $(x_{i-1}, x_i)$ leads to

$$\int_{x_{i-1}}^{x_i}(f(x)-(x-x_{i-1})f'(x))dx - h_i f_{i-1} = -\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^s (x-x_{i-1})f''(x)dxds.$$

Since $f(x) \in \mathcal{C}^2(\Omega)$, we have

$$\left| \int_{x_{i-1}}^{x_i} (f(x) - (x - x_{i-1})f'(x))dx - h_i f_{i-1} \right| \leq \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{s} (x - x_{i-1})|f''(x)|dxds$$

$$\leq Ch_i^3. \tag{15}$$

Hence,

$$\left| \int_{x_i}^{1} \left( f(x) - h(x)f'(x) \right)dx - \sum_{p=i+1}^{N} h_p f_{p-1} \right|$$

$$\leq \sum_{p=i+1}^{N} \left| \int_{x_{p-1}}^{x_p} (f(x) - (x - x_{p-1})f'(x))dx - h_p f_{p-1} \right|$$

$$\leq C \sum_{p=i+1}^{N} h_p^3 \leq C \max_{p} h_p^2 \sum_{p=i+1}^{N} h_p \leq C \max_{i} h_i^2$$

$$\leq C \max_{i} \max_{[x_{i-1},x_i]} h_i^2 [1 + |w''(x)|^{1/2}]^2. \tag{16}$$

To bound the fifth term of the right-hand side expression in (13), we need the derivative bound of the leading error term $\chi(x)$. The following technical lemma is very useful to derive the derivative bounds of $\chi(x)$.

**Lemma 4.2** *Let $x \in (x_{i-1}, x_i)$. Then, we have*

$$h(x)(1 + \varepsilon^{-1}\exp(-ax/2\varepsilon)) \leq \int_{x_{i-1}}^{x} \left( 1 + \varepsilon^{-1}\exp(-at/2\varepsilon) \right)dt.$$

**Proof.** It is easy to check that the difference between the two functions appearing in either sides of the above inequality is monotonic in the interval $(x_{i-1}, x_i)$. ∎

The following lemma provides the derivative estimates of $\chi(x)$.

**Lemma 4.3** *Assume that $a(x), f(x) \in \mathcal{C}^2(\Omega)$. Then the solution of (12) and its derivatives satisfy*

$$|\chi^{(p)}(x)| \leq C\left(1 + \varepsilon^{-p}\exp(-ax/2\varepsilon)\right) \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + \varepsilon^{-1}\exp(-at/2\varepsilon) \right)dt,$$

*for $p = 0, 1$, and*

$$\varepsilon|\chi''(x)| \leq C\left(1 + \varepsilon^{-1}\exp(-ax/2\varepsilon)\right) \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + \varepsilon^{-1}\exp(-at/2\varepsilon) \right)dt$$

$$+ h(x)(1 + \varepsilon^{-1}\exp(-ax/\varepsilon)), \quad for \quad x \in \Omega \backslash \Omega^N. \tag{17}$$

**Proof.** The proof of this lemma is given by Linß (2004). ∎

Now, the first derivative bound of $\chi(x)$ and the mean-value theorem yield

$$
\begin{aligned}
\varepsilon([D^-\chi]_i - \chi_i') &= \varepsilon\left|\frac{\chi_i - \chi_{i-1}}{h_i} - \chi_i'\right| \\
&\leq \varepsilon \max_{(x_{i-1}, x_i)} |\chi'(x)| \\
&\leq C \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1}\exp(-ax/2\varepsilon)\right) dx \\
&\leq C \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left(1 + |w''(x)|^{1/2}\right) dx.
\end{aligned}
$$

This implies that the proposed technique will lead to first-order convergence if the function $M(x, w(x))$ is equidistributed as a monitor function. Nevertheless, we can improve the convergence rate by using the second-order derivative bound of $\chi(x)$ from Lemma 4.3. Again, Taylor series expansion with integral form of the remainder leads to

$$
\begin{aligned}
\varepsilon([D^-\chi]_i - \chi_i') &= \frac{\varepsilon}{h_i}\left|\int_{x_{i-1}}^{x_i} (x - x_{i-1})\chi''(x)dx\right| \\
&\leq \frac{C}{h_i}\int_{x_{i-1}}^{x_i} h^2(x)\left(1 + \varepsilon^{-1}\exp(-ax/\varepsilon)\right) dx \\
&\quad + \frac{C}{h_i}\max_{i=1,\cdots,N}\int_{x_{i-1}}^{x_i}\left(1 + \varepsilon^{-1}\exp(-ax/2\varepsilon)\right) dx \times \\
&\quad \int_{x_{i-1}}^{x_i} (x - x_{i-1})\left(1 + \varepsilon^{-2}\exp(-ax/2\varepsilon)\right) dx \\
&= I_1 + I_2.
\end{aligned}
$$

Here

$$
\begin{aligned}
I_1 &= \frac{C}{h_i}\int_{x_{i-1}}^{x_i} h^2(x)\left(1 + \varepsilon^{-1}\exp(-ax/\varepsilon)\right) dx \\
&\leq C\int_{x_{i-1}}^{x_i} (x - x_{i-1})dx + \int_{x_{i-1}}^{x_i} (x - x_{i-1})\varepsilon^{-2}\exp(-ax/\varepsilon)dx.
\end{aligned}
$$

Now, we shall use the following identity from Pereyra and Sewell (1975): For any positive monotonically decreasing function $\psi(x)$ defined on $[a, b]$ and arbitrary

$k \in \mathbb{N}^+$, we have

$$\int_a^b \psi(t)(t-a)^{(k-1)} dt \leq \frac{1}{k} \left[ \int_a^b \psi(t)^{1/k} dt \right]^k.$$

Hence, the above identity for $k = 2$ implies that

$$
\begin{aligned}
I_1 \quad &\leq Ch_i^2 + C \left[ \int_{x_{i-1}}^{x_i} \left( \varepsilon^{-1} \exp(-ax/2\varepsilon) \right) dx \right]^2 \\
&\leq Ch_i^2 + C \left[ \int_{x_{i-1}}^{x_i} \left( 1 + \varepsilon^{-1} \exp(-ax/2\varepsilon) \right) dx \right]^2 \\
&\leq Ch_i^2 + C \left[ \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + |w''(x)|^{1/2} \right) dx \right]^2.
\end{aligned}
$$

A similar technique used to derive the bound for $I_1$ can be applied to get

$$
\begin{aligned}
I_2 \quad &= \quad \frac{C}{h_i} \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + \varepsilon^{-1} \exp(-ax/2\varepsilon) \right) dx \times \\
&\qquad \int_{x_{i-1}}^{x_i} (x - x_{i-1}) \left( 1 + \varepsilon^{-1} \exp(-ax/2\varepsilon) \right) dx \\
&\leq \quad C \left[ \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + |w''(x)|^{1/2} \right) dx \right]^2.
\end{aligned}
$$

It should be noted that $\max_i \max_{[x_{i-1},x_i]} h_i^2 [1 + |w''(x)|^{1/2}]^2$ is the discrete analogue of continuous form of the error estimate $\left[ \max_{i=1,\cdots,N} \int_{x_{i-1}}^{x_i} \left( 1 + |w''(x)|^{1/2} \right) dx \right]^2$. From these two expressions, it is clear that the monitor function which is being equidistributed to obtain the error estimate is $M(x, w(x)) = 1 + |w''(x)|^{1/2}$, where $w(x)$ is the singular component of the solution $u(x)$.

Henceforth, combining all these above estimates, it is clear from the equality (13) that

$$||\mathcal{L}^N(u - \chi - U)||_{*,\Omega^N} \leq C \max_i \max_{[x_{i-1},x_i]} h_i^2 [1 + |w''(x)|^{1/2}]^2. \tag{18}$$

Hence, the stability estimate (7) implies that

$$||u - \chi - U||_{*,\Omega^N} \leq C \max_i \max_{[x_{i-1},x_i]} h_i^2 [1 + |w''(x)|^{1/2}]^2. \tag{19}$$

The key idea of Richardson extrapolation is to provide better numerical approximation of the exact solution, by considering an average of the numerical solutions in two embedded meshes. To explain this method, let us define a mesh $\bar{\Omega}^N \equiv \{0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_{2N} = 1\}$, which is obtained by bisecting the original mesh $\Omega^N$ with the step size $\bar{h}_i = \bar{x}_i - \bar{x}_{i-1}$.

Now consider the discrete problem

$$\begin{cases} [\bar{\mathfrak{L}}^N \bar{U}]_i \equiv -\varepsilon[D^+ D^- \bar{U}]_i - [D^+(\bar{a}\bar{U})]_i = \bar{f}_i, & \text{for} \quad i = 1, \cdots, 2N-1, \\ \bar{U}_0 = \bar{U}_N = 0. \end{cases} \tag{20}$$

Extrapolation technique will be used to improve the solutions at the points $x_i \in \Omega^N$, with the help of $\bar{U}$. As like (7), the stability estimate of the solution $\bar{U}$ implies that

$$||v||_{\infty,\Omega^N} \leq 2a^{-1} \max_{i=1,\cdots,N-1} \left| \sum_{p=2i}^{2N-1} \bar{h}_{p+1} [\bar{\mathfrak{L}}^N v]_p \right|, \quad \text{where} \quad v \in R_0^{2N+1}. \tag{21}$$

Hence, following the technique of obtaining the expression (13), we deduce that

$$\sum_{p=2i}^{2N-1} \bar{h}_{p+1} [\bar{\mathfrak{L}}^N (u - \frac{\chi}{2} - \bar{U})]_p = \varepsilon([D^- u]_{2i} - u'_{2i} + \frac{h_i}{4} u''_i) - \varepsilon([D^- u]_{2N} - u'_{2N}$$

$$+ \frac{h_N}{4} u''_N) + \int_{x_i}^1 \left( f(x) - \frac{h(x)}{2} f'(x) \right) dx - \sum_{p=2i}^{2N-1} \bar{h}_{p+1} \bar{f}_p$$

$$+ \frac{\varepsilon}{2} \left( [D^- \chi]_{2N} - \bar{\chi}'_{2N} \right) - \frac{\varepsilon}{2} \left( [D^- \chi]_{2i} - \chi'_{2i} \right), \tag{22}$$

where $\bar{h}_{2i} = h_i/2$ for $i = 1, \cdots, N-1$. The procedure of bounding the first and fifth terms of (13) can be extended to find the bounds of the corresponding terms in (22). Hence, we shall be considering only the expressions involving $f(x)$.

Denoting $f_{p+1/2} = f(x_p + x_{p+1})/2$, observe that

$$\int_{x_p}^{x_{p+1}} \left( f(x) - \frac{h(x)}{2} f'(x) \right) dx - \bar{h}_{2p+1} \bar{f}_{2p} - \bar{h}_{2p+2} \bar{f}_{2p+1}$$

$$= \int_{x_p}^{x_{p+1}} \left( f(x) - \frac{h(x)}{2} f'(x) \right) dx - \frac{h_{p+1}}{2} (f_p + f_{p+1/2})$$

$$= \frac{1}{2} \left[ \int_{x_p}^{x_{p+1}} (f(x) - h(x) f'(x)) dx - h_{p+1} f_p \right] + \frac{1}{2} \left[ \int_{x_p}^{x_{p+1}} f(x) dx - h_{p+1} f_{p+1/2} \right]. \tag{23}$$

The first term of the above expression is bounded from (15), *i.e.,*

$$\left| \int_{x_p}^{x_{p+1}} (f(x) - h(x)f'(x))dx - h_{p+1}f_p \right| \leq Ch_{p+1}^3.$$

To bound the second term, the Taylor series expansion of $f(x)$ with respect to the point $x_{p+1/2}$, up to second-order derivative implies that

$$\left| \int_{x_p}^{x_{p+1}} f(x)dx - h_{p+1}f_{p+1/2} \right| \leq Ch_{p+1}^3.$$

By combining the above two inequalities, we get

$$\int_{x_i}^{1} \left( f(x) - \frac{h(x)}{2}f'(x) \right)dx - \sum_{p=2i}^{2N-1} \bar{h}_{p+1}\bar{f}_p \quad \leq C\max_i h_i^2$$

$$\leq C\max_i \max_{[x_{i-1},x_i]} h_i^2[1 + |w''(x)|^{1/2}]^2.$$

$$\tag{24}$$

Hence, the stability estimate for the discrete operator (20) leads to

$$||u - \frac{\chi}{2} - \bar{U}||_{\infty,\Omega^N} \leq C\max_i \max_{[x_{i-1},x_i]} h_i^2[1 + |w''(x)|^{1/2}]^2. \tag{25}$$

Now, we are in a position to define the extrapolated solution. Let $U_{extp}$ be the solution obtained through Richardson extrapolation, which is defined as

$$U_{i,extp} = 2\bar{U}_{2i} - U_i, \quad \text{for} \quad i = 0, \cdots, N. \tag{26}$$

Then, from the triangle inequality, we have

$$\begin{aligned} ||u - U_{extp}||_{\infty,\Omega^N} &= ||(2u - \chi - 2\bar{U}) - (u - \chi - U)||_{\infty,\Omega^N} \\ &\leq 2||u - \frac{1}{2}\chi - \bar{U}||_{\infty,\Omega^N} + ||u - \chi - U||_{\infty,\Omega^N}. \end{aligned} \tag{27}$$

Therefore, by combining (19) and (25) in (27), we get

$$||u - U_{extp}||_{\infty,\Omega^N} \leq C\max_i \max_{[x_{i-1},x_i]} h_i^2[1 + |w''(x)|^{1/2}]^2. \tag{28}$$

Note that the error estimator appearing in the right-hand side of the above expression depends on the singular component $w(x)$ of the solution $u(x)$. In reality, from the *apriori* analysis, it is observed that the boundary layer phenomena occurs actually from the singular component of the solution. Assuming sufficient smoothness

of the given data $a(x)$ and $f(x)$, it should also be noted that the derivatives of the decomposed solution's smooth component $v(x)$ can be uniformly bounded irrespective of the perturbation parameter $\varepsilon$ from Lemma 2.2. This fact can be used to get a proper error estimator to improve the numerical solution, which can avoid of finding the singular component of the solution at the time of generating a new mesh through equidistribution. Now observe that the solution decomposition and Lemma 2.2 lead to

$$|w''(x)| \leq |u''(x)| + |v''(x)| \leq C[1 + |u''(x)|].$$

Hence the expression (28) reduces to

$$\|u - U_{extp}\|_{\infty,\Omega^N} \leq C \max_i \max_{[x_{i-1},x_i]} h_i^2 [1 + |u''(x)|^{1/2}]^2. \tag{29}$$

Therefore, we can state the main theorem of this chapter as follows.

**Theorem 4.4** *If u is the solution of the convection-diffusion problem (1) and $U_{extp}$ is the extrapolated solution obtained through Richardson extrapolation formula (26), then we have*

$$\|u - U_{extp}\|_{\infty,\Omega^N} \leq C \max_i \max_{[x_{i-1},x_i]} h_i^2 [1 + |u''(x)|^{1/2}]^2, \tag{30}$$

*where C is independent of the perturbation parameter $\varepsilon$ and the number of mesh intervals N.*

## 5  Numerical Computation

This section computationally verifies the theoretical findings with the proposed monitor functions. The generation of the finite difference solution using adaptive technique requires two steps; firstly the adaptive mesh has to be determined by a mesh generation algorithm and thereafter, the finite difference solution will be computed on that mesh. We consider the well-known de Boor algorithm to generate the adaptive mesh.

### 5.1  *Adaptive mesh generation algorithm*

The following iterative algorithm will be used for equidistributing the proposed monitor functions (4). Here, we have stated the algorithm by considering $M(x, w(x))$ as a monitor function. This algorithm is applied by Kopteva and Stynes (2001) for convection-diffusion problems and by Das and Natesan (2012) for Robin type reaction-diffusion problems. Our main aim is to construct a mesh that solves the equidistribution problem (3). Observe that instead of solving the discretized

equidistribution problem (3) for (1) exactly, it is sufficient that this algorithm can be stopped when the weakly equidistribution principle

$$\widehat{M}_i h_i \leq \frac{C_0}{N} \sum_{j=1}^{N} \widehat{M}_j h_j, \quad \text{for} \quad i = 1(1)N, \tag{31}$$

is satisfied with a user chosen constant $C_0 > 1$. $C_0$ will be chosen larger enough to get fewer iterations for the convergence of the algorithm. As $C_0$ approaches to 1, this algorithm produces more accurate solution with several iterations.

### 5.1.1 Algorithm-

**Step 1:** Define the initial uniform mesh $\{x_i^{(0)} : 0 \leq i \leq N, \, x_i^{(0)} = i/N\}$ and go to Step 2 with $p = 0$.

**Step 2:** Solve the discretized problem $[\mathfrak{L}^N U^{(p)}]_i = f_i^{(p)}$ with $U_0^{(p)} = U_N^{(p)} = 0$ at the mesh $\{x_i^{(p)} : 0 \leq i \leq N\}$ for $(U_0^{(p)}, \cdots, U_N^{(p)})$ and define $h_i^{(p)} = x_i^{(p)} - x_{i-1}^{(p)}$ for $i = 1, \cdots, N$.

**Step 3:** Find the smooth part $V_i^{(p)}$ of the numerical solution $U_i^{(p)}$, by solving the discretized problem (6) for $\varepsilon = 0$. Denote the discrete layer part of the solution $U_i^{(p)}$ as $W_i^{(p)}$, which is defined by $W_i^{(p)} = U_i^{(p)} - V_i^{(p)}$. Define $D^2 = D^+ D^-$. Find the discretized monitor function

$$\psi_i^{(p)} = \left[ 1 + |\overline{D}^2 W_i^{(p)}|^{1/2} \right] \text{ for } i = 1, \cdots, N, \tag{32}$$

by defining $\overline{D}^2 W_i = (D^2 W_i + D^2 W_{i-1})/2$ with $\overline{D}^2 W_1 = D^2 W_1$ and $\overline{D}^2 W_N = D^2 W_{N-1}$. Compute

$$\Psi_j^{(p)} = \sum_{i=1}^{j} h_i^{(p)} \psi_i^{(p)}.$$

**Step 4:** Choose a constant $C_0 \geq 1$. The stopping criteria for the iterative technique is

$$\frac{\max_{i=1,\dots,N} h_i^{(p)} \psi_i^{(p)}}{\Psi_N^{(p)}} \leq \frac{C_0}{N}. \tag{33}$$

If it holds true, then go to Step 6, else continue with Step 5.

**Step 5:** Generate a new mesh by equidistributing the proposed monitor function using current computed solution from Step 2 and $\Psi_j^{(p)}$ from Step 3: Set $Y_i^{(p)} = i\Psi_N^{(p)}/N$ for $i = 0, \cdots, N$. Now interpolate $(Y_i^{(p)}, x_i^{(p+1)})$ to $(\Psi_i^{(p)}, x_i^{(p)})$ using

piecewise linear interpolation. Generate a new mesh $x_i^{(p+1)} = \{0 = x_0^{(p+1)} < x_1^{(p+1)} < \cdots < x_N^{(p+1)} = 1\}$ and return to Step 2.

Step 6:  Set $x^* = \{0 = x_0^* < x_1^* < \cdots < x_N^* = 1\} = x_i^{(p+1)}$ and $U^* = U^{(p+1)}$, where $U^*$ is our desired solution. Stop.

It is easy to observe that this technique is also used for the numerical experiments provided in Beckett and Mackenzie (2000).

### 5.2  Numerical results

Here, two numerical examples are presented to confirm the theoretical findings. For these two text problems, layer-adapted meshes are obtained by the equidistribution of two monitor functions stated in (4).

**Example 5.1** *Consider the following singularly perturbed two-point BVP:*

$$\begin{cases} -\varepsilon u''(x) - u'(x) + 2u(x) = \exp(x-1), & x \in \Omega, \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

The exact solution of this problem is

$$u(x) = c_1 \exp(m_1 x) + c_2 \exp(m_2 x) - \exp(x-1)/(\varepsilon(1-m_1)(1-m_2)),$$

where

$$c_2 = \frac{1 - \exp(-1+m_1)}{(1-m_1)(1-m_2)\varepsilon(\exp(m_2) - \exp(m_1))}, \quad \text{and}$$

$$c_1 = -c_2 + \frac{\exp(-1)}{\varepsilon(1-m_1)(1-m_2)},$$

with $m_1 = (-1 + \sqrt{1+8\varepsilon})/2\varepsilon, \quad m_2 = (-1 - \sqrt{1+8\varepsilon})/2\varepsilon.$

The maximum point-wise errors and the corresponding rates of convergence are calculated by using the exact solution. The maximum point-wise errors are obtained by

$$E_\varepsilon^N = \max_{0 \le i \le N} |U_i^N - u_i|,$$

where $u_i$ denotes the exact solution at $x_i$ and $U_i^N$ denotes the numerically approximated solution at the point $x_i$ with $N$ number of mesh intervals.

**Example 5.2** *Consider the singularly perturbed convection-diffusion BVP:*

$$\begin{cases} \varepsilon u''(x) + ((1+x(1-x))u(x))' = \exp(x), & x \in \Omega, \\ u(0) = 0, & u(1) = 0. \end{cases}$$

As the exact solution for the Example 5.2 is not available, so the accuracy of its numerical solution will be computed using double mesh principle. This principle is defined as follows: For any fixed value of $N$, the maximum point-wise error $E_\varepsilon^N$ of the numerical solution before and after extrapolation will be calculated by

$$\max_{0 \le i \le N} |U_i^N - \bar{U}_{2i}^{2N}|, \text{ and } \max_{0 \le i \le N} |U_{extp,i}^N - \bar{U}_{extp,2i}^{2N}|,$$

where $U_i^N$ is the computed solution at $x_i$ with $N$ number of intervals, $\bar{U}_{2i}^{2N}$ is the numerical solution at $x_i$, on a mesh obtained by bisecting the original mesh with $2N$ number of mesh intervals and $U_{extp}^N$ is the extrapolated solution at $\Omega^N$ obtained by the formula $U_{extp}^N = 2\bar{U}^{2N} - U^N$. In a similar way, $\bar{U}_{extp}^{2N}$ can be defined at $\Omega^N$.

For both problem, the uniform errors for each fixed $N$ and the corresponding parameter uniform rates of convergence are calculated by the following formulas

$$E^N = \max_{\varepsilon \in S} E_\varepsilon^N, \text{ and } p^N = \log_2 \left( \frac{E^N}{E^{2N}} \right).$$

For these two problems, we take $\varepsilon$ from the set $S$ defined as

$$S = \left\{ \varepsilon | \varepsilon = 2^{-2}, \cdots, 2^{-30} \right\},$$

and for the numerical computation, the adaptively generated meshes are constructed using the constant $C_0 = 1.6$ in the adaptive algorithm.

Table 1: *Improved uniform errors and orders of convergence using the monitor function $M(x, w(x))$ for Example 5.1.*

| Extrapolation | Number of intervals $N$ | | | | | | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| Before | 9.38e-3 | 4.72e-3 | 2.35e-3 | 1.18e-3 | 5.50e-4 | 2.98e-4 | 1.48e-4 |
| | 0.99069 | 1.0088 | 0.9903 | 1.1011 | 0.8857 | 1.0081 | - |
| After | 2.64e-4 | 6.95e-5 | 1.78e-5 | 4.59e-6 | 9.64e-7 | 3.02e-7 | 6.98e-8 |
| | 1.9229 | 1.9670 | 1.9547 | 2.2508 | 1.6740 | 2.1130 | - |

For Examples 5.1 and 5.2, we displayed the maximum uniform errors and the corresponding rates of convergence row-wise respectively, using maximum norm. In

Table 2: *Improved uniform errors and orders of convergence using the monitor function $M(x, u(x))$ for Example 5.1.*

| Extrapolation | Number of intervals $N$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| Before | 1.04e-2 | 4.55e-3 | 2.21e-3 | 1.11e-3 | 5.23e-4 | 2.82e-4 | 1.34e-4 |
| | 1.1951 | 1.0439 | 0.99127 | 1.0858 | 0.88872 | 1.0711 | - |
| After | 3.04e-4 | 5.88e-5 | 1.50e-5 | 3.95e-6 | 8.35e-7 | 2.59e-7 | 5.37e-8 |
| | 2.3706 | 1.9744 | 1.9209 | 2.2418 | 1.6882 | 2.2706 | - |

Table 1, we presented the uniform errors for Example 5.1 before and after extrapolation where the mesh is obtained by the equidistribution of the monitor function $M(x, w(x))$. From Table 2, one can see that there is strong correlation between the two monitor functions $M(x, w(x))$ and $M(x, u(x))$. In fact, Table 2 suggests that one can use $M(x, u(x))$ as a monitor function to get better convergence rate, in order to avoid the reduced problem (*i.e.,* by taking $\varepsilon = 0$) solving each time. It should also be noted that the derivatives of the smooth component of the solution $u(x)$ are uniformly bounded from Lemma 2.2. A similar observation is noticed for Example 5.2, where the uniform errors and the corresponding rates of convergence are displayed in Table 3 and Table 4 for the two monitor functions $M(x, w(x))$ and $M(x, u(x))$ respectively.

For the above two examples, extrapolation technique has improved the order of convergence from first-order to second-order. A rapidly decreasing behavior of $\varepsilon$-uniform errors after extrapolation can be noticed for these two problems, as $N$ increases. Some variations in the numerical results from actual theoretical findings is expected, since instead of solving the equidistribution problem (3) exactly, we have solved the weakly equidistribution problem (31) for $C_0 = 1.6$. Although the errors are more uniform for smaller values of $C_0$, but the improvements are insignificant. Observe that a similar result with weakly equidistribution principle (31) is obtained by Kopteva and Stynes (2001) and also by Das and Natesan (2012).

The proposed improvement can be compared with the results obtained by Natividad and Stynes (2003), where a second-order up to a logarithmic factor is achieved through the extrapolation technique on Shishkin meshes. As like us, Linß (2004) obtained a similar result with *apriori* chosen Shishkin and Bakhvalov meshes for Example 5.1. But the plus point of adaptive technique with the monitor function $M(x, u(x))$ is that it does not need any *apriori* information about the location and width of the boundary layer.

In Figure 1 and Figure 2, we have plotted the maximum point-wise errors versus number of mesh intervals for Example 5.1 with the monitor functions $M(x, w(x))$

and $M(x, u(x))$ respectively. These figures are drawn in logarithmic scale for $\varepsilon = 2^{-30}$. Graphically these also suggest that the computed errors are decreasing with the rate of $O(N^{-1})$ and $O(N^{-2})$ approximately before and after extrapolation. Figure 3 and Figure 4 show the similar behavior for the Example 5.2.
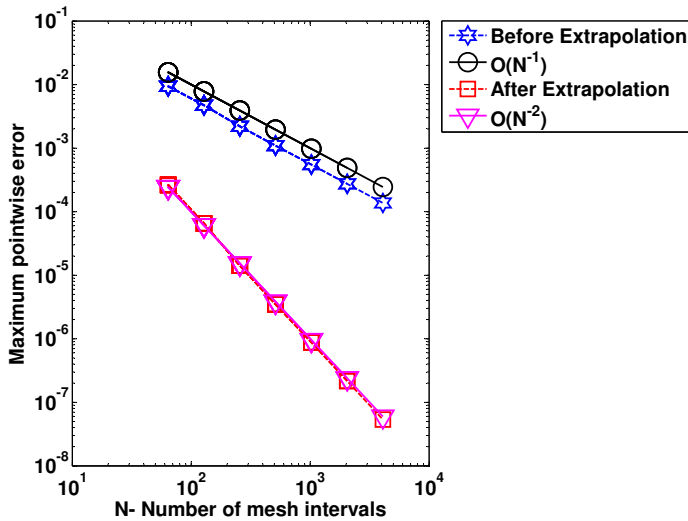


Figure 1: *Loglog plot of the maximum point-wise errors before and after extrapolation for Example 5.1 for $\varepsilon = 2^{-30}$ with the monitor function $M(x, w(x))$.*

Table 3: *Improved uniform errors and orders of convergence using the monitor function $M(x, w(x))$ for Example 5.2.*

| Extrapolation | Number of intervals $N$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| Before | 3.17e-2 | 1.55e-2 | 7.54e-3 | 3.74e-3 | 1.87e-3 | 9.35e-4 | 4.67e-4 |
| | 1.0285 | 1.0422 | 1.0098 | 1.0010 | 1.0009 | 1.0005 | - |
| After | 5.89e-4 | 1.31e-4 | 3.06e-5 | 8.06e-6 | 1.87e-6 | 5.35e-7 | 1.32e-7 |
| | 2.1684 | 2.1003 | 1.9236 | 2.1044 | 1.8073 | 2.0170 | - |

Table 4: *Improved uniform errors and orders of convergence using the monitor function $M(x, u(x))$ for Example 5.2.*

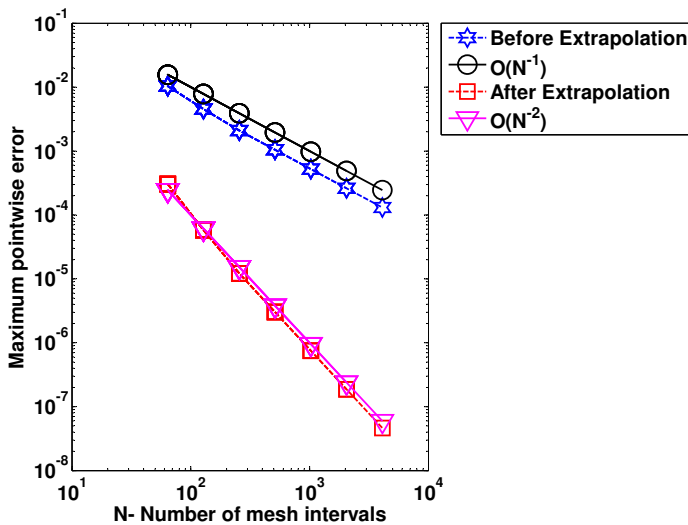| Extrapolation | Number of intervals $N$ | | | | | | |
| | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|---|---|---|---|---|---|---|---|
| Before | 2.60e-2 | 1.28e-2 | 6.39e-3 | 3.19e-3 | 1.56e-3 | 7.82e-4 | 3.97e-4 |
| | 1.0213 | 1.0036 | 1.0009 | 1.0306 | 1.0001 | 0.97830 | - |
| After | 6.02e-4 | 1.43e-4 | 3.70e-5 | 1.02e-5 | 2.18e-6 | 5.95e-7 | 1.43e-7 |
| | 2.0675 | 1.9576 | 1.8587 | 2.2219 | 1.8751 | 2.0533 | - |



Figure 2: *Loglog plot of the maximum point-wise errors before and after extrapolation for Example 5.1 for $\varepsilon = 2^{-30}$ with the monitor function $M(x, u(x))$.*
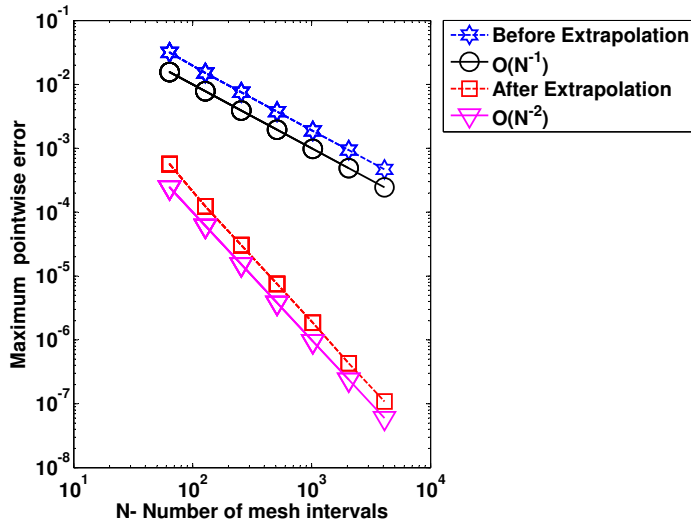
Figure 3: *Loglog plot of the maximum point-wise errors before and after extrapolation for Example 5.2 for $\varepsilon = 2^{-30}$ with the monitor function $M(x, w(x))$.*
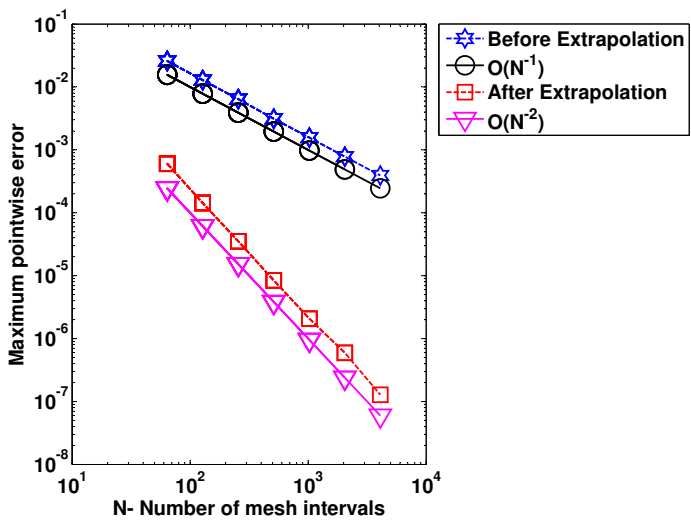


Figure 4: *Loglog plot of the maximum point-wise errors before and after extrapolation for Example 5.2 for $\varepsilon = 2^{-30}$ with the monitor function $M(x, u(x))$.*

## 6 Conclusion

In this paper, a post-processing technique is considered to obtain higher-order convergent numerical approximate solution for convection-diffusion singular perturbation problems on adaptively generated mesh. First, a monitor function is generated from the error analysis, which provides first-order convergence for the discrete solution. This monitor function is a variant of the monitor function proposed by Beckett and Mackenzie (2000). Using this monitor function, it is shown that the Richardson extrapolation technique can be used to obtain higher-order (in this case second-order) convergence on equidistributed nonuniform mesh. Though the analysis provided here is for a simple model problem, it gives us a useful insight about one possible way (using post-processing technique) to obtain a higher-order convergent solution.

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