# Fracture & Fatigue Analyses: SGBEM-FEM or XFEM? Part 2: 3D Solids

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The SGBEM-FEM alternating method is compared with the recently Abstract: popularized XFEM, for analyzing mixed-mode fracture and fatigue growth of 3D nonplanar cracks in complex solid and structural geometries. A large set of 3D examples with different degrees of complexity is analyzed by the SGBEM-FEM alternating method, and the numerical results are compared with those obtained by XFEM available in the open literature. It is clearly shown that: (a) SGBEM-FEM alternating method gives extremely high accuracy for the stress intensity factors; but the XFEM gives rather poor computational results, even for the most simple 3D cracks; (b) while SGBEM-FEM alternating method requires very coarse meshes, which are independent of each other, for both the uncracked solid as well as the non-planar crack-surface, XFEM requires, on the other hand, an extremely fine mesh for 3D solids, which can sometimes be un-usable on a normal PC; (c) the SGBEM-FEM alternating method requires very minimal computational as well as human-labor costs for modeling the non-planar fatigue growth of 3D cracks; on the other hand, fatigue analysis by XFEM requires intensive computational as well as human-labor costs even for the most simple problems; (d) because of the very poor accuracy for the stress intensity factors as computed by XFEM, the number fatigue cycles for crack-growth and failure as predicted by XFEM are meaningless for the most part, even for the most simple 3D problems computed even with extremely fine meshes; (e) with very low computational as well as human-labor costs, the SGBEM-FEM alternating method can very accurately model complex 3D cracked-solids easily, even for those cases which are too complex to be solved by XFEM. It is thus concluded that the SGBEM-FEM alternating method, among the many alternating methods developed in the past 20-30 years by Atluri and his many collaborators, are far more efficient, far more accurate, and far more reliable than XFEM for analyzing fracture and 3D non-planar fatigue crack propagation

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in complex structures. The implementation of the SGBEM-related method as presented in this study, as well as those presented in its companion Part 1 [Dong and Atluri (2013b)], in general-purpose off-the-shelf commercial software, is greatly valuable and is thus being pursued by the authors.

**Keywords:** SGBEM, Alternating Method, XFEM, stress intensity factor, fatiguecrack-propagation

## 1 Introduction

In a previous study [Dong and Atluri (2013b)], brief reviews of the historical development & the theoretical/algorithmic formulations of XFEM and SGBEM-FEM coupling/alternating method were given, and a comprehensive evaluation of their performances in fracture and fatigue analysis of 2D non-collinear cracks was carried out. It was pointed out that, in spite of the current wild-popularity of the Extended Finite Element Method (XFEM), XFEM differs very little in mathematical theory from the embedded-singularity elements developed in the 1970s by [Tong, Pian and Lasry (1973); Atluri, Kobayashi and Nakagaki (1975)], and many others as summarized in [Atluri (1986)]. Many numerical examples for 2-d problems, presented in[ Dong and Atluri (2013b)], demonstrated that the XFEM is unsuitable for fracture and fatigue analyses, due to the following reasons: 1. the low accuracy for the computed stress intensity factors (and correspondingly the very highly inaccurate results for crack growth rates and fatigue cycles to failure); 2. the necessity for very fine high-quality meshes in XFEM; and 3. the complicated procedures necessary in XFEM for advancing the crack-fronts to simulate crack-propagation. In contrast, the 2D SGBEM and SGBEM super element based methods [Dong and Atluri (2012, 2013a)] show great advantages in modeling 2D cracks in complex structures, and their non-collinear fatigue growth.

In this companion Part 2, a further comparison of XFEM and SGBEM-based methods is carried out, in the context of fracture and fatigue growth of 3D nonplanar cracks in complex 3-dimensional solid and structural geometries.

The theory and formulation of the 3D XFEM does not differ much from the 2D XFEM. As described in [Dong and Atluri (2013b)], XFEM uses the Heaviside Function to enrich the discontinuous displacement field across the crack surface, and uses Asymptotic Singular Field to enrich the crack-tip/crack-front displacement field. XFEM uses the primitive field variational principle, or the equivalent symmetric Galerkin weak form [Atluri (2005a)] to develop finite element equations. XFEM uses path-independent/ domain-independent integrals as in [Eshelby (1951); Rice (1968); Atluri (1982); Nishioka and Atluri (1983); Nikishkov and Atluri (1987a,b)], or their interaction- integral variants [Chen and Shield (1977),



Figure 1: A local coordinate system at a 3D crack-front

Atluri (1998)], among other techniques, to extract and to evaluate the stress intensity factors from the computed displacement/strain/stress solutions. XFEM requires special techniques, such as Level Set Functions and Fast Marching Methods, to track the crack-surface and crack-front, when crack-propagation analyses are carried out. However, it is emphasized that, in the 3D version of XFEM (see [Sukumar, Moës, Moran, Belytschko (2000)] for example), the enriched fields at the crack-front are the same as those for 2D cracks:

$$\left[\sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin\left(\theta\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin\left(\theta\right)\right]$$
(1)

where  $r = \sqrt{x_2^2 + x_3^2}$ ,  $\theta = \arctan\left(\frac{x_3}{x_2}\right)$ , are determined in a crack-front local coordinate system as in Fig. 1. While this Singular Enrichment corresponds to the asymptotic crack-tip fields of the 2D plane stress/strain problems, its validity for general non-smooth 3D cracks is of questionable validity.

The SGBEM used in this study is based on the weakly-singular BIEs developed in [Han and Atluri (2003,2007)], using the non-hyper-singular BIEs developed by [Okada, Rajiyah and Atluri (1988,1989)]. In a procedure similar to those in [Vijayakumar and Atluri (1981), Nishioka and Atluri (1983)], the SGBEM for the sub-domain containing the crack is coupled with the FEM for the otherwise uncracked solid, indirectly, by using the Schwartz-Neumann Alternating Method, in [Nikishkov, Park and Atluri (2001), Han and Atluri (2002), Atluri (2005b)]. As shown in Fig. 2, a simple FEM is used to model the global uncracked structure, and SGBEM is used to model the local cracked subdomain. By imposing residual stresses at the global and the local boundaries in an alternating procedure, the solution of the original problem is obtained by superposing the solutions for each



(c) (d) Figure 2: Superposition principle for FEM-SGBEM alternating method: (a) the uncracked body modeled by simple FEM, (b) the local domain containing cracks modeled by SGBEM, (c) FEM model subjected to residual loads, (d) alternating solution for the original problem, taken from [Han and Atluri (2002)]

individual sub-problem. *There are several clear advantages of the SGBEM-FEM alternating method:* (a) the SGBEM mesh for the cracked sub-domain is totally independent of the crude FEM mesh of the un-cracked global structure; (b) the SGBEM for the local cracked subdomain can capture crack-front stress/strain singularities very accurately; (c) very complex large-scale structures, such as a *Digital Twin of an Aerospace Vehicle*, can be modeled easily by very crude FEM, without a local mesh-refinement at the crack fronts; (d) the crack-front advancement during

crack-propagation requires minimal effort, by simply adding a layer of boundaryelements to the crack-fronts, with an appropriate size, and in the direction as determined by appropriate crack-growth criteria.

In the present paper, the performance of SGBEM-FEM alternating method is compared to that of XFEM in several 3D examples of fracture and fatigue crack propagation. Various highly-cited papers on 3D crack analyses by XFEM are first chosen. The same examples as presented in these highly cited XFEM papers are then solved by the SGBEM-FEM alternating method developed by [Han and Atluri (2002)]. In section 2, the numerical results are demonstrated and discussed in detail. Numerical results clearly show that the SGFEM-FEM alternating method is far more efficient, far more accurate, and far more reliable than XFEM for analyzing fracture and 3D non-planar fatigue crack propagation in complex structures. For some complex problems, XFEM gives unacceptably poor results, even with an extremely fine mesh that is almost unviable for a normal PC.

## 2 Numerical Examples

In this section, the SGBEM-FEM alternating method is compared to the XFEM, by analyzing several 3D examples. All the XFEM examples are chosen from the most-cited papers on XFEM in the open literature, up to the current time (2013). All the SGBEM-based results are generated by using a simple code as developed by [Han and Atluri (2002)], on a PC with an Intel Core i5 Processor.

### Example 1. An Embedded Penny-shaped Crack

First, we solve the simplest problem of an embedded penny-shaped crack in an infinite domain. The radius of the crack is *a*, and far field uniaxial tension  $\sigma = 1$  is applied. The stress intensity factor can be computed from the closed-form solution of [Green and Sneddon (1950)]:

$$K_1 = 2\sigma \sqrt{\frac{a}{\pi}} \tag{2}$$

This problem was solved by XFEM in [Sukumar, Moës, Moran, Belytschko (2000)], who consider a penny-shaped crack with radius a = 0.1 in a bi-unit cube. Two different meshes were used: (a) Mesh 1 consists of  $24 \times 24 \times 24 = 13824$  hexahedral elements, and (b) Mesh 2 has  $24 \times 24 \times 25 = 14400$  hexahedral elements. In the mesh 1, the crack lies on element faces, whereas in mesh 2, the crack is located in the center of the elements (no intersection with any element faces). Both meshes have graded refinement towards the center of the cube where the crack is located, see Fig. 3.

We also solve this problem by SGBEM-FEM alternating method. Two different meshes are used, as shown in Fig. 4-5. In mesh A, 8 finite elements are used for the uncracked global structure, 16 boundary elements are used for the crack surface. In mesh B, 64 finite elements are used for the uncracked global structure, 24 boundary elements are used for the crack surface.

For this specific problem, the analytical solution gives  $K_1 = 0.3568$ . The computed stress intensity factors for this penny-shaped crack are listed in Tab. 1 for XFEM, and in table 2 for the SGBEM-FEM alternating method. Relative errors of the computed stress intensity factors are also shown in Fig. 6. As can be clearly seen, even for this simplest problem of 3D fracture mechanics, XFEM gives an error up to 5%, even with a very fine mesh. On the other hand, the SGBEM-FEM alternating method gives almost the exact solution, even with an extremely coarse mesh.



Figure 3: The XFEM mesh 1 of the embedded circular crack, with  $24 \times 24 \times 24 = 13824$  finite elements, taken from [Sukumar, Moës, Moran, and Belytschko (2000)]



(a) (b) Figure 4: "Mesh A" for the problem of penny-shaped crack by the SGBEM-FEM alternating method: (a) the uncracked global structure is modeled with 8 FE; and (b) the crack surface is modeled independently, using 16 BE



(a) (b) Figure 5: "Mesh B" for the problem of penny-shaped crack by the SGBEM-FEM alternating method: (a) the uncracked global structure is modeled with 64 FE; (b) the crack surface is modeled independently, with 24 BE

$\theta$ (degree)	XFEM with Mesh 1		XFEM with Mesh 2		Applytical
	$K_1$	error	<i>K</i> <sub>1</sub>	error	Anarytical
0	0.3508	-1.68%	0.3617	1.37%	0.3568
10	0.3583	0.42%	0.3503	-1.82%	0.3568
20	0.3592	0.67%	0.3559	-0.25%	0.3568
30	0.3467	-2.83%	0.3595	0.76%	0.3568
40	0.3535	-0.92%	0.3383	-5.19%	0.3568
50	0.3535	-0.92%	0.3383	-5.19%	0.3568
60	0.3467	-2.83%	0.3595	0.76%	0.3568
70	0.3592	0.67%	0.3559	-0.25%	0.3568
80	0.3583	0.42%	0.3503	-1.82%	0.3568
90	0.3508	-1.68%	0.3617	1.37%	0.3568

Table 1: The computed SIFs for the penny-shaped crack, Using XFEM: [Sukumar, Moës, Moran, Belytschko (2000)]

Table 2: The computed SIFs for the penny-shaped crack, Using the SGBEM-FEM alternating method

0 (desmas)	SGBEM-FEM		SGBEM-FEM		A
o (degree)	with Mesh A	A	with Mesh B		Anarytical
	<i>K</i> <sub>1</sub>	error	<i>K</i> <sub>1</sub>	error	
0.0	0.3557	-0.30%	0.3569	0.04%	0.3568
22.5	0.3557	-0.30%	0.3569	0.04%	0.3568
45.0	0.3557	-0.30%	0.3569	0.04%	0.3568
67.5	0.3557	-0.30%	0.3569	0.04%	0.3568
90.0	0.3557	-0.30%	0.3569	0.04%	0.3568



Figure 6: Computational errors for the penny-shaped crack: A comparison of XFEM and SGBEM-FEM alternating methods

#### **Example 2.** An Embedded Elliptical Crack

The problem of an embedded elliptical crack was solved by XFEM in [Sukumar, Moës, Moran, Belytschko (2000)]. An elliptical crack with semi-axes a = 0.1, b = 0.05 was considered. A far field uniaxial tension of  $\sigma = 1$  is applied. The same mesh of the bi-unit cube, as for the penny-shaped crack was used, as shown in Fig. 3, with  $24 \times 24 \times 24 = 13824$  finite elements. In this study, we also use the SGBEM-FEM alternating method to solve this problem, by using 64 finite elements to model the un-cracked global structure, and 144 boundary elements to model the crack surface, independently, as shown in Fig. 7. The computed results are listed in Tab. 3 and are plotted in Fig. 8, showing the high-accuracy of the current SGBEM-based method.



(a) (b) Figure 7: Meshes for the problem of an embedded elliptical crack, as solved by the SGBEM-FEM alternating method: (a) the uncracked global structure is modeled with 64 FE; (b) the crack surface is modeled independently with 144 BE

Table 3: The computed results for an embedded elliptical crack, by using XFEM and SGBEM-FEM alternating methods, respectively

$\theta$ (degree)	XFEM		SGBEM-FEM		Apolytical
	<i>K</i> <sub>1</sub>	error	$K_1$	error	Analytical
0	0.2358	1.90%	0.2307	-0.29%	0.2314
10	0.2378	0.55%	0.2358	-0.30%	0.2365
20	0.2459	-1.44%	0.2489	-0.26%	0.2495
30	0.2564	-3.68%	0.2657	-0.20%	0.2662
40	0.2776	-1.91%	0.2825	-0.18%	0.2830
50	0.2965	-0.60%	0.2969	-0.46%	0.2983
60	0.3089	-0.58%	0.3083	-0.79%	0.3107
70	0.3163	-1.09%	0.3171	-0.85%	0.3198
80	0.3194	-1.82%	0.3227	-0.80%	0.3253
90	0.3202	-2.17%	0.3246	-0.82%	0.3273



Figure 8: Relative errors in the computed K-factors, for an embedded elliptical crack, using the XFEM and SGBEM-FEM alternating methods, respectively.

#### Example 3. An Embedded Lens-shaped Crack

Now we solve the problem of an embedded lens-shaped crack subjected to far field hydrostatic tension, see Fig. 9. This problem was solved by [Moës, Gravouil, and Belytschko (2002)] using XFEM. A finite cube with h/R = 5 was considered. An embedded lens-shaped crack with  $\alpha = \pi/4$  was placed in the middle of the cube. Young's modulus and Poisson's ratio were taken to be E = 68.9GPa and v = 0.22. The mesh was not shown in [Moës, Gravouil, and Belytschko (2002)]. However, it was stated that 145,000 degrees of freedom were used.

In this study, we also use SGBEM-FEM alternating method to solve this lensshaped crack problem. As shown in Fig. 10, 64 finite elements and 80 boundary elements are used to independently model the un-cracked global structure, and the crack face, respectively. h = 1, R = 0.2 is used, and  $\sigma = 50$  is applied to each face of the cube.

For this mixed mode problem, analytical solution of [Martynenko and Ulitko (1978)]



Figure 9: The problem of an embedded lens-shaped crack (the crack-surface is in the shape of a lens) in a cube, subjected to hydrostatic tension



(a) (b) Figure 10: Meshes for the problem of an embedded lens-shaped crack, as solved by the SGBEM-FEM alternating method: (a) the uncracked global structure is modeled with 64 FE; (b) the crack surface is modeled independently, using 80 BE

gives:

$$K_1 = 1.754\sigma \sqrt{\frac{a}{\pi}}$$

$$K_2 = 0.470\sigma \sqrt{\frac{a}{\pi}}$$

$$a = R \cos \alpha$$
(3)

The exact values of computed stress intensity factors were not listed in [Moës, Gravouil, and Belytschko (2002)]. *However, it was reported that the relative errors in the computed results were about 2 % for*  $K_1$  *and 10% for*  $K_2$ , *by using XFEM*. Compared to these reported results by XFEM, the SGBEM-BEM alternating method gives much more accurate solutions of the computed SIFs, as shown in Tab. 4.

Table 4: Computed SIFs for the embedded lens-shaped crack, by using XFEM and the SGBEM-FEM alternating method (the computed values for SIFs marked as \* were not given in [Moës, Gravouil, and Belytschko (2002)])

	Computed Results for Stress-Intensity Factors			
SIFs	$K_1$ Error of $K_1$ $K_2$ Error of $K_2$			
XFEM	*	2%	*	10%
SGBEM-FEM	18.627	0.11%	4.997	0.22%

We also study the non-planar fatigue growth of this lens-shaped crack. The crack is allowed to advance a total length of 0.6 in 15 steps. In each crack-growth step, after computing stress intensity factors, the corresponding Eshelby-force-vector is determined by its relation with SIFs, see [Nikishokv, Park and Atluri (2001)]. The crack-growth direction is taken to be that of the Eshelby-force-vector, which has the physical meaning of "force on the singularity" [Eshelby (1951)]. The Paris-law  $da/dN = 5.6 \times 10^{-14} K^{3.5}$  is used to predict the number of fatigue cycles, for a specified crack growth increment. It is found that, in 9,866,070 cycles, the lens-shaped crack grows into a mode 1 dominated crack, as shown in Fig. 11-12.

It should be emphasized that an accurate prediction of fatigue cycles to failure of a structure, is contingent on an accurate computation of stress intensity factors. For an error of 10% as in XFEM in the computed stress intensity factors for this problem, the Paris growth-law as used in this example will give an error about 30%-40% for the crack-growth-rate, which will cumulatively give a very large error for the estimated number of loading cycles it takes for the crack to grow to a certain length or to failure.



Figure 11: The final shape of the embedded lens-shaped crack, after fatigue growth, as predicted by the SGBEM-FEM alternating method: a 3D view



Figure 12: The final shape of the embedded lens-shaped crack, after fatigue growth, as predicted by the SGBEM-FEM alternating method: a plane view

#### Example 4. Fatigue Growth of an Inclined Penny Crack, Embedded in a Cube

In [Sukumar, Chopp, Béchet, and Moës, (2008)], the fatigue-growth of an inclined penny crack, embedded in a cube, was studied. The crack grows from an initially 45°-inclined crack, to a mode 1 dominated crack, as shown in Fig. 13. Although not shown in [Sukumar, Chopp, Béchet, and Moës, (2008)], a mesh consisting of 17000 nodes (51,000 dofs) was used to study this problem. Interestingly, the authors did not report the exact computer-time for this computation, but stated: *the time taken to compute a single crack growth step is between 30 and 45min.* Therefore a quick estimation gives *10-15 hours* for a total 20 crack-growth-steps.



Figure 13: The growth of an inclined penny crack, predicted by XFEM, taken from [Sukumar, Chopp, Béchet, and Moës, (2008)]

We use the SGBEM-FEM alternating method to study this example. As shown in Fig. 14, 1000 finite elements, and 144 boundary elements, are used to independently model the uncracked global structure, and the crack face, respectively. After fatigue growth, the shape of the crack is shown in Fig. 15. The computer time for



Figure 14: Independent Meshes for the problem of inclined penny crack, as solved by the SGBEM-FEM alternating method: (a) the uncracked global structure with 1000 FE; (b) the crack surface 144 BE

analyzing this problem, by the SGBEM-FEM alternating method, adds up to **7.9** *minutes*! We thus conclude that the computational efficiency of the SGBEM-FEM alternating method, roughly measured in computational time, is around 100 times better than the XFEM.

The crack-growth path predicted by XFEM, for this inclined penny-shaped crack, is somehow similar to the crack path predicted by the SGBEM-FEM alternating method. However, in [Sukumar, Chopp, Béchet, and Moës, (2008)] and many other studies on fatigue crack growth by XFEM, the computed fatigue cycles for a given amount of crack-growth are not given. As is pointed out in [Dong and Atluri (2013b)], for power-function type of fatigue laws, such as the Paris Law used in this example, a 10% error in the computed SIFs can cause 30%-40% error in predicted fatigue cycles for a given crack increment. This will cumulatively lead to meaningless predictions of fatigue lives, in XFEM. It is hoped that some more details will be given on predicted fatigue cycles by XFEM by researchers in the future, so that a direct comparison of the predicted fatigue cycles by XFEM and SGBEM-based methods can be made.



(c) (d) Figure 15: The growth of the initially inclined crack by the SGBEM-FEM alternating method

#### **Example 5.** A Through-Thickness Crack

All the examples presented above are for embedded cracks. In this and several following examples, we study through-thickness cracks and *surface cracks*. It is also found that, surface cracks are rarely studied in the literature published so far, by using XFEM, which might be due to the fact that surface cracks are more complicated.

In this example, a through-thickness crack with length 2a, located in a  $B \times W \times H$  plate, is studied. Uniform tension is applied to the upper and low surfaces of the plate, as shown in Fig. 16.

[Levén and Rickert (2012)] studied this problem, by using XFEM. Geometrical parameters H/a = 10, W/a = 10 are considered. Three different thicknesses are considered: B/a = 10, 20, 30 respectively. In each case, half of the structure was modeled, considering symmetry conditions. Also, 211,050, 422,100, and 634,800 finite elements were used to mesh half of the structure, respectively, for these three cases. It is expected that, when the thickness is very large, the stress intensity factor at the middle point should converge to the plane strain solution:

$$K_1 = \sigma \sqrt{\pi a} \sqrt{\sec\left(\frac{\pi a}{W}\right)} = 1.0254 \sigma \sqrt{\pi a} \tag{4}$$

However, it was found that such an expected convergence was not reached in XFEM, and a relatively large error is obtained, as shown in Tab. 5.

B/a	Number of elements	$K_1/\sigma\sqrt{\pi a}$	Error
10	211,050	1.0653	3.89%
20	422,100	1.0679	4.14%
30	634,800	1.0693	4.28%

Table 5: Normalized SIFs computed using XFEM by [Levén and Rickert (2012)]

We also use SGBEM-FEM alternating method to study this simple example. As shown in Fig. 18, for B = 10a, 125 finite elements and 48 boundary elements are used to independently model the uncracked global structure, and the crack face, respectively. For the other two cases with twice and three times of the current thickness, the element number is doubled and tripled respectively. Computed stress intensity factors are given in Tab. 6, showing that the SGBEM-FEM alternating method is much more accurate than XFEM, even with very coarse meshes.



Figure 16: A through-thickness crack in a tension plate



Figure 17: The mesh of the through-thickness crack considering symmetric conditions, by using XFEM, taken from [Levén and Rickert (2012)].

B/a	Number of elements	$K_1/\sigma\sqrt{\pi a}$	Error
10	125 FE, 48 BE	1.0284	0.29%
20	250 FE, 96 BE	1.0310	0.55%
30	375 FE, 144 BE	1.0323	0.67%

Table 6: Normalized SIFs computed by using the SGBEM-FEM alternating method



(a) (b) Figure 18: Meshes for the problem of a through-thickness crack by using the SGBEM-FEM alternating method with B = 10a: (a) the uncracked global structure is modeled with 125 FE; (b) the crack surface is modeled independently with 48 BE

# Example 6. A Semi-Elliptical Surface Crack in A Finite Plate



Figure 19: A semi-elliptical crack



Figure 20: The mesh of the semi-elliptical crack in a finite plate by using XFEM, taken from [Levén and Rickert (2012)]

In this example, the problem of a semi-elliptical crack in a finite plate is considered, as shown in Fig. 19. Geometric parameters are given as: a = 8, c = 3, w = h = 50, t = 21. Uniform tension is applied to the upper and lower surface.

[Levén and Rickert (2012)] studied this problem, by using XFEM. Three different meshes were considered, with different number of degrees of freedoms. A typical mesh is shown in Fig. 20. The computed stress intensity factor at the deepest point are compared to the solutions of [Newman and Raju (1981)], and the errors are listed in Tab. 7. *Even with over 1 million elements in XFEM*, the error of the stress intensity factor is still greater than 2.65%.

Table 7: Computational errors for the problem of a semi-elliptical surface-crack,by using XFEM in [Levén and Rickert (2012)]

Number of elements	Error of SIF
140,950	-4.46%
486,112	-4.77%
1,260,330	-2.65%



Figure 21: Independent meshes for the problem of semi-elliptical crack in a finite plate by using the SGBEM-FEM alternating method: (a) the un-cracked global structure is modeled with 108 FE; (b) the crack surface is modeled with 24 BE

solution.

## Example 7. A Very Small Semi-Elliptical Crack at a U-notch

We now consider a slightly more complicated problem, that of a very small semielliptical crack at the U-notch. As shown in Fig. 22, the geometric parameters of the U-notch are: W = 1000, H = 400, B = 500, d = 150. The long semi-axis of the crack is a = 10, and the short semi-axis of the crack is c = 2.

This problem was modeled by XFEM in [Levén and Rickert (2012)]. A very fine mesh of 686,331 elements was used, as shown in Fig. 23. We also solve this problem with SGBEM-FEM alternating method, the meshes for which are shown in Fig. 24, with 1,660 finite elements for the uncracked solid, and 80 boundary elements for the crack, independently. We see in Tab. 8, similar SIFs are obtained by XFEM and by SGBEM-FEM alternating method. However, *the number of elements used by the SGBEM-FEM alternating method is only 0.25% of the number of elements used by XFEM*; the computational time used by SGBEM-FEM alternating method is only 33 seconds, which should be significantly less than that of XFEM with about 0.7 million elements; and of course, the human labor spent on preparing the crude mesh for SGBEM-FEM is minimal, while the high-end mesh by XFEM, with graded refinement near the notch and at the crack-front, requires meticulous preparation with much time and energy, as is pointed out by [Levén and Rickert (2012)].

• •		•
	XFEM	SGBEM-FEM
Mesh	686,331 FEs	1,660 FEs, 80 BEs
$K_1/\sigma\sqrt{\pi c}$	6.52	6.523

Table 8: Normalized SIFs at the deepest point of semi-elliptical crack at U-notch, by using XFEM, and by using the SGBEM-FEM alternating method, respectively

We also study the fatigue growth of this semi-elliptical crack. A total of 19 fatigue growth steps are carried out with a maximum increment of 20. It is shown in Fig. 25-26 that the semi-elliptical crack becomes almost a quasi-circular crack, after fatigue growth, as modeled by using SGBEM-FEM alternating method.



Figure 22: A semi-elliptical crack at U-notch



Figure 23: The mesh of the semi-elliptical crack at U-notch, by using XFEM with 686,331 elements, taken from [Levén and Rickert (2012)]

It should be noted that, the total time used for the fatigue analysis of this problem by SGBEM-FEM alternating method is 18 minutes. On the other hand, it was pointed out in Example 4, the fatigue analysis of a simple inclined penny crack with 17,000 nodes takes 15-20 hours. Therefore, for this slightly complex problem, with about 0.7 million elements (which should add up to more than 1 million nodes), the fatigue analysis becomes almost impossible with reasonable computational resources.



Figure 24: Meshes for the problem of semi-elliptical crack at a U-notch, by using the SGBEM-FEM alternating method: (a) the uncracked global structure is modeled with 1660 FE; (b) the crack surface is modeled independently with 80 BE



Figure 25: The fatigue growth of the semi-elliptical crack emanating from U-notch, as predicted by SGBEM-FEM Alternation



Figure 26: The shape of the semi-elliptical crack at U-notch, after fatigue growth, predicted by the SGBEM-FEM alternating method

Example 8. Fatigue Growth of a Semi-Circular Surface Crack in a Thin Plate



(a) (b) Figure 27: Meshes for the problem of semi-circular surface crack in a tension plate, by the SGBEM-FEM alternating method : (a) the uncracked global structure is modeled with 300 FE; (b) the crack surface is modeled with 64 BE, independently.

In this example, we model the fatigue growth of a semi-circular surface crack in a thin plate. Referring to Fig. 19, the geometric parameters a = c = 0.5, h = w = 10, t = 1 are considered. A uniform tension of  $\sigma = 10$  is applied to the upper and lower surfaces of the plate. The meshes for the SGBEM-FEM alternating method are shown in Fig. 27, with 300 finite elements and 64 boundary elements, respectively. After fatigue growth, we see that the surface crack breaks through the thickness of the plate, and becomes a through-thickness crack. Then, it keeps growing in the width direction, as shown in Fig. 28-30.



Figure 28: The growth of the semi-circular crack in a thin plate as analyzed by the SGBEM-FEM Alternating Method: 3D view, Part A



Figure 29: The growth of the semi-circular crack in a thin plate, as analyzed by the SGBEM-FEM Alternating Method: 3D view, Part B



Figure 30: The growth of the semi-circular crack in a thin plate, as analyzed by the SGBEM-FEM Alternating Method: 2D view, in the plane of the crack

## Example 9: A Surface Crack Emanating from a hole in an Attachment Lug

We also solve a more complicated problem, of the propagation under fatigue, of a semi-circular surface crack emanating from a hole in an attachment lug. No one has attempted to solve such a simple practical problem by using XFEM, in spite of its long history, and more than 3500 publications on XFEM. As shown in Fig. 31-32, the meshes for the SGBEM-FEM alternating method consists of 4392 finite elements and 45 boundary elements. Fatigue analysis of the attachment lug crack is carried out, giving the final crack shape as shown in Fig. 33.



Figure 31: A Quarter-Circular Surface crack emanating from the hole in an Attachment-Lug



(a) (b) Figure 32: Meshes for the problem of a Quarter-Circular Surface Crack Emanating from a hole in an attachment lug : (a) the uncracked global structure is modeled with 4392 FE; (b) the crack surface is modeled independently, with 45 BE



Figure 33: The growth of attachment lug crack: 2D view

### 3 Conclusion

By carefully analyzing several numerical examples, we demonstrated that SGBEM-FEM alternating method is much more suitable for fracture and fatigue analysis of 3D nonplanar cracks in complex structures, because: (a) the SGBEM-FEM alternating method gives extremely high accuracy for stress intensity factors; but XFEM gives rather poor computational results, even for the most simple problems of 3D cracks; (b) while SGBEM-FEM alternating method requires independent and very coarse meshes for both the uncracked solid and the crack, the XFEM requires extremely fine and high-quality meshes for 3D solids, which can sometimes be unviable for a normal PC; (c) the SGBEM-FEM alternating method requires minimal computational as well as human-labor efforts for modeling the non-planar fatigue growth of 3D cracks; but fatigue analysis by XFEM involves unreasonable computational and human efforts even for those most simple problems; (d) because of the low accuracy of computed stress intensity factors by XFEM, the fatigue cycles to failure, as predicted by XFEM should be far from correct and meaningless, even for those most simple 3D problems, such as an inclined embedded penny-shaped crack, as computed in several hours with an extremely fine mesh; (e) with very low computational and human-labor burden, the SGBEM-FEM alternating method can accurately solve complex 3D cracked-solid problems easily, even in those cases which are too complex to be even attempted by XFEM with a reasonable cost. It is thus concluded that the SGBEM-FEM alternating method, among the many alternating methods developed in the past 20-30 years by Atluri and his many collaborators, are far more efficient, far more accurate, and far more reliable than the XFEM, for analyzing fracture and 3D non-planar fatigue crack propagation in complex structures. The implementation of the SGBEM-related method as presented in this study, as well as those presented in its companion Part 1 [Dong and Atluri (2013b)], in general-purpose, off-the-shelf commercial software, is greatly valuable and is being pursued by the authors.

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