

Numerical Solution of Space-Time Fractional Convection-Diffusion Equations with Variable Coefficients Using Haar Wavelets

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Abstract: In this paper, we present a computational method for solving a class of space-time fractional convection-diffusion equations with variable coefficients which is based on the Haar wavelets operational matrix of fractional order differentiation. Haar wavelets method is used because its computation is simple as it converts the original problem into Sylvester equation. Error analysis is given that shows efficiency of the method. Finally, a numerical example shows the implementation and accuracy of the approach.

Keywords: Haar wavelets, operational matrix, fractional convection-diffusion equation, Sylvester equation, error analysis, numerical solution.

1 Introduction

Fractional differential equations are generalized from classical integer order ones, which are obtained by replacing integer order derivatives by fractional ones. In the last few decades, fractional calculus and fractional differential equations have found a wide area of applications in several different fields [Hilfer (2000); He (1998); Chen, Yi, Chen, and Yu (2012)]. For example, one could mention the problem of anomalous diffusion [El-Sayed (1996); Gafiychuk, Datsun, and Meleshko (2008)], the nonlinear oscillation of earthquake can be modeled with fractional derivative [Delbosco and Rodino (1996)] and many other [Ryabov and Puzenko (2002)] recent developments in the description of anomalous transport by fractional dynamics. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an perfect framework for modeling real problems. Owing to the increasing applications, a considerable attention has been given to exact and numerical solutions of frac-

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tional differential equations. They have been solved by means of the numerical and analytical methods such as variational iteration method [Odibat (2010)], Adomian decomposition method [EI-Sayed (1998); EI-Kalla (2011)], generalized differential transform method [Odibat and Momani (2008); Momani and Odibat (2007)], wavelet method [Yi and Chen (2012)].

In this paper, our study focuses on the following space-time fractional convection-diffusion equation with variable coefficients:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -b(x) \frac{\partial u(x,t)}{\partial x} + a(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} - c(x)u(x,t) + q(x,t) \quad (1)$$

$$0 < x < 1, \quad 0 < t \leq 1$$

subject to the initial conditions

$$u(x,0) = 0, \quad 0 \leq x \leq 1 \quad (2)$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \leq 1 \quad (3)$$

where $\partial^\alpha u(x,t)/\partial t^\alpha$ is fractional derivative of Caputo sense, $\partial^\beta u(x,t)/\partial x^\beta$ is fractional derivative Riemann-Liouville sense [Podlubny (1999)]. $a(x), b(x), c(x), q(x,t)$ are the known continuous functions, $u(x,t)$ is the unknown function, $0 < \alpha < 1$, $1 < \beta < 2$.

Many scholars have studied the time fractional diffusion equations in recent years. Lin and Xu [Lin and Xu (2007)] used finite difference method to solve the time fractional diffusion equation. Meerschaert et al. [Meerschaert, Scheffler, and Tadjeran (2006)] applied the finite difference methods to solve two-dimensional fractional dispersion equation. Zhang [Zhang (2009)] discussed a practical implicit method to solve a class of initial boundary value space-time fractional convection-diffusion equations with variable coefficients.

Recently, the operational matrices of fractional order integration for the Legendre wavelets [Rehman and Khan (2011)], Chebyshev wavelets [Li (2010)], Haar wavelets [Li and Zhao (2010)], CAS wavelets [Saeedi, Moghadam, Mollahasani, and Chuev (2011)] and the second kind Chebyshev wavelets [Wang and Fan (2012)] have been developed to solve the fractional differential equations. Our purpose is to proposed Haar wavelets operational matrix method to solve a class of space-time fractional convection-diffusion equations with variable coefficients.

2 Definitions of fractional derivatives and integrals

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this article [Podlubny (1999)].

Definition 1. Riemann-Liouville definition of fractional differential operator is given by

$$D_t^\alpha u(t) = \begin{cases} \frac{d^r u(t)}{dt^r}, & \alpha = r \in N \\ \frac{1}{\Gamma(r-\alpha)} \frac{d^r}{dt^r} \int_0^t \frac{u(T)}{(t-T)^{\alpha-r+1}} dT, & 0 \leq r-1 < \alpha < r \end{cases} \quad (4)$$

The Riemann-Liouville fractional integral operator J^α of order α is defined as

$$J^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} u(T) dT, \quad t > 0, \quad J^0 u(t) = u(t) \quad (5)$$

Definition 2. The Caputo definition of fractional differential operator is given by

$$D_*^\alpha u(t) = \begin{cases} \frac{d^r u(t)}{dt^r}, & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{u^{(r)}(T)}{(t-T)^{\alpha-r+1}} dT, & 0 \leq r-1 < \alpha < r. \end{cases} \quad (6)$$

The Caputo fractional derivatives of order α is also defined as $D_*^\alpha u(t) = J^{r-\alpha} D^r u(t)$, where D^r is the usual integer differential operator of order r . The relation between the Riemann- Liouville operator and Caputo operator is given by the following expressions:

$$D_*^\alpha J^\alpha u(t) = u(t) \quad (7)$$

$$J^\alpha D_*^\alpha u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0 \quad (8)$$

3 Haar wavelets and function approximation

For $t \in [0, 1]$, Haar wavelets functions are defined as follows [Ray (2012)]:

$$h_0(t) = \frac{1}{\sqrt{m}} \quad (9)$$

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{j/2}, & \frac{k-1}{2^j} \leq t < \frac{k-1/2}{2^j} \\ -2^{j/2}, & \frac{k-1/2}{2^j} \leq t < \frac{k}{2^j} \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

where $i = 0, 1, 2, \dots, m-1$, $m = 2^{p+1}$ and p is a positive integer which is called the maximum level of resolution. j and k represent integer decomposition of the index i , i.e. $i = 2^j + k - 1$.

For arbitrary function $u(x,t) \in L^2([0, 1) \times [0,1))$, it can be expanded into Haar series by

$$u(x,t) \cong \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} u_{ij} h_i(x) h_j(t) \tag{11}$$

where $u_{ij} = \langle h_i(x), \langle u(x,t), h_j(t) \rangle \rangle$ are wavelets coefficients, $\langle h_i(x), h_j(x) \rangle = \int_0^1 h_i(x) h_j(x) dx$.

Let $H_m(x) = [h_0(x), h_1(x), \dots, h_{m-1}(x)]^T$, $H_m(t) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T$, then Eq.(11) will be written as $u(x,t) \cong H_m^T(x) \cdot U \cdot H_m(t)$.

In this paper, we use wavelet collocation method to determine the coefficients u_{ij} . These collocation points are shown in the following:

$$x_l = t_l = (l - 1/2)/m, \quad l = 1, 2, \dots, m. \tag{12}$$

Discreting Eq.(11) by the step (12), we can obtain the matrix form of Eq.(11)

$$C = H^T \cdot U \cdot H \tag{13}$$

where $U = [u_{ij}]_{m \times m}$ and $C = [u(x_i, t_j)]_{m \times m}$. H is called Haar wavelets matrix of order m , i.e.

$$H = \begin{bmatrix} h_0(t_0) & h_0(t_1) & \cdots & h_0(t_{m-1}) \\ h_1(t_0) & h_1(t_1) & \cdots & h_1(t_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_0) & h_{m-1}(t_1) & \cdots & h_{m-1}(t_{m-1}) \end{bmatrix}.$$

From the definition of Haar wavelets functions, we may know easily that H is a orthogonal matrix, then we have

$$U = H \cdot C \cdot H^T \tag{14}$$

4 Haar wavelets operational matrix of fractional order integration and differentiation

The integration of the $H_m(t)$ can be approximated by Chen and Hsiao [Chen and Hsiao (1997)]:

$$\int_0^t H_m(s) ds \cong PH_m(t) \tag{15}$$

where P is called the Haar wavelets operational matrix of integration.

Now, we are able to derive the Haar wavelets operational matrix of fractional order integration. For this purpose, we may make full use of the definition of Riemann-Liouville fractional integral operator J^α which is given by **Definition 1**.

The Haar wavelets operational matrix of fractional order integration P^α will be deduced by

$$\begin{aligned} P^\alpha H_m(t) &= J^\alpha H_m(t) \\ &= [J^\alpha h_0(t), J^\alpha h_1(t), \dots, J^\alpha h_{m-1}(t)]^T \\ &= \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} h_0(T) dT, \frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} h_1(T) dT, \dots, \right. \\ &\quad \left. \frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} h_{m-1}(T) dT \right]^T \\ &= [Ph_0(t), Ph_1(t), \dots, Ph_{m-1}(t)]^T \end{aligned}$$

where

$$Ph_0(t) = \frac{1}{\sqrt{m}} \frac{t^\alpha}{\Gamma(\alpha+1)} \quad t \in [0, 1) \tag{16}$$

$$Ph_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 0, & 0 \leq t < \frac{k-1}{2^j} \\ 2^{j/2} \lambda_1(t), & \frac{k-1}{2^j} \leq t < \frac{k-1/2}{2^j} \\ 2^{j/2} \lambda_2(t), & \frac{k-1/2}{2^j} \leq t < \frac{k}{2^j} \\ 2^{j/2} \lambda_3(t), & \frac{k}{2^j} \leq t < 1 \end{cases} \tag{17}$$

where

$$\lambda_1(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j} \right)^\alpha ;$$

$$\lambda_2(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j} \right)^\alpha - \frac{2}{\Gamma(\alpha+1)} \left(t - \frac{k-1/2}{2^j} \right)^\alpha ;$$

$$\lambda_3(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j} \right)^\alpha - \frac{2}{\Gamma(\alpha+1)} \left(t - \frac{k-1/2}{2^j} \right)^\alpha + \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k}{2^j} \right)^\alpha .$$

The derived Haar wavelets operational matrix of fractional integration is $P^\alpha = (P^\alpha H) \cdot H^T$.

Let D^α is the Haar wavelets operational matrix of fractional differentiation. According to the property of fractional calculus $D^\alpha P^\alpha = I$, we can obtain the matrix D^α by inverting the matrix P^α .

For instance, if $\alpha = 0.5, m = 8$, we have

$$D^{1/2} = \begin{bmatrix} 1.1229 & 0.4694 & 0.4589 & 0.0396 & 0.6488 & 0.0568 & 0.0185 & 0.0108 \\ -0.4694 & 2.0678 & 0.4589 & -0.8783 & 0.6488 & 0.0568 & -1.2790 & -0.1028 \\ -0.0396 & -0.8783 & 2.8964 & 0.4711 & 0.9175 & -1.7547 & 0.7831 & 0.0432 \\ -0.4589 & 0.4589 & 0 & 2.8964 & 0 & 0 & 0.9175 & -1.7547 \\ -0.0108 & -0.1028 & -1.7547 & 0.0432 & 4.8424 & 1.5241 & 0.0671 & 0.0051 \\ -0.0185 & -1.2790 & 0.9175 & 0.7831 & 0 & 4.8424 & 1.5241 & 0.0671 \\ -0.0568 & 0.0568 & 0 & -1.7547 & 0 & 0 & 4.8424 & 1.5241 \\ -0.6488 & 0.6488 & 0 & 0.9175 & 0 & 0 & -0 & 4.8424 \end{bmatrix}$$

The fractional order differentiation of the function t was selected to verify the correctness of matrix D^α . The fractional order differentiation of the function $u(t) = t$ is obtained in the following:

$$D_*^\alpha u(t) = \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \tag{18}$$

When $\alpha = 0.5, m = 32$, the comparison result for fractional differentiation is shown in **Fig. 1**.

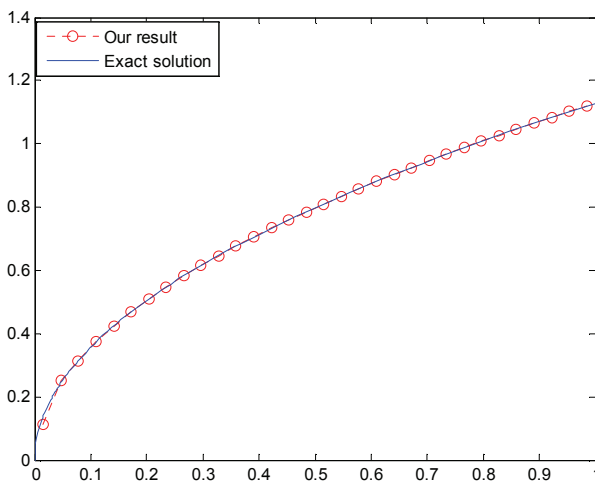


Figure 1: 0.5-order differentiation of the function $u(t) = t$.

5 Numerical solution of the fractional partial differential equations

Consider the space-time fractional convection-diffusion equations with variable coefficients Eq.(1). If we approximate the function $u(x,t)$ by using Haar wavelets, we have

$$u(x,t) \cong H_m^T(x) \cdot U \cdot H_m(t) \tag{19}$$

Then we can get

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \cong \frac{\partial^\alpha (H_m^T(x)UH_m(t))}{\partial t^\alpha} = H_m^T(x)U \frac{\partial^\alpha (H_m(t))}{\partial t^\alpha} = H_m^T(x)UD^\alpha H_m(t) \quad (20)$$

$$\frac{\partial u(x,t)}{\partial x} \cong \frac{\partial (H_m^T(x)UH_m(t))}{\partial x} = \left[\frac{\partial H_m(x)}{\partial x} \right]^T UH_m(t) = H_m^T(x)[D^1]^T UH_m(t) \quad (21)$$

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} \cong \frac{\partial^\beta (H_m^T(x)UH_m(t))}{\partial x^\beta} = \left[\frac{\partial^\beta H_m(x)}{\partial x^\beta} \right]^T UH_m(t) = H_m^T(x)[D^\beta]^T UH_m(t) \quad (22)$$

The function $q(x,t)$ of Eq.(1) can be also expressed as

$$q(x,t) \cong H_m^T(x) \cdot Q \cdot H_m(t) \quad (23)$$

where $Q = [q_{ij}]_{m \times m}$.

Substituting Eq.(19), Eq.(20), Eq.(21), Eq.(22) and Eq.(23) into Eq.(1), we have

$$H^T(x)UD^\alpha H(t) = -b(x)H^T(x)[D^1]^T UH(t) + a(x)H^T(x)[D^\beta]^T UH(t) - c(x)H^T(x)UH(t) + q(x,t) \quad (24)$$

Dispersing Eq.(24) by the points (x_i, t_j) , $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$, we can obtain

$$H^T UD^\alpha H = -A_1 H^T [D^1]^T UH + A_2 H^T [D^\beta]^T UH - A_3 H^T UH + H^T QH \quad (25)$$

namely

$$\left\{ HA_1 H^T [D^1]^T - HA_2 H^T [D^\beta]^T + HA_3 H^T \right\} U + UD^\alpha = Q \quad (26)$$

where

$$A_1 = \begin{bmatrix} b(x_0) & 0 & \dots & 0 \\ 0 & b(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b(x_{m-1}) \end{bmatrix}, \quad A_2 = \begin{bmatrix} a(x_0) & 0 & \dots & 0 \\ 0 & a(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(x_{m-1}) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} c(x_0) & 0 & \dots & 0 \\ 0 & c(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c(x_{m-1}) \end{bmatrix}.$$

Eq.(26) is a Sylvester equation. The Sylvester equation can be solved easily by using Matlab software.

6 Error analysis

In this section, we assume that $\frac{\partial u(x,t)}{\partial x}$ is continuous and bounded on $(0, 1) \times (0, 1)$, there is

$$\exists M > 0, \quad \forall x, t \in (0, 1) \times (0, 1), \quad \left| \frac{\partial u(x,t)}{\partial x} \right| \leq M \tag{27}$$

Suppose $u_m(x, t)$ is the following approximation of $u(x, t)$

$$u_m(x, t) = \sum_{n=0}^{m-1} \sum_{l=0}^{m-1} u_{nl} h_n(x) h_l(t) \tag{28}$$

where $m = 2^{p+1}$, $p = 0, 1, 2, \dots$. Then

$$u(x, t) - u_m(x, t) = \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} u_{nl} h_n(x) h_l(t) = \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl} h_n(x) h_l(t) \tag{29}$$

Theorem 6.1 Assume $u(x, t) \in L^2([0, 1] \times [0, 1])$ and $u_m(x, t)$ be defined by Eq.(28), then we have $\|u(x, t) - u_m(x, t)\|_E \leq \frac{M}{\sqrt{3}} \frac{1}{m^3}$, where $\|u(x, t)\|_E = \left(\int_0^1 \int_0^1 u^2(x, t) dx dt \right)^{1/2}$.

Proof. The orthonormality of the sequence $\{h_i(t)\}$ on $[0, 1)$ implies that

$$\int_0^1 h_n(x) h_{n'}(x) dx = \begin{cases} 1/m, & n = n' \\ 0, & n \neq n' \end{cases} \tag{30}$$

Then we have

$$\begin{aligned} \|u(x, t) - u_m(x, t)\|_E^2 &= \int_0^1 \int_0^1 [u(x, t) - u_m(x, t)]^2 dx dt \\ &= \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} \sum_{n'=2^{p+1}}^{\infty} \sum_{l'=2^{p+1}}^{\infty} u_{nl} u_{n'l'} \left(\int_0^1 h_n(x) h_{n'}(x) dx \right) \left(\int_0^1 h_l(t) h_{l'}(t) dt \right) \\ &= \frac{1}{m^2} \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl}^2 \end{aligned} \tag{31}$$

where $u_{nl} = \langle h_n(x), \langle u(x, t), h_l(t) \rangle \rangle$.

According to Eq.(9) and Eq.(10), we obtain

$$\langle u(x, t), h_l(t) \rangle = \int_0^1 u(x, t) h_l(t) dt = \frac{2^{j/2}}{\sqrt{m}} \left(\int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}} u(x, t) dt - \int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}} u(x, t) dt \right) \tag{32}$$

Using mean value theorem of integrals:

$$\exists t_1, t_2 \quad (k-1) \cdot 2^{-j} \leq t_1 < (k-\frac{1}{2}) \cdot 2^{-j}, \quad (k-\frac{1}{2}) \cdot 2^{-j} \leq t_2 < k \cdot 2^{-j}$$

such that

$$\begin{aligned} & \langle u(x, t), h_l(t) \rangle \\ &= \frac{2^{j/2}}{\sqrt{m}} \left\{ [(k-\frac{1}{2})2^{-j} - (k-1)2^{-j}]u(x, t_1) - [k2^{-j} - (k-\frac{1}{2})2^{-j}]u(x, t_2) \right\} \\ &= \frac{2^{-j/2-1}}{\sqrt{m}} (u(x, t_1) - u(x, t_2)) \end{aligned} \tag{33}$$

hence

$$\begin{aligned} u_{nl} &= \left\langle h_n(x), \frac{2^{-j/2-1}}{\sqrt{m}} (u(x, t_1) - u(x, t_2)) \right\rangle \\ &= \frac{2^{-j/2-1}}{\sqrt{m}} \int_0^1 h_n(x) (u(x, t_1) - u(x, t_2)) dx \\ &= \frac{2^{-j/2-1}}{\sqrt{m}} \left(\int_0^1 h_n(x) u(x, t_1) dx - \int_0^1 h_n(x) u(x, t_2) dx \right) \\ &= \frac{1}{2m} \left(\int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}} u(x, t_1) dx - \int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}} u(x, t_1) dx - \int_{(k-1)2^{-j}}^{(k-\frac{1}{2})2^{-j}} u(x, t_2) dx \right. \\ & \quad \left. + \int_{(k-\frac{1}{2})2^{-j}}^{k2^{-j}} u(x, t_2) dx \right) \end{aligned}$$

Using mean value theorem of integrals again:

$$\exists x_1, x_2, x_3, x_4 \quad (k-1) \cdot 2^{-j} \leq x_1, x_3 < (k-\frac{1}{2}) \cdot 2^{-j}, \quad (k-\frac{1}{2}) \cdot 2^{-j} \leq x_2, x_4 < k \cdot 2^{-j}$$

such that

$$\begin{aligned} u_{nl} &= \frac{1}{2m} \left\{ [(k-\frac{1}{2})2^{-j} - (k-1)2^{-j}]u(x_1, t_1) - [k2^{-j} - (k-\frac{1}{2})2^{-j}]u(x_2, t_1) \right. \\ & \quad \left. - [(k-\frac{1}{2})2^{-j} - (k-1)2^{-j}]u(x_3, t_2) + [k2^{-j} - (k-\frac{1}{2})2^{-j}]u(x_4, t_2) \right\} \\ &= \frac{1}{2^{j+2}m} [(u(x_1, t_1) - u(x_2, t_1)) - (u(x_3, t_2) - u(x_4, t_2))] \end{aligned} \tag{34}$$

therefore

$$u_{nl}^2 = \frac{1}{2^{2j+4}m^2} [(u(x_1, t_1) - u(x_2, t_1)) - (u(x_3, t_2) - u(x_4, t_2))]^2 \tag{35}$$

Using mean value theorem of derivatives:

$$\exists \xi_1, \xi_2 \quad x_1 \leq \xi_1 < x_2, \quad x_3 \leq \xi_2 < x_4$$

such that

$$\begin{aligned} u_{nl}^2 &= \frac{1}{2^{2j+4}m^2} \left[(x_2 - x_1) \frac{\partial u(\xi_1, t_1)}{\partial x} - (x_4 - x_3) \frac{\partial u(\xi_2, t_2)}{\partial x} \right]^2 \\ &\leq \frac{1}{2^{2j+4}m^2} \left\{ (x_2 - x_1)^2 \left[\frac{\partial u(\xi_1, t_1)}{\partial x} \right]^2 + (x_4 - x_3)^2 \left[\frac{\partial u(\xi_2, t_2)}{\partial x} \right]^2 \right. \\ &\quad \left. + 2(x_2 - x_1)(x_4 - x_3) \left| \frac{\partial u(\xi_1, t_1)}{\partial x} \right| \left| \frac{\partial u(\xi_2, t_2)}{\partial x} \right| \right\} \end{aligned} \tag{36}$$

Putting Eq.(27) and Eq.(36) together, we get

$$u_{nl}^2 \leq \frac{4M^2}{2^{4j+4}m^2} = \frac{M^2}{2^{4j+2}m^2} \tag{37}$$

Substituting Eq.(37) into Eq.(31), then we have

$$\begin{aligned} \|u(x, t) - u_m(x, t)\|_E^2 &= \frac{1}{m^2} \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl}^2 \\ &= \frac{1}{m^2} \sum_{j=p+1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} \sum_{l=2^j}^{2^{j+1}-1} u_{nl}^2 \right) \\ &\leq \frac{1}{m^2} \sum_{j=p+1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} \sum_{l=2^j}^{2^{j+1}-1} \frac{M^2}{2^{4j+2}m^2} \right) \\ &= \frac{M^2}{m^4} \sum_{j=p+1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} \sum_{l=2^j}^{2^{j+1}-1} \frac{1}{2^{4j+2}} \right) \\ &= \frac{M^2}{3m^4} \frac{1}{2^{2(p+1)}} \\ &= \frac{M^2}{3} \frac{1}{m^6} \end{aligned} \tag{38}$$

Therefore

$$\|u(x, t) - u_m(x, t)\|_E \leq \frac{M}{\sqrt{3}} \frac{1}{m^3} \tag{39}$$

This theorem is complete.

From the Eq.(39), we can see that $\|u(x, t) - u_m(x, t)\|_E \rightarrow 0$ when $m \rightarrow \infty$. A conclusion is drawn that Haar wavelets method is convergent when it is used to solve the numerical solution of fractional differential equations.

7 A numerical example

In this section, to demonstrate the validity and applicability of the approach, we consider the space-time fractional convection-diffusion equation with variable coefficients Eq.(1).

Let $a(x) = \Gamma(2.8)x/2$, $b(x) = x^{0.8}$, $c(x) = x^{1.5}$, when $\alpha = 0.8$, $\beta = 1.5$, and $q(x, t) = \frac{\Gamma(3)x^2(1-x)t^{1.2}}{\Gamma(2.2)} + \left[2x^{1.8} - 3x^{2.8} - \frac{\Gamma(2.8)\Gamma(3)}{2\Gamma(1.5)}x^{1.5} + \frac{\Gamma(2.8)\Gamma(4)}{2\Gamma(2.5)}x^{2.5} + x^{3.5} - x^{4.5} \right] t^2$ the exact solution is $u(x, t) = x^2(1-x)t^2$. **Figs. 2-5** show the numerical solutions for various m . The absolute error for different m is shown in **Table 1**.

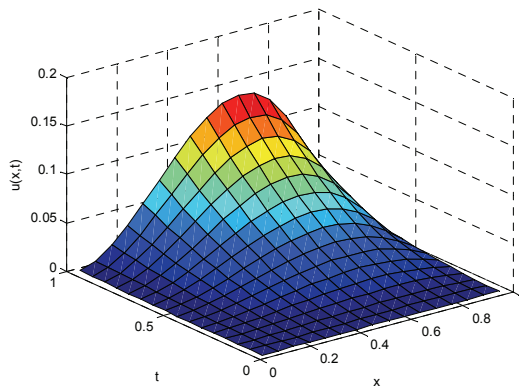


Figure 2: Numerical solution of $m = 16$

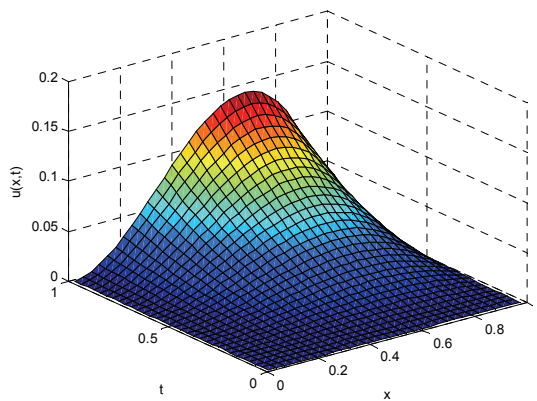


Figure 3: Numerical solution of $m = 32$

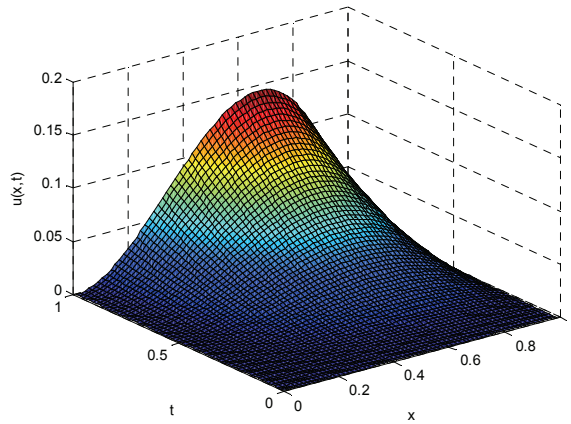
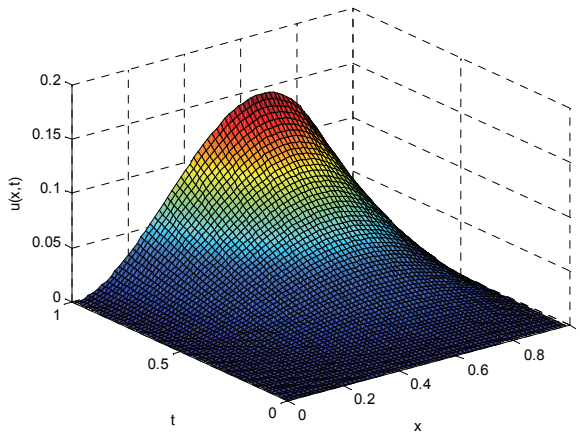
Figure 4: Numerical solution of $m = 64$ 

Figure 5: Exact solution

Table 1: The absolute error of different m

(x, t)	$m = 8$	$m = 16$	$m = 32$	$m = 64$
(0,0)	1.356227e-005	8.797021e-007	5.587532e-008	3.517159e-009
(1/8,1/8)	2.251419e-005	1.548394e-005	2.700108e-006	6.962682e-007
(2/8,2/8)	1.451388e-004	2.937151e-005	7.827678e-006	2.057847e-006
(3/8,3/8)	2.330821e-004	4.762675e-005	1.346391e-005	3.758644e-006
(4/8,4/8)	2.607993e-004	6.915614e-005	2.064872e-005	6.122976e-006
(5/8,5/8)	3.335308e-004	1.024516e-004	3.244374e-005	1.017015e-005
(6/8,6/8)	4.942812e-004	1.650434e-004	5.426294e-005	1.760939e-005
(7/8,7/8)	8.335541e-004	2.814749e-004	9.355213e-005	3.074167e-005

From the **Figs. 2-5** and **Table 1**, we can conclude that the numerical solutions are more and more close to the exact solution when m increases. Compared with the finite difference method in Ref. [Zhang (2009)], taking advantage of above method can greatly reduce the computation. Moreover, the method in this paper is easy implementation.

8 Conclusion

A operational matrix for the Haar wavelets operational matrix of fractional differentiation has been derived. This matrix is used to solve the numerical solutions of a class of space-time fractional convection-diffusion equations with variable coefficients effectively. We transform the fractional partial differential equation into a Sylvester equation which is easily to solve. Numerical example illustrates the powerful of the proposed method. The solutions obtained using the suggested method show that numerical solutions are in very good coincidence with the exact solution. From **Theorem 6.1**, we have illustrated the convergence of this algorithm.

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