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The Second-Order Two-Scale Method for Heat Transfer Performances of Periodic Porous Materials with Interior Surface Radiation

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Abstract: In this paper, a new second-order two-scale (SOTS) method is developed to predict heat transfer performances of periodic porous materials with interior surface radiation. Firstly, the second-order two-scale formulation for computing temperature field of the problem is given by means of construction way. Then, the error estimation of the second-order two-scale approximate solution is derived on some regularity hypothesis. Finally, the corresponding finite element algorithms are proposed and some numerical results are presented. They show that the SOTS method in this paper is feasible and valid for predicting the heat transfer performances of periodic porous materials.

Keywords: Second-order two-scale method, Interior surface radiation, Periodic porous materials.

1 Introduction

With the rapid advance of material science and technology, porous materials are of importance in engineering and industry owing to their high heat resistance and light weight. Especially, with rapid development of space aircraft, porous materials have attracted wide interest in thermal engineering. Therefore, it is essential to accurately predict the heat transfer performances of the porous materials. Up to now, some methods have been proposed to predict those of porous materials, such as the Maxwell-Eucken model [Hashin and Shtrikman (1962)], effective medium theory [Landauer (1952); Kirkpatrick (1973)], the self-consistent method [Torquato (2002)] and so on. The methods mentioned above are usually used to predict the heat conductivity without the effect of radiation. But, radiative heat transfer plays a

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significant rule in modern technology, especially when the temperature on surface is high enough [Modest (2003)].

Solving the radiative heat transfer problem by classical numerical methods becomes very difficult because it would require fine meshes and great amount of computation. Based on the mathematical homogenization method [Bensoussan, Lions, and Papanicolaou (1978); Oleinik, Shamaev, and Yosifian (1992)], scientists have made some worthwhile contributions to thermal radiation calculation of porous materials. Liu and Zhang (2004) predicted the effective macroscopic properties of heat conduction-radiation problem. Bakhvalov (1981) obtained the formal expansions for the solution of those problems, but had not theoretical justification. Later, Allaire and El Ganaoui (2009) discussed the heat conduction model with ε^{-1}-order radiation boundary conditions by two-scale asymptotic expansion, and justified the convergence. It should be noticed that this non-classical model can not be used for the materials with low porosity, because it over-estimates the radiation behavior on the interior surface of cavities.

In addition, various multi-scale approaches for periodic problems have been proposed, refer to E [E and Engquist (2003); E and Engquist (2005)], Hou [Hou and Wu (1997); Hou, Wu, and Cai (1999)], and Zhang [Zhang, Lv, and Zheng (2010)]. They only considered the first order asymptotic expansions. In recent years, Cui et al. [Cao and Cui (2004); Cui, Shin, and Wang (1999); Cui and Yu (2006)] presented a high-order multi-scale method to predict the physical and mechanical properties of composite materials, and solved some practical problems. By high-order correctors, the microscopic fluctuation of physical and mechanical behaviors inside the material can be captured more accurately.

We recall that the homogenization method only describes the asymptotic behavior of the problems as $\varepsilon \to 0$. But, in practical engineering computation, the period ε is a fixed smaller constant, not tends to zero. If substituting the first-order two-scale solution into original equation, one can find out that the residual is $O(1)$ even though H^1 norm of its error is $O(\varepsilon^{1/2})$. The local error $O(1)$ is not accepted for engineer who wants to capture the local behavior inside materials. Therefore we would like to say that it is necessary to seek higher-order two-scale approximation.

In this paper, we will mainly discuss the heat transfer behavior of periodic porous materials with interior surface radiation. The heat radiation boundary condition was investigated by Bakhvalov (1981), Liu and Zhang (2004) and Amosov [Amosov (2011); Amosov (2010)], and it is a classical model in physics. This paper is to establish a new high order two-scale method to give a better approximation. We introduce correction terms into the first-order two-scale expansion of the temperature field, define a family of cell functions, and then obtain second-order two-scale approximate solution. It should be pointed that the error estimation in H^1-norm is still $O(\varepsilon^{1/2})$ due to its boundary error.

The remainder of this paper is outlined as follows. Section 2 is devoted to the formulations of the second-order two-scale method. In Section 3 the error estimation on the approximate solution is analyzed. Finally, the SOTS algorithm and the numerical results for the heat transfer problem are shown.

Throughout the paper the Einstein summation convention on repeated indices is adopted. C denotes a positive constant independent of ε.

2 Second-order two-scale method

In this section, a new second-order two-scale formulation is derived for solving the heat transfer problem of periodic porous materials with interior surface radiation.

Let $Y = \{y : 0 \leq y_j \leq 1, j = 1...3\}$ and ω be an unbounded domain of R^3 which satisfies following conditions:

(B1)ω is a smooth unbounded domain of R^3 with a 1-periodic structure.

(B2) The cell of periodicity $Y^* = \omega \cap Y$ is a subdomain with a Lipschitz boundary, where Y^* is a reference periodicity cell, shown in Fig.1 (b).

(B3) The set $Y \backslash \bar{\omega}$ and the intersection of $Y \backslash \bar{\omega}$ with the δ_0 neighborhood of ∂Y consist of finite number of Lipschitz domains separated from each other and from the edges of the cube Y by a positive distance.

(B4) The cavities are convex.

Then, the domain Ω^ε, as shown in Fig.1 (a), has the form: $\Omega^\varepsilon = \Omega \cap \varepsilon\omega$, where Ω is a bounded Lipschitz convex domain of R^3. Moreover, suppose that the radiative surfaces are diffuse and grey, that is, the emissivity e of the surfaces does not depend on the wavelength of the radiation, and surface emits, absorb and reflect radiation in the same manner in all directions.

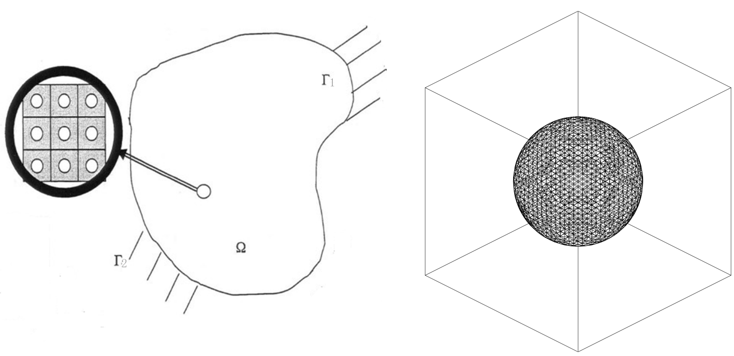
The heat conduction equation with surface radiation is firstly studied by Bakhvalov (1981), and later by Liu and Zhang (2004) and Amosov [Amosov (2011); Amosov (2010)], in which the radiation boundary condition in a closed cavity is essentially expressed as

$$-\nu_i k_{ij}^{\varepsilon}(x)\frac{\partial T_\varepsilon(x)}{\partial x_j} = e\sigma T_\varepsilon^4(x) - e\int_{\Gamma_{\varepsilon,i}^c} R_\varepsilon(x)F(x,z)dz \tag{1}$$

The classical radiation boundary condition is considered in this paper. Allaire and El Ganaoui (2009) considered the radiation condition with ε^{-1} scaling factor on Γ_ε^c. It must be noted that it is a mathematical hypothesis and the scaling ε^{-1} may not actually reflect the physical behavior of the material when the porosity is low.

We consider the heat transfer problem with interior surface radiation for given structure as follows

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(k_{ij}^{\varepsilon}(x)\dfrac{\partial T_\varepsilon(x)}{\partial x_j}\right) = f(x) & x \in \Omega^\varepsilon \\ T_\varepsilon(x) = \bar{T}(x) & x \in \Gamma_1 \\ \nu_i k_{ij}^{\varepsilon}(x)\dfrac{\partial T_\varepsilon(x)}{\partial x_j} = \bar{q}(x) & x \in \Gamma_2 \\ -\nu_i k_{ij}^{\varepsilon}(x)\dfrac{\partial T_\varepsilon(x)}{\partial x_j} = G_\varepsilon\left(\sigma T_\varepsilon^4(x)\right) & x \in \Gamma_\varepsilon^c \end{cases} \tag{2}$$



(a) (b)

Figure 1: Periodic distribution of porous materials

where $k_{ij}^{\varepsilon}(x)$ is the coefficients of thermal conductivity, and $f(x)$ the internal thermal source. The boundary can be expressed as $\partial\Omega^\varepsilon = \partial\Omega \cup \Gamma_\varepsilon^c$; Γ_1 and Γ_2 denote the boundary portions where temperature and heat flux are prescribed, respectively, such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$; Γ_ε^c is the boundary that is composed of interior surfaces of cavity $\Gamma_{\varepsilon,i}^c$, $\Gamma_\varepsilon^c = \bigcup\limits_{i=1}^{m(\varepsilon)} \Gamma_{\varepsilon,i}^c$. G_ε is the operator defined as follows (see [Allaire and El Ganaoui (2009); Bakhvalov (1981); Qatanani, Barham, and Heeh (2007); Tiihonen (1997)])

$$G_\varepsilon(\sigma T_\varepsilon^4(x)) = e\sigma T_\varepsilon^4(x) - e\int_{\Gamma_{\varepsilon,i}^c} R_\varepsilon(x)F(x,z)dz \tag{3}$$

R_ε is the intensity of emitted radiation, and has the following relationship

$$R_\varepsilon(x) = e\sigma T_\varepsilon^4(x) + (1-e)\int_{\Gamma_{\varepsilon,i}^c} R_\varepsilon(z)F(x,z)dz \tag{4}$$

From [Qatanani, Barham, and Heeh (2007); Tiihonen (1997)], we know that (4) has the unique solution $R_\varepsilon(x)$.

σ is the Stefan-Boltzmann constant, $F(x,z)$ is the view factor between two different points x and z on $\Gamma_{\varepsilon,i}^c$, and is defined as follows (see [Allaire and El Ganaoui (2009); Qatanani, Barham, and Heeh (2007); Tiihonen (1997)])

$$F(x,z) = \frac{n_z \cdot (x-z) n_x \cdot (z-x)}{\pi |z-x|^4}$$

n_z denotes the unit normal at the point z, and for any $(x,z) \in (\Gamma_{\varepsilon,i}^c)^2$ (for a closed surface), it satisfies the following properties

$$F(x,z) \geqslant 0, F(x,z) = F(z,x), \int_{\Gamma_{\varepsilon,i}^c} F(x,z)dz = 1$$

At first, suppose that:

(i) $k_{ij}(\frac{x}{\varepsilon}) = k_{ji}(\frac{x}{\varepsilon})$.

(ii) $k_{ij}(\frac{x}{\varepsilon})$ are bounded, and there exist two positive constants c_1, c_2 such that

$$c_1 \eta_i \eta_i \leqslant k_{ij}(\frac{x}{\varepsilon}) \eta_i \eta_j \leqslant c_2 \eta_i \eta_i$$

(iii) $k_{ij}(y)$ is 1-periodic functions in y, and $y = \frac{x}{\varepsilon}$.

(iiii) $k_{ij}^\varepsilon(x) \in L^\infty(\Omega^\varepsilon)$.

By supposition (i) and (ii), [Qatanani, Barham, and Heeh (2007); Tiihonen (1997)] proved the existence and uniqueness of Eq.(2).

By analogy in [Bensoussan, Lions, and Papanicolaou (1978); Oleinik, Shamaev, and Yosifian (1992)], $T_\varepsilon(x)$ can be expanded into a series in the following form:

$$T_\varepsilon(x) = T_0(x, \frac{x}{\varepsilon}) + \varepsilon T_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 T_2(x, \frac{x}{\varepsilon}) + \cdots \tag{5}$$

Then

$$\begin{aligned} T_\varepsilon^4(x) &= (T_0(x, \tfrac{x}{\varepsilon}) + \varepsilon T_1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 T_2(x, \tfrac{x}{\varepsilon}) + \cdots)^4 \\ &= T_0^4 + \varepsilon(4T_0^3 T_1) + \varepsilon^2(6T_0^2 T_1^2 + 4T_0^3 T_2) + \cdots \end{aligned} \tag{6}$$

Let $y = \frac{x}{\varepsilon}$, where x is macroscopic coordinate of the structure, and y the local coordinate of 1-normalized cell.

Let

$$A^{\varepsilon} = \varepsilon^{-2}A_1 + \varepsilon^{-1}A_2 + \varepsilon^{0}A_3$$

where

$$\begin{aligned}
A_1 &= -\frac{\partial}{\partial y_i}(k_{ij}(y)\frac{\partial}{\partial y_j}) \\
A_2 &= -\frac{\partial}{\partial y_i}(k_{ij}(y)\frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_i}(k_{ij}(y)\frac{\partial}{\partial y_j}) \\
A_3 &= -\frac{\partial}{\partial x_i}(k_{ij}(y)\frac{\partial}{\partial x_j})
\end{aligned} \tag{7}$$

Taking into account that

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$$

Substituting (5) into (2), and considering the coefficients of ε^{-2}, ε^{-1}, ε^{0}, one obtains that

$$A_1T_0 = 0 \tag{8}$$

$$A_1T_1 + A_2T_0 = 0 \tag{9}$$

$$A_1T_2 + A_2T_1 + A_3T_0 = f \tag{10}$$

$$\begin{aligned}
&A_1T_3 + A_2T_2 + A_3T_1 = 0 \\
&\qquad \ldots
\end{aligned} \tag{11}$$

For the interior radiative boundary term of (2), we have

$$-v_ik_{ij}(y)\left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right) = -v_i\frac{1}{\varepsilon}k_{ij}(y)\frac{\partial}{\partial y_i} - v_ik_{ij}(y)\frac{\partial}{\partial x_i} \tag{12}$$

Let

$$B_{\varepsilon} = \varepsilon^{-1}B_0 + B_1 \tag{13}$$

where

$$\begin{aligned}
B_0 &= -v_ik_{ij}(y)\frac{\partial}{\partial y_i} \\
B_1 &= -v_ik_{ij}(y)\frac{\partial}{\partial x_i}
\end{aligned} \tag{14}$$

From [Allaire and El Ganaoui (2009)], $R_\varepsilon(x)$ has the following expansion

$$R_\varepsilon(x) = R_0(x) + \varepsilon R_1(x,y) + \varepsilon^2 R_2(x,y) + \cdots \tag{15}$$

Let $s = \frac{z}{\varepsilon}$, where z denotes macroscopic coordinate, s the local coordinate. Substituting (6) and (15) into (3) and (4), which leads to the following identities

$$\begin{aligned} B_\varepsilon T_\varepsilon &= \varepsilon^{-1} B_0 T_0 + B_1 T_0 + B_0 T_1 + \varepsilon \left(B_1 T_1 + B_0 T_2\right) \\ &= G_\varepsilon(\sigma T_\varepsilon^4) = e\sigma T_\varepsilon^4(x) - e\int_{\Gamma^c_{\varepsilon,i}} R_\varepsilon(x) F(x,z)dz \\ &= e\sigma T_0^4(x,y) - e\int_{\Gamma^c} R_0(x) F(y,s)ds \\ &+\varepsilon(4e\sigma T_0^3(x,y)T_1(x,y) - e\int_{\Gamma^c} R_1(x,s)F(y,s)ds) \\ &+\ldots \end{aligned} \tag{16}$$

$$\begin{aligned} &R_0(x) + \varepsilon R_1(x,y) + \varepsilon^2 R_2(x,y) \\ &= e\sigma T_0^4(x,y) + (1-e)\int_{\Gamma^c} R_0(x) F(y,s)ds \\ &+\varepsilon(4e\sigma T_0^3(x,y)T_1(x,y) + (1-e)\int_{\Gamma^c} R_1(x,s)F(y,s)ds) \\ &+\ldots \end{aligned} \tag{17}$$

Using (17) and $\int_{\Gamma^c} F(s,y)dy = 1$, we obtain the equalities as follows

$$R_0(x) = \sigma T_0^4(x,y) \tag{18}$$

$$R_1(x,y) = 4e\sigma T_0^3(x,y)T_1(x,y) + (1-e)\int_{\Gamma^c} R_1(x,s)F(y,s)ds \tag{19}$$

Considering the coefficients of ε^{-1}, ε^0, ε in (16), one obtains that

$$B_0 T_0 = 0 \tag{20}$$

$$B_1 T_0 + B_0 T_1 = e\sigma T_0^4(x,y) - e\int_{\Gamma^c} R_0(x) F(y,s)ds \tag{21}$$

$$B_1 T_1 + B_0 T_2 = 4e\sigma T_0^3(x,y)T_1(x,y) - e\int_{\Gamma^c} R_1(x,s)F(y,s)ds \tag{22}$$

Thus, from (8), (9) and (10), a series of the mixed boundary value problems are obtained:

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_0(x,y)}{\partial y_j}\right) = 0 & in \quad Y^* \\ -v_i k_{ij}(y)\dfrac{\partial T_0(x,y)}{\partial y_j} = 0 & y \in \Gamma^c \\ T_0(x,y) \qquad is \quad Y-periodic \end{cases} \tag{23}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_1(x,y)}{\partial y_j}\right) = \dfrac{\partial}{\partial y_i}(k_{ij}(y)\dfrac{\partial T_0(x,y)}{\partial x_j}) \\ +\dfrac{\partial}{\partial x_i}(k_{ij}(y)\dfrac{\partial T_0(x,y)}{\partial y_j}) \qquad\qquad in \quad Y^* \\ -v_i k_{ij}(y)(\dfrac{\partial T_1(x,y)}{\partial y_j} + \dfrac{\partial T_0(x,y)}{\partial x_j}) \\ = e\sigma T_0^4(x,y) - e\displaystyle\int_{\Gamma^c} R_0(x)F(y,s)ds \; y \in \Gamma^c \\ T_1(x,y) \qquad\qquad\qquad is \quad Y-periodic \end{cases} \tag{24}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_2(x,y)}{\partial y_j}\right) = f + \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_1(x,y)}{\partial x_j}\right) \\ +k_{ij}(y)\dfrac{\partial^2 T_0(x,y)}{\partial x_i \partial x_j} + k_{ij}(y)\dfrac{\partial}{\partial x_i}\dfrac{\partial T_1(x,y)}{\partial y_j} \qquad in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\dfrac{\partial T_2(x,y)}{\partial y_j} + k_{ij}(y)\dfrac{\partial T_1(x,y)}{\partial x_j}\right) \\ = 4e\sigma T_0^3(x,y)T_1(x,y) - e\displaystyle\int_{\Gamma^c} R_1(x,s)F(y,s)ds \; y \in \Gamma^c \\ T_2(x,y) \qquad\qquad\qquad is \quad Y-periodic \end{cases} \tag{25}$$

where Γ^c is the boundary of the cavities contained in Y^*.
From Eq.(23) it follows that $T_0(x,y)$ is independent of y.
Then, (24) can be rewritten as

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_1(x,y)}{\partial y_j}\right) = \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_0(x)}{\partial x_j}\right) \; in \; Y^* \\ -\nu_i k_{ij}(y)\left(\dfrac{\partial T_1(x,y)}{\partial y_j} + \dfrac{\partial T_0(x)}{\partial x_j}\right) \\ = e\sigma T_0^4(x) - e\displaystyle\int_{\Gamma^c} R_0(x)F(y,s)ds \qquad\qquad y \in \Gamma^c \\ T_1(x,y) \qquad\qquad\qquad is \quad Y-periodic \end{cases} \tag{26}$$

From (18) and $\int_{\Gamma^c} F(s,y)dy = 1$, it is easy to verify that

$$e\sigma T_0^4(x) - e\int_{\Gamma^c} R_0(x)F(y,s)ds = 0 \tag{27}$$

By virtue of (27), (26) can be written as follows:

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_1(x,y)}{\partial y_j}\right) = \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_0(x)}{\partial x_j}\right) \ in\ Y^* \\ -\nu_i k_{ij}(y)\left(\dfrac{\partial T_1(x,y)}{\partial y_j} + \dfrac{\partial T_0(x)}{\partial x_j}\right) = 0 \qquad y \in \Gamma^c \\ T_1(x,y) \qquad is \quad Y-periodic \end{cases} \tag{28}$$

Suppose that the solution of Eq.(28) has the following form

$$T_1(x,y) = N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} + \tilde{T}_1(x) \tag{29}$$

$\tilde{T}_1(x)$ is only dependent of x, $T_1(x,y)$ is y-periodicity. Then, we obtain the auxiliary function $N_{\alpha_1}(y)$ defined on 1-normalized cell Y^*.
$N_{\alpha_1}(y)$ is the solution of the following elliptic partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial N_{\alpha_1}(y)}{\partial y_j}\right) = -\dfrac{\partial k_{i\alpha_1}(y)}{\partial y_i} \qquad in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\dfrac{\partial N_{\alpha_1}(y)}{\partial y_j} + k_{i\alpha_1}(y)\right) = 0 \qquad y \in \Gamma^c \\ N_{\alpha_1}(y) \qquad is \quad Y-periodic \end{cases} \tag{30}$$

From theorem 2.1 in [Cao and Cui (1999)], Eq.(30) has one unique solution.
From Eq.(25), one obtains that

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y)\dfrac{\partial T_2(x,y)}{\partial y_j}\right) = f + \dfrac{\partial}{\partial y_i}\left(k_{ij}(y)N_{\alpha_1}(y)\right)\dfrac{\partial^2 T_0}{\partial x_j \partial x_{\alpha_1}} \\ +k_{ij}(y)\dfrac{\partial^2 T_0}{\partial x_i \partial x_j} + k_{ij}(y)\dfrac{\partial N_{\alpha_1}(y)}{\partial y_j}\dfrac{\partial^2 T_0}{\partial x_i \partial x_{\alpha_1}} \qquad in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\dfrac{\partial T_2(x,y)}{\partial y_j} + k_{ij}(y)N_{\alpha_1}(y)\dfrac{\partial^2 T_0}{\partial x_j \partial x_{\alpha_1}}\right) \\ = 4e\sigma T_0^3(x)N_{\alpha_1}(y)\dfrac{\partial T_0}{\partial x_{\alpha_1}} - e\displaystyle\int_{\Gamma^c} R_1(x,s)F(y,s)ds \qquad y \in \Gamma^c \\ T_2(x,y) \qquad is \quad Y-periodic \end{cases} \tag{31}$$

$f(x)$ is an integrable function on domain $G \in R^n$ and let$\sim$ denote the homogenization operator defined by

$$\tilde{f} = \frac{1}{|G|}\int_G f(x)dv \tag{32}$$

We impose the homogenization operator to both sides of equality (31), and then obtain the homogenized equation associated with Eq.(2) as follows:

$$
\begin{aligned}
&-\int_{Y^*} k_{ij}(y)\frac{\partial N_{\alpha_1}(y)}{\partial y_j}dy\frac{\partial^2 T_0}{\partial x_i \partial x_{\alpha_1}} - \int_{Y^*} k_{ij}(y)dy\frac{\partial^2 T_0}{\partial x_i \partial x_j} \\
&+\int_{\Gamma^c}\left(4e\sigma T_0^3(x)T_1(x,y) - e\int_{\Gamma^c} R_1(x,s)F(y,s)ds\right)dy = |Y^*|f
\end{aligned}
\tag{33}
$$

Further, taking into account (19) we have

$$
R_1(x,y) = 4e\sigma T_0^3(x,y)T_1(x,y) + (1-e)\int_{\Gamma^c} R_1(x,s)F(y,s)ds \tag{34}
$$

After integrating on both sides of the above equation with respect to y, (34) can be rewritten as

$$
\int_{\Gamma^c} R_1(x,y)dy = \int_{\Gamma^c} 4e\sigma T_0^3(x)T_1(x,y)dy + (1-e)\int_{\Gamma^c}\int_{\Gamma^c} R_1(x,s)F(y,s)dsdy \tag{35}
$$

Then, taking the property $\int_{\Gamma^c} F(s,y)dy = 1$ yields the equality

$$
\int_{\Gamma^c} 4\sigma T_0^3(x)T_1(x,y)dy = \int_{\Gamma^c} R_1(x,y)dy \tag{36}
$$

Therefore, we obtain the result

$$
\int_{\Gamma^c}\left(4e\sigma T_0^3(x)T_1(x,y) - e\int_{\Gamma^c} R_1(x,s)F(y,s)ds\right)dy = 0 \tag{37}
$$

Then, by using (33) and (37), one can obtain the homogenized equation associated with Eq.(2) as follows

$$
\begin{cases}
-\dfrac{\partial}{\partial x_i}(\hat{k}_{ij}\dfrac{\partial T_0}{\partial x_j}) = \dfrac{|Y^*|}{|Y|}f & in \quad \Omega \\
T_0 = \bar{T} & on \quad \Gamma_1 \\
\nu_i \hat{k}_{ij}\dfrac{\partial T_0}{\partial x_j} = \bar{q} & on \quad \Gamma_2
\end{cases}
\tag{38}
$$

where

$$
\hat{k}_{ij} = \frac{1}{|Y|}\int_{Y^*}(k_{ip}(y)\frac{\partial N_j(y)}{\partial y_p} + k_{ij}(y))dy \tag{39}
$$

From supposition (ii) and [Oleinik, Shamaev, and Yosifian (1992)] it follows that $\hat{k}_{ij}$ is symmetrical and positive definite. By Lax-Milgram theorem, Poincare's inequality, the homogenization problem (38) has a unique solution.

Further, $f(x)$ can be substituted by Eq.(38), Eq.(31) can be written as

$$\begin{cases} -\frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial T_2(x,y)}{\partial y_j}\right) = \left(-\frac{|Y|}{|Y^*|}\hat{k}_{\alpha_1\alpha_2} + \frac{\partial}{\partial y_i}\left(k_{i\alpha_2}(y)N_{\alpha_1}(y)\right) + k_{\alpha_1\alpha_2}(y) \right. \\ \left. + k_{\alpha_2 j}(y)\frac{\partial N_{\alpha_1}(y)}{\partial y_j}\right)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} & in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\frac{\partial T_2(x,y)}{\partial y_j} + k_{i\alpha_2}(y)N_{\alpha_1}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\right) \\ = 4e\sigma T_0^3(x)N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} - e\int_{\Gamma^c} R_1(x,s)F(y,s)ds & y\in\Gamma^c \\ T_2(x,y) & is \quad Y-periodic \end{cases} \tag{40}$$

From (19), one obtains that

$$R_1(x,y) = 4e\sigma T_0^3(x)N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} + (1-e)\int_{\Gamma^c} R_1(x,s)F(y,s)ds \tag{41}$$

So we define that the solution of $R_1(x,y)$ has the following form

$$R_1(x,y) = M_{\alpha_1}(y)T_0^3(x)\frac{\partial T_0}{\partial x_{\alpha_1}} \tag{42}$$

and the auxiliary function $M_{\alpha_1}(y)$ satisfies the equality

$$M_{\alpha_1}(y) = 4e\sigma N_{\alpha_1}(y) + (1-e)\int_{\Gamma^c} M_{\alpha_1}(s)F(y,s)ds \tag{43}$$

Similar to (4), we can prove that (43) has one unique solution $M_{\alpha_1}(y)$.
Then, to satisfy (40), we seek a reasonable expression for $T_2(x,y)$

$$T_2(x,y) = N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}$$

where $T_0(x)$ is the homogenization solution on Ω. $N_{\alpha_1\alpha_2}(y)$ and $C_{a_1}(y)$ are the local functions defined on Y^*. One can define them as follows:

$N_{\alpha_1\alpha_2}(y)$ is the solution of the following problem:

$$\begin{cases} \frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_j}\right) = \tilde{k}_{\alpha_1\alpha_2} - \frac{\partial}{\partial y_i}\left(k_{i\alpha_2}(y)N_{\alpha_1}(y)\right) - k_{\alpha_1\alpha_2}(y) \\ -k_{\alpha_2 j}(y)\frac{\partial N_{\alpha_1}(y)}{\partial y_j} \qquad in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_j} + k_{i\alpha_2}(y)N_{\alpha_1}(y)\right) = 0 \qquad y \in \Gamma^c \\ N_{\alpha_1\alpha_2}(y) \qquad is \quad Y-periodic \end{cases} \tag{44}$$

$C_{\alpha_1}(y)$ is the solution of the following problem:

$$\begin{cases} \frac{\partial}{\partial y_i}\left(k_{ij}(y)\frac{\partial C_{\alpha_1}(y)}{\partial y_j}\right) = 0 \qquad in \quad Y^* \\ -\nu_i\left(k_{ij}(y)\frac{\partial C_{\alpha_1}(y)}{\partial y_j}\right) \\ = 4e\sigma N_{\alpha_1}(y) - e\int_{\Gamma^c} M_{\alpha_1}(s)F(y,s)ds \qquad y \in \Gamma^c \\ C_{\alpha_1}(y) \qquad is \quad Y-periodic \end{cases} \tag{45}$$

where $\tilde{k}_{\alpha_1\alpha_2} = \frac{|Y|}{|Y^*|}\hat{k}_{\alpha_1\alpha_2}$.

Lemma 2.1. Each of the cell problem (44)-(45) admits a unique solution, up to a constant, in $H^1(Y^*)/R$

Proof. Similar to (30), it is easy to prove that the problems (44) has the unique solution $N_{\alpha_1\alpha_2}(y)$. For equation (45), and taking the property $\int_{\Gamma^c} F(s,y)dy = 1$ we obtain that

$$\begin{aligned} &\int_{\Gamma^c} 4e\sigma N_{\alpha_1}(y) - e\int_{\Gamma^c} M_{\alpha_1}(s)F(y,s)dsdy \\ &= \int_{\Gamma^c} e4\sigma N_{\alpha_1}(y)dy - \int_{\Gamma^c} eM_{\alpha_1}(s)ds \end{aligned} \tag{46}$$

Then, by virtue of (43) and integrating on both sides of the equation with respect to y, (43) can be rewritten as

$$\int_{\Gamma^c} M_{\alpha_1}(y)dy = \int_{\Gamma^c} (4e\sigma N_{\alpha_1}(y) + (1-e)\int_{\Gamma^c} M_{\alpha_1}(s)F(y,s))dsdy \tag{47}$$

Therefore, we obtain the result

$$\int_{\Gamma^c} M_{\alpha_1}(y)dy = \int_{\Gamma^c} 4\sigma N_{\alpha_1}(y)dy \tag{48}$$

So from (46), (48), and theorem 2.1 in [Cao and Cui (1999)], the Fredholm alternative yields existence and uniqueness in $H^1(Y^*)/R$ of the cell problem (45).

Now we can define the two-scale approximate solution of problem (2) as follows

$$
\begin{aligned}
T_1^\varepsilon(x) &= T_0(x) + \varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} \\
T_2^\varepsilon(x) &= T_0(x) + \varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} + \varepsilon^2\left(N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\right)
\end{aligned}
\tag{49}
$$

$T_1^\varepsilon(x)$ and $T_2^\varepsilon(x)$ are called as the first-order and the second-order two-scale approximate solutions, respectively. To compare $T_1^\varepsilon(x)$ with the original solution, we substitute $T_\varepsilon(x) - T_1^\varepsilon(x)$ into (2), and have

$$
\begin{aligned}
L_\varepsilon\left(T_\varepsilon(x) - T_1^\varepsilon(x)\right) = f - k_{ij}(y)\frac{\partial N_{\alpha_1}(y)}{\partial y_j}\frac{\partial^2 T_0}{\partial x_i \partial x_{\alpha_1}} - \frac{\partial}{\partial y_i}(k_{ij}(y)N_{\alpha_1}(y))\frac{\partial^2 T_0}{\partial x_j \partial x_{\alpha_1}} \\
-k_{ij}(y)\frac{\partial^2 T_0}{\partial x_i \partial x_j} - \varepsilon k_{ij}(y)\frac{\partial^2}{\partial x_i \partial x_j}(N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}})
\end{aligned}
$$

It is not difficult to see that the residual is the order $O(1)$. But in the practical engineering computation, ε is a fixed smaller constant rather than tending to zero. The error $O(1)$ is not accepted for the engineers who want to capture the local behavior inside materials. So the first order solution is not adopted in engineering computation.

Substituting $T_\varepsilon(x) - T_2^\varepsilon(x)$ into (2) we obtain that

$$
\begin{aligned}
L_\varepsilon\left(T_\varepsilon(x) - T_2^\varepsilon(x)\right) &= -\frac{\partial}{\partial x_i}(k_{ij}^\varepsilon(x)\frac{\partial T_\varepsilon(x)}{\partial x_j} \\
&+\frac{\partial}{\partial x_i}(k_{ij}^\varepsilon(x)\frac{\partial}{\partial x_j}(T_0(x) + \varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} + \varepsilon^2(N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}))) \\
&= \varepsilon F_0 + \varepsilon\frac{\partial}{\partial x_i}F_i
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
F_0 &= k_{ij}(y)N_{\alpha_1}(y)\frac{\partial^3 T_0}{\partial x_i \partial x_j \partial x_{\alpha_1}} + k_{ij}(y)\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_j}\frac{\partial^3 T_0}{\partial x_i \partial x_{\alpha_1}\partial x_{\alpha_2}} \\
&+ \frac{\partial}{\partial y_i}(k_{ij}(y)N_{\alpha_1\alpha_2}(y))\frac{\partial^3 T_0}{\partial x_j \partial x_{\alpha_1}\partial x_{\alpha_2}} + k_{ij}(y)\frac{\partial C_{\alpha_1}(y)}{\partial y_j}\frac{\partial}{\partial x_i}(T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}) \\
&+ \frac{\partial}{\partial y_i}(k_{ij}(y)C_{\alpha_1}(y))\frac{\partial}{\partial x_j}(T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}) + \varepsilon k_{ij}(y)N_{\alpha_1\alpha_2}(y)\frac{\partial^4 T_0}{\partial x_i \partial x_j \partial x_{\alpha_1}\partial x_{\alpha_2}} \\
F_i &= \varepsilon k_{ij}(y)C_{\alpha_1}(y)\frac{\partial}{\partial x_j}(T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}})
\end{aligned}
\tag{51}
$$

Note that the residual of (50) is the order $O(\varepsilon)$. The second-order solution is equivalent to the solution of original problem in $O(\varepsilon)$-order pointwise sense. It is the reason that we consider the second order expansions in this paper.

Theorem 2.1. Temperature field for the heat transfer problem (2) of periodic porous materials with interior surface radiation has a second-order two-scale asymptotic expansion as follows

$$
\begin{aligned}
T_\varepsilon(x) = T_0(x) &+ \varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} + \varepsilon^2\left(N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\right) \\
&+ \varepsilon^3 P_1(\varepsilon, x, y)
\end{aligned}
\tag{52}
$$

where $T_0(x)$ is the solution of the homogenized (38) with the parameters (39). $N_{\alpha_1}(y)$, $N_{\alpha_1\alpha_2}(y)$ and $C_{\alpha_1}(y)$ are the local solutions satisfying (30), (44) and (45), respectively.

Further, the temperature gradient can be evaluated as follows.

$$
\begin{aligned}
\frac{\partial T_2^\varepsilon(x)}{\partial x_i} &= \frac{\partial T_0(x)}{\partial x_i} + \frac{\partial N_{\alpha_1}(y)}{\partial y_i}\frac{\partial T_0(x)}{\partial x_{\alpha_1}} + \varepsilon N_{\alpha_1}(y)\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_i} \\
&+ \varepsilon\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_i}\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + \varepsilon^2 N_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} + \varepsilon\frac{\partial C_{\alpha_1}(y)}{\partial y_i}T_0^3\frac{\partial T_0(x)}{\partial x_{\alpha_1}} \\
&+ \varepsilon^2 C_{\alpha_1}(y)\left(T_0^3\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_i} + 3T_0^2\frac{\partial T_0(x)}{\partial x_i}\frac{\partial T_0(x)}{\partial x_{\alpha_1}}\right)
\end{aligned}
\tag{53}
$$

3 Error estimation of H^1-norm

In this section we give the error estimation for the second-order two-scale approximate solution, and suppose that $e = 1$. Now the following equations is considered.

$$\begin{cases} L_\varepsilon(T_\varepsilon(x)) = -\dfrac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon(x)\dfrac{\partial T_\varepsilon(x)}{\partial x_j}\right) = f(x) & x \in \Omega^\varepsilon \\ T_\varepsilon(x) = \bar{T}(x) & x \in \Gamma_1 \\ -\nu_i k_{ij}^\varepsilon(x)\dfrac{\partial T_\varepsilon(x)}{\partial x_j} = G_\varepsilon\left(\sigma T_\varepsilon^4(x)\right) & x \in \Gamma_\varepsilon^c \end{cases} \tag{54}$$

where $\Gamma_1 = \partial\Omega$. And the homogenized equation is obtained as follows

$$\begin{cases} \hat{L}(T_0(x)) = -\dfrac{\partial}{\partial x_i}(\hat{k}_{ij}\dfrac{\partial T_0}{\partial x_j}) = \dfrac{|Y^*|}{|Y|}f & in \quad \Omega \\ T_0 = \bar{T}(x) & on \quad \Gamma_1 \end{cases} \tag{55}$$

At first, we give a lemma.

Lemma 3.1. Let Ω be a bounded domain with a smooth boundary and $B_\delta = \{x \in \Omega, \rho(x,\partial\Omega) < \delta\}, \delta > 0$. Then there exists a constant $\delta_0 > 0$ such that for $\forall\delta \in (0,\delta_0)$ and $\forall v \in H^1(\Omega)$ we have

$$\|v\|_{L^2(B_\delta)} \leqslant C\delta^{1/2}\|v\|_{H^1(\Omega)}$$

where C is a constant independent of δ and v.

The proof of lemma 3.1 can be found in [Oleinik, Shamaev, and Yosifian (1992)].

Theorem 3.1. Suppose that $\Omega^\varepsilon \subset R^3$ is the union of entire periodic cells. Let $T_\varepsilon(x)$ be the solution of (54), $T_0(x)$ is the solution of homogenized equation of (55). $T_2^\varepsilon(x)$ is the approximate solution stated in (49). Under assumptions (B1)-(B4) and (i)-(iiii), if $f \in L^2(\Omega)$, for sufficiently smooth homogenized solutions $T_0(x)$, we obtain the following error estimation:

$$\|T_\varepsilon(x) - T_2^\varepsilon(x)\|_{H^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{1/2} \tag{56}$$

C is positive constant independent of ε.

Proof: From (50) and (51), we obtain the residual inside Ω^ε.

On the boundary Γ_1 we have

$$T_\varepsilon(x) - T_2^\varepsilon(x) = -\varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}} - \varepsilon^2 N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \varepsilon^2 C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}} = \varphi_\varepsilon(x) \tag{57}$$

It is similar to [Oleinik, Shamaev, and Yosifian (1992); Cioranescu and Donato (1999)], we can obtain that

$$\|\varphi_\varepsilon(x)\|_{H^{1/2}(\Gamma_1)} \leqslant C\varepsilon^{1/2}\|T_0\|_{H^3(\Omega)} \tag{58}$$

For sufficiently smooth homogenized solutions $T_0(x)$, We have that $\|P_1(\varepsilon,x,y)\|_{L^2(\Omega)} \leqslant C\varepsilon^l \;\; l \geqslant 0$. Then, on the boundary $\Gamma^c_{\varepsilon,i}$, let $y=\frac{x}{\varepsilon}$, $s=\frac{z}{\varepsilon}$, we have

$$\begin{aligned}
&-v_i k^\varepsilon_{ij}(x)(\frac{\partial T_\varepsilon(x)}{\partial x_j}-\frac{\partial T^\varepsilon_2(x)}{\partial x_j})\\
&=\sigma(T_0+\varepsilon N_{\alpha_1}(y)\frac{\partial T_0}{\partial x_{\alpha_1}}+\varepsilon^2 N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+\varepsilon^2 C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}+\varepsilon^3 P_1(\varepsilon,x,y))^4\\
&-\sigma\int_{\Gamma^c}(T_0+\varepsilon N_\alpha(s)\frac{\partial T_0}{\partial x_\alpha}+\varepsilon^2 N_{\alpha_1\alpha_2}(s)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+\varepsilon^2 C_{\alpha_1}(s)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}+\varepsilon^3 P_1(\varepsilon,x,y))^4 F(y,s)ds\\
&+v_i k_{ij}(y)(\frac{\partial T_0}{\partial x_j}+\frac{\partial N_{\alpha_1}(y)}{\partial y_j}\frac{\partial T_0}{\partial x_{\alpha_1}}+\varepsilon N_{\alpha_1}(y)\frac{\partial^2 T_0}{\partial x_j\partial x_{\alpha_1}}+\varepsilon\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+\varepsilon^2 N_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}+\varepsilon\frac{\partial C_{\alpha_1}(y)}{\partial y_j}T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}+\varepsilon^2 C_{\alpha_1}(y)\frac{\partial}{\partial x_j}(T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}))\\
&=4\varepsilon\sigma N_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}-\int_{\Gamma^c}(4\varepsilon\sigma N_{\alpha_1}(s)F(y,s)dsT_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\\
&+\varepsilon v_i k_{ij}(y)\frac{\partial C_{\alpha_1}(y)}{\partial y_j}T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}+v_i(k_{i\alpha_1}(y)+\frac{\partial N_{\alpha_1}(y)}{\partial y_j})\frac{\partial T_0}{\partial x_{\alpha_1}}\\
&+\varepsilon v_i\left(k_{ij}(y)\frac{\partial N_{\alpha_1\alpha_2}(y)}{\partial y_j}+k_{i\alpha_2}(y)N_{\alpha_1}(y)\right)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+\varepsilon^2 v_i k_{ij}(y)(N_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}+C_{\alpha_1}(y)\frac{\partial}{\partial x_j}(T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}))\\
&+\varepsilon^2 4T_0^3\sigma(N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}})\\
&-\varepsilon^2\int_{\Gamma^c}4T_0^3\sigma(N_{\alpha_1\alpha_2}(s)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+C_{\alpha_1}(s)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}})F(y,s)ds\\
&+\varepsilon^2 6T_0^2\sigma(N_\alpha(y)\frac{\partial T_0}{\partial x_\alpha})^2-\varepsilon^2\int_{\Gamma^c}6T_0^2\sigma(N_{\alpha_1}(s)\frac{\partial T_0}{\partial x_{\alpha_1}})^2 F(y,s)ds+O(\varepsilon^3)
\end{aligned} \tag{59}$$

By virtue of the boundary condition on Γ^c for $N_{\alpha_1}(y)$, $N_{\alpha_1\alpha_2}(y)$ and $C_{\alpha_1}(y)$ and the regularity of T_0 it follows that

$$-v_i k_{ij}^{\varepsilon}(x)\left(\frac{\partial T_\varepsilon(x)}{\partial x_j} - \frac{\partial T_2^{\varepsilon}(x)}{\partial x_j}\right) = \varepsilon^2 F \tag{60}$$

where

$$\begin{aligned}
F &= \nu_i k_{ij}(y)\left(N_{\alpha_1\alpha_2}(y)\frac{\partial^3 T_0}{\partial x_j \partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)\frac{\partial}{\partial x_j}\left(T_0^3 \frac{\partial T_0}{\partial x_{\alpha_1}}\right)\right)\\
&+4T_0^3\sigma\left(N_{\alpha_1\alpha_2}(y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(y)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\right)\\
&-\int_{\Gamma^c} 4T_0^3\sigma\left(N_{\alpha_1\alpha_2}(s)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + C_{\alpha_1}(s)T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\right)F(y,s)ds\\
&+6T_0^2\sigma\left(N_\alpha(y)\frac{\partial T_0}{\partial x_\alpha}\right)^2 - \int_{\Gamma^c} 6T_0^2\sigma\left(N_{\alpha_1}(s)\frac{\partial T_0}{\partial x_{\alpha_1}}\right)^2 F(y,s)ds + O(\varepsilon)
\end{aligned}$$

We conclude that $T_\varepsilon(x) - T_2^{\varepsilon}(x)$ is a weak solution of the following boundary value problem

$$\begin{cases}
L_\varepsilon\left(T_\varepsilon(x) - T_2^{\varepsilon}(x)\right) = \varepsilon F_0 + \varepsilon \dfrac{\partial}{\partial x_i}F_i & x \in \Omega^\varepsilon\\
T_\varepsilon(x) - T_2^{\varepsilon}(x) = \varphi_\varepsilon(x) & x \in \Gamma_1\\
-\nu_i k_{ij}^{\varepsilon}(x)\dfrac{\partial\left(T_\varepsilon(x) - T_2^{\varepsilon}(x)\right)}{\partial x_j} = \varepsilon^2 F & x \in \Gamma_\varepsilon^c
\end{cases} \tag{61}$$

Noting, after integration and summation over all cells, we obtain a remainder term given by

$$\sum_{i=1}^{m(\varepsilon)} \left|\Gamma_{\varepsilon,i}^c\right| O(\varepsilon^2) = O(\varepsilon^{-d})O(\varepsilon^{d-1})O(\varepsilon^2) = O(\varepsilon) \tag{62}$$

where d=2 or 3 in applications. According to (58), (60), (61) and (62), we complete the proof. Because of the residual (57) on Γ_1, we can only get the approximate order of $O(\varepsilon^{1/2})$.

4 FE algorithms and numerical examples

In this section we describe the algorithm procedure to study the asymptotic behavior of the heat transfer problem (2), and performed some numerical examples.

4.1 FE Algorithms for SOTS method

The algorithm procedure of SOTS method for predicting the thermal properties of the heat transfer problem with interior surface radiation is stated as follows

1. Form and verify the distribution of cavities in reference cell, and the geometry of the structure. Further, partition Y^* into FE mesh set.

2. Obtain the FE solutions of $N_{\alpha_1}(y)$ according to the problem (30) with given material properties based on the FE model of the unit cell. Furthermore, the homogenization coefficient $\hat{k}_{ij}$ is evaluated by the formula (39).

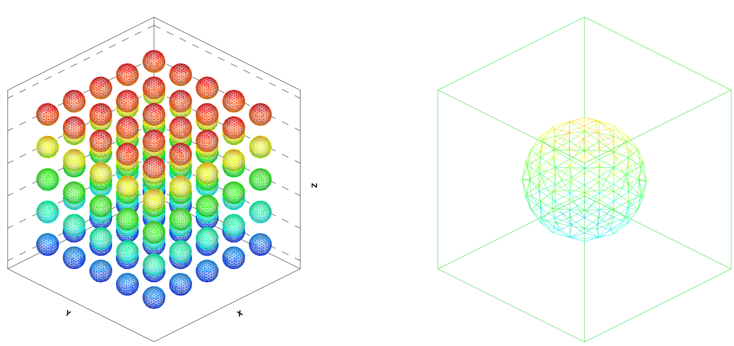
3. According $\hat{k}_{ij}$ we have known from 2, the homogenization solution $T_0(x)$ is obtained by solving problem (38) in Ω.

4. With the same meshes to 2, we evaluate $N_{\alpha_1\alpha_2}(y)$ and $C_{a_1}(y)$ by solving the cell problems (44) and (45).

5. From (49) and (53), the temperature and the temperature gradient are evaluated.

4.2 Numerical examples

Consider the mixed boundary value problem (2), where Ω^ε is composed of entire cells shown in Fig.2(a), and the reference cell Y^* is shown in Fig.2(b). The heat flux $\bar{q}(x)$ on lateral surface is set to zero. The boundary temperatures in the z-direction are set as T_1 and T_2. $\sigma = 5.669996 \times 10^{-8} W/\mathrm{m}^2 K^4$, and the radius of the cavity in Y^* is 0.3.



(a) (b)

Figure 2: (a) Domain $\Omega^\varepsilon = [0, 0.25]^3$ (b) Unit cell $Y^* = [0,1]^3$

Since it is difficult to find the analytical solution of the above problem, we have to take $T_\varepsilon(x)$ to be its FE solution Te in the very fine mesh. The information of the FE meshes is listed in Table 1

Table 1: Comparison of computational cost

	Original equation	Unit cell	Homogenized equation
Elements	685500	5078	93750
Nodes	131526	1403	17576

The following four cases are investigated:

Case 1: $e = 0.2$, k_{ij}=100δ_{ij}, T_1 =100, $T_2 = 500$, $f = 0$

Case 2: $e = 1.0$, k_{ij} =100δ_{ij}, T_1 =100, $T_2 = 1000$, $f = 10000000$

Case 3: $e = 1.0$, k_{ij}=δ_{ij}, T_1 =100, $T_2 = 500$, $f = 0$

Case 4: $e = 1.0$, k_{ij} =δ_{ij}, T_1 =100, $T_2 = 800$, $f = 0$

It should be noted that $T_0(x)$ denotes the numerical solution of the homogenized equations (38), $\hat{T}_1^\varepsilon(x)$ and $\hat{T}_2^\varepsilon(x)$ the first-order and the second-order two-scale numerical solutions based on (49). Set $error_0 = Te - T_0(x)$, $error_1 = Te - \hat{T}_1^\varepsilon(x)$ and $error_2 = Te - \hat{T}_2^\varepsilon(x)$. For convenience, we introduce the following notation

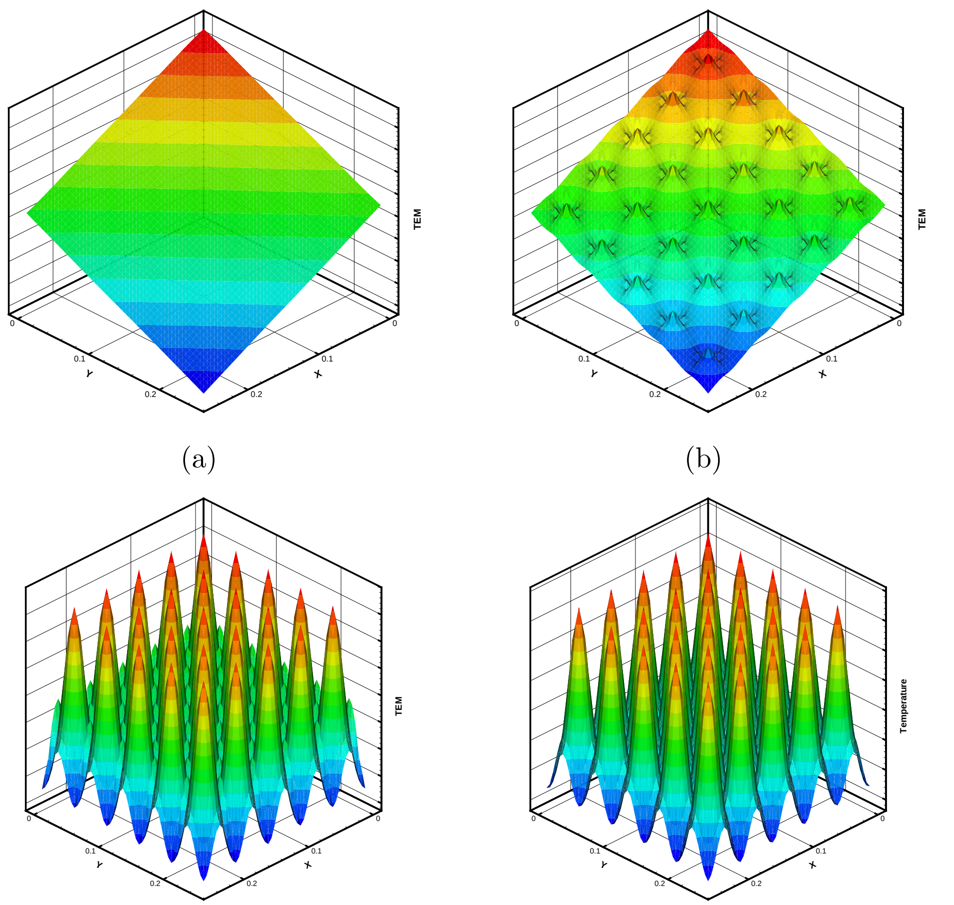
$$\|v\|_{L^2(\Omega^\varepsilon)} = (\int_{\Omega^\varepsilon} |v|^2 dx)^{1/2}, |v|_{H^1(\Omega^\varepsilon)} = (\int_{\Omega^\varepsilon} (|\nabla v|^2) dx)^{1/2}$$

The relative numerical errors of the homogenization, first-order two-scale, and second-order two-scale methods in H^1 for examples are listed in Table 2.

Table 2: Comparison with computing results of semi-norm H^1.

	$\lvert error_0\rvert_{H^1} / \lvert Te\rvert_{H^1}$	$\lvert error_1\rvert_{H^1} / \lvert Te\rvert_{H^1}$	$\lvert error_2\rvert_{H^1} / \lvert Te\rvert_{H^1}$
Case1	0.21749172	0.00984979	0.00984899
Case2	0.22164915	0.10700699	0.07142537
Case3	0.18443035	0.04943442	0.03569669
Case4	0.14772598	0.14737135	0.09423171

Figures 3(a)-(d) illustrate the numerical results for $T_0(x)$, $\hat{T}_1^\varepsilon(x)$, $\hat{T}_2^\varepsilon(x)$ and Te at the intersection z=0.15 in Case 2.



(a) (b)

(c) (d)

Figure 3: (a) Case 2, $T_0(x)$; (b) Case 2, $\hat{T}_1^{\varepsilon}(x)$; (c) Case 2, $\hat{T}_2^{\varepsilon}(x)$; (d) Case 2, Te

Figures 4(a)-(d) illustrate the numerical results for $\hat{T}_1^{\varepsilon}(x)$, $\hat{T}_2^{\varepsilon}(x)$ and Te at the intersection x=0.125, and for $\frac{\partial \hat{T}_1^{\varepsilon}(x)}{\partial y}$, $\frac{\partial \hat{T}_2^{\varepsilon}(x)}{\partial y}$, $\frac{\partial Te}{\partial y}$ at the intersection z=0.15 in Case 4.

From the numerical simulations for the above case studies, we note that when the $k_{ij}^{\varepsilon}(\frac{x}{\varepsilon})$ of the unit cell Y^* is small or the source term is varying with a large amplitude, the homogenization method and the first-order two-scale method fail to provide satisfactory results. The SOTS method, however, clearly is the best among the three methods, and it results in accurate numerical solutions.

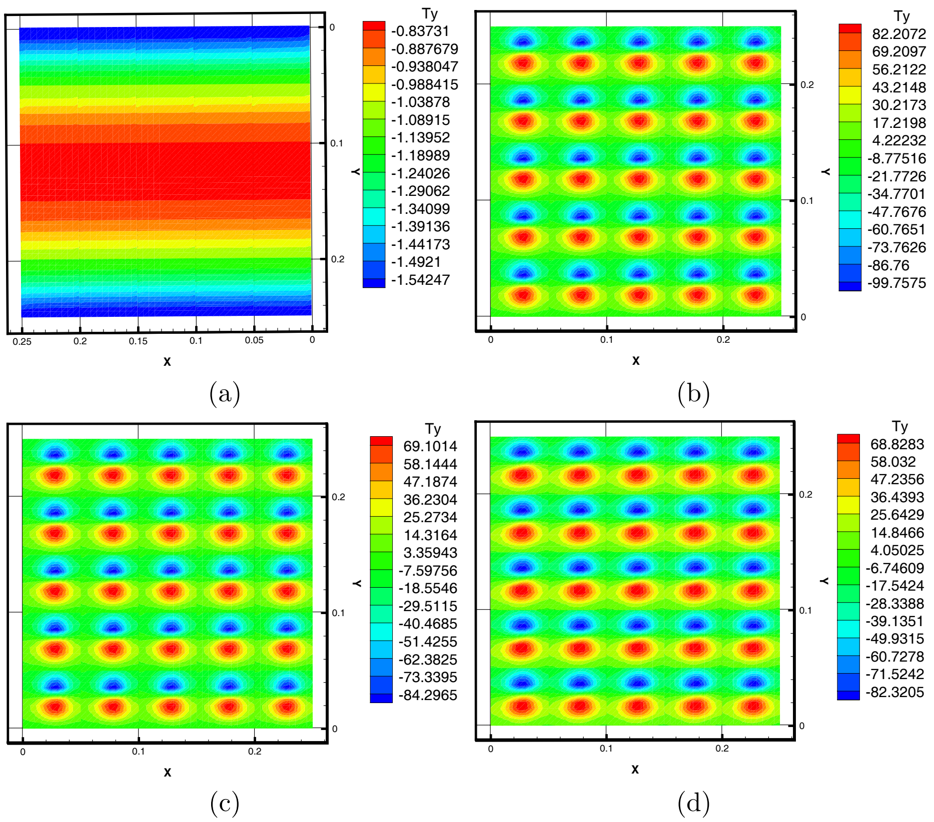


Figure 4: (a) Case 4, $\frac{\partial T_0(x)}{\partial y}$; (b) Case 4, $\frac{\partial \hat{T}_1^{\varepsilon}(x)}{\partial y}$; (c) Case 4, $\frac{\partial \hat{T}_2^{\varepsilon}(x)}{\partial y}$; (d) Case 4, $\frac{\partial Te}{\partial y}$

Eventually, to check the convergence of the homogenized process as the small parameters ε, we display a example for different ε. The periodic structure is the same with that the above problem and we choose $e = 1$. The internal heat source $f(x)$ is set to zero.

The following case is investigated:

Case : k_{ij}=$100\delta_{ij}$, T_1 =100, $T_2 = 1000$

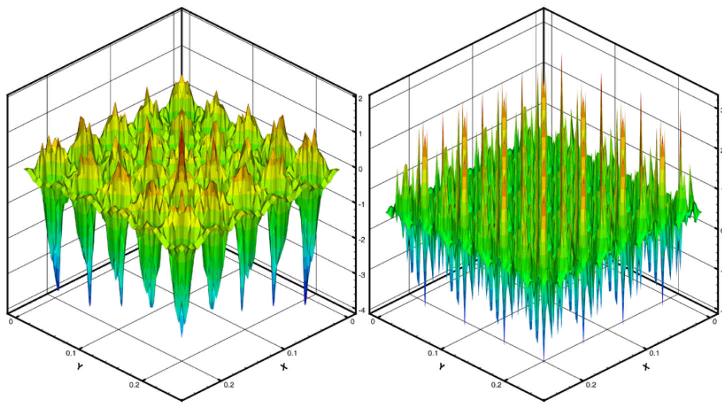
The relative numerical errors of the different methods in L^2 and H^1 for examples are listed in Table 3.

Figures 5 illustrate the numerical results for $\hat{T}_2^{\varepsilon}(x)$ -*Te* at the intersection *z*=0.109375.

As a result of our numerical analysis, we claim that, though the SOTS method does not improve the approximate estimation, for a fixed value of ε it improves the qualitative behavior of the reconstructed solution. The relative errors between different

Table 3: Comparison with computing results of norm L^2 and semi-norm H^1.

	$\|error_0\|_{L^2} / \|Te\|_{L^2}$	$\|error_2\|_{L^2} / \|Te\|_{L^2}$	$\lvert error_2\rvert_{H^1} / \lvert Te\rvert_{H^1}$
ε =1/5	0.00637718	0.00099632	0.05547819
ε =1/8	0.00418188	0.00071414	0.05179645



(a) (b)

Figure 5: (a) ε =1/5, $\hat{T}_2^{\varepsilon}(x)$ -Te; (b) ε =1/8, $\hat{T}_2^{\varepsilon}(x)$ -Te

approximate solutions and FE solutions in a refined mesh are shown in Tables 2 and 3. All the results show that the SOTS method is effective in approximating the heat transfer problem with interior surface radiation.

From Tables 1, it can be seen that the mesh partition numbers of second-order two-scale approximate solution are much less than that of the refined FE solution, especially for small ε. It means that the SOTS method can greatly save computer memory and CPU time.

5 Conclusions

In this paper, a new second-order two-scale method is developed for predicting the heat transfer performances of periodic porous materials with interior surface radiation. The second-order two-scale formulation for the heat transfer problem is given, and the error estimation with $\varepsilon^{1/2}$ order is derived under the regularity assumption of the homogenized solution.

Finally, some numerical results are reported, which support the theoretical results of this paper and show that the SOTS method is effective.

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