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Haar Wavelet Operational Matrix Method for Solving Fractional Partial Differential Equations

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Abstract: In this paper, Haar wavelet operational matrix method is proposed to solve a class of fractional partial differential equations. We derive the Haar wavelet operational matrix of fractional order integration. Meanwhile, the Haar wavelet operational matrix of fractional order differentiation is obtained. The operational matrix of fractional order differentiation is utilized to reduce the initial equation to a Sylvester equation. Some numerical examples are included to demonstrate the validity and applicability of the approach.

Keywords: Haar wavelet, operational matrix, fractional partial differential equation, Sylvester equation, numerical solution.

1 Introduction

Wavelet analysis is a relatively new area in different fields of science and engineering. It is a developing of Fourier analysis. Wavelet analysis has been applied widely in time-frequency analysis, signal analysis and numerical analysis. It permits the accurate representation of a variety of functions and operators, and establishes a connection with fast numerical algorithms [Beylkin, Coifman, and Rokhlin (1991)]. Functions are decomposed into summation of "basic functions", and every "basic function" is achieved by compression and translation of a mother wavelet function with good properties of smoothness and locality, which makes people analyse the properties of locality and integer in the process of expressing functions [Li and Luo (2005); Ge and Sha (2007)]. Consequently, wavelet analysis can describe the properties of functions more accurate than Fourier analysis.

Fractional differential equations are generalized from classical integer order ones, which are obtained by replacing integer order derivatives by fractional ones. Fractional calculus is an old mathematical topic with history as long as that of integer order calculus. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing nu-

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merical schemes for their solution. Fractional calculus and many fractional differential equations have been found applications in several different disciplines, both physicists and mathematicians have also engaged in studying the numerical methods for solving fractional differential equations in recent years. These methods include variational iteration method (VIM) [Odibat (2010)], Adomian decomposition method (ADM) [EI-Sayed (1998); EI-Kalla (2011)], generalized differential transform method (GDTM) [Odibat and Momani (2008); Momani and Odibat (2007)], generalized block pulse operational matrix method [Li and Sun (2011)] and wavelet method [Chen and Wu et al. (2010)]. The operational matrix of fractional order integration for the Legendre wavelet and the Chebyshev wavelet [Jafari and Yousefi (2011); Wang and Fan (2012)] have been derived to the fractional differential equations. In [Saeedi and Moghadam et al. (2011); Saeedi and Moghadam (2011)], a CAS wavelet operational matrix of fractional order integration has been used to solve integro- differential equations of fractional order.

In this paper, our study focuses on a class of fractional partial differential equations:

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}=f(x,t) \tag{1}$$

subject to the initial conditions

$$u(0,t)=u(x,0)=0 \tag{2}$$

where $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ and $\frac{\partial^\beta u(x,t)}{\partial t^\beta}$ are fractional derivative of Caputo sense, $f(x,t)$ is the known continuous function, $u(x,t)$ is the unknown function, $0<\alpha,\beta\leq 1$.

There have been several methods for solving the fractional partial differential equations. Podlubny [Podlubny (1999)] used the Laplace Transform method to solve the fractional partial differential equations with constant coefficients. Zaid Odibat and Shaher Momani [Odibat and Momani (2008)] applied generalized differential transform method to solve the numerical solution of linear partial differential equations of fractional order.

Our purpose is to proposed Haar wavelet operational matrix method to solve a class of fractional partial differential equations. We introduce Haar wavelet operational matrix of fractional order integration without using the block pulse functions. Here, we adopt the orthogonal Haar wavelet matrix which is different from the Haar wavelet matrix in the Ref. [Ray (2012)]. We need not calculate the inverse of Haar wavelet matrix in this way.

2 Definitions of fractional derivatives and integrals

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this article [Podlubny (1999) and Odibat (2006)].

Definition 1. The Riemann-Liouville fractional integral operator $J^{\alpha}$of order α is given by

$$J^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1}u(T)dT, \quad \alpha > 0 \tag{3}$$

$$J^0u(t) = u(t) \tag{4}$$

Its properties as following:
(i) $J^{\alpha}J^{\beta}u(t) = J^{\alpha+\beta}u(t)$, (ii) $J^{\alpha}J^{\beta}u(t) = J^{\beta}J^{\alpha}u(t)$, (iii) $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$

Definition 2. The Caputo fractional differential operator D_*^{α} is given by

$$D_*^{\alpha}u(t) = \begin{cases} \frac{d^r u(t)}{dt^r}, & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)}\int_0^t \frac{u^{(r)}(T)}{(t-T)^{\alpha-r+1}}dT, & 0 \le r-1 < \alpha < r. \end{cases} \tag{5}$$

The Caputo fractional derivatives of order α is also defined as $D_*^{\alpha}u(t) = J^{r-\alpha}D^r u(t)$, where D^r is the usual integer differential operator of order r. The relation between the Riemann- Liouville operator and Caputo operator is given by the following expressions:

$$D_*^{\alpha}J^{\alpha}u(t) = u(t) \tag{6}$$

$$J^{\alpha}D_*^{\alpha}u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0 \tag{7}$$

3 Haar wavelet and function approximation

For $t \in [0,1]$, Haar wavelet functions are defined as follows [Chen and Wu et al. (2010)]:

$$h_0(t) = \frac{1}{\sqrt{m}}$$

$$h_i(t) = \frac{1}{\sqrt{m}}\begin{cases} 2^{j/2}, & \frac{k-1}{2^j} \le t < \frac{k-1/2}{2^j} \\ -2^{j/2}, & \frac{k-1/2}{2^j} \le t < \frac{k}{2^j} \\ 0, & otherwise \end{cases}$$

where $i = 0,1,2,\ldots,m-1$, $m = 2^M$ and M is a positive integer. j and $k$represent integer decomposition of the index i, i.e. $i = 2^j + k - 1$.

For arbitrary function $u(x,t) \in L^2([0,1)\times[0,1))$, it can be expanded into Haar series by

$$u(x,t) \cong \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{i,j}h_i(x)h_j(t) \tag{8}$$

where $c_{i,j} = \int_0^1 u(x,t)h_i(x)dx \cdot \int_0^1 u(x,t)h_j(t)dt$ are wavelet coefficients, m is a power of 2.

Let $H_m(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T$, $H_m(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T$, then Eq.(8) will be written as $u(x,t) \cong H_m^T(x)\cdot C\cdot H_m(t)$.

In this paper, we use wavelet collocation method to determine the coefficients $c_{i,j}$. These collocation points are shown in the following:

$$x_l = t_l = (l-1/2)/m, \quad l = 1,2,\ldots,m. \tag{9}$$

Discreting Eq.(8) by the step (9), we can obtain the matrix form of Eq.(8)

$$U = H^T \cdot C \cdot H \tag{10}$$

where $C = [c_{i,j}]_{m\times m}$ and $U = [u(x_i,t_j)]_{m\times m}$. H is called Haar wavelet matrix of order m, i.e.

$$H = \begin{bmatrix} h_0(t_0) & h_0(t_1) & \cdots & h_0(t_{m-1}) \\ h_1(t_0) & h_1(t_1) & \cdots & h_1(t_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_0) & h_{m-1}(t_1) & \cdots & h_{m-1}(t_{m-1}) \end{bmatrix}.$$

From the definition of Haar wavelet functions, we may know easily that H is a orthogonal matrix, then we have

$$C = H \cdot U \cdot H^T \tag{11}$$

4 Haar wavelet operational matrix of fractional order integration and differentiation

The integration of the $H_m(t)$ can be approximated by Chen and Hsiao [Chen and Hsiao]:

$$\int_0^t H_m(s)ds \cong PH_m(t) \tag{12}$$

where P is called the Haar wavelet operational matrix of integration.

Now, we are able to derive the Haar wavelet operational matrix of fractional order integration. For this purpose, we may make full use of the definition of Riemann-Liouville fractional integral operator J^α which is given by **Definition 1.**

Haar wavelet operational matrix of fractional order integration P^α will be deduced by

$$\begin{aligned}
P^\alpha H_m(t) =& J^\alpha H_m(t)\\
=& [J^\alpha h_0(t), J^\alpha h_1(t), \ldots, J^\alpha h_{m-1}(t)]^T\\
=& \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1} h_0(T)dT, \frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1} h_1(T)dT, \ldots,\right.\\
& \left.\frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1} h_{m-1}(T)dT\right]^T\\
=& [Ph_0(t), Ph_1(t), \ldots, Ph_{m-1}(t)]^T
\end{aligned}$$

where

$$Ph_0(t) = \frac{1}{\sqrt{m}} \frac{t^\alpha}{\Gamma(\alpha+1)} \quad t \in [0,1) \tag{13}$$

$$Ph_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 0, & 0 \le t < \frac{k-1}{2^j} \\ 2^{j/2}\lambda_1(t), & \frac{k-1}{2^j} \le t < \frac{k-1/2}{2^j} \\ 2^{j/2}\lambda_2(t), & \frac{k-1/2}{2^j} \le t < \frac{k}{2^j} \\ 2^{j/2}\lambda_3(t), & \frac{k}{2^j} \le t < 1 \end{cases} \tag{14}$$

where

$$\lambda_1(t) = \frac{1}{\Gamma(\alpha+1)}\left(t - \frac{k-1}{2^j}\right)^\alpha;$$

$$\lambda_2(t) = \frac{1}{\Gamma(\alpha+1)}\left(t - \frac{k-1}{2^j}\right)^\alpha - \frac{2}{\Gamma(\alpha+1)}\left(t - \frac{k-1/2}{2^j}\right)^\alpha;$$

$$\lambda_3(t) = \frac{1}{\Gamma(\alpha+1)}\left(t - \frac{k-1}{2^j}\right)^\alpha - \frac{2}{\Gamma(\alpha+1)}\left(t - \frac{k-1/2}{2^j}\right)^\alpha + \frac{1}{\Gamma(\alpha+1)}\left(t - \frac{k}{2^j}\right)^\alpha.$$

The derived Haar wavelet operational matrix of fractional integration is $P^\alpha = (P^\alpha H) \cdot H^T$. Let D^α is the Haar wavelet operational matrix of fractional differentiation. According to the property of fractional calculus $D^\alpha P^\alpha = I$, we can obtain the matrix D^α by inverting the matrix P^α. For instance, if $\alpha = 0.5, m = 8$, we have

$P^{1/2} =$

$$\begin{bmatrix} 0.7549 & -0.2180 & -0.1072 & -0.0579 & -0.0516 & -0.0289 & -0.0223 & -0.0189 \\ 0.2180 & 0.3190 & -0.1072 & 0.1565 & -0.0516 & -0.0289 & 0.0809 & 0.0389 \\ 0.0579 & 0.1565 & 0.2337 & -0.0312 & -0.0730 & 0.1052 & -0.0229 & -0.0044 \\ 0.1072 & -0.1072 & 0 & 0.2337 & 0 & 0 & -0.0730 & 0.1052 \\ 0.0189 & 0.0389 & 0.1052 & -0.0044 & 0.1788 & -0.0189 & -0.0025 & -0.0009 \\ 0.0223 & 0.0809 & -0.0730 & -0.0229 & 0 & 0.1788 & -0.0189 & -0.0025 \\ 0.0289 & -0.0289 & 0 & 0.1052 & 0 & 0 & 0.1788 & -0.0189 \\ 0.0516 & -0.0516 & 0 & -0.0730 & 0 & 0 & 0 & 0.1788 \end{bmatrix},$$

$D^{1/2} =$

$$\begin{bmatrix} 1.1229 & 0.4694 & 0.4589 & 0.0396 & 0.6488 & 0.0568 & 0.0185 & 0.0108 \\ -0.4694 & 2.0678 & 0.4589 & -0.8783 & 0.6488 & 0.0568 & -1.2790 & -0.1028 \\ -0.0396 & -0.8783 & 2.8964 & 0.4711 & 0.9175 & -1.7547 & 0.7831 & 0.0432 \\ -0.4589 & 0.4589 & 0 & 2.8964 & 0 & 0 & 0.9175 & -1.7547 \\ -0.0108 & -0.1028 & -1.7547 & 0.0432 & 4.8424 & 1.5241 & 0.0671 & 0.0051 \\ -0.0185 & -1.2790 & 0.9175 & 0.7831 & 0 & 4.8424 & 1.5241 & 0.0671 \\ -0.0568 & 0.0568 & 0 & -1.7547 & 0 & 0 & 4.8424 & 1.5241 \\ -0.6488 & 0.6488 & 0 & 0.9175 & 0 & 0 & -0 & 4.8424 \end{bmatrix}.$$

The fractional order integration and differentiation of the function t was selected to verify the correctness of matrix P^{α} and D^{α}. The fractional order integration and differentiation of the function $u(t) = t$ is obtained in the following:

$$J^{\alpha}u(t) = \frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}$$

and

$$D_*^{\alpha}u(t) = \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}.$$

When $\alpha = 0.5, m = 32$, the comparison results for the fractional integration and differentiation are shown in **Fig. 1** and **Fig. 2**, respectively.

5 Numerical solution of the fractional partial differential equations

Consider the fractional partial differential equation Eq.(1). If we approximate function $u(x,t)$ by using Haar wavelet, we have

$$u(x,t) \cong H_m^T(x) \cdot C \cdot H_m(t) \tag{15}$$

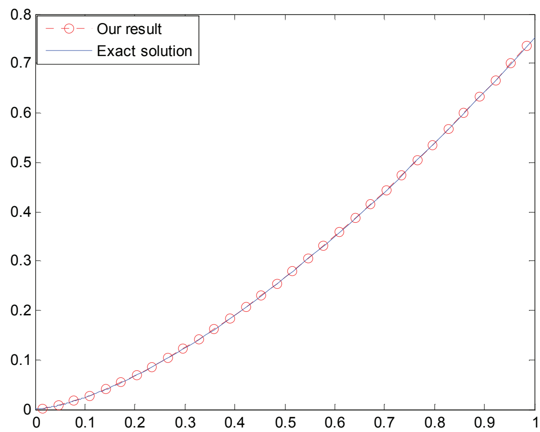

Figure 1: 0.5-order integration of the function $u(t) = t$.

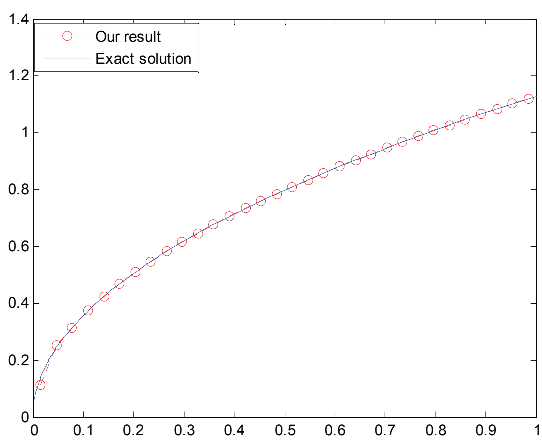

Figure 2: 0.5-order differentiation of the function $u(t) = t$.

Then we can get

$$\begin{aligned}\frac{\partial^\alpha u}{\partial x^\alpha} \cong \frac{\partial^\alpha (H_m^T(x)CH_m(t))}{\partial x^\alpha} &= \left[\frac{\partial^\alpha H_m(x)}{\partial x^\alpha}\right]^T CH_m(t) \\ &= [D^\alpha H_m(x)]^T CH_m(t) \\ &= H_m^T(x)[D^\alpha]^T CH_m(t)\end{aligned} \tag{16}$$

$$\frac{\partial^\beta u}{\partial t^\beta} \cong \frac{\partial^\beta (H_m^T(x)CH_m(t))}{\partial t^\beta} = H_m^T(x)C\frac{\partial^\beta (H_m(t))}{\partial t^\beta} = H_m^T(x)CD^\beta H_m(t) \tag{17}$$

The function $f(x,t)$ of Eq.(1) can be also expressed as

$$f(x,t) \cong H_m^T(x)\cdot F \cdot H_m(t) \tag{18}$$

where $F = [f_{i,j}]_{m\times m}$.

Substituting Eq.(16), Eq.(17) and Eq.(18) into Eq.(1), we have

$$H_m^T(x)[D^\alpha]^T CH_m(t) + H_m^T(x)CD^\beta H_m(t) = H_m^T(x)FH_m(t) \tag{19}$$

Dispersing Eq.(19) by the points (x_i,t_j), $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,m$, we can obtain

$$[D^\alpha]^T C + CD^\beta = F \tag{20}$$

Eq.(19) is a Sylvester equation. The Sylvester equation can be solved easily by using Matlab software.

6 Numerical examples

To demonstrate the efficiency and the practicability of the proposed method based on Haar wavelet operational matrix method, we consider some examples.

Example 1: Consider the following nonhomogeneous partial differential equation

$$\frac{\partial^{1/4} u}{\partial x^{1/4}} + \frac{\partial^{1/4} u}{\partial t^{1/4}} = f(x,t), x,t \geq 0,$$

such that $u(0,t) = u(x,0) = 0$ and $f(x,t) = \frac{4(x^{3/4}t + xt^{3/4})}{3\Gamma(3/4)}$. The numerical results for $m = 8$, $m = 16$, $m = 32$ are shown in **Fig. 3**, **Fig. 4**, **Fig. 5**. The exact solution of the partial differential equation is given by xt which is shown in **Fig. 6**. From the **Fig. 3-6**, we can see clearly that the numerical solutions are in very good agreement with the exact solution.

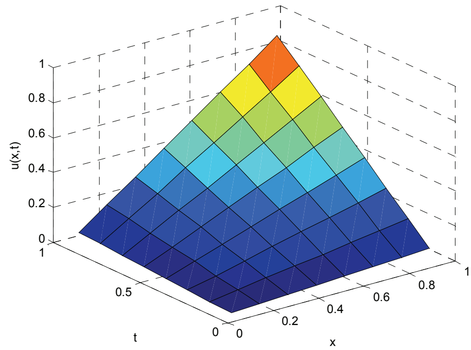

Figure 3: Numerical solution of $m = 8$

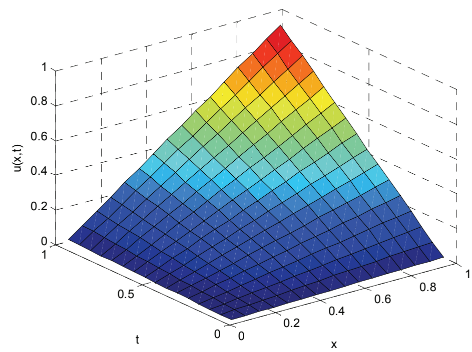

Figure 4: Numerical solution of $m = 16$

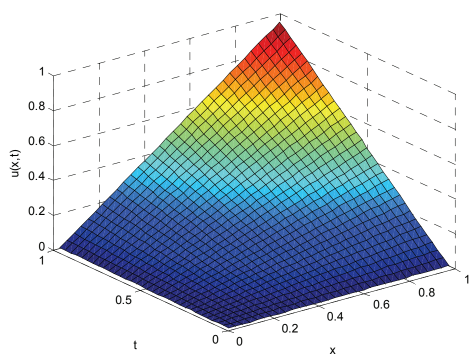

Figure 5: Numerical solution of $m = 32$

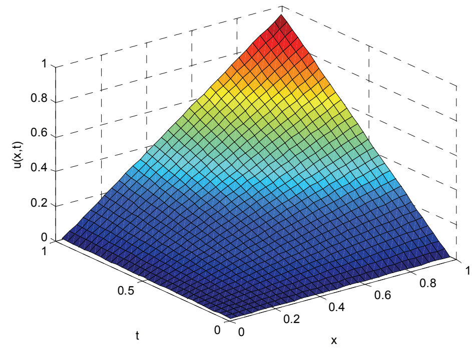

Figure 6: Exact solution for **Example 1**

Example 2: Consider the following fractional partial differential equation

$$\frac{\partial^{1/3}u}{\partial x^{1/3}}+\frac{\partial^{1/2}u}{\partial t^{1/2}}=f(x,t), \quad x,t\geq 0,$$

subject to the initial conditions $u(0,t)=u(x,0)=0$, $f(x,t)=\frac{9x^2t^{5/3}}{5\Gamma(2/3)}+\frac{8x^{3/2}t^2}{3\Gamma(1/2)}$. **Fig. 7- 10** show the numerical solutions for various m and the exact solution x^2t^2. The absolute error for different m is shown in **Table 1**. From the **Fig. 7-10** and **Table 1**, we can conclude that the numerical solutions are more and more close to the exact solution when m increases.

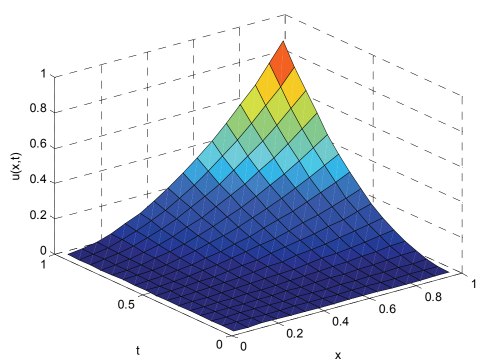

Figure 7: Numerical solution of $m=16$

Example 3: Consider the below fractional partial differential equation

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}}+\frac{\partial^{\beta}u}{\partial t^{\beta}}=\sin(x+t), \quad x,t\geq 0$$

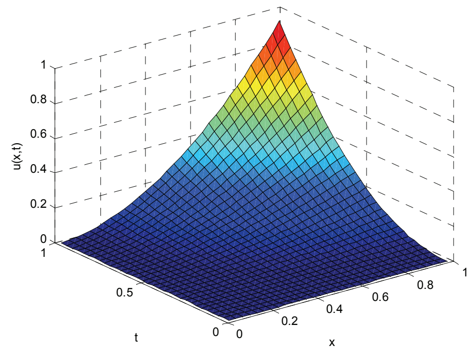

Figure 8: Numerical solution of $m = 32$

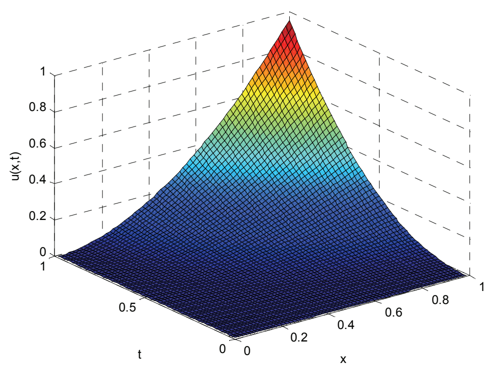

Figure 9: Numerical solution of $m = 64$

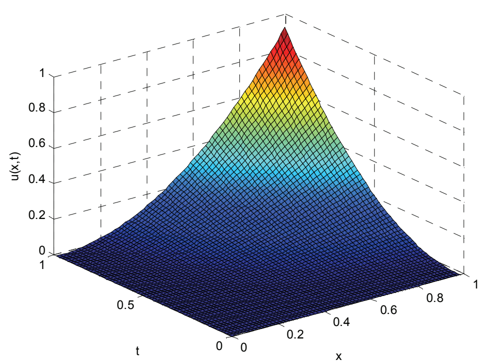

Figure 10: Exact solution for **Example 2**

Table 1: The absolute error of different m for **Example 2**

(x,t)	Exact solution	$m=8$	$m=16$	$m=32$	$m=64$
(0,0)	0	2.4165e-006	5.9331e-007	3.7408e-008	4.3981e-009
(1/8,1/8)	0.0002	8.3295e-005	1.8652e-005	1.4567e-006	2.3435e-007
(2/8,2/8)	0.0039	4.4467e-004	4.1953e-005	2.2503e-005	1.0247e-005
(3/8,3/8)	0.0198	1.3310e-003	1.7432e-005	9.6749e-006	6.5843e-006
(4/8,4/8)	0.0625	2.9517e-003	1.2828e-004	8.7434e-005	7.0127e-005
(5/8,5/8)	0.1526	5.4704e-003	4.8133e-004	3.1627e-004	1.3465e-004
(6/8,6/8)	0.3164	8.9731e-003	1.1364e-003	8.9431e-004	6.6143e-004
(7/8,7/8)	0.5862	1.3454e-002	2.1938e-003	1.0468e-003	8.3421e-004

such that $u(0,t) = u(x,0) = 0$. When $\alpha = \beta = 1$, the exact solution of this partial differential equation is $\sin x \sin t$. We can achieve its numerical solution which is shown in **Fig. 11**, and the exact solution is shown in **Fig. 12**. **Fig.13** and **Fig.14** show the numerical solutions for different values of α, β. Here, we may take $m = 32$.

They demonstrate the simplicity, and powerfulness of the proposed method. Compared with the generalized differential transform method in the Ref. [16], taking advantage of above method can greatly reduce the computation. Moreover, the method in this paper is easy implementation.

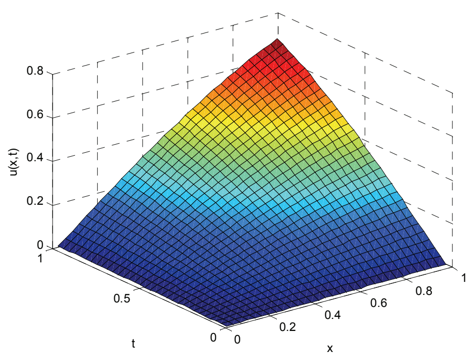

Figure 11: Numerical solution of $\alpha = \beta = 1$

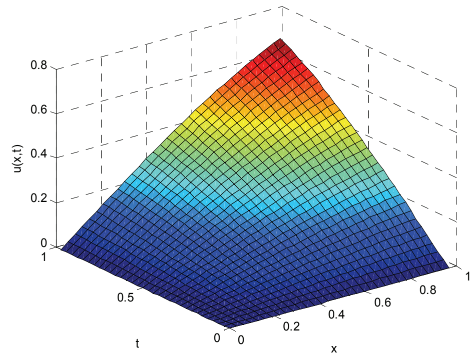

Figure 12: Exact solution of $\alpha = \beta = 1$

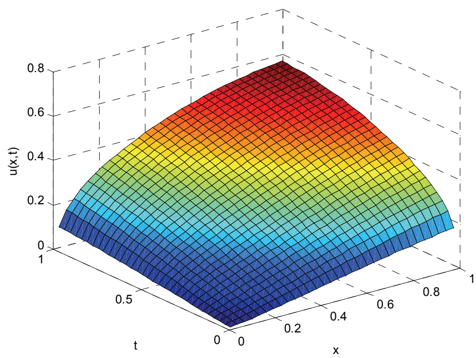

Figure 13: Numerical solution of $\alpha = 1/2, \beta = 1/3$

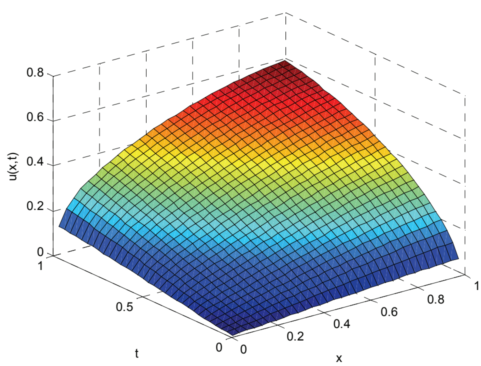

Figure 14: Numerical solution of $\alpha = 3/7, \beta = 3/5$

7 Conclusion

Another operational matrix for the Haar wavelet operational matrix of fractional differentiation has been derived. The fractional derivatives are described in the Caputo sense. This matrix is used to solve the numerical solutions of a class of fractional partial differential equations effectively. We translate the fractional partial differential equation into a Sylvester equation which is easily to solve. Numerical examples illustrate the powerful of the proposed method. The solutions obtained using the suggested method show that numerical solutions are in very good coincidence with the exact solution. The method can be applied by developing for the other fractional problem.

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