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Boundary Knot Method: An Overview and Some Novel Approaches

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Abstract: The boundary knot method (BKM) is a kind of boundary-type meshless method, only boundary points are needed in the solution process. Since the BKM is mathematically simple and easy to implement, it is superior in dealing with Helmholtz problems with high wavenumbers and high dimensional problems. In this paper, we give an overview of the traditional BKM with collocation approach and provide three novel approaches for the BKM, as far as they are relevant for the other boundary-type techniques. The promising research directions are expected from an improved BKM aspect.

Keywords: Collocation; least-square; Galerkin; variational; Helmholtz equation; non-singular general solution.

1 Introduction

The boundary knot method (BKM), named by Chen and Tanaka [Chen and Tanaka (2000a);Chen and Tanaka (2000b)], was pioneered by Kang et al. [Kang, Lee, and Kang (1996)] for the vibration analysis of arbitrarily shaped membranes. Since only boundary points are needed in the solution procedure, the BKM belongs to the category of boundary-type meshless methods which also include the method of fundamental solutions [Fairweather and Karageorghis (1998);Chen, Karageorghis, and Smyrlis (2008);Liu (2008);Lin, Chen, and Wang (2011);Wang, Chen, and Ling (2012)] and regularized meshless method [Young, Chen, and Lee (2005)] as typical examples.

Compared with the method of fundamental solutions, the BKM do not need the fictitious boundary since the basis functions used in the BKM has no singularity at

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the origin. In contrast with the regularized meshless method, the BKM has higher accurate solutions. Meanwhile, the BKM is superior in dealing with Helmholtz problems with high wavenumbers and high dimensional problems in terms of stability and solution accuracy [Chen and Hon (2003)].

Although the BKM has the above-mentioned various advantages, it is less popular than traditional numerical methods such as the finite difference, finite element and boundary element methods. The aim of this paper is to overview the traditional BKM approach and provide three other new approaches for the BKM which is shown in Section 2. Followed by Section 3, we review the development of the traditional BKM with collocation approach. In Section 4, we make some concluding remarks and expect some research directions.

2 Boundary knot method

For brevity, we consider the Helmhotz equation

$$\nabla^2 u(\mathbf{x}) + \lambda^2 u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{1}$$

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_D \tag{2}$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = q(\mathbf{x}) = \bar{q}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_N \tag{3}$$

where $\nabla^2 = \Delta$ is Laplace operator, $\Omega \in \mathbb{R}^d$, d is the dimensionality, $\bar{u}(\mathbf{x})$ and $\bar{q}(\mathbf{x})$ are known boundary conditions on the Dirichlet boundary Γ_D and Neumann boundary Γ_N ($\partial\Omega = \Gamma_D \cup \Gamma_N$), respectively. $\lambda = \omega/c$ represents the wavenumber, where ω, c and n are sound speed, frequency and normal dirivative, respectively.

The fundamental theory of the BKM is similar with the method of fundamental solutions. However, the superiority of the BKM lies in that all collocation points and source points can be located on the physical boundary simultaneously. This advantage attributes to the use of non-singular general solutions which can circumvent the singularity generated by the superposition of the source and collocation points.

By using the non-singular general solutions, we have the approximate solution $u_N(\mathbf{x})$ in the form of [Chen and Tanaka (2000a)]

$$u_N(\mathbf{x}) = \sum_{j=1}^{N} c_j Q_H(\mathbf{x}, \mathbf{y}_j), \qquad \mathbf{y}_j \in \partial\Omega \tag{4}$$

where $\mathbf{y}_j$ are source points on the physical boundary, c_j the unknown coefficients, N the total number of source points,

$$Q_H(\mathbf{x}, \mathbf{y}) = (\frac{\lambda}{2\pi r})^{(d/2)-1} J_{(d/2)-1}(\lambda r), \;\; d \geq 2 \tag{5}$$

the non-singular general solution for the Helmholtz equation, where J is the Bessel function of the first kind, d the dimensionality, $r = \| \mathbf{x} - \mathbf{y} \|$ the distance between points $\mathbf{x}$ and $\mathbf{y}$ with $\| \cdot \|$ denoting the Euclidean norm.
Especially, we have the following non-singular general solution

$$Q_H(\mathbf{x},\mathbf{y}) = J_0(\lambda r), \ \ \mathbf{x} \in \mathbb{R}^2 \tag{6}$$

for 2D Helmholtz equation, and

$$Q_H(\mathbf{x},\mathbf{y}) = \frac{\sin(\lambda r)}{r}, \ \ \mathbf{x} \in \mathbb{R}^3, \tag{7}$$

for 3D Helmholtz equation. Table 1 shows non-singular general solutions for the other two types of commonly-used operators.

Table 1: Non-singular general solutions for general operators

Operator	2D	3D
$\Delta - \lambda^2$	$\frac{1}{2\pi} I_0(\lambda r)$	$\frac{\sinh(\lambda r)}{r}$
$\mathbb{D}\Delta + \nu\nabla - k$	$\frac{1}{2\pi} e^{(-\nu r)/(2\mathbb{D})} I_0(\lambda r)$	$e^{(-\nu r)/(2\mathbb{D})} \frac{\sinh(\lambda r)}{r}$

where $\mathbb{D}$ represents the diffusivity, k means the reaction coefficient, and ν stands for the velocity vector.

The normal derivative of Eq. (4) leads to

$$q_N(\mathbf{x}) = \frac{\partial u_N(\mathbf{x})}{\partial n} = \sum_{j=1}^{N} c_j \frac{\partial Q_H(\mathbf{x},\mathbf{y}_j)}{\partial n}. \qquad \mathbf{x} \in \partial\Omega \tag{8}$$

Generally speaking, the approximate solution $u_N(\mathbf{x})$ can not match the boundary conditions Eqs. (2)-(3) exactly. Therefore, some residuals will appear

$$\mathbf{x}_i \in \Gamma_1: \ R_1(\mathbf{x}) = \sum_{j=1}^{N} c_j Q_H(\mathbf{x},\mathbf{y}_j) - \bar{u}(\mathbf{x}) \ = \alpha^T \tilde{u}(\mathbf{x}) - \bar{u}(\mathbf{x}), \tag{9}$$

$$\mathbf{x} \in \Gamma_2: \ R_2(\mathbf{x}) = \sum_{j=1}^{N} c_j \frac{\partial Q_H(\mathbf{x},\mathbf{y}_j)}{\partial n} - \bar{q}(\mathbf{x}) = \alpha^T \tilde{q}(\mathbf{x}) - \bar{q}(\mathbf{x}), \tag{10}$$

where $\alpha = (c_1, c_2, \ldots, c_N)^T$, $\tilde{u}(\mathbf{x}) = (Q_H(\mathbf{x},\mathbf{y}_1), Q_H(\mathbf{x},\mathbf{y}_2), \ldots, Q_H(\mathbf{x},\mathbf{y}_N))^T$ with ^{T} denoting the transpose of a matrix or vector, $\tilde{q}(\mathbf{x}) = \frac{\partial \tilde{u}(\mathbf{x})}{\partial n}$, R_1 and R_2 represent residuals on boundary Γ_1 and Γ_2, respectively.

2.1 Traditional approach: collocation

So far, all researches related to the BKM is based on the collocation approach, that is, the residuals on points $\{\mathbf{x}_i\}_{i=1}^{N}$ are forced to vanish. More specifically, Eqs. (9)-(10) leads to

$$\mathbf{x}_i \in \Gamma_1 : R_1(\mathbf{x}_i) = \sum_{j=1}^{N} c_j Q_H(\mathbf{x}_i, \mathbf{y}_j) - \bar{u}(\mathbf{x}_i) = 0,\ i = 1, \ldots, N_1 \tag{11}$$

$$\mathbf{x}_k \in \Gamma_2 : R_2(\mathbf{x}_k) = \sum_{j=1}^{N} c_j \frac{\partial Q_H(\mathbf{x}_k, \mathbf{y}_j)}{\partial n} - \bar{q}(\mathbf{x}_k) = 0,\ k = 1, \ldots, N_2 \tag{12}$$

where N_1 and N_2 are numbers of collocation points on boundary Γ_D and Γ_N ($N_1 + N_2 = N$), respectively.

Eqs. (11)-(12) can be rewritten in the following matrix system

$$Q\alpha = b, \tag{13}$$

where

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,N} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{N_1,1} & Q_{N_1,2} & \cdots & Q_{N_1,N} \\ \frac{\partial Q_{1,1}}{\partial n} & \frac{\partial Q_{1,2}}{\partial n} & \cdots & \frac{\partial Q_{1,N}}{\partial n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial Q_{N_2,1}}{\partial n} & \frac{\partial Q_{N_2,2}}{\partial n} & \cdots & \frac{\partial Q_{N_2,N}}{\partial n} \end{bmatrix} \tag{14}$$

is a $(N \times N)$ coefficient matrix, $\alpha = (c_1, c_2, \ldots, c_N)^T$ is a $(N \times 1)$ coefficient vector to be determined, $b = (\bar{u}_1, \ldots, \bar{u}_{N_1}, \bar{q}_1, \ldots, \bar{q}_{N_2})^T$ is a $(N \times 1)$ vector composed by boundary conditions.

The collocation approach has the following characters. For each point, only shape functions and their derivations are needed, while integration is excluded during the whole solution procedure. This can be easily implemented and has high computation efficiency. No need to worry about the boundary conditions that can be implemented freely. However, the coefficient matrix is non-symmetric, and sometimes is ill-conditioned which will lead to the instability of the numerical solution [Wang, Chen, and Jiang (2010)].

As is known to all, there are several approaches for numerical methods [Kita and Kamiya (1995)]. In addition to the collocation approach, we provide three other approaches for the BKM, i.e., the least-square approach, Galerkin approach and variational approach. Detailed expressions are given in the remaining part.

2.2 Least-square approach

Here, we define a function

$$
\begin{aligned}
F(\alpha) &= \int_{\Gamma_1} R_1^2 d\Gamma + \tau \int_{\Gamma_2} R_2^2 d\Gamma \\
&= \int_{\Gamma_1} (\tilde{u}^T \alpha - \bar{u})^2 d\Gamma + \tau \int_{\Gamma_2} (\tilde{q}^T \alpha - \bar{q})^2 d\Gamma,
\end{aligned}
$$

where τ is an weight parameter which is used to balance the numerical value between the first and second terms in right-hand of the above equation. In the least-square approach, we impose the derivative related to α to 0.

$$
\begin{aligned}
\frac{\partial F}{\partial \alpha} &= \frac{\partial}{\partial \alpha}[\int_{\Gamma_1} (\tilde{u}^T \alpha - \bar{u})^2 d\Gamma + \tau \int_{\Gamma_2} (\tilde{q}^T \alpha - \bar{q})^2 d\Gamma] \\
&= 2\int_{\Gamma_1} \tilde{u}(\tilde{u}^T \alpha - \bar{u}) d\Gamma + 2\tau \int_{\Gamma_2} \tilde{q}^T (\tilde{q}^T \alpha - \bar{q}) d\Gamma \\
&= 0.
\end{aligned}
$$

Rearrange the above equation, we have

$$
[\int_{\Gamma_1} \tilde{u}\tilde{u}^T d\Gamma + \tau \int_{\Gamma_2} \tilde{q}\tilde{q}^T d\Gamma]\alpha = \int_{\Gamma_1} \tilde{u}\bar{u} d\Gamma + \tau \int_{\Gamma_2} \tilde{q}\bar{q} d\Gamma, \tag{15}
$$

or

$$
Q\alpha = b, \tag{16}
$$

where

$$
Q_{ij} = \int_{\Gamma_1} \tilde{u}_i \tilde{u}_j d\Gamma + \tau \int_{\Gamma_2} \tilde{q}_i \tilde{q}_j d\Gamma, \tag{17}
$$

$$
b_i = \int_{\Gamma_1} \tilde{u}_i \bar{u} d\Gamma + \tau \int_{\Gamma_2} \tilde{q}_i \bar{q} d\Gamma. \tag{18}
$$

It should be noted that it is difficult to pre-define a proper weight parameter τ in the least-square approach [Kita and Kamiya (1995)].

2.3 Galerkin approach

In this approach, q_N and $-u_N$ are chosen as the weight function for residuals R_1 and R_2, respectively [Nguyen, Rabczuk, Bordas, and Duflot (2008)]. Under such condition, we can get the following weight residual equation

$$
\int_{\Gamma_1} q_N R_1 d\Gamma - \int_{\Gamma_2} u_N R_2 d\Gamma = 0. \tag{19}
$$

Substitute Eqs. (4) and (8) into the above equation, we have

$$\int_{\Gamma_1} \tilde{q}^T \alpha(\tilde{u}^T \alpha - \bar{u}) d\Gamma - \int_{\Gamma_2} \tilde{u}^T \alpha(\tilde{q}^T \alpha - \bar{q}) d\Gamma = 0, \tag{20}$$

$$\alpha^T [\int_{\Gamma_1} \tilde{q}^T (\tilde{u}^T \alpha - \bar{u}) d\Gamma - \int_{\Gamma_2} \tilde{u}^T (\tilde{q}^T \alpha - \bar{q}) d\Gamma] = 0. \tag{21}$$

Therefore,

$$\int_{\Gamma_1} \tilde{q}(\tilde{u}^T \alpha - \bar{u}) d\Gamma - \int_{\Gamma_2} \tilde{u}(\tilde{q}^T \alpha - \bar{q}) d\Gamma = 0. \tag{22}$$

Rearrange the above equation lead to

$$[\int_{\Gamma_1} \tilde{q}\tilde{u}^T d\Gamma - \int_{\Gamma_2} \tilde{u}\tilde{q}^T d\Gamma]\alpha = \int_{\Gamma_1} \tilde{q}\bar{u} d\Gamma - \int_{\Gamma_2} \tilde{u}\bar{q} d\Gamma, \tag{23}$$

or

$$Q\alpha = b, \tag{24}$$

where

$$Q_{ij} = \int_{\Gamma_1} \tilde{q}_i \tilde{u}_j d\Gamma - \int_{\Gamma_2} \tilde{u}_i \tilde{q}_j d\Gamma, \tag{25}$$

$$b_i = \int_{\Gamma_1} \tilde{q}_i \bar{u} d\Gamma - \int_{\Gamma_2} \tilde{u}_i \bar{q} d\Gamma. \tag{26}$$

Motivated by the work of Cheung and his coworkers [Jin, Cheung, and Zienkiewicz (1993)], we can prove that the coefficient matrix Q, generated by the BKM with Galerkin approach, is symmetric. Subtracting the element Q_{ij} from the element Q_{ji} results in

$$\begin{aligned} Q_{ij} - Q_{ji} &= \int_{\Gamma_1} \tilde{q}_i \tilde{u}_j d\Gamma - \int_{\Gamma_2} \tilde{u}_i \tilde{q}_j d\Gamma - (\int_{\Gamma_1} \tilde{q}_j \tilde{u}_i d\Gamma - \int_{\Gamma_2} \tilde{u}_j \tilde{q}_i d\Gamma) \\ &= \int_{\Gamma_1+\Gamma_2} \tilde{q}_i \tilde{u}_j d\Gamma - \int_{\Gamma_2+\Gamma_1} \tilde{u}_i \tilde{q}_j d\Gamma \\ &= 0, \end{aligned}$$

or

$$Q_{ij} = Q_{ji}. \tag{27}$$

Note that the Galerkin approach ensures that the coefficient matrix is symmetric. For this reason, the corresponding solution accuracy and computational efficiency is higher than the other approaches.

Furthermore, we can propose another Galerkin approach based on the research of Hochard and Proslier [Hochard and Proslier (1992)]. The weight function q_N and u_N are considered for the residual R_1 and R_2 which generates

$$\int_{\Gamma_1} q_N R_1 d\Gamma + \int_{\Gamma_2} u_N R_2 d\Gamma = 0. \tag{28}$$

Hochard and Proslier proved that this approach can ensure the uniqueness of solution.

2.4 Variational formulation

Based on the energy functional [Zienkiewicz, Kelly, and Bettes (1979)], we can propose the variational formulation

$$\Phi = \int_{\Gamma} \frac{1}{2} q_N u_N d\Gamma - \int_{\Gamma_2} u_N \bar{q} d\Gamma - \int_{\Gamma_1} (u_N - \bar{u}) \bar{q} d\Gamma. \tag{29}$$

Substituting Eqs. (4) and (8) into the above equation gives rise to

$$\begin{aligned}\Phi &= \int_{\Gamma} \frac{1}{2} (\alpha^T \tilde{q})^T (\alpha^T \tilde{u}) d\Gamma - \int_{\Gamma_2} (\alpha^T \tilde{u})^T \bar{q} d\Gamma - \int_{\Gamma_1} (\alpha^T \tilde{u} - \bar{u})^T (\alpha^T \tilde{q}) d\Gamma \\ &= \alpha^T \int_{\Gamma_2} \frac{1}{2} \tilde{q} \tilde{u}^T d\Gamma \alpha - \alpha^T \int_{\Gamma_1} \frac{1}{2} \tilde{u} \tilde{q}^T d\Gamma \alpha - \alpha^T \int_{\Gamma_2} \bar{q} \tilde{u} d\Gamma + \alpha^T \int_{\Gamma_1} \bar{u} \tilde{q} d\Gamma.\end{aligned}$$

Vanishing the first variation of this equation gives

$$\delta\Phi = \delta\alpha^T [\int_{\Gamma_2} \tilde{q} \tilde{u}^T d\Gamma \alpha - \int_{\Gamma_1} \tilde{u} \tilde{q}_T d\Gamma \alpha - \int_{\Gamma_2} \bar{q} \tilde{u} d\Gamma + \int_{\Gamma_1} \bar{u} \tilde{q} d\Gamma] = 0, \tag{30}$$

or

$$[\int_{\Gamma_2} \tilde{q} \tilde{u}^T d\Gamma - \int_{\Gamma_1} \tilde{u} \tilde{q}_T d\Gamma] \alpha = \int_{\Gamma_2} \bar{q} \tilde{u} d\Gamma - \int_{\Gamma_1} \bar{u} \tilde{q} d\Gamma. \tag{31}$$

This equation is identical with Eq. (23) in the Galerkin formulation.

Once the coefficient α is derived, we can calculate the value of each point on the whole physical domain Ω through Eqs. (4) and (8).

3 Overview of the BKM with collocation approach

Up to now, all BKM-related literatures, mainly focused on numerical algorithms and practical applications, are based on the traditional collocation approach.

Chen and He [Chen and He (2001)] introduced a new scheme employing the BKM to deal with nonlinear convection-diffusion problem. Chen and Tanaka [Chen and Tanaka (2002)] extends the BKM to general problems such as Laplace and convection-diffusion problems by a combined use of the dual reciprocity method [Patridge, Brebbia, and Wrobel (1992)]. Furthermore, Chen proposed the symmetric BKM [Chen (2002)] to deal with boundary value problems with mixed boundary conditions where the coefficient matrix is non-symmetric. Hon and Chen [Hon and Chen (2003)] extended the BKM to solve 2D Helmholtz and convection-diffusion problems under rather complicated irregular geometry. They also applied the BKM to 3D problems for the first time and found that some inner points are necessary to guarantee accuracy and stability for inhomogeneous cases. Chen and Hon [Chen and Hon (2003)] made a numerical study of convergence properties of the BKM by solving 2D and 3D homogeneous Helmholtz, modified Helmholtz, and convection-diffusion problems. Since there is no non-singular general solutions for the Laplace equation, Chen and his coworkers [Hon and Chen (2003)] used the high-order general solutions of the Helmholtz and modified Helmholtz equations to evaluate the particular solution for the Laplacian equation.

Jin and Yao [Jin and Zheng (2005a);Jin and Zheng (2005b)] applied the BKM to the solution of some inverse problems for the homogeneous and inhomogeneous Helmholtz equation, including the highly ill-posed Cauchy problem. Since the resulting matrix equation is badly ill-conditioned, they employed the truncated singular value decomposition under parameter choice of L-curve method to get a regularized solution. Zhang and Tan [Zhang and Tan (2005)] solved homogeneous and nonhomogeneous partial differential equations using BKM combined with overlapped DDM which is a good choice to avoid singular fundamental solution and ill-conditioned coefficient matrix. Jin and Chen [Jin and Chen (2006)] made the first attempt to use the geodesic distance with the BKM to solve 2D and 3D anisotropic Helmholtz-type and convection-diffusion problems.

Recently, Shi *et al.* [Shi, Chen, and Wang (2009)] used the BKM to calculate the free vibration of free and simply-supported thin plates of complex shape subjected to different boundary conditions. Canelas and Sensale [Canelas and Berardi (2010)] derived specialized radial basis function for harmonic elastic and viscoelastic problems, and proposed a boundary knot method for the solution of these problems. They also discussed the completeness issue regarding the proposed set of radial basis functions, and presented a formal proof of incompleteness for the circular ring problem.

Wang *et al.* [Wang, Chen, and Jiang (2010)] used the BKM, together with three regularization techniques [Hansen (1994);Tikhonov, Goncharsky, Stepanov, and Yagola (1995);Liu (2007);Liu, Hong, and Atluri (2010); Liu and Kuo (2011);Liu, Kuo, and Liu (2011)] and two algorithms for selecting regularization parameters, to investigate the numerical instability induced from highly dense and ill-conditioned

BKM interpolation matrix. On the other hand, the effective condition number [Drombosky, Meyer, and Ling (2009);Wang (2011)] is considered to be a superior indicator, to the traditional L^2 condition number, to scale the ill-conditioned interpolation system. Wang *et al.* [Wang, Ling, and Chen (2009)] also pointed that the ECN is roughly inversely proportional to the solution accuracy. And one can fine-tune the user-defined parameters (without the knowledge of exact solution) to ensure high numerical accuracy from the BKM. Dehghan and Salehi [Dehghan and Salehi (2011)] successfully used the analog equation method [Katsikadelis (1994)] accompanied with the BKM to solve the Eikonal equation. Recently, Zheng *et al.* [Zheng and Ma (in press)] used an improved the analogy equation method together with the BKM to the nonlinear problems. Fu *et al.* [Fu, Chen, and Qin (2011)] firstly derived the nonsingular general solution of heat conduction in nonlinear functionally graded materials(FGMs), and then presents the BKM in conjunction with Kirchhoff transformation and various variable transformations in the solution of nonlinear FGM problems.

4 Conclusions

In this paper, we have presented an overview of the BKM focusing on the collocation approach. Meanwhile, three novel approaches, which are promising for the future investigations, are proposed for the BKM. Compared to the traditional finite and boundary element methods, the advantages of the BKM is that there's no need to worry about the mesh-generation and the solution accuracy is very high. On the other hand, the BKM has great challenges in developing the theory, speed and robustness. Details include the adaptivity for choosing the optimal location and number of collocation points, the theoretical investigation on the error analysis and efficient algorithms for high-dimensional or large-scale problems [Liu (2009)]. Breakthroughs in these directions will have considerable impact on the BKM.

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