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A Novel Method for Solving One-, Two- and Three-Dimensional Problems with Nonlinear Equation of the Poisson Type

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Abstract: The paper presents a new meshless numerical technique for solving nonlinear Poisson-type equation $\nabla^2 u = f(\mathbf{x}) + F(u,\mathbf{x})$ for $\mathbf{x} \in R^d$, $d = 1,2,3$. We assume that the nonlinear term can be represented as a linear combination of basis functions $F(u,\mathbf{x}) = \sum_m^M q_m \varphi_m$. We use the basis functions φ_m of three types: the monomials, the trigonometric functions and the multiquadric radial basis functions. For basis functions φ_m of each kind there exist particular solutions of the equation $\nabla^2 \phi_m = \varphi_m$ in an analytic form. This permits to write the approximate solution in the form $u_M = u_f + \sum_m^M q_m \Phi_m$, where $\Phi_m = \phi_m + \omega_m$. The term ω_m provides that Φ_m satisfies the homogeneous conditions on the boundary of the domain. Substituting u_M into the equation for F, we transform it to the system of nonlinear equations $F(u_M,\mathbf{x}_n) = \sum_m^M q_m \varphi_m(\mathbf{x}_n)$, $n = 1,...,M$ for the unknown coefficients q_m. Then the nonlinear system is solved numerically. Numerical experiments are carried out for accuracy and convergence investigations. A comparison of the numerical results obtained in the paper with the exact solutions or other numerical methods indicates that the proposed method is accurate and efficient in dealing with complicated geometry and strong nonlinearity.

Keywords: Nonlinear boundary value problems, Numerical solution, Dual reciprocity method, Particular solutions, Linear combination.

1 Introduction

In this paper we present a meshless method for solving boundary value problems (BVP) of the type:

$$\nabla^2 u = f(\mathbf{x}) + F(u, \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u, \mathbf{x}), \ \mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z) \in \Omega \subset \mathbf{R}^3, \tag{1}$$

$$\mathscr{B} u = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega. \tag{2}$$

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One and two-dimensional analogues of the equation (1) are also considered in this paper. We assume that g , f and F are smooth enough functions of each argument. Such problems often arise in many branches of applied science. If one encounters a nonlinear problem in an irregular domain, one faces both the geometric complexity and nonlinearity, and naturally one needs to resort to a numerical technique for solving this problem. Thus, during the last decades many numerical techniques have been developed in this field. The boundary element method (BEM) [Kasab, Karur and Ramachandran (1995)], the method of fundamental solutions (MFS) in combination with the analog equation method developed by Li and Zhu (2009), the hybrid Trefftz finite element method suggested by Wang, Qin and Arounsavat (2007), the hybrid finite element model based on fundamental solutions by Wang, Qin and Liang (2012), the homotopy method combined with the MFS proposed by Tsai (2012) and the combination of the asymptotic method with MFS studied by Tri, Zahrouni and Potier-Ferry (2011) are the most important techniques developed in this field recently. In the last two decades, there has been an increasing interest in the idea of meshless or mesh-free numerical methods for solving partial differential equations (PDEs). These methods are nowadays the main stream in numerical computations, as strongly advocated by many researchers, for example, Zhu, Zhang and Atluri (1998a,b); Atluri and Zhu (1998a,b); Atluri, Liu, and Kuo (2009); Atluri and Shen (2002); Cho, Golberg, Muleshkov, and Li (2004); Jin (2004); Li, Lu, Huang and Cheng (2007); Liu (2007a,b); Tsai, Lin, Young and Atluri (2007); Young, Chen, Chen and Kao (2007)

In [Tsai, Liu and Yeih (2010)] the fictitious time integration method (FTIM) previously developed by Liu and Atluri (2008a,b) is combined with the method of fundamental solutions and the Chebyshev polynomials to solve Poisson-type nonlinear PDEs.

For the past two decades radial basis functions (RBFs) have played an important role in the development of meshless methods for solving partial differential equations (PDEs): Kansa (1990a,b); Kansa and Hon (2000); Golberg and Chen (1997); Golberg, Chen and Bowman (1999); Power and Barraco (2002); Larsson and Fornberg (2003); Li, Cheng and Chen (2003); Cheng and Cabral (2005). A significant place among these techniques is taken up by methods based on the use of particular solutions.

In this approach, RBFs have been used to approximate the particular solution corresponding to the given f and the original inhomogeneous PDE has been converted into a homogeneous one, so that one can apply the MFS or other boundary methods developed by Golberg and Chen (1997); Golberg, Chen and Bowman (1999); Cheng (2000). This is the so-called two-stage scheme: $f \simeq \sum_{i=1}^{N_0} p_i \varphi(r_i)$, $L[\Phi(r_i)] = \varphi(r_i)$, $u = u_h + \sum_{i=1}^{N_0} p_i \Phi(r_i)$, $L[u_h] = 0$. Note that similar technique has been developed with the use of the Chebishev polynomials instead of the RBFs by Cheng (2000); Golberg, Muleshkov, Chen and Cheng (2003); Cheng, Ahtes, and Ortner (1994); Tsai (2008) and for the spline approximation of f by Tsai, Cheng and Chen (2009).

The scheme which combines the MFS and RBFs approximation has been proposed for further improvement of the MFS for solving PDEs with variable coefficients. This is the so-called one-stage scheme or the MFS-MPS technique [Chen, Fan and Monroe (2008)]: $u = \sum_{i=1}^{N_0} p_i \Phi(r_i) + \sum_{j=1}^{N_b} q_j G_j(r_j)$, $L[G_j] = 0$. Recently this technique has been transformed into the method of approximate particular solutions (MAPS) Chen, Fan and Wen (2010, 2011). Applying it to the problem

$$\nabla^2 u + b_1(\mathbf{x}) \frac{\partial u}{\partial x_1} + b_2(\mathbf{x}) \frac{\partial u}{\partial x_2} + q(\mathbf{x}) u = f(\mathbf{x}), \, \mathbf{x} \in \Omega, \tag{3}$$

$$\mathscr{B} u(\mathbf{x}) = g(\mathbf{x}), \, \mathbf{x} \in \partial\Omega, \tag{4}$$

one rearranges (3) into Poisson-type equation

$$\nabla^2 u = h\left(x, w, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = -b_1(\mathbf{x}) \frac{\partial u}{\partial x_1} - b_2(\mathbf{x}) \frac{\partial u}{\partial x_2} - q(\mathbf{x}) u + f(\mathbf{x}). \tag{5}$$

The solution is approximated by

$$u \simeq \sum_{i=1}^{N} p_i \Phi(r_i), \tag{6}$$

where Φ is obtained by analytical solution of

$$\nabla^2 \Phi(r_i) = \varphi(r_i). \tag{7}$$

and $\varphi(r_i)$ are RBF functions. Substituting (6) and (7) in (5), one gets

$$h\left(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) \simeq \sum_{i=1}^{N} p_i \varphi(r_i). \tag{8}$$

Similar to Kansa's approach the unknowns p_i are determined by the collocation at the inner points of the solution domain and by the collocation of the boundary conditions. The collocation at the inner points is performed for equation (8) and this technique utilizes expansion (6) to approximate the boundary condition (4). More detailed information on the method can be found in the original papers cited above. In [Li, Chen and Tsai (2012)] the MAPS is applied for solving the Cauchy problems of elliptic partial differential equations with variable coefficients.

The method proposed in this paper is as follows. We assume that there exist such the basis functions $\varphi_m(\mathbf{x})$ that the nonlinear term on the right hand side of (1) can be approximated by the linear combination

$$F\left(u,\partial_{x_1}u,\partial_{x_2}u,\partial_{x_3}u,\mathbf{x}\right)\simeq\sum_{m=1}^{M}q_m\varphi_m(\mathbf{x}). \tag{9}$$

We denote $u_M(\mathbf{x},\mathbf{q})$ as the solution of the problem

$$\nabla^2 u_M(\mathbf{x},\mathbf{q})=\sum_{m=1}^{M}q_m\varphi_m(\mathbf{x})+f(\mathbf{x}),\ \mathbf{q}=(q_1,\ldots,q_M), \tag{10}$$

$$\mathscr{B}u_M=g(\mathbf{x}),\ \mathbf{x}\in\partial\Omega, \tag{11}$$

where the nonlinear term is replaced by the linear combination (9).
We assume that for each $\varphi_m(\mathbf{x})$ there exists a particular solution

$$\nabla^2\phi_m(\mathbf{x})=\varphi_m(\mathbf{x}),\,\mathbf{x}\in\Omega \tag{12}$$

in a closed analytic form.
Let $\omega_m(\mathbf{x}),\,m=1,\ldots,M$ be the solutions of the problems

$$\nabla^2\omega_m(\mathbf{x})=0,\,\mathbf{x}\in\Omega, \tag{13}$$

$$\mathscr{B}\omega_m(\mathbf{x})=-\mathscr{B}\phi_m(\mathbf{x}),\ \mathbf{x}\in\partial\Omega. \tag{14}$$

So, the sum

$$\Phi_m(\mathbf{x})=\phi_m(\mathbf{x})+\omega_m(\mathbf{x}) \tag{15}$$

satisfies equation

$$\nabla^2\Phi_m(\mathbf{x})=\varphi_m(\mathbf{x}),\,\mathbf{x}\in\Omega \tag{16}$$

and the homogeneous boundary conditions

$$\mathscr{B}\Phi_m(\mathbf{x})=0,\ \mathbf{x}\in\partial\Omega. \tag{17}$$

Let $u_f(\mathbf{x})$ be the solution of the problem

$$\nabla^2 u_f(\mathbf{x})=f(\mathbf{x}),\,\mathbf{x}\in\Omega, \tag{18}$$

$$\mathscr{B}u_f(\mathbf{x})=g(\mathbf{x}),\ \mathbf{x}\in\partial\Omega. \tag{19}$$

As a result of these assumptions we can write the solution of (10) in the form

$$u_M(\mathbf{x},\mathbf{q}) = u_f(\mathbf{x}) + \sum_{m=1}^{M} q_m \Phi_m(\mathbf{x}). \tag{20}$$

It is easy to prove that (20) satisfies the boundary condition (11) for any choice of $\mathbf{q} = (q_1, ..., q_M)$. The free parameters $q_1, ..., q_M$ are determined from the condition (9), where u is replaced by u_M:

$$F(u_M, \partial_{x_1} u_M, \partial_{x_2} u_M, \partial_{x_3} u_M, \mathbf{x}) = \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}). \tag{21}$$

The standard MFS procedure is applied to solve problems (13), (14). It is described with more details in the next sections. Solving (18), (19) we split u_f into u_p and u_h:

$$u_f = u_p + u_h, \tag{22}$$

where u_p is a particular solution of the equation

$$\nabla^2 u_p(\mathbf{x}) = f(\mathbf{x}) \tag{23}$$

and u_h satisfies the BVP

$$\nabla^2 u_h(\mathbf{x}) = 0, \mathbf{x} \in \Omega, \tag{24}$$

$$\mathscr{B} u_h(\mathbf{x}) = g(\mathbf{x}) - \mathscr{B} u_p(\mathbf{x}), \mathbf{x} \in \partial\Omega. \tag{25}$$

Then we apply the MFS to solve (24), (25). This is the standard procedure to handle the right-hand-side of the Poissone equation applying the MFS. Note that solving problems (13), (14) and (24), (25) with the help of the MFS, we get $M+1$ linear systems with the same matrix and with different right-hand-sides. So, all these $M+1$ problems are solved by a single call of the standard procedure.

The difficulties arise when there is no simple analytic solution u_p. In this case it is possible to find an approximate solution of (23) by using different approximations of $f(\mathbf{x})$. Instead we use the second version of the proposed method, which is as follows. Instead of (9) we apply the approximation

$$f(\mathbf{x}) + F(u, \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u, \mathbf{x}) \simeq \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}). \tag{26}$$

Here the term $f(\mathbf{x})$ is joined with the non linear term F. As a result, instead of (18), (19) we get

$$\nabla^2 u_f(\mathbf{x}) = 0, \tag{27}$$

$$\mathscr{B}u_f(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega. \tag{28}$$

In this version of the method $u_p \equiv 0$ in the splitting (22) So, we again should solve $M+1$ Laplace's equations in the same domain Ω with zero right-hand sides and with different boundary conditions: M equations are used to get $\omega_m(\mathbf{x})$ and one to get $u_f(\mathbf{x})$. And, as mentioned above, all these $M+1$ problems are solved by a single call of the standard procedure. As a result, here we use equation

$$f(\mathbf{x}) + F(u_M, \partial_{x_1}u_M, \partial_{x_2}u_M, \partial_{x_3}u_M, \mathbf{x}) = \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}) \tag{29}$$

instead of (21) to find the free parameters $q_1, ..., q_M$.

The outline of this paper is as follows: for the sake of simplicity we begin with a one-dimensional analog of the problem (1), (2) - the two-point boundary value problems (TPBVP) in Section 2. Here we consider three different basis functions: x^m, trigonometric functions and Multiquadric (MQ) RBFs. We consider the two-dimensional problems in Section 3 and three-dimensional problems in Section 4. Finally, in Section 5, we give the conclusion and describe the directions for future development of the method presented.

2 One dimensional problems

In this section we consider the two-point boundary value problem (TPBVP) of the form

$$u''(x) = F\left(u, u', x\right) + f(x), \tag{30}$$

$$(\alpha_0 + \beta_0\partial x)\,u(0) = a,\ (\alpha_1 + \beta_1\partial x)\,u(1) = b \tag{31}$$

This is a one-dimensional analog of the initial problem (1), (2). Besides, this problem is interesting in itself. We assume that F and f are smooth enough functions of each argument. There is a vast amount of literature on analytic and numerical solutions of such problems. The homotopy analysis method and its modifications [Abbasbandy and Shivanian (2011)], the finite difference method [Erdogan and Ozis (2011)], the Adomian decomposition method [Wazwaz (2005)] and versions of the Sinc method [El-Gamel and Zayed (2004)] are the most popular in this field.

2.1 Main algorithm

The method presented in application to TPBVP is as follows. Let $\varphi_m(x)$ be some system of basis functions on $[0,1]$. In this section we consider the following three ones:

1) the monomials:

$$\varphi_m^{(1)}(x) = x^{m-1}, m = 1, 2 ..., M. \tag{32}$$

2) the trigonometric basis functions:

$$\varphi_m^{(2)}(x) = \sin(0.5m\pi(x+0.5)), m = 1, 2 ..., M. \tag{33}$$

3) the multiquadric (MQ) RBFs

$$\varphi_m^{(3)}(x) = \psi(x - y_m), \ \psi(x) = \sqrt{x^2 + c^2}, \tag{34}$$

where the centers y_m are placed in the solution domain and c is the shape parameter. The particular solutions of the equation

$$\phi_m''(x) = \varphi_m(x), \tag{35}$$

which correspond to the basis functions (32), (33), (34) are:

$$\phi_m^{(1)}(x) = \frac{x^{m+1}}{m(m+1)}, \tag{36}$$

$$\phi_m^{(2)}(x) = -\frac{4\sin(0.5m\pi(x+0.5))}{\pi^2 m^2}, \tag{37}$$

$$\begin{aligned} \phi_m^{(3)}(x) = {} & \frac{1}{6}\left((x-y_m)^2 + c^2\right)^{3/2} + \\ & + \frac{c^2}{2}\left[(x-y_m)\ln\left((x-y_m) + \sqrt{(x-y_m)^2 + c^2}\right) - \sqrt{(x-y_m)^2 + c^2}\right]. \end{aligned} \tag{38}$$

The formula is taken from [Chen, Fan and Wen (2010, 2011)].
We denote

$$\Phi_m = \phi_m + ax + b,$$

where the free constants a, b are chosen in accordance with the boundary conditions (31) in such a way that

$$(\alpha_0 + \beta_0 \partial x)\Phi_m(0) = 0, \ (\alpha_1 + \beta_1 \partial x)\Phi_m(1) = 0.$$

Following the algorithm described in Introduction, we assume that the nonlinear term can be approximated by the linear combinations of the basis functions $\varphi_m(x)$:

$$F(u, u', x) \simeq \sum_{m=1}^{M} q_m \varphi_m(x). \tag{39}$$

Substituting this approximation in the initial equation (30), one gets

$$u''_M(x) = \sum_{m=1}^{M} q_m \varphi_m(x) + f(x). \tag{40}$$

Let $u_f(x)$ satisfy the problem

$$u''_f(x) = f(x), \tag{41}$$

$$(\alpha_0 + \beta_0 \partial x) u_f(0) = a, \ (\alpha_1 + \beta_1 \partial x) u_f(1) = b. \tag{42}$$

Then any combination

$$u_M(x, \mathbf{q}) = u_f(x) + \sum_{m=1}^{M} q_m \Phi_m(x), \ \mathbf{q} = (q_1, ..., q_M) \tag{43}$$

satisfies exactly the equation (40) and the boundary conditions (31). To get unknowns $q_1, ..., q_M$ we substitute $u_M(x, \mathbf{q})$ in (39)

$$F\left(u_M(x, \mathbf{q}), u'_M(x, \mathbf{q}), x\right) = \sum_{m=1}^{M} q_m \varphi_m(x). \tag{44}$$

Note that we can always get the solution of (41) in the analytic way when $f(x)$ is a simple combination of elementary functions, e.g., quasipolynomial $(a_0 + a_1 x + ... + a_p x^p) \exp(\mu x)$. Otherwise we can use the well known formula

$$u_f(x) = \int_{x_0}^{x} (x - t) f(t) \, dt + c_0 + c_1 x$$

and evaluate the integral numerically.

Another approach to handle $f(x)$ is to take $u_f(x) = c_0 + c_1 x$ as a particular solution of the problem

$$u''_f(x) = 0, \tag{45}$$

$$(\alpha_0 + \beta_0 \partial x) u_f(0) = a, \ \ (\alpha_1 + \beta_1 \partial x) u_f(1) = b. \tag{46}$$

We get the linear system $\alpha_0 c_0 + \beta_0 c_1 = a$, $\alpha_1 c_0 + (\alpha_1 + \beta_1) c_1 = b$ to determine c_0, c_1. In this case, $u_M(x, \mathbf{q})$ given in (43), satisfies the equation

$$u''_M(x) = \sum_{m=1}^{M} q_m \varphi_m(x), \tag{47}$$

and the boundary conditions (31). The free parameters $q_1, ..., q_M$ are determined from the condition

$$F\left(u_M(x, \mathbf{q}), u'_M(x, \mathbf{q}), x\right) + f(x) = \sum_{m=1}^{M} q_m \varphi_m(x). \tag{48}$$

This is the second version of the method.

2.2 Numerical implementation

To solve (44) or (48) we use the following algorithm. Let $0 \le x_1 < x_2 < ... < x_M \le 1$ be collocation points. In particular we use the following distributions of the collocation points:

1) the uniform distribution

$$x_n = \frac{n-1}{M-1}, \tag{49}$$

2) the Chebishev collocation points

$$x_n = \frac{1}{2}\left[1 + \cos\left(\frac{\pi(n-1)}{M-1}\right)\right]. \tag{50}$$

When the MQ are used as the basis functions, we take the centers ξ_n which coincide with the collocation points x_n.

We write the collocation of (44) at these points and get the system of M nonlinear equations

$$F\left(u_M(x_n,\mathbf{q}), u'_M(x_n,\mathbf{q}), x_n\right) = \sum_{m=1}^{M} q_m \varphi_m(x_n), \; n = 1,...,M. \tag{51}$$

The similar system can be written for (48).

We solve this system using the NEQNF procedure from the IMSL Fortran Numerical Math Library based on the use of a modified Powell hybrid algorithm and a finite-difference approximation to the Jacobian. This is an iteration procedure and we take $\mathbf{q}^{(0)} = (0,0,..,0)$ as the initial guess. After determining $q_1,...,q_M$ we get the approximate solution $u_M(x,\mathbf{q})$ (43). We use the root mean square error

$$e_{rms} = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left[u_M(x_j) - u_{exact}(x_j)\right]^2}$$

to evaluate the exactness of the solution. To calculate e_{rms} we use $N = 1001$ test points uniformly distributed inside $[0,1]$.

Example 1 Consider the problem

$$u'' = u^2 + 2\pi^2\cos(2\pi x) - \sin^4(\pi x), \, u(0) = u(1) = 0$$

with the exact solution

$$u_{exact}(x) = \sin^2(\pi x).$$

This problem is taken from [Erdogan and Ozis (2011)]. Here $f(x) = 2\pi^2 \cos(2\pi x) - \sin^4(\pi x)$ and the exact solution of equation (41) can be written in the form

$$u_f(x) = -\left(\frac{1}{2} + \frac{1}{8\pi^2}\right)\cos 2\pi x + \frac{\cos 4\pi x}{128\pi^2} - 3\frac{x^2}{16} + c_1 x + c_0. \tag{52}$$

The parameters c_1, c_0 are obtained from the boundary conditions $u_f(0) = u_f(1) = 0$.

The results of the calculations with the basis functions $\varphi_m^{(1)}(x) = x^{m-1}$ are placed in Table 1. The data in the first and second rows correspond to the uniform and Chebyshev's distributions of the collocation points. For $M > 14$ the NEQNF iteration procedure diverges with both distributions. The absolute values of the coefficients $q_1, ..., q_M$ grow as M increases: $\max|q_i| = 58$, 3.3×10^3 and 3.5×10^4 for $M = 5$, 10 and 14 correspondingly.

Next we perform the same calculation with the use of the trigonometric functions (33) as the basis functions. In this case the iterative process of solving the system (51) is stable for more basis functions and this allows for higher accuracy of the solution. Some results are placed in the second part of the table. The absolute values of coefficients $q_1, ..., q_M$ here are much less: $\max|q_i| = 0.35$, 0.4 and 0.41 for $M = 10$, 20 and 30 correspondingly.

The results of the solved problem with the use of the MQ RBFs (34) as the basis functions are placed in the last row of the table. The iterative process of solving the system (51) is stable for $M \leq 200$. The absolute values of coefficients $q_1, ..., q_M$ here are: $\max|q_i| = 3.3$, 0.22 and 0.14 for $M = 50$, 100 and 150 correspondingly. Here we use only the uniform distribution (49) of the collocation points because when using Chebyshev's distribution, one gets a very small distance between the collocation points in the neighborhood of the endpoints of the interval $[0, 1]$. This leads to very small values of the shape parameter c in (34). And this also leads to a decrease in the accuracy of the calculations. The data presented in the table correspond to $c = 0.095$, 0.03 and 0.02 for $M = 50$, 100 and 150.

In Table 2 we compare the method proposed with the nonstandard finite difference method (FD) and the classical Numerov method. These data are also taken from [Erdogan and Ozis (2011)]. We place the absolute errors of the solutions obtained by the use of the trigonometric functions (33) as the basis functions in the first two columns. The data correspond to $M = 20$ and $M = 30$. The last two columns of the table contain the absolute errors of the solutions obtained by the FD and by Numerov's method.

All the calculations presented above were performed with the use of the exact particular solution u_f given in (52). As it is mentioned above, it is possible to find an

Table 1: Example 1. Solutions with three different basis functions and with different M.

$\varphi_m^{(1)}(x) = x^{m-1}$			
M	5	10	14
$e_{rms}(uni)$	1.8×10^{-3}	2.6×10^{-5}	2.5×10^{-7}
$e_{rms}(Cheb)$	5.8×10^{-3}	8.8×10^{-6}	2.0×10^{-8}
$\varphi_m^{(2)}(x) = \sin(0.5m\pi(x+0.5))$			
M	10	20	30
$e_{rms}(uni)$	1.4×10^{-6}	2.0×10^{-10}	4.0×10^{-13}
$e_{rms}(Cheb)$	1.4×10^{-6}	2.2×10^{-11}	1.7×10^{-15}
$\varphi_m^{(3)}(x) = \psi(x - y_m),\ \psi(x) = \sqrt{x^2 + c^2}$			
M	50	100	150
$e_{rms}(uni)$	1.1×10^{-10}	1.1×10^{-11}	1.8×10^{-12}

approximate solution with the use of the second version of the proposed method, where f is joined with the non linear term F. In the framework of this version there is no need to look for an exact particular solution like (52). Some results of the calculations performed with the use the second version are presented in Table 3.

Example 2 Consider the problem

$$u'' = 0.16u^{-3}, \tag{53}$$

$$u'(0) = 0,\ u(1) = 1. \tag{54}$$

This BVP arises in the description of the heat transfer on a finned surface. As it is shown in [Abbasbandy and Shivanian (2011)], there exist two exact solutions of the problem. They are given by the formula $0.25\lambda\sqrt{u^2 - \lambda^2} = x$, where $\lambda_1 = 0.4472135954$ and $\lambda_2 = 0.8944271909$ for the first and the second solutions.

Here we use the monomials (32) as the basis functions. It is easy to prove that

$$\Phi_m(x) = \frac{x^{(m+1)} - 1}{m(m+1)} \tag{55}$$

is the solution of the BVP

$$\Phi_m''(x) = \varphi_m(x),\ \Phi_m'(0) = \Phi_m(1) = 0.$$

Table 2: Example 1. Solution with the use the trigonometric functions (33) as the basis functions. The data in the last two columns correspond to the nonstandard finite difference method and the Numerov method [Erdogan and Ozis (2011)].

x	$M=20$	$M=30$	I	II
0.1	2×10^{-10}	4×10^{-13}	1.6×10^{-7}	2.9×10^{-9}
0.3	2×10^{-10}	4×10^{-13}	1.4×10^{-6}	4.2×10^{-9}
0.5	2×10^{-10}	4×10^{-13}	2.2×10^{-6}	7.6×10^{-9}
0.7	2×10^{-10}	4×10^{-13}	1.4×10^{-6}	4.2×10^{-9}
0.9	2×10^{-11}	5×10^{-13}	1.6×10^{-7}	2.9×10^{-9}

Table 3: Example 1. Solution with the use of the second version of the method (see (48)).

$\varphi_m^{(1)}(x)=x^{m-1}$			
M	8	10	12
$e_{rms}\,(Cheb)$	1.6×10^{-4}	2.6×10^{-6}	3.2×10^{-8}
$\varphi_m^{(2)}(x)=\sin(0.5m\pi(x+0.5))$			
M	15	20	25
$e_{rms}\,(Cheb)$	1.4×10^{-8}	4.0×10^{-10}	6.6×10^{-13}
$\varphi_m^{(3)}(x)=\psi(x-y_m),\ \psi(x)=\sqrt{x^2+c^2}$			
M	50	100	150
$e_{rms}\,(uni)$	1.7×10^{-7}	3.6×10^{-8}	9.2×10^{-9}

The function $u_f(x)=1$ satisfies the equation $u_f''=0$ and the boundary conditions (54). Thus, we sought the solution of (53), (54) in the form (43) with $\Phi_m(x)$ given in (55).

First, we have solved the problem (51) with $M=1$ and with the sole collocation point $x_1=0.5$. Thus, we solve the nonlinear equation $0.16\left[1+q_1\Phi_1(0.5)\right]^{-3}=q_1$ using the different initial estimates of the root $q_1^{(ini)}=-10,\,-9.9,\ldots,0.9,10$. For all these initial approximations we get only two roots $Q_1=0.2028$ and $Q_2=1.3600$. Next, the calculation for $M>1$ were performed with the two types of the initial data: $\mathbf{q}_1^{(ini)}=(Q_1,0,0,\ldots,0)$ and $\mathbf{q}_2^{(ini)}=(Q_2,0,0,\ldots,0)$. As a result we get the both branches of the solution which are shown in Table 4.

Table 4: Example 2. The two branches of the solution of (53), (54).

	$\lambda_1 = 0.4472135954$, $Q_1 = 0.2028$		
x	$u_{ex}^{(1)}$	$M = 10$	$M = 15$
0.1	0.456070165	0.4560703	0.456070164
0.3	0.521536192	0.5215350	0.521536190
0.5	0.632455537	0.6324558	0.632455533
0.7	0.769415373	0.7694154	0.769415367
0.9	0.920869171	0.9208692	0.843800933
	$\lambda_2 = 0.8944271909$, $Q_2 = 1.3600$		
x	$u_{ex}^{(2)}$	$M = 10$	$M = 15$
0.1	0.895544516	0.8955445	0.89554453
0.3	0.904433514	0.9044335	0.90443352
0.5	0.921954437	0.9219545	0.92195445
0.7	0.947628612	0.9476286	0.94762862
0.9	0.980815982	0.9808160	0.98081598

Example 3 Consider the problem

$$u'' + \frac{1}{x}u' = -\delta \exp(u), \tag{56}$$

$$u'(0) = 0,\ u(1) = 1. \tag{57}$$

The equation (56) has the singular point at $x = 0$. As it is shown by Liu (2006), the analytic solution of the problem is

$$u(x) = \ln\left[\frac{8\rho}{\delta(1+\rho x^2)^2}\right], \tag{58}$$

where the integration constant ρ is determined by

$$\frac{8\rho}{\delta(1+\rho)^2} = 1 \tag{59}$$

It can be seen that for a given δ in the range of $0 < \delta < 2$, there are exist two distinct real roots of (59):

$$\rho_1 = \frac{1}{2}\left[\frac{8}{\delta} - 2 + \sqrt{\left(\frac{8}{\delta} - 2\right)^2 - 4}\right] \tag{60}$$

$$\rho_2 = \frac{1}{2}\left[\frac{8}{\delta} - 2 - \sqrt{\left(\frac{8}{\delta} - 2\right)^2 - 4}\right] \tag{61}$$

and correspondingly, there are two solutions of (56), (57) given in (58).

We rewrite (56) in the form

$$xu'' = -u' - x\delta \exp(u), \tag{62}$$

and approximate the right hand side by the linear combinations

$$-u' - x\delta \exp(u) = \sum_{m=1}^{M} q_m \varphi_m(x) \tag{63}$$

We write the approximate solution

$$u_M = u_f + \sum_{m=1}^{M} q_m \Phi_m(x), \tag{64}$$

where $u_f = 1$ satisfies equation $xu_f'' = 0$ and the boundary conditions (57), Φ_m is a solution of the problem

$$x\Phi_m'' = \varphi_m(x), \tag{65}$$

$$\Phi_m'(0) = \Phi_m(1) = 0. \tag{66}$$

To handle the singularity the basis functions φ_m should be modernized. Considering the monomials (32), the first basis function $\varphi_1(x) = 1$ should be excluded, because it generates a singular solution $\sim x\ln x$ with a derivative which tends to infinity when x tends to zero. Thus we take

$$\varphi_m(x) = x^m = \varphi_{m+1}^{(1)}(x). \tag{67}$$

Substituting (67) in (65), (66) one gets:

$$\Phi_m(x) = \frac{x^{m+1} - 1}{m(m+1)}.$$

We modify the trigonometric basis (33) in the similar way

$$\varphi_m(x) = x\sin(0.5m\pi(x+0.5)) = x\varphi_m^{(2)}(x). \tag{68}$$

Substituting (68) in (65), (66) one gets

$$\Phi_m(x) = \phi_m^{(2)}(x) + ax + b,$$

where $\phi_m^{(2)}(x)$ is given in (37) and a,b are determined from the boundary conditions (66).

The rest part of the algorithm is the same as the one in Example 2. We set $M=1$ and substitute (64) in (63) using collocation in the sole point $x_1=0.5$. Then we solve the resulting nonlinear equation

$$-u_1'(0.5,q_1)-0.5\delta\exp(u_1(0.5,q_1))=q_1\varphi_1(0.5),$$

using the different initial estimates of the root $q_1^{(ini)}=-1000,\,-999,\ldots,999,1000$. For all these initial estimates we get only two roots: $Q_1=-7.0604$, $Q_2=-0.63426$ when the monomials (67) are used as the basis functions and $Q_1=-7.8677$, $Q_2=-0.66034$ for trigonometric basis (68). This data correspond to $\beta=1$, $\rho_1=5.828427124746$ and $\rho_2=0.171572875254$. The calculation for $M>1$ were performed with the two types of the initial data: $\mathbf{q}_1^{(ini)}=(Q_1,0,0,\ldots,0)$ and $\mathbf{q}_2^{(ini)}=(Q_2,0,0,\ldots,0)$. As a result, we get the both branches of the solution which are shown in Table 5.

Table 5: Example 3. Solution of the singular problem (56), (57) using basis functions of the two kinds.

$\varphi(x)=x^m$			
M	8	10	12
$e_{rms}(u1)$	4.6×10^{-3}	1.3×10^{-3}	6.5×10^{-5}
$e_{rms}(u2)$	6.5×10^{-8}	4.9×10^{-10}	2.8×10^{-12}
$\varphi_m(x)=x\sin(0.5m\pi(x+0.5))$			
M	15	20	25
$e_{rms}(u1)$	2.8×10^{-6}	1.8×10^{-8}	1.1×10^{-10}
$e_{rms}(u2)$	3.0×10^{-8}	7.8×10^{-11}	3.1×10^{-12}

3 Two-dimensional problems

In application to 2D problems

$$\nabla^2u=f(\mathbf{x})+F(u,\partial_{x_1}u,\partial_{x_2}u,\mathbf{x}),\ \mathbf{x}=(x_1,x_2)\equiv(x,y)\in\Omega\subset\mathbf{R}^2, \tag{69}$$

$$\mathscr{B}u=g(\mathbf{x}),\ \mathbf{x}\in\partial\Omega. \tag{70}$$

the method presented is as follows. We consider the basis functions of the three kinds

1) the monomials:

$$\varphi_m^{(1)}(x,y) = x^{m_1-1}y^{m_2-1},\ m_1,m_2 = 1,2,... \tag{71}$$

2) the trigonometric functions:

$$\varphi_m^{(2)}(x,y) = \sin(0.5m_1\pi(x+1))\sin(0.5m_2\pi(y+1)) \tag{72}$$

3) the Multiquadric (MQ) RBFs

$$\varphi_m^{(3)}(x,y) = \psi(\mathbf{r}-\xi_m),\ \psi(\mathbf{r}) = \sqrt{|\mathbf{r}|^2+c^2}, \tag{73}$$

where the centers ξ_m are placed in the solution domain and c is the shape parameter. The particular solutions of the equation

$$\nabla^2\phi(x,y) = \varphi(x,y)$$

corresponding to the basis functions (71), (72), (73) are:
1)

$$\phi_m^{(1)}(x,y) = (m_1)!\,(m_2)!\sum_{l=0}^{[m_2/2]}(-1)^l\frac{x^{(m_1+2l+2)}y^{(m_2-2l)}}{(m_1+2l+2)!\,(m_2-2l)} \tag{74}$$

The formula is taken from [Tsai, Cheng and Chen (2009)].
2)

$$\phi_m^{(2)}(x,y) = -\frac{4\varphi_m^{(2)}(x,y)}{\pi^2\left(m_1^2+m_2^2\right)} \tag{75}$$

3)

$$\phi_m^{(3)}(x,y) = \frac{1}{9}\left(4c^2+|\mathbf{r}-\xi_m|^2\right)\sqrt{|\mathbf{r}-\xi_m|^2+c^2}-\frac{c^3}{3}\ln\left(c+\sqrt{|\mathbf{r}-\xi_m|^2+c^2}\right) \tag{76}$$

The formula is taken from [Chen, Hon and Schaback (2005)].

According to the key idea of the method proposed we assume that the nonlinear term of the equation can be approximated by the linear combinations of the basis functions:

$$F(u,\partial_{x_1}u,\partial_{x_2}u,\mathbf{x}) = \sum_{m=1}^{M} q_m\varphi_m(\mathbf{x}). \tag{77}$$

Instead of (69) we get:

$$\nabla^2 u_M(\mathbf{x},\mathbf{q}) = f(\mathbf{x}) + \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}). \tag{78}$$

Thus, the approximate solution can be written in the form:

$$u_M(\mathbf{x},\mathbf{q}) = u_f(\mathbf{x}) + \sum_{m=1}^{M} q_m \Phi_m(\mathbf{x}). \tag{79}$$

To get the functions

$$\Phi_m(\mathbf{x}) = \phi_m(\mathbf{x}) + \omega_m(\mathbf{x}),$$

which satisfy the boundary conditions

$$\mathscr{B}\Phi_m(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega$$

we solve M Laplace's equations with different boundary conditions

$$\nabla^2 \omega_m(\mathbf{x}) = 0,\ m = 1,...,M, \tag{80}$$

$$\mathscr{B}\omega_m(\mathbf{x}) = -\mathscr{B}\phi_m(\mathbf{x}),\ \mathbf{x} \in \partial\Omega \tag{81}$$

by the use of the MFS.
To get a particular solution u_f we solve

$$\nabla^2 u_f(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega \tag{82}$$

$$\mathscr{B}u_f(\mathbf{x}) = g(\mathbf{x}),\ \ \mathbf{x} \in \partial\Omega. \tag{83}$$

When there exists an analytic particular solution $u_p(\mathbf{x})$ satisfying (82), one uses the splitting $u_f = u_p + \omega_f$ and get the problem

$$\nabla^2 \omega_f(\mathbf{x}) = 0,\ \mathbf{x} \in \Omega \tag{84}$$

$$\mathscr{B}\omega_f(\mathbf{x}) = g(\mathbf{x}) - \mathscr{B}u_p(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \tag{85}$$

which is solved by the MFS similar (80), (81).
It is easy to verify that (79) satisfies the boundary conditions (70) for any $\mathbf{q} = q_1, q_2, ..., q_M$. We get the unknowns $\mathbf{q}$ from the equation

$$F\left(u_M(\mathbf{x},\mathbf{q}), \partial_{x_1} u_M(\mathbf{x},\mathbf{q}), \partial_{x_2} u_M(\mathbf{x},\mathbf{q}), \mathbf{x}\right) = \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}) \tag{86}$$

When there is no simple analytic solution of (82), we take u_f as a solution of the problem

$$\nabla^2 u_f(\mathbf{x}) = 0, \mathbf{x} \in \Omega, \tag{87}$$

$$\mathscr{B} u_f(\mathbf{x}) = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega \tag{88}$$

and use

$$F\left(u_M(\mathbf{x},\mathbf{q}), \partial_{x_1} u_M(\mathbf{x},\mathbf{q}), \partial_{x_2} u_M(\mathbf{x},\mathbf{q}), \mathbf{x}\right) + f(\mathbf{x}) = \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}) \tag{89}$$

to find the unknowns $q_1, q_2, ..., q_M$ according to the second version of the method proposed.

3.1 Numerical implementation

As it is shown above, the algorithm of the method presented includes solution of the problems (80), (81) and (84), (85) (or (87), (88)) with the same Laplace equation and with different boundary conditions. In application to these problems the MFS is as follows. The approximation solution is sought in the form

$$\omega(\mathbf{x}) = \sum_{i=1}^{K} p_k \ln|\mathbf{x} - \varsigma_k|. \tag{90}$$

Here $\ln|\mathbf{x} - \varsigma_k|$ up to a constant multiplier coincides with the fundamental solutions and the source points ς_k are placed outside the solution domain. For problem (80), (81) the unknowns p_k are determined from the boundary condition on the boundary $\partial\Omega$

$$\sum_{i=1}^{K} p_k \mathscr{B} \ln|\mathbf{y}_i - \varsigma_k| = -\mathscr{B}\phi_m(\mathbf{y}_i), \ \mathbf{y}_i \in \partial\Omega, \ i = 1, ..., N. \tag{91}$$

In the same way we get the system

$$\sum_{i=1}^{K} p_k \mathscr{B} \ln|\mathbf{y}_i - \varsigma_k| = g(\mathbf{y}_i) - \mathscr{B} u_p(\mathbf{y}_i), \ \mathbf{y}_i \in \partial\Omega, \ i = 1, ..., N \tag{92}$$

for problem (84), (85). Here $\mathbf{y}_i$ are the MFS collocation points distributed on the boundary. We set $u_p(\mathbf{x}) \equiv 0$ in (92) when the second version of the method is applied. As it is mentioned above, solving these problems by the MFS, we get

$M+1$ linear systems with the same matrix $\mathscr{B}\ln\left|\mathbf{y}_j-\varsigma_k\right|$ and with different right-hand-sides $-B\phi_m(\mathbf{y}_i)$, $g(\mathbf{y}_i)-\mathscr{B}u_p(\mathbf{y}_i)$. So, all these $M+1$ problems are solved by a single call of the standard procedure.

We take the number of the collocation points N approximately twice as many as the number of free parameters K. As a result, we obtain an overdetermined linear system which can be solved by the standard least squares procedure.

To solve (86) we use the following algorithm. Let $\mathbf{x}_n\in\Omega$, $n=1,...,M$ be collocation points distributed inside the solution domain Ω. We write the collocation at these points and get the system of M nonlinear equations

$$F\left(u_M(\mathbf{x}_n,\mathbf{q}),\partial_{x_1}u_M(\mathbf{x}_n,\mathbf{q}),\partial_{x_2}u_M(\mathbf{x}_n,\mathbf{q}),\mathbf{x}_n\right)=\sum_{m=1}^{M}q_m\varphi_m(\mathbf{x}_n),n=1,...,M. \quad (93)$$

We solve this system using the same NEQNF iteration procedure mentioned above and we take $\mathbf{q}^{(0)}=(0,0,..,0)$ as the initial guess. When the MQ RBFs (73) are used as the basis functions, the centers ξ_m are chosen to coincide with the collocation points $\mathbf{x}_n$ and are distributed inside Ω with control of the minimal distance d between them.

Example 4 Consider the equation

$$\nabla^2u=4u^3. \quad (94)$$

The Dirichlet boundary condition

$$u(\mathbf{x})=u_{ex}(\mathbf{x}),\ \mathbf{x}\in\partial\Omega \quad (95)$$

is set up by using the exact solution

$$u_{exact}(\mathbf{x})=\frac{1}{4+x_1+x_2}$$

on the boundary of the peanut-shaped computational domain

$$\rho(\theta)=0.3\sqrt{\cos2\theta+\sqrt{1.1-\sin^2 2\theta}},\ 0\leq\theta\leq2\pi \quad (96)$$

depicted in Fig. 1, where (ρ,θ) are polar coordinates. Here $f(\mathbf{x})=0$ and we have $u_p(\mathbf{x})=0$ in (85). The MFS source points are placed in the following way

$$\varsigma_k=(1+\beta)\rho(\theta_k)(\cos\theta_k,\sin\theta_k),k=1,...,K, \quad (97)$$

where $\rho(\theta)$ is given in (96).

Table 6: Example 4. Solution in the peanut-shaped domain with different basis functions.

	$\varphi_m^{(1)}(x,y)$		
M	10	21	36
e_{rms}	3.7×10^{-9}	9.1×10^{-10}	7.4×10^{-11}
	$\varphi_m^{(2)}(x,y)$		
M	15	21	45
e_{rms}	5.0×10^{-7}	3.3×10^{-7}	3.0×10^{-8}
	$\varphi_m^{(3)}(x,y)$		
M	10	40	70
e_{rms}	1.6×10^{-7}	4.4×10^{-9}	8.6×10^{-10}
c	0.7	0.4	0.25

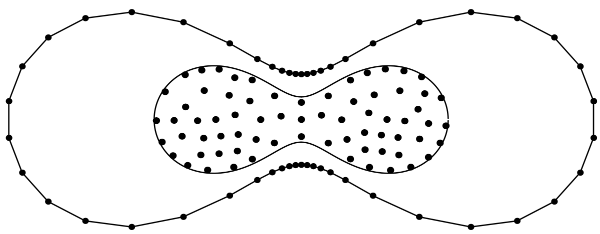

Figure 1: Example 4. The peanut domain. The collocation points $\mathbf{x}_n$ are shown inside the domain and the MFS source points are placed outside it.

Some results of the calculations are placed in Table 6. In Fig. 1 we show the MFS source points ς_k distributed with $\beta = 1.2$ and the collocation points $\mathbf{x}_n$. To calculate the root mean square errors $e_{msr}(u)$ we use 200 test points distributed inside Ω. Considering the basis functions (71), let us introduce I_l - the number of monomials $x^i y^j$ with $i+j \leq l$. It is easy to prove that $I_5 = 10$, $I_7 = 21$ and $I_9 = 36$. The data presented in the first part of the table correspond to these three numbers of the basis functions. Further increase of M leads to a divergence of the NEQNF iteration procedure.

Using the basis functions $\varphi_m^{(2)}(x,y)$, we introduce I_l - the number of trigonometric products $\sin(0.5m_1(x_1+1))\sin(0.5m_2(x_2+1))$ with $m_1+m_2 \leq l$. The data presented in the second part of the table correspond to $M = I_6 = 15$, $M = I_7 = 21$ and

$M = I_{10} = 45$.

The last part of the table contains the data obtained with the use the MQ RBFs as the basis functions. For each M we place the maximal c which provides convergence of the NEQNF procedure and it also provides the minimal error in the solution with given M. Generally, when M is increased, then one should decrease c in order to provide the convergence of the procedure NEQNF and thus, reduce the precision of the approximation by RBFs. Comparing the errors placed in the table with the data reported in [Tsai, Liu and Yeih (2010)] and [Tsai (2012)], we can state that the method presented is more accurate. In [Tsai, Liu and Yeih (2010)] the better result in solving this problem (see Example 5 of this paper) is $e_{msr}(u) = 4.12 \times 10^{-7}$ and in [Tsai (2012)] the better approximation has the maximum error $e_{\max}(u) = 1.89 \times 10^{-6}$(see point 4.5 of this paper).

Example 5 Consider the equation

$$\nabla^2 u = u^2 + 6x_1 - x_1^6 - 4x_1^4 x_2 - 4x_1^2 x_2^2 \tag{98}$$

with the boundary condition

$$u(\mathbf{x}) = u_{exact}(\mathbf{x}), \mathbf{x} \in \partial\Omega \tag{99}$$

corresponding to the exact solution

$$u_{exact}(\mathbf{x}) = x_1^3 + 2x_1 x_2. \tag{100}$$

The solution domain is the ameba-like irregular shape

$$\rho(\theta) = \exp(\sin\theta)\sin^2(2\theta) + \exp(\cos\theta),\ 0 \le \theta \le 2\pi \tag{101}$$

depicted in Fig. 2. Here

$$f(\mathbf{x}) = 6x_1 - x_1^6 - 4x_1^4 x_2 - 4x_1^2 x_2^2. \tag{102}$$

Using the formulae obtained in [Cheng (2000)] , it is easy to find a particular solution of $\nabla^2 u_p = f$:

$$u_p(\mathbf{x}) = x_1^3 - \frac{x_1^8}{56} - \frac{2x_1^6 x_2}{15} - \frac{x_1^4 x_2^2}{3} + \frac{x_1^6}{45}. \tag{103}$$

In Fig. 2 we show the collocation points $\mathbf{x}_n$ inside the domain and the MFS source points ς_k outside it. The source points ς_k are distributed according to formula (97) with $\beta = 0.2$.

We take the number of the MFS sources $K = 200$. Some results of the calculations are placed in Table 7. The trigonometric basis functions (72) are not used here

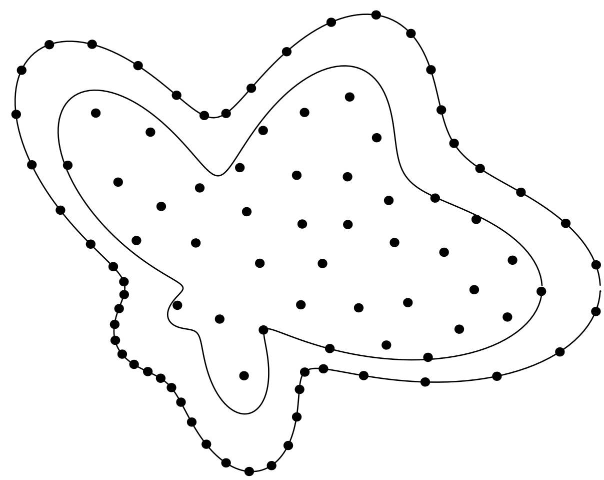

Figure 2: Example 5. The ameba-like irregular shape. The collocation points $\mathbf{x}_n$ are shown inside the domain and the MFS source points ς_k are placed outside it.

because the solution domain is much bigger than the square $[-1,1]\times[-1,1]$. The monomials (71) demonstrate a good precision when M increases but the MQ RBF does not provide a good approximation for the exact solution.

Then we apply the second version of the method and join the homogeneous term $f(\mathbf{x})$ given in (102) with the nonlinear term u^2 . In this case $u_p(\mathbf{x})=0$. The rest part of the algorithm is the same. The results of the calculations using the modified algorithm are shown in Table 8.

This problem was considered by Liu (2009) with the use of the fictitious time integration method. The best result in solving this problem obtained by the method presented in [Liu (2009)] is $e_{\max}=0.025$.

Example 6 We also demonstrate the applicability of the presented method to equations which include the derivatives in the nonlinear term. Consider the equation

$$\nabla^2 u = 4u^3 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 + \frac{2}{(4+x_1+x_2)^4} \tag{104}$$

in the ellipse

$x_1 = 1.5\cos\theta,\ x_2 = \sin\theta,\ 0\le\theta\le 2\pi.$

Table 7: Example 5. Solution in the ameba-like irregular shape.

$\varphi_m^{(1)}(x,y)$			
M	15	21	36
e_{rms}	0.42	0.04	4.4×10^{-5}
$\varphi_m^{(3)}(x,y)$			
M	10	20	30
e_{rms}	3.3	0.2	3.3
c	0.75	0.55	0.45

Table 8: Example 5. Solution in the ameba-like irregular shape using the second version of the method.

$\varphi_m^{(1)}(x,y)$			
M	15	21	55
e_{rms}	4.0×10^{-4}	4.0×10^{-4}	3.0×10^{-4}
$\varphi_m^{(3)}(x,y)$			
M	10	20	30
e_{rms}	2.3×10^{-3}	7.0×10^{-4}	6.0×10^{-4}
c	0.75	0.55	0.45

The Dirichlet boundary conditions on $\partial\Omega$ correspond to the exact solution

$$w_{exact}(x_1,x_2)=\frac{1}{(4+x_1+x_2)}.$$

We use the monomials (71) and the MQ RBFs (73) as the basis functions and the second version of the method when $f(\mathbf{x})$ is joined with the nonlinear term. This means that $F(u,u_{x_1},u_{x_2},\mathbf{x})$ is equal to the right hand side of (104) and $u_p(\mathbf{x})=0$ in (92). Some results are shown in Table 9. The number of the MFS source points is: $K=100$. The source points are distributed in accordance with formula (97), where $\beta=2.5$.

Example 7 Then, the applicability of the present method to problems with more general boundary conditions can also be demonstrated. Consider the equation

$$\nabla^2 u=3u^2 \tag{105}$$

Table 9: Example 6. Solution of equation (104) with the derivatives in the nonlinear term.

	$\varphi_m^{(1)}(x,y)$		
M	15	36	66
e_{rms}	2.3×10^{-4}	1.0×10^{-5}	5.3×10^{-8}
	$\varphi_m^{(3)}(x,y)$		
M	10	30	30
e_{rms}	4.3×10^{-4}	1.6×10^{-5}	2.7×10^{-6}
c	2.5	3.0	2.0

in the circle with the radius 0.5. The boundary condition is of the mixed (or Robin) type:

$$u(\mathbf{x}) + \upsilon(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}} = h(\mathbf{x}),\ \mathbf{x} \in \partial\Omega,$$

where $\mathbf{n}$ is the outward unit normal and $h(\mathbf{x})$ corresponds to the exact solution

$$w_{exact}(x_1,x_2) = \frac{4}{(3+x_1+x_2)^2}.$$

The data placed in Table 10 correspond to $\upsilon(\mathbf{x}) = \left(1+\sin^2(x_1x_2)\right)$

Table 10: Example 7. Solution of equation (105).

	$\varphi_m^{(1)}(x,y)$		
M	15	36	66
e_{rms}	6.9×10^{-5}	2.4×10^{-6}	2.5×10^{-9}
	$\varphi_m^{(2)}(x,y)$		
M	15	36	66
e_{rms}	4.7×10^{-4}	2.5×10^{-5}	7.7×10^{-7}
	$\varphi_m^{(3)}(x,y)$		
M	10	30	70
e_{rms}	6.0×10^{-4}	7.3×10^{-6}	1.5×10^{-6}
c	2.5	1.0	0.51

The number of the MFS source points $K = 200$ and the source points are distributed in accordance with formula (97) with $\beta = 2.3$.

4 Three dimensional case

In application to 3D problems

$$\nabla^2 u = f(\mathbf{x}) + F(u, \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u, \mathbf{x}), \ \mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z) \in \Omega \subset \mathbf{R}^3, \quad (106)$$

$$\mathscr{B} u = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega \quad (107)$$

the method is analogous to the 2D case. We use the following basis functions:

1) Monomials

$$\varphi_m^{(1)}(x, y, z) = x^{m_1} y^{m_2} z^{m_3}$$

2) Trigonometric functions

$$\varphi_m^{(2)}(x, y, z) = \sin\left(m_1 \pi \frac{x+1}{2}\right) \sin\left(m_2 \pi \frac{y+1}{2}\right) \times \sin\left(m_3 \pi \frac{z+1}{2}\right)$$

3) MQ RBF

$$\varphi_m^{(3)}(x, y, z) = \sqrt{|\mathbf{x} - \xi_m|^2 + c^2} = \sqrt{(x_1 - \xi_{m,1})^2 + (x_2 - \xi_{m,2})^2 + (x_3 - \xi_{m,3})^2 + c^2}$$

The corresponding particular solutions are

1)

$$\phi_m^{(1)} = (m_1)!\,(m_2)!\,(m_3)! \sum_{l=0}^{[m_2/2]} \sum_{k=0}^{[m_3/2]} \frac{(-1)^{l+k}(l+k)!\,x^{(m_1+2l+2k+2)} y^{(m_2-2l)} z^{(m_3-2k)}}{l!k!\,(m_1+2l+2k+2)!\,(m_2-2l)!\,(m_3-2k)!}$$

The formula is taken from [Tsai, Cheng and Chen (2009)].

2)

$$\phi_m^{(2)} = -\frac{4\varphi^{(2)}(x, y, z)}{\pi^2 \left(m_1^2 + m_2^2 + m_3^2\right)}$$

3)

$$\phi_m^{(3)} = \frac{1}{12}\left(|\mathbf{x} - \xi_m|^2 + c^2\right)^{3/2} + \frac{c^2}{8}\sqrt{|\mathbf{x} - \xi_m|^2 + c^2} + \\ + \frac{c^8}{8\,|\mathbf{x} - \xi_m|} \ln\left(|\mathbf{x} - \xi_m| + \sqrt{|\mathbf{x} - \xi_m|^2 + c^2}\right)$$

for $|\mathbf{x}-\xi_m| \neq 0$ and

$$\phi_m^{(3)} = \frac{c^3}{3},$$

when $|\mathbf{x}-\xi_m| = 0$. The formula is taken from [Chen, Hon and Schaback (2005)].

To get $\Phi_m(\mathbf{x}) = \phi_m(\mathbf{x}) + \omega_m(\mathbf{x})$ we solve the three-dimensional analogous of the problems (80), (81) and (84), (85) using the MFS. The approximation solution is sought in the form

$$\omega(\mathbf{x}) = \sum_{i=1}^{K} p_k \frac{1}{|\mathbf{x}-\varsigma_k|}.$$

For problem (80), (81) the unknowns p_k are determined from the boundary condition on the boundary $\partial\Omega$

$$\sum_{i=1}^{K} p_k \mathscr{B}\left[\frac{1}{|\mathbf{x}-\varsigma_k|}\right] = -\mathscr{B}\phi_m(\mathbf{y}_i),\ \mathbf{y}_i \in \partial\Omega,\ i = 1,...,N.$$

In the same way we get the system

$$\sum_{i=1}^{K} p_k \mathscr{B}\left[\frac{1}{|\mathbf{x}-\varsigma_k|}\right] = g(\mathbf{y}_i) - \mathscr{B}u_p(\mathbf{y}_i)$$

for problem (84), (85). Here $\mathbf{y}_i$ are the MFS collocation points distributed on the boundary. Having Φ_m and $u_f(\mathbf{x})$ we seek the approximate solution in the form (79) and get the unknowns $\mathbf{q}$ from the equation

$$F\left(u_M, \partial_{x_1}u_M, \partial_{x_2}u_M, \partial_{x_3}u_M, \mathbf{x}\right) = \sum_{m=1}^{M} q_m \varphi_m(\mathbf{x}) \tag{108}$$

written in collocation points $\mathbf{x}_n$ (see (93)). So, the algorithm is the same as the one described in the previous section.

Example 8 Consider the equation

$$\nabla^2 u = \frac{2}{u} + \frac{3}{u^3} \tag{109}$$

with the boundary condition

$$u(\mathbf{x}) = u_{exact}(\mathbf{x}),\ \mathbf{x} \in \partial\Omega \tag{110}$$

corresponding to the exact solution

$$u_{exact}(\mathbf{x}) = \sqrt{3 + x_1^2 + x_2^2 + x_3^2}. \tag{111}$$

Table 11: Example 8. Solution of the 3D problem (109), (110) in the sphere with the radius $R = 0.5$.

	$\varphi_m^{(1)}(x,y,z)$		
M	56	120	220
e_{rms}	7.9×10^{-6}	1.3×10^{-8}	1.7×10^{-9}
	$\varphi_m^{(2)}(x,y,z)$		
M	56	165	283
e_{rms}	8.2×10^{-3}	4.4×10^{-6}	3.0×10^{-7}
	$\varphi_m^{(3)}(x,y,z)$		
M	10	50	100
e_{rms}	8.0×10^{-4}	1.2×10^{-4}	5.4×10^{-5}
c	0.05	0.05	0.075

The solution domain is the sphere with the radius $R = 0.5$.

Some results of the calculations are shown in Table 11. The MFS source points ς_k, $k = 1,...,K$ are randomly distributed on the sphere with the radius 5 and the MFS collocation points $\mathbf{y}_i$ are randomly distributed on $\partial\Omega$. The number of the MFS source points $K = 200$ in all the calculations shown in the table. The root mean square error e_{rms} placed in the table is computed with the use $N_t = 500$ test points randomly distributed in Ω.

5 Conclusion

This paper has presented a truly meshless numerical method for solving one- two- and three-dimensional nonlinear equations of the Poisson type. In the solution procedure, a nonlinear part of the equation is first replaced by a linear combination (9) of the basis functions $\varphi_m(\mathbf{x})$ for whose corresponding analytical particular solutions $\phi_m(\mathbf{x})$ are known. These basis functions can be chosen in different ways. Throughout this paper we use the monomials, trigonometric functions and MQ RBFs. Traditionally these particular solutions $\phi_m(\mathbf{x})$ are used to approximate the solution of the problem considered in the framework of the DRM approach. The key idea of the method presented is the use of the sums $\Phi_m(\mathbf{x}) = \phi_m(\mathbf{x}) + \omega_m(\mathbf{x})$ to this goal. The additional term $\omega_m(\mathbf{x})$ provides satisfaction of homogeneous boundary conditions on $\partial\Omega$. As a result, assuming approximation of the nonlinear term by the linear combination of $\varphi_m(\mathbf{x})$ (see (9), (26) and the corresponding formulae in Section 2, Section 3 and Section 4), we can write an approximate solution $u_M(\mathbf{x},\mathbf{q})$

which satisfies the boundary condition of the problem considered with any choice of the free parameters $\mathbf{q} = (q_1, ..., q_M)$. We substitute u_M back into the equation for F and transform it to the system of nonlinear equations $F(u_M, \mathbf{x}_n) = \sum_m^M q_m \varphi_m(\mathbf{x}_n)$, $n = 1, ..., M$ for the unknown coefficients q_m. Then the nonlinear system is solved numerically.

Numerical experiments are carried out for accuracy and convergence investigations. A comparison of the numerical results obtained in the paper against the exact solutions or other numerical methods indicates that the proposed method is more accurate and efficient in dealing with complicated geometry and strong nonlinearity. The method introduced in this paper can obviously be extended onto problems with differential operators of higher orders and to non stationary problems. This will be the subject of further studies.

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