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Determination of an Unknown Heat Source Term from Boundary Data

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Abstract: This paper employ the method of fundamental solutions for determining an unknown heat source term in a heat equation from overspecified boundary measurement data. By a function transformation, the inverse source problem is changed into an inverse initial data problem which is solved by a method of fundamental solutions. The standard Tikhonov regularization technique with the generalized cross-validation criterion for choosing the regularization parameter is adopted for solving the resulting ill-conditioned system of linear algebraic equations. The effectiveness of the algorithm is illustrated by five numerical examples in one-dimensional and two-dimensional cases.

Keywords: Method of fundamental solutions; Inverse heat source; Inverse initial problem

1 Introduction

In the process of transportation, diffusion and heat conduction of natural materials, the following heat equation is always introduced:

$$u_t - \triangle u = F(\boldsymbol{x},t;u), \quad (\boldsymbol{x},t) \in \Omega \times (0,t_{max}), \tag{1.1}$$

where u represents state variable and the right-hand side F denotes source (sink) terms.

Since in many branches of science and engineering, e.g. crack identification, geophysical prospecting and pollutant detection, the characteristics of sources are often unknown and need to be determined, this is the so-called inverse source identification problem which is a typical ill-posed problem in the sense of Hadamard and many researches have been dedicated to this topic from 1970s. It is well known that a general source cannot be determined uniquely from practical boundary measurements except that suitable a priori knowledge is assumed. If the source term is

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assumed to have an a priori function form, the inverse source problem from a final observation has been investigated in Choulli and Yamamoto (1997); Tadi (1997). For $F = f(u)$, the inverse source problem by additional Dirichlet-Neumann data was studied by many researchers, e.g. Cannon and DuChateau (1998); Fatullayev (2004).Nanda and Das (1996) restored the source depending not only on the unknown function u but also the space variable. Coles and Murio (2001); Wang and Liu (2008) consider the source term that is a function of both space and time variables but is additive or separable. We note that most of papers focused on a source depending on space or time variable only Farcas and Lesnic (2006); Yan, Fu, and Yang (2008); Yan, Yang, and Fu (2009). Recently,Wei and Wang (2012) simultaneously constructed the spacewise dependent source function and initial temperature.

In this paper, we employ the method of fundamental solutions (MFS) to solve an inverse problem for determining the source function $f = f(\boldsymbol{x})$ from $F = \lambda(t)f(\boldsymbol{x})$ when $\lambda(t)$ is given.

A number of numerical methods have been proposed for solving the inverse source problem, such as the boundary element method (BEM) Farcas and Lesnic (2006), iterative regularization methods Johansson and Lesnic (2007a,b, 2008) and mollification methods Yi and Murio (2004a,b). Besides, a sequential method Yang (1998a) and linear least-squares error method Yang (1998b) have also been used for solving the inverse source problem. In all mentioned methods, the partial differential equation must be discretized. The traditional mesh-dependent finite difference method (FDM) and finite element method (FEM) require a mesh on the domain to support the solution process, and the boundary element method only need a mesh on the boundary. However, the BEM needs singular integrals on boundary which requires an additional computational effort.

The MFS was first introduced by Kupradze and Aleksidze in Kupradze and Aleksidze (1964) and the basic idea is to approximate the solution of the problem by a linear combination of fundamental solutions for the governing differential equation. The MFS is an inherently meshless, integration-free technique for solving partial differential equations which has been used extensively for solving various direct and inverse problems, e.g., Kress and Mohsen (1986); Fairweather and Karageorghis (1998); Marin and Lesnic (2005); Hon and Wei (2004, 2005); Young and Ruan (2005); Young, Chen, Chen, and Kao (2007). One possible disadvantage of the MFS is that the resulting system of linear equation is always ill-conditioned, even for a well-posed problem, see Golberg and Chen (1999). Therefore, special regularization methods are required in order to solve this system of algebraic equations.

In this paper, we successfully transform the inverse source problem into an inverse initial data problem and then applied the MFS technique on the resulted equation.

The resulting system of linear equations is solved by employing the Tikhonov regularization method, while the choice of the regularization parameter is based on the generalized cross-validation (GCV) criterion.

The rest of paper is organized as follows. In Section 2, we present the formulation of the problem and transform it into an inverse initial data problem. In Section 3, we use the MFS combined with the Tikhonov regularization (TR) to solve the ill-posed inverse initial data problem. The generalized cross-validation (GCV) criterion is used to choose a suitable regularization parameter. In Section 4, we show some numerical examples which include both analytical and non-analytical solution cases . Section 5 ends this paper with a brief conclusion.

2 Formulation of the problem and transformation into an inverse initial data problem

In this paper, we consider an inverse source problem for heat equation

$$u_t = \triangle u + \lambda(t) f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,\ t \in (0, t_{max}], \tag{2.1}$$

$$u(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \bar{\Omega}, \tag{2.2}$$

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega,\ t \in (0, t_{max}], \tag{2.3}$$

with the overspecified condition

$$u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Gamma,\ t \in [0, t_{max}]. \tag{2.4}$$

where Γ is a part of boundary of Ω, $\lambda(t)$ is given satisfying $\lambda(0) \neq 0$ and $f(x)$ is unknown to be determined from the boundary measured data $g(\boldsymbol{x}, t)$ on Γ. In the case of $\lambda(t) = \mathrm{e}^{-ct}$ with $c > 0$, system (2.1)–(2.3) describes a heat conduction process where a radioactive isotope with the decay rate c supplies heat and the spatial density is given by $f(\boldsymbol{x})$.

The inverse source problem (2.1)–(2.4) is ill-posed. Under an additional a priori condition, the unique solvability and conditional stability can be obtained, see Yamamoto (1993). In order to use the MFS to solve this problem, the first goal is to find a transformation to change equation (2.1) into a homogeneous heat equation with only one unknown function and then apply MFS technique on the resulted problem.

Define

$$u(\boldsymbol{x}, t) = \int_0^t \lambda(t - s) v(\boldsymbol{x}, s)\, \mathrm{d}s. \tag{2.5}$$

Let $v(\boldsymbol{x},t)$ be the solution of the following problem

$$v_t = \triangle v, \quad \boldsymbol{x} \in \Omega, t \in (0, t_{max}], \tag{2.6}$$

$$\frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial\Omega, t \in (0, t_{max}], \tag{2.7}$$

$$\int_0^t \lambda(t-s)v(\boldsymbol{x},s)\,\mathrm{d}s = g(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma, t \in (0, t_{max}]. \tag{2.8}$$

If $v(\boldsymbol{x},0)$ is obtained, then it is easy to know the unknown heat source term $f(\boldsymbol{x})$ can be calculated through the identity

$$f(\boldsymbol{x}) = v(\boldsymbol{x},0). \tag{2.9}$$

Suppose that the given noisy data $\tilde{g}$ representing the measurement of the exact g satisfies

$$\|\tilde{g} - g\|_{L^2(\Omega)} \leqslant \delta, \tag{2.10}$$

where δ is a positive constant representing the noise level of the input data. We aim at finding an approximate function $v^*(\boldsymbol{x},t)$ of $v(\boldsymbol{x},t)$ such that $\|v^*(\boldsymbol{x},0) - f(\boldsymbol{x})\|$ converges to zero, as δ tends to zero. In the following section, we develop a numerical method based on the MFS with regularization to solve the problem (2.6)–(2.8).

3 Method of fundamental solutions and regularization

Suppose the solution v for (2.6) can be extended to a larger domain $D \times (-T, t_{max})$ with $\Omega \subset D$ and $-T < 0$ is a fictitious past time, and assume $v(\boldsymbol{x}, -T) = 0$, then by the single layer heat potential Noon (1988), there exists a density function Φ such that v can be expressed by

$$\begin{aligned} v(\boldsymbol{x},t) &= \int_{\partial D}\int_{-T}^{t} G(\boldsymbol{x}-\boldsymbol{y}, t-\tau)\Phi(\boldsymbol{y},\tau)\,\mathrm{d}\tau\mathrm{d}s(\boldsymbol{y}) \\ &= \int_{\partial D}\int_{0}^{t+T} G(\boldsymbol{x}-\boldsymbol{y}, t+T-\tau)\Phi(\boldsymbol{y},\tau-T)\,\mathrm{d}\tau\mathrm{d}s(\boldsymbol{y}) \\ &= \int_{\partial D}\int_{0}^{t_{max}+T} G(\boldsymbol{x}-\boldsymbol{y}, t+T-\tau)\Phi(\boldsymbol{y},\tau-T)\,\mathrm{d}\tau\mathrm{d}s(\boldsymbol{y}) \end{aligned} \tag{3.1}$$

where $G(\boldsymbol{x}-\boldsymbol{y}, t-\tau)$ is the fundamental solution of heat equation with

$$G(\boldsymbol{x}-\boldsymbol{y}, t-\tau) = \frac{1}{(4\pi(t-\tau))^{\frac{d}{2}}}\mathrm{e}^{-|\boldsymbol{x}-\boldsymbol{y}|^2/4(t-\tau)}H(t-\tau),$$

where $H(t-\tau)=1$ if $t\geq\tau$ and $H(t-\tau)=0$ if $t<\tau$.

Choose source points $(\boldsymbol{y}_j,\tau_j)$, $j=1,2,\ldots,n_s$ on $\partial D\times[0,t_{max}+T)$, then from (3.1) we can obtain an approximate solution given by a linear combination of fundamental solutions as

$$v(\boldsymbol{x},t)\approx\sum_{j=1}^{n_s}\lambda_j G(\boldsymbol{x}-\boldsymbol{y}_j,t+T-\tau_j). \tag{3.2}$$

We choose collocation points $(x_i,t_i)\in\partial\Omega\times[0,t_{max}]$ for $i=1,\ldots,n_1$, $(x_i,t_i)\in\Gamma\times[0,t_{max}]$ for $i=n_1,\ldots,n_1+n_2$. Let $n_c=n_1+n_2$ and let $n_c>n_s$ in computations.

Since function (3.2) satisfies the heat equation, we impose conditions (2.7)–(2.8) at the collocation points. Then the unknown coefficients λ_j satisfy the following linear system of equations:

$$A\lambda=b \tag{3.3}$$

where A is a $n_c\times n_s$ matrix:

$$\begin{pmatrix}\dfrac{\partial G}{\partial\boldsymbol{n}}(\boldsymbol{x}_i-\boldsymbol{y}_j,t_i+T-\tau_j)\\ \int_0^{t_k}\lambda(t-s)G(\boldsymbol{x}_k-\boldsymbol{y}_j,s+T-\tau_j)\,\mathrm{d}s\end{pmatrix} \tag{3.4}$$

and b is a n_c vector:

$$\begin{pmatrix}\mathbf{0}\\ \tilde{g}(\boldsymbol{x}_k,t_k)\end{pmatrix} \tag{3.5}$$

where $i=1,\ldots,n_1,k=n_1+1,\ldots,n_1+m_2,j=1,\ldots,n_s$.

Since the original inverse heat source problem is ill-posed, the ill-conditioning of the matrix A in equation (3.3) still persists. In other words, most standard numerical methods cannot achieve good accuracy in solving the matrix equation (3.3) due to the bad condition number of the matrix A. In fact, the condition number of matrix A increases dramatically with respect to the total number of collocation points. Several regularization methods have been developed for solving these kinds of ill-conditional problems Hansen (1998). In our computation we adapt the Tikhonov regularization Engl and Hanke (1996) to solve the matrix equation (3.3). The Tikhonov regularized solution λ_α for equation (3.3) is defined as the solution of the following least squares problem:

$$\min_{\lambda}\{\|A\lambda-b\|^2+\alpha^2\|\lambda\|^2\}, \tag{3.6}$$

where $\|\cdot\|$ denotes the Euclidean norm and $\alpha>0$ is called the regularization parameter. The choice of a suitable value of the regularization parameter α is crucial for the accuracy of the final numerical solution and is still under intensive research Tautenhahn and Hämarik (1999). For the TR method, several heuristical approaches have been proposed, including the L-curve criterion Hansen (1998), cross-validation (CV), and generalized cross validation (GCV) Golub, Heath, and Wahba (1979). In this paper, we use the GCV to choose the regularization parameter. In GCV the regularization parameter α is chosen to minimize the GCV function:

$$G(\alpha) = \frac{\|A\lambda_\alpha - b\|^2}{(trace(I_{n+m} - AA^I))^2}, \quad \alpha > 0, \tag{3.7}$$

where $A^I = (A^{tr}A + \alpha I_{n+m})^{-1}A^{tr}$
In our computation, we used the Matlab code developed by Hansen Hansen (1992) for solving the discrete ill-conditioned system of equations (3.3). Denote the regularized solution of equation (3.3) by λ^{α^*}, where α^* is the positive minimizer of (3.7). The approximated solution v_α^* for the problem (2.6)–(2.8) is then given as

$$v_\alpha^*(\boldsymbol{x},t) = \sum_{j=1}^{n_s} \lambda_j^{\alpha^*} G(\boldsymbol{x} - \boldsymbol{y}_j, t + T - \tau_j), \tag{3.8}$$

The solution of problem (2.1)–(2.3) is then given by

$$u^*(\boldsymbol{x},t) = \int_0^t \lambda(t-s) v_\alpha^*(\boldsymbol{x},s)\,\mathrm{d}s \tag{3.9}$$

and

$$f^*(\boldsymbol{x}) = \sum_{j=1}^{n_s} \lambda_j^{\alpha^*} G(\boldsymbol{x} - \boldsymbol{y}_j, 0 + T - \tau_j), \tag{3.10}$$

4 Numerical experiments

For simplicity, we set $t_{max} = 1$ in all the following examples. We use the function rand given in Matlab to generate the noisy data $\tilde{g}_i = g_i \times (1 + 2\delta(rand(i) - 0.5))$, where g_i is the exact data and $rand(i)$ denotes a random number from the uniform distribution of interval $(0,1)$. The magnitude δ indicates the noise level of measurement data.

To test the accuracy of the approximate solution, we use the root mean square error

(*RMS*) and the relative root mean square error (*RES*) defined as

$$RMS(f) = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(f(\boldsymbol{x}_i) - f^*(\boldsymbol{x}_i))^2}, \tag{4.1}$$

$$RES(f) = \frac{\sqrt{\sum_{i=1}^{N_t}(f(\boldsymbol{x}_i) - f^*(\boldsymbol{x}_i))^2}}{\sqrt{\sum_{i=1}^{N_t}(f(\boldsymbol{x}_i))^2}}, \tag{4.2}$$

where N_t is the total number of testing points in the domain $[0,1]\times[0,t_{max}]$, $f(x_i), f^*(x_i)$ are, respectively, the exact and approximated value at these points. The *RMS* and *RES* for the heat temperature $RMS(u)$ and $RES(u)$ are also similarly defined.

4.1 One-dimensional examples

We fix $T=3, \Omega=(0,1), D=(-1,2)$ and take Γ is $x=0$ unless otherwise specified. The numbers of source points on $x=-1\times[0,t_{max}+T)$ and $x=2\times[0,t_{max}+T)$ are both 30, the numbers of collocation points on $x=0\times[0,t_{max}]$,$x=1\times[0,t_{max}]$ and $\Gamma\times[0,t_{max}]$ are both 20 unless otherwise specified. All points on each line are uniformly distributed.

Example 1. The exact solution of problem (2.1)–(2.4) is given by

$$u(x,t) = (\mathrm{e}^{-\pi^2 t} - \mathrm{e}^{-t})\cos\pi x, \quad (x,t)\in[0,1]\times[0,1], \tag{4.3}$$

$$f(x) = (1-\pi^2)\cos\pi x, \quad x\in[0,1], \tag{4.4}$$

$$\lambda(t) = e^{-t}. \tag{4.5}$$

Figure 1 presents the GCV function $G(\alpha)$ obtained for the inverse heat source problem using the Tikhonov regularization method to solve the MFS systems of equations (3.3). The numerical results with various levels of noise δ for Example 1 are shown in Figure 2. From this figure, the numerical results are quite satisfactory, even with the noise level up to $\delta=0.05$. Furthermore, by comparing Figure 1 and 2, we can see that the choice of the regularization parameter α^* according to the GCV is fully justified.

In order to investigate the influence of the parameter T on the accuracy and stability of the numerical solutions for the temperature and the heat source, we consider Example 1 with noise data ($\delta=1\%$). In Figure 3(a) we present the errors *RMS* and *RES* for Example 1 as functions of the parameter T. It is noted that our proposed method is stable to the parameter T which is useful in the MFS.

To verify the relationship between the accuracy of solution and ds where ds is a parameter in $D=(-ds,1+ds)$ representing the distance of the source points from

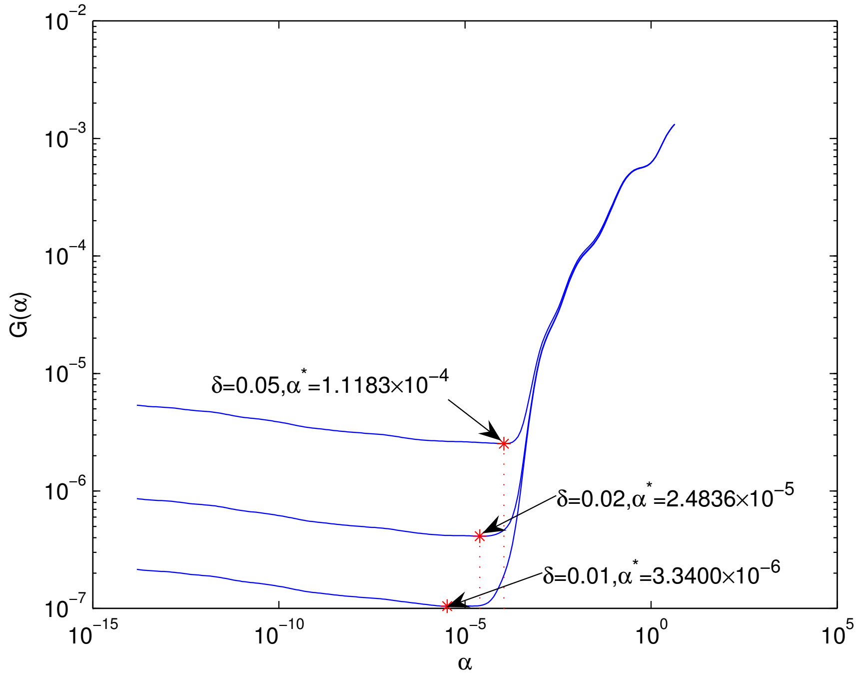


Figure 1: The GCV functions for various noise levels for Example 1.

$\partial\Omega$, we compute the error RMS and RES as functions of ds with a fixed noise level $\delta = 1\%$, and the numerical results are shown in Figure 3(b). We can see that the numerical errors keep a stable level when the ds is in a certain range.

Example 2. The exact solution of problem (2.1)–(2.4) is given by

$$u(x,t) = t\cos \pi x, \quad (x,t) \in [0,1] \times [0,1], \tag{4.6}$$

$$f(x) = \cos \pi x, \quad x \in [0,1], \tag{4.7}$$

$$\lambda(t) = 1 + \pi^2 t. \tag{4.8}$$

The source function $f(x)$ and the approximation $f^*(x)$ are displayed in Figure 4. From this figure, we can see that the numerical approximations are good agreement with the exact solution. The stability of the numerical solution with the parameter T and ds is studied in Figure 5. The insensitivity of the solutions to T and ds over fairly large ranges of the parameters is a favorable feature of MFS because there is no need to search for optimal values of parameters.

Example 3. To further explore the applicability of the proposed method for solving

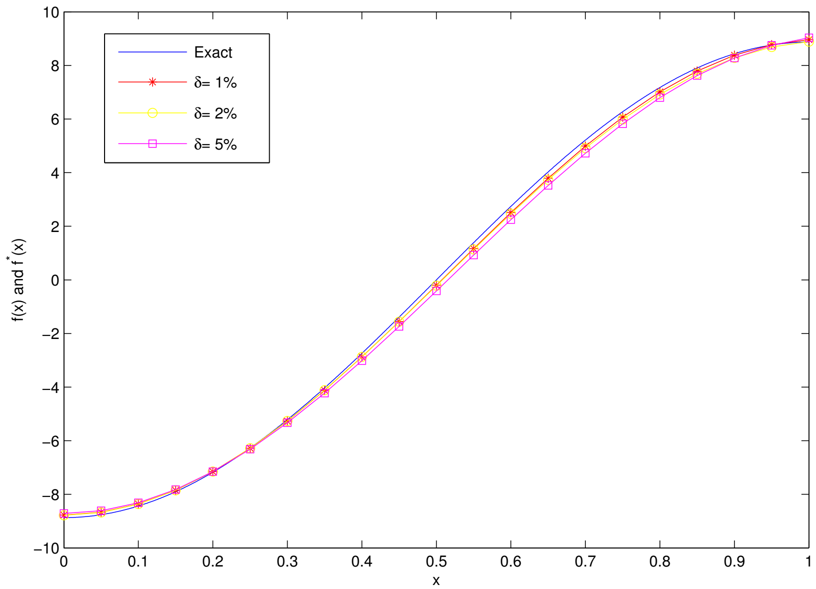


Figure 2: The numerical results with various noise levels for Example 1.

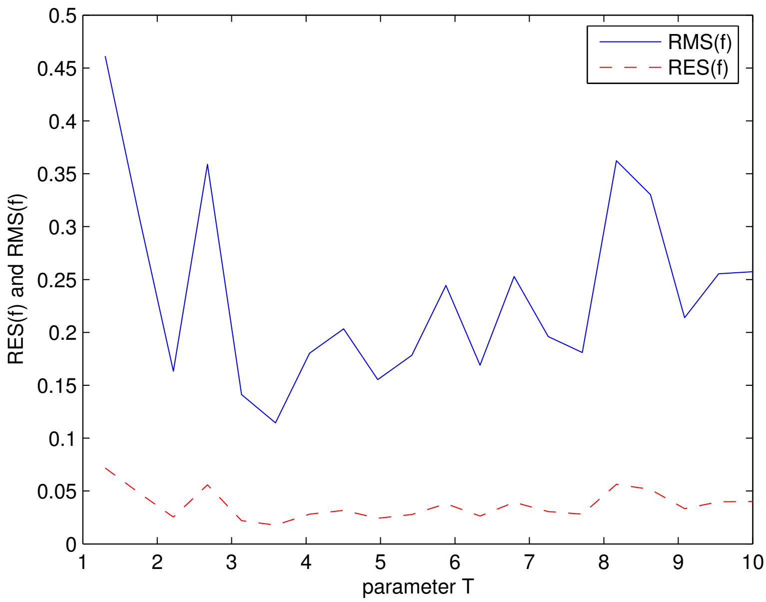
the inverse heat source problem, we examine reconstruction of a Gaussian normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}, \tag{4.9}$$

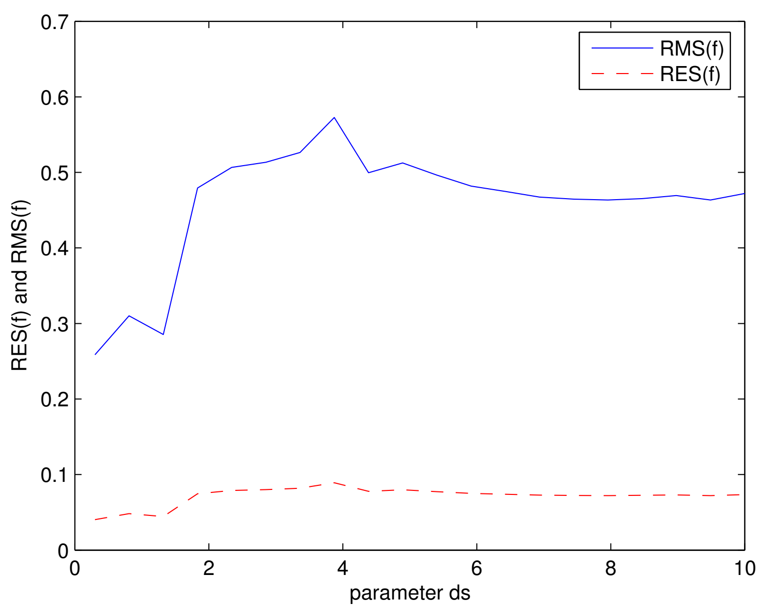
where $\mu = 0.5$ is the mean and $\sigma = 0.2$ is the standard deviation. Since the direct problem given by equations (2.1)–(2.3) with f given by (4.9) does not have an analytical solution the data (2.4) is obtained by solving the direct problem using the Crank–Nicholson (CN) difference scheme:

$$u_t = u_{xx} + \mathrm{e}^{-t} f(x), \quad (x,t) \in (0,1) \times (0,1],$$
$$u(x,0) = 0, \quad x \in [0,1],$$
$$u_x(0,t) = 0, \quad t \in (0,1],$$
$$u_x(1,t) = 0, \quad t \in (0,1].$$

The numerical results obtained for Example 3 using various amounts of noise added into the data are presented in Figure 6. From this figure it can be seen that the



(a) RES and RMS versus T



(b) RES and RMS versus ds

Figure 3: The errors of numerical solutions for Example 1 with $\delta = 1\%$ with respect to the parameter T and ds.

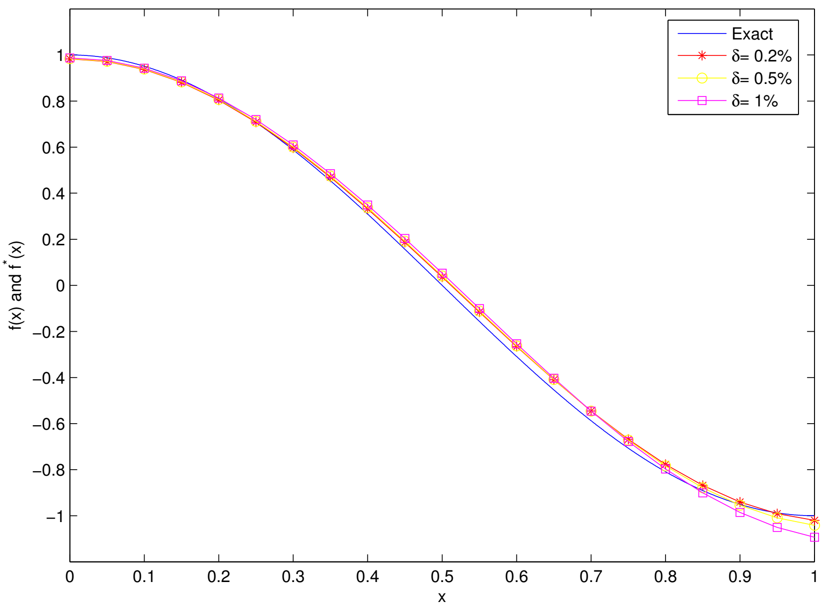


Figure 4: The numerical results with various noise levels for Example 2.

numerical approximation is not as good as in the previous examples, but it is in reasonable agreement with (4.9)

4.2 Two-dimensional examples

In the following two examples, we set $\Omega = \{(x_1,x_2)|0 < x_1 < 1, 0 < x_2 < 1,\}, \Gamma = \{(x_1,x_2)|0 < x_1 < 1, x_2 = 0\}$. We fix $T = 2.4, ds = 1, n_c = 825, n_s = 800$ if no other specification.

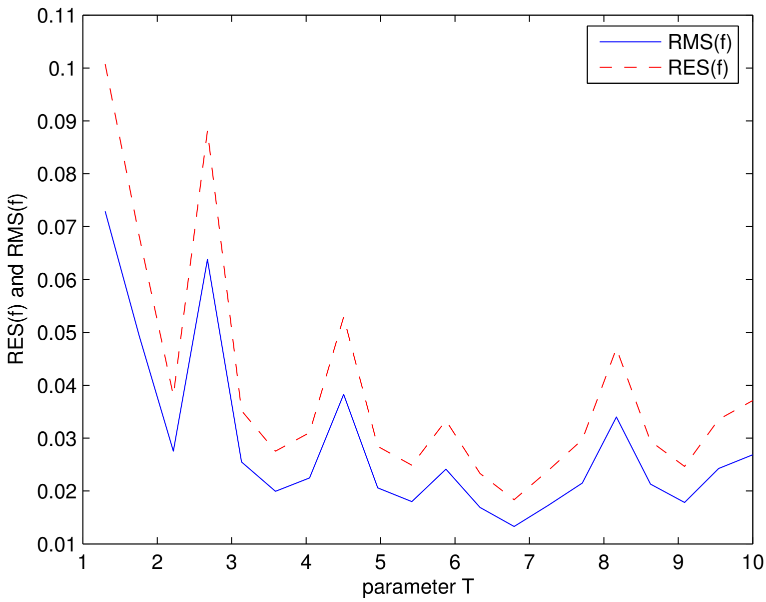
Example 4. The exact solution of problem (2.1)–(2.4) is given by

$$u(\boldsymbol{x},t) = t(\cos\pi x_1 + \cos\pi x_2), \quad (x,t) \in \Omega \times [0,1], \tag{4.10}$$

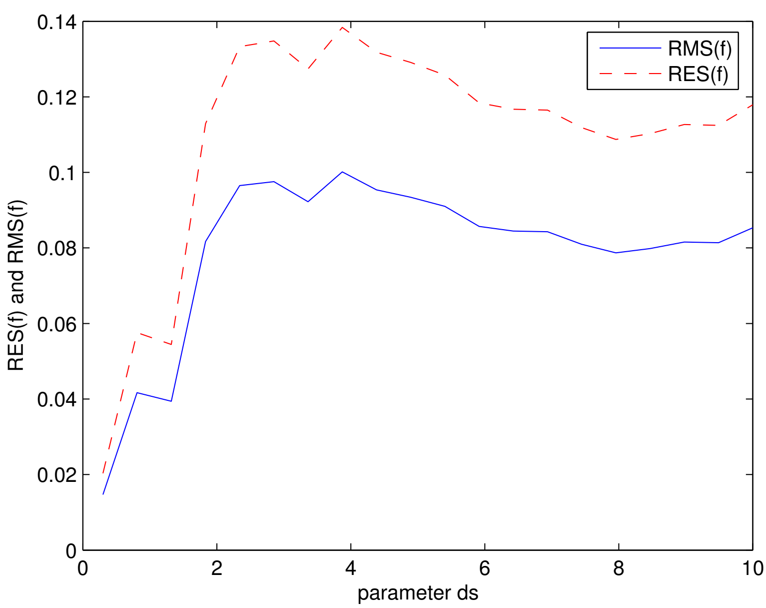
$$f(\boldsymbol{x}) = \cos\pi x_1 + \cos\pi x_2, \quad x \in \Omega, \tag{4.11}$$

$$\lambda(t) = 1 + \pi^2 t. \tag{4.12}$$

Example 5. We consider an example where there is no analytical solution available. To obtain the data (2.4), we first solve the following direct problem by using the



(a) RES and RMS versus T



(b) RES and RMS versus ds

Figure 5: The errors of numerical solutions for Example 2 with $\delta = 1\%$ with respect to the parameter T and ds.

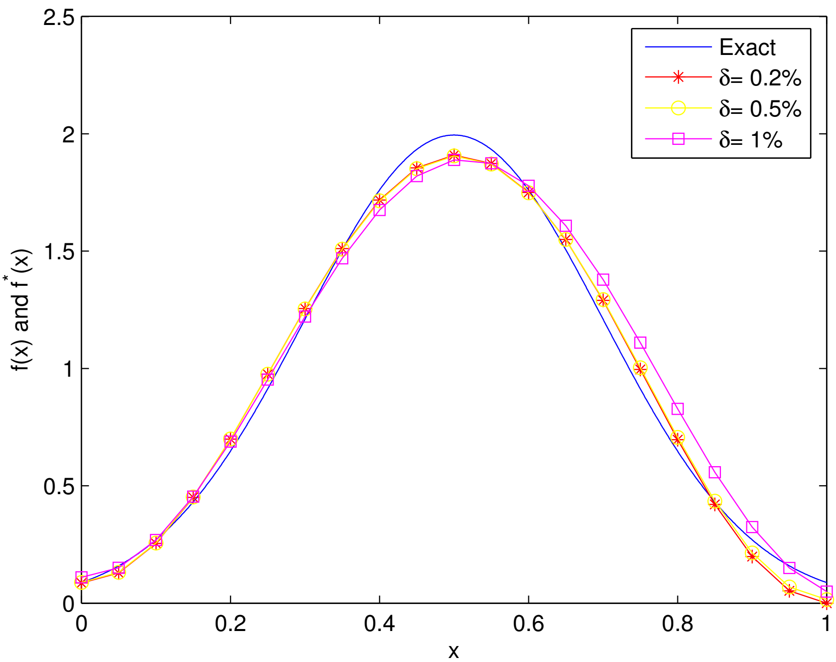


Figure 6: The numerical results with various noise levels for Example 3.

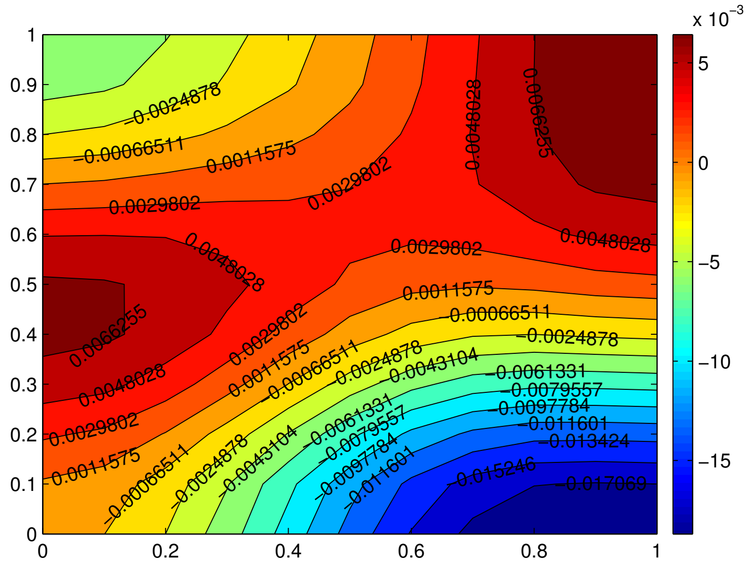
alternating direction implicit method(ADI) Morton and Mayers (2005):

$$u_t = \triangle u + \mathrm{e}^{-t} f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, t \in (0,1],$$
$$u(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \bar{\Omega},$$
$$\frac{\partial u}{\partial n}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, t \in (0,1],$$

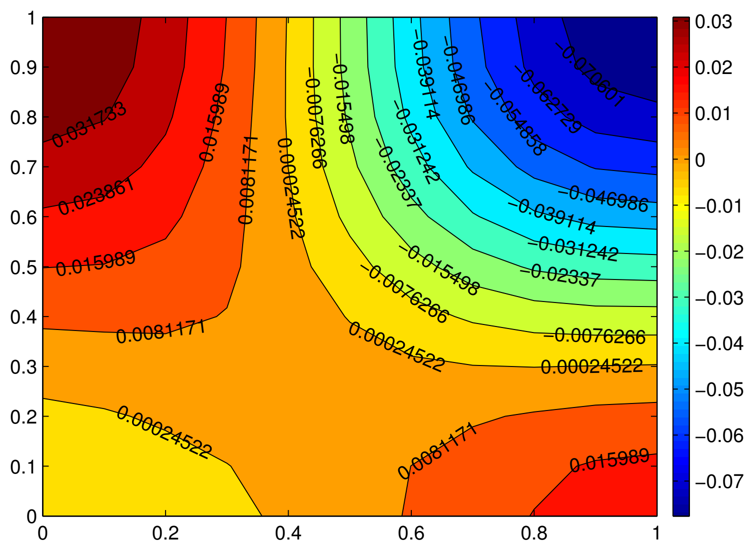
where

$$f(\boldsymbol{x}) = \ln(x_1 + x_2 + 2), \quad \boldsymbol{x} \in \Omega.$$

The error distribution for the numerical heat sources of Example 4 by using $\delta = 1\%, 5\%$, are presented in Figure 7. It can be seen from these figures that the numerical results retrieved for the heat source represent good approximations for their analytical values. Furthermore, the numerical heat sources converge towards their corresponding exact solutions as the amount of noise decreases. Similar results

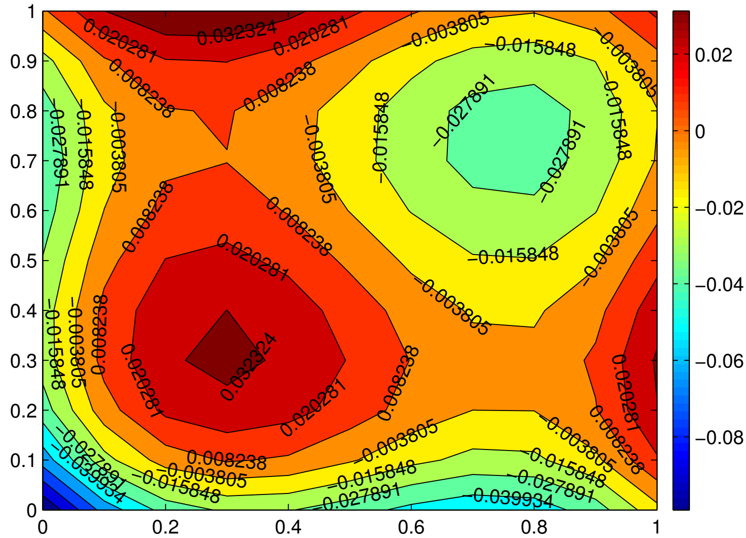


(a)

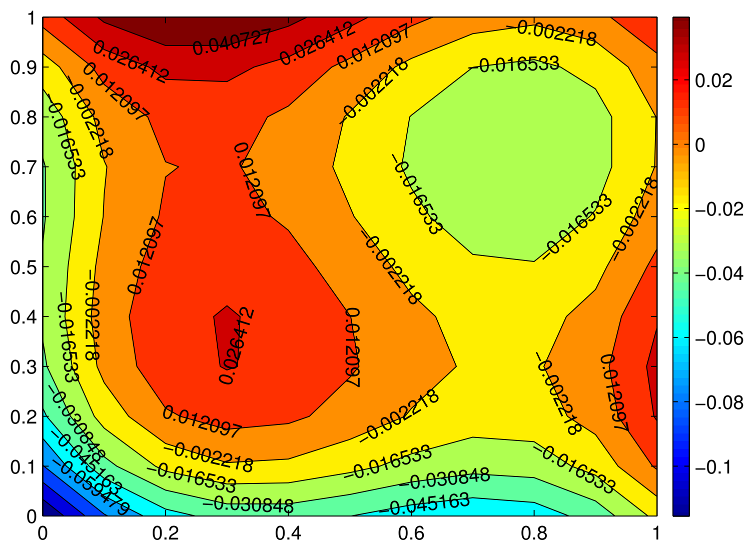


(b)

Figure 7: The error distribution for $f(\boldsymbol{x})$ with various noise levels, namely: (a)1%; and (b)5%, for Example 4.



(a)



(b)

Figure 8: The error distribution for $f(\boldsymbol{x})$ with various noise levels, namely: (a)1%; and (b)3%, for Example 5.

have been obtained for Example 5 are shown in Figure 8. Hence the MFS, in conjunction with the TR method, provides stable numerical solutions to the 2-D inverse source problem .

5 Conclusion

In this paper, we have implemented the MFS to solve a nonhomogeneous heat source problem on the Tikhonov regularization method with the GCV criterion. We successfully transform the inverse source problem into an inverse initial data problem and then applied the MFS technique to the resulted problem. The numerical results show that the MFS is an accurate and reliable numerical technique for the solution of the inverse heat source problem. The proposed scheme can be adapted to higher dimensional problems with complication domains.

References

Cannon, J. R.; DuChateau, P. (1998): Structural identification of an unknown source term in a heat equation. *Inverse Problems*, vol. 14, no. 3, pp. 535–551.

Choulli, M.; Yamamoto, M. (1997): An inverse parabolic problem with non-zero initial condition. *Inverse Problems*, vol. 13, no. 1, pp. 19–27.

Coles, C.; Murio, D. (2001): Simultaneous space diffusivity and source term reconstruction in 2D IHCP. *Computers & Mathematics with Applications*, vol. 42, no. 12, pp. 1549–1564.

Engl, H.; Hanke, A. (1996): *Regularization of inverse problems*. Kluwer Academic Publishers.

Fairweather, G.; Karageorghis, A. (1998): The method of fundamental solutions for elliptic boundary value problems. *Advances in Computational Mathematics*, vol. 9, pp. 69–95.

Farcas, A.; Lesnic, D. (2006): The boundary-element method for the determination of a heat source dependent on one variable. *Journal of Engineering Mathematics*, vol. 54, no. 4, pp. 375–388.

Fatullayev, A. G. (2004): Numerical solution of the inverse problem of determining an unknown source term in a two-dimensional heat equation. *Applied Mathematics and Computation*, vol. 152, no. 3, pp. 659–666.

Golberg, M.; Chen, C. (1999): The method of fundamental solutions for potential, Hemholtz and diffusion problems. *Boundary integral methods: numerical and mathematical aspects.Computational engineering, Boston, MA: WIT Press, Computational Mechanics Publications;*, vol. 1, no. 1, pp. 103–176.

Golub, G.; Heath, M.; Wahba, G. (1979): Generalized cross-validation as a method for choosing a good ridge parameter. *Technometrics*, vol. 21, no. 2, pp. 215–223.

Hansen, P. C. (1992): Regularization tools: a Matlab package for analysis and solution of discrete ill-posed problems. *Numer. Algorithms*, vol. 6, pp. 1–35.

Hansen, P. C. (1998): Rank-deficient and discrete ill-posed problems. *Philadelphia: SIAM*.

Hon, Y.; Wei, T. (2004): A fundamental solution method for inverse heat conduction problem. *Engineering Analysis with Boundary Elements*, vol. 28, no. 5, pp. 489–495.

Hon, Y.; Wei, T. (2005): The method of fundamental solution for solving multidimensional inverse heat conduction problem. *CMES:Computer Modeling in Engineering & Sciences*, vol. 7, pp. 119–132.

Johansson, B. T.; Lesnic, D. (2007): A variational method for identifying a spacewise-dependent heat source. *IMA Journal of Applied Mathematics*, vol. 72, no. 6, pp. 748–760.

Johansson, B. T.; Lesnic, D. (2008): A procedure for determining a spacewise dependent heat source and the initial temperature. *Applicable Analysis*, vol. 87, no. 3, pp. 265–276.

Johansson, T.; Lesnic, D. (2007): Determination of a spacewise dependent heat source. *Journal of Computational and Applied Mathematics*, vol. 209, no. 1, pp. 66–80.

Kress, R.; Mohsen, A. (1986): On the Simulation Source Technique for Exterior Problems in Acoustics. *Math. Meth.in the Appl. Sci.*, vol. 8, pp. 585–597.

Kupradze, V.; Aleksidze, M. (1964): The method of functional equations for the approximate solution of certain boundary value problems. *USSR Computational Mathematics and Mathematical Physics*, vol. 4, no. 4, pp. 82 – 126.

Marin, L.; Lesnic, D. (2005): The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations. *Computers & Structures*, vol. 83, no. 4-5, pp. 267–278.

Morton, K.; Mayers, D. (2005): *Numerical Solution of Partial Differential Equations(2nd)*. Cambridge University Press.

Nanda, A.; Das, P. C. (1996): Determination of the source term in the heat conduction equation. *Inverse Problems*, vol. 12, pp. 325–339.

Noon, P. (1988): The single layer heat potential and Galerkin boundary element methods for the heat equation. *Doctor of philosophy.University of Maryland, the Faculty of the Graduate School of The University of Maryland*.

Tadi, M. (1997): Inverse heat conduction based on boundary measurement. *Inverse Problems*, vol. 13, pp. 1585–1605.

Tautenhahn, U.; Hämarik, U. (1999): The use of monotonicity for choosing the regularization parameter in ill-posed problems. *Inverse Problems*, vol. 15, pp. 1487–1505.

Wang, Z.; Liu, J. J. (2008): Identification of the pollution source from one-dimensional parabolic equation models. *Applied Mathematics and Computation*.

Wei, T.; Wang, J. C. (2012): Simultaneous determination for a space-dependent heat source and the initial data by the MFS. *Engineering Analysis with Boundary Elements*, vol. 36, no. 12, pp. 1848–1855.

Yamamoto, M. (1993): Conditional stability in determination of force terms of heat equations in a rectangle. *Mathematical and Computer Modelling*, vol. 18, no. 1, pp. 79–88.

Yan, L.; Fu, C. L.; Yang, F. L. (2008): The method of fundamental solutions for the inverse heat source problem. *Engineering Analysis with Boundary Elements*, vol. 32, no. 3, pp. 216–222.

Yan, L.; Yang, F. L.; Fu, C. L. (2009): A meshless method for solving an inverse spacewise-dependent heat source problem. *Journal of Computational Physics*, vol. 228, no. 1, pp. 123–136.

Yang, C. Y. (1998): A sequential method to estimate the strength of the heat source based on symbolic computation. *International Journal of Heat and Mass Transfer*, vol. 41, no. 14, pp. 2245–2252.

Yang, C. Y. (1998): Solving the two-dimensional inverse heat source problem through the linear least- squares error method. *International Journal of Heat and Mass Transfer*, vol. 41, no. 2, pp. 393–398.

Yi, Z.; Murio, D. A. (2004): Identification of Source terms in 2-D IHCP. *Computers & Mathematics with Applications*, vol. 47, pp. 1517–1533.

Yi, Z.; Murio, D. A. (2004): Source Term Identification in 1-D IHCP. *Computers & Mathematics with Applications*, vol. 47, pp. 1921–1933.

Young, D. L.; Chen, K. H.; Chen, J. T.; Kao, J. H. (2007): A Modified Method of Fundamental Solutions with Source on the Boundary for Solving Laplace Equations with Circular and Arbitrary Domains. *CMES:Computer Modeling in Engineering & Sciences*, vol. 19, pp. 197–221.

Young, D. L.; Ruan, J. W. (2005): Method of fundamental solutions for scattering problems of electromagnetic waves. *CMES:Computer Modeling in Engineering & Sciences*, vol. 17, pp. 223–232.