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Combinations of the Boundary Knot Method with Analogy Equation Method for Nonlinear Problems

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Abstract: Based on the analogy equation method and method of particular solutions, we propose a combined boundary knot method (CBKM) for solving nonlinear problems in this paper. The principle of the CBKM lies in that the analogy equation method is used to convert the nonlinear governing equation into a corresponding linear inhomogeneous one under the same boundary conditions. Then the method of particular solutions and boundary knot method are, respectively, used to construct the particular and homogeneous solutions for the newly-introduced inhomogeneous equation. Finally, the field function and its derivatives involved in the nonlinear governing equation are expressed via the unknown coefficients, which are established by collocating the equations at discrete knots on the physical domain. A classical nonlinear problem, among numerous examples, is chosen to validate the convergence, stability and accuracy of the proposed method.

Keywords: Boundary knot method, analogy equation method, nonlinear, radial basis function.

1 Introduction

Many realistic problems encountered in engineering practice always perform the nonlinear character [Liu, Hong and Atluri (2010);Liu and Kuo (2011);Liu and Atluri (2011)]. Thus, numerical methods are inevitably introduced to solve these nonlinear problems, such as the finite difference method, finite element method and boundary element methods [Liu, Zhang, Li, Lam and Kee (2006);Partridge, Brebbia and Wrobel (1992)]. However, the boundary element method has a major obstacle, just like the finite element method, surface mesh or re-mesh requires expensive computation, especially for moving boundary and nonlinear problems. The

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boundary-type meshless methods are regarded as alternative techniques to alleviate these drawbacks, take the method of fundamental solutions (MFS) [Fairweather and Karageorghis (1998);Young, Jane, Fan, Murugesan and Tsai (1998);Chen, Karageorghis and Smyrlis (2008);Liu (2008); Chen, Lee, Yu and Shieh (2009);Lin, Chen and Wang (2011)] and boundary knot method (BKM) [Chen and Tanaka (2002);Jin and Zheng (2005);Wang, Chen and Jiang (2010)] for example.

The MFS is an attractive method since it is integration-free, truly meshless, and easy-to-use. By using the MFS, Wang and his coworkers [Wang, Qin and Kang (2006)] proposed a meshless method which combines the analogy equation method (AEM) and radial basis functions to solve nonlinear Poisson-type problems. Recently, Li and Zhu [Li and Zhu (2009)] investigated nonlinear elliptic problems by using the similar procedure described in [Wang, Qin and Kang (2006)]. It is stated that the numerical solution is not sensitive to the locations of source points. However, their numerical results of the MFS are just confined to regular-shaped boundary problems. It still has difficult in dealing with complex-shaped boundary problems due to the requirement of fictitious boundary [Wang, Chen and Jiang (2010);Lin, Chen and Wang (2011);Chen and Wang (2010);Chen, Lin and Wang (2011)].

To circumvent the fictitious boundary, Kang et al. [Kang, Lee and Kang (1999)] proposed a new method, named as BKM [Chen and Tanaka (2002)], which uses the non-singular general solutions. In the BKM, the collocation and source points are simultaneously placed on the physical boundary of the problem. Particularly, the BKM is found to produce high accurate solution to complex-shaped boundary problems.

Based on the above-mentioned work, we propose the CBKM which is composed of the AEM, the method of particular solutions (MPS) and the BKM. The structure of the paper is as follows. In Section 2, we briefly describe the AEM, the MPS and the BKM. In Section 3, we examine a classical nonlinear boundary value problem to demonstrate the convergence, stability and accuracy of the current method. Followed by Section 4, we make some concluding remarks.

2 Formulation of AEM in combination with BKM

Consider a non-homogeneous body occupying the two-dimensional domain Ω, whose dynamic response is governed by the following boundary value problem

$$\mathscr{N}(u) = f(X), \qquad X \in \Omega, \tag{1}$$

$$B_1 u + B_2 q = g, \qquad X \in \Gamma, \tag{2}$$

where $u = u(X)$ is the unknown field function, $X = (x,y) \in \Omega$, $q = q(X) = \partial u(X)/\partial \bar{n}$ and

$$\mathscr{N}(u) = \mathscr{N}(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \tag{3}$$

represents a nonlinear second order differential operator defined in Ω. Moreover, $B_i = B_i(X), i = 1,2$ and $g = g(X)$ are functions specified on the physical boundary Γ, $f(X)$ the forcing function.

2.1 The analogy equation method

The AEM is first proposed by Katsikadelis to solve nonlinear problems [Katsikadelis (1994)]. After that, some applications have been done in many other areas, see other works done by Katsikadelis [Katsikadelis and Nerantzaki (1999);Chinnaboon, Katsikadelis and Chucheepsakul (2007);Katsikadelis (2002); Katsikadelis and Tsiatas (2003);Nerantzaki and Katsikadelis (2003);Tsiatas and Katsikadelis (2006)] and references therein. It is well known that the fundamental solution of the Laplace equation has the singular property. For this reason, the nonsingular general solutions of the Helmholtz equation is used as an alternative [Wang, Ling and Chen (2009)]. Here, the AEM is established following a procedure similar to that presented in [Katsikadelis (1994)]. Let $u = u(X)$ be the sought solution to the problem (1)-(2). This function is two times continuously differentiable in Ω. Thus, if the Helmholtz operator $\Delta + I = \partial^2/\partial x^2 + \partial^2/\partial y^2 + I$ is applied to it, we have

$$\Delta u + u = b(X), \tag{4}$$

where I is the identity operator.

Eq. (4) indicates that the solution of Eq. (1) can be established by solving this equation under the boundary condition (2), if the source density function $b(X)$ were known. The establishment of this unknown source density function is one of the essential ingredients of AEM [Katsikadelis (1994)]. By using the Dual Reciprocity Method [Partridge, Brebbia and Wrobel (1992)], we can approximate b with a finite series of basis functions

$$b \simeq \sum_{j=1}^{N+M} \alpha_j \varphi_j, \tag{5}$$

where N and M are boundary knot number and inner knot number, respectively, $\varphi_j = \varphi_j(X)$ denote a set of approximate basis functions and α_j the coefficients to be determined.

2.2 The method of particular solutions

Using the method of particular solutions [Tsai, Chen and Hsu (2009)], we can split the solution of Eq. (4) together with boundary condition (2) into a homogeneous solution $u_h = u_h(X)$ and a particular solution $u_p = u_p(X)$ of the nonhomogeneous equation, that is,

$$u = u_h + u_p, \tag{6}$$

where u_p is a particular solution satisfying the nonhomogeneous equation

$$\triangle u_p + u_p = \sum_{j=1}^{N+M} \alpha_j \varphi_j, \tag{7}$$

which yields

$$u_p = \sum_{j=1}^{N+M} \alpha_j \Psi_j, \tag{8}$$

where $\Psi_j (j = 1, 2, ..., M)$ are particular solutions of the equation

$$\triangle \Psi_j + \Psi_j = \varphi_j, \qquad j = 1, 2, ..., N+M. \tag{9}$$

The approximate particular solution Ψ_j is always determined beforehand, and then we evaluate the corresponding RBF φ_j by a simple differentiation process, The particular solution of Eq. (9) can always be determined, if φ_j is specified.

In this study, the chosen approximate particular solution Ψ_j is [Power and Barraco (2002)]

$$\Psi_j = (r_j^2 + c^2)^{3/2}, \tag{10}$$

where c is the Multiquadrics (MQ) shape parameter and $r_j = \| X - X_j \|$ the Euclidean norm distance. The corresponding RBF φ_j is

$$\varphi_j = 6(r_j^2 + c^2)^{1/2} + 3r_j^2 / \sqrt{(r_j^2 + c^2)} + (r_j^2 + c^2)^{3/2}. \tag{11}$$

The homogeneous solution u_h is obtained from the boundary value problem

$$\begin{aligned} \triangle u_h + u_h &= 0, & X \in \Omega, \quad (12) \\ B_1 u_h + B_2 q_h &= g - (B_1 \sum_{j=1}^{N+M} \alpha_j \hat{u}_j + B_2 \sum_{j=1}^{N+M} \alpha_j \hat{q}_j), & X \in \Gamma, \quad (13) \end{aligned}$$

where $q_h = \partial u_h / \partial \bar{n}$ and $\hat{q}_j = \partial \hat{u}_j / \partial \bar{n}$.

The boundary value problem (12)-(13) is solved by using the BKM, which is briefly illustrated in the following part.

2.3 The boundary knot method

For the homogeneous Helmholtz equation (12), the non-singular general solution is given by

$$u_d^*(r) = \left(\frac{\lambda}{2\pi r}\right)^{(d/2)-1} J_{(d/2)-1}(\lambda r), \;\; d \geq 2, \tag{14}$$

where λ is the wave number, d the dimensionality of the problem considered, J represents the Bessel function of the first kind, I denotes the modified Bessel functions of the first kind, and r means the Euclidean norm distance.

Using the non-singular general solution (14) as the interpolation basis function, we can approximate the solution of equation (12) by

$$u_h(X) = \sum_{j=1}^{N} \beta_j u_d^*(\|X - X_j\|), \tag{15}$$

where j represents the index of source knots $\{X_1, X_2, \ldots, X_N\}$ on physical boundary Γ, N denotes the total number of boundary knots and β_j $(j = 1, \ldots, N)$ are the unknown expansion coefficients. For more details, we refer readers to [Wang (2011);Wang, Chen and Jiang (2010)].

2.4 Numerical implementation

On the basis of Eqs. (6), (8) and (15), the solution of Eq. (4) can be illustrated as

$$u(X) = \sum_{j=1}^{N+M} \alpha_j \Psi_j(X) + \sum_{j=1}^{N} \beta_j u_d^*(\|X - X_j\|). \tag{16}$$

According to the AEM, the approximate solution for boundary value problem (1)-(2) is expressed as the form in Eq. (16).

Differentiation Eq. (16) yields

$$
\begin{aligned}
q(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,n}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,n}(\|X - X_j\|), && (17)\\
u_{,x}(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,x}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,x}(\|X - X_j\|), && (18)\\
u_{,y}(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,y}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,y}(\|X - X_j\|), && (19)\\
u_{,xx}(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,xx}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,xx}(\|X - X_j\|), && (20)\\
u_{,xy}(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,xy}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,xy}(\|X - X_j\|), && (21)\\
u_{,yy}(X) &= \sum_{j=1}^{N+M} \alpha_j \Psi_{j,yy}(X) + \sum_{j=1}^{N} \beta_j u^*_{d,yy}(\|X - X_j\|), && (22)
\end{aligned}
$$

where $\Psi_{j,n} = \partial \hat{u}_j / \partial \bar{n}$, $u^*_{d,n} = \partial u^*_d / \partial \bar{n}$.

Substituting Eqs. (16), (18) – (22) into Eq. (1) and collocating at $N+M$ collocation knots, then forcing the boundary conditions Eq. (2) at N boundary knots, we can obtain $M+2N$ equations to determine the unknowns α_j and β_j. Note that the boundary conditions are linear, thus there have N linear equations. If the $M+2N$ simultaneous equations are solved directly, a considerable computational time may be wasted. In the following, we use an indirect approach to save computational time.

Applying Eq. (2) to N boundary knots $X_i (i = 1, 2, ..., N)$, we have the matrix form

$$
([B_1][K] + [B_2][K_n])\{\alpha\} + ([B_1][H] + [B_2][H_n])\{\beta\} = \{g\}, \tag{23}
$$

where $B_1 = B_1(X_i)$ and $B_2 = B_2(X_i)$ are $N \times N$ diagonal matrices, $[K] = \Psi_j(X_i)$ and $[K_n] = \Psi_{j,n}(X_i)$ are $N \times (N+M)$ matrices, $[H] = u^*_d(\|X_i - X_j\|)$ and $[H_n] = u^*_d(\|X_i - X_j\|)$ are $N \times N$ matrices, the vector $\{\alpha\}$ and $\{\beta\}$ are unknown vectors.

Solving Eq. (23), we get

$$
\{\beta\} = ([B_1][H] + [B_2][H_n])^{-1}(\{g\} - ([B_1][K] + [B_2][K_n])\{\alpha\}), \tag{24}
$$

then applying Eq. (16) to M collocation knots yields

$$
\{u\} = [\bar{K}]\{\alpha\} + [\bar{H}]\{\beta\} \triangleq [T]\{\alpha\} + \{f\}, \tag{25}
$$

where $[\bar{K}]$ and $[\bar{H}]$ are $(N+M)\times(N+M)$ and $(N+M)\times N$ known matrix, respectively. The matrix $[T]=[\bar{K}]-[\bar{H}]([B_1][H]+[B_2][H_n])^{-1}([B_1][K]+[B_2][K_n])$ and $\{f\}=[\bar{H}]([B_1][H]+[B_2][H_n])^{-1}\{g\}$.
Similarly, applying Eq. (18)-(22) to M collocation knots, we have

$$\{u_{,x}\} = [T_{,x}]\{\alpha\}+\{f_{,x}\}, \tag{26}$$

$$\{u_{,y}\} = [T_{,y}]\{\alpha\}+\{f_{,y}\}, \tag{27}$$

$$\{u_{,xx}\} = [T_{,xx}]\{\alpha\}+\{f_{,xx}\}, \tag{28}$$

$$\{u_{,xy}\} = [T_{,xy}]\{\alpha\}+\{f_{,xy}\}, \tag{29}$$

$$\{u_{,yy}\} = [T_{,yy}]\{\alpha\}+\{f_{,yy}\}, \tag{30}$$

where $[T_{,x}]$, $[T_{,y}]$, $[T_{,xx}]$, $[T_{,xy}]$ and $[T_{,yy}]$ are $M\times M$ matrices, $\{f_{,x}\}$, $\{f_{,y}\}$, $\{f_{,xx}\}$, $\{f_{,xy}\}$ and $\{f_{,yy}\}$ are known vectors.
Finally, collocating Eq. (1) at collocation knots $X_i(i=1,2,...,N+M)$ yields

$$\{\mathscr{N}(\{u\},\{u_{,x}\},\{u_{,y}\},\{u_{,xx}\},\{u_{,xy}\},\{u_{,yy}\})\}=\{f\}, \tag{31}$$

then substituting Eqs. (25)-(30) into Eq. (31), one gets

$$\{\mathscr{N}(\{\alpha\})\}=\{f\}. \tag{32}$$

Eq. (32) is a system with M nonlinear equations and M unknowns $\{\alpha\}$, which can be solved using the iteration method. Then, substituting $\{\alpha\}$ into Eq. (22), the unknowns $\{\beta\}$ can be solved easily. Once the coefficients $\{\alpha\}$ and $\{\beta\}$ are calculated, the field function u and its derivatives at any point X inside the domain or on its boundary can be determined by using Eqs. (16)-(17).

3 Numerical results

Like the other traditional numerical methods, the adoption of iteration procedure in the CBKM is a key to the solution of nonlinear problems. However, the construction of efficient iterative algorithms for solving nonlinear problems is still an important task of computational mathematics [Farhat, Lacour and Rixen (1998)]. One of the most popular methods used for such problems is the Newton's method. In this study, the solution of Eq. (32) is obtained by implementing the 'fslove' subroutine of MATLAB. If the initial value α in Eq. (32) is too far from the true zero, 'fslove' may fail to converge. For this reason, the initial value guess will be investigated in the following example.

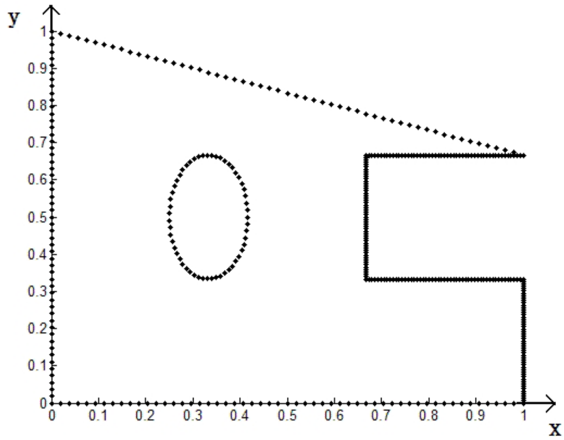

Figure 1: Configuration of 2D multiply connected domain.

Using the analytical and numerical procedure presented in the previous sections, we examine a classical boundary value problem stated as below:

$$\Delta u + u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} - k(1 + u_x^2 + u_y^2)^3 = 0, \quad (x,y) \in \Omega, \tag{33}$$

$$u = g, \quad (x,y) \in \Gamma, \tag{34}$$

from which we can determine a surface $u = u(x,y)$ bounded by one or more nonintersecting space curves and having constant mean curvature k.

Eq. (33) is solved on an irregular physical domain which is depicted in Fig. 1 under Dirichlet boundary condition and mean curvature $k = -\sqrt{2}/5$. The exact solution for this problem is

$$u = (50 - x^2 - y^2)^{1/2}. \tag{35}$$

The relative average error(root mean-square relative error: RMSE) used in the following figures is defined as below :

$$\text{RMSE} = \sqrt{\frac{1}{N_t} \sum_{j=1}^{N_t} \text{Rerr}^2}, \tag{36}$$

where $\text{Rerr} = \left| \frac{u(X_j) - \tilde{u}(X_j)}{u(X_j)} \right|$ for $|u(X_j)| \geq 10^{-3}$ and $\text{Rerr} = \left| u(X_j) - \tilde{u}(X_j) \right|$, for $|u(X_j)| < 10^{-3}$, respectively, j is the index of inner point of interest, $u(X_j)$ and $\overline{u}(X_j)$ denote the analytical and numerical solutions at the j-th inner point, respectively, and N_t represents the total number of test points of interest.

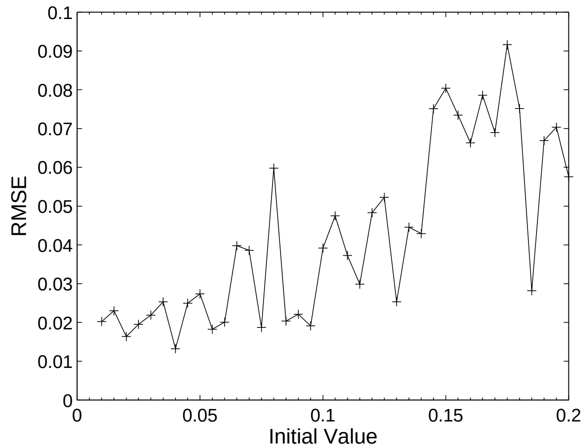

Figure 2: Effect of initial guess of iterations to numerical results.

For completeness, we investigate three aspects which may have effect to the numerical results, that is, the initial value choice encountered in the 'fsolve' subroutine, MQ parameter and inner knot number.

3.1 Initial value effect

Due to the importance of the initial vector value α in the 'fsolve' subroutine, we consider this issue first. All initial vector elements α_i $(i = 1,2,\dots,N+M)$ are assumed equal. Fig. 2 describes the influence of initial vector value in the interval $\alpha_i \in (0,0.2)$ to numerical results when the MQ parameter $c = 1.2$ and boundary knot number $N = 105$ without inner knots. It is noted that the numerical solution is sensitive to the choice of initial vector value. However, we observe that initial vector value $\alpha_i = 0.04$ $(i = 1,2,\dots,N+M)$ corresponds with the best solution accuracy RMSE $= 0.013$. In the following investigation, we will choose $\alpha_i = 0.04$ $(i = 1,2,\dots,N+M)$ to get better numerical results.

3.2 MQ parameter effect

Here, we consider the influence of MQ parameter to the numerical results. For the case that initial vector value $\alpha_i = 0.04$ $(i = 1,2,\dots,N+M)$ and boundary knot number $N = 105$ without inner knots, Fig. 3 illustrates the effect of MQ parameter to the numerical results in the interval $c \in (0,3)$. From which we can see that the numerical result is not sensitive to the MQ parameter in the interval $c \in (0.1,0.4)$ but sensitive to $c \in (0.4,3)$. Nevertheless, the best numerical result RMSE $= 0.012$

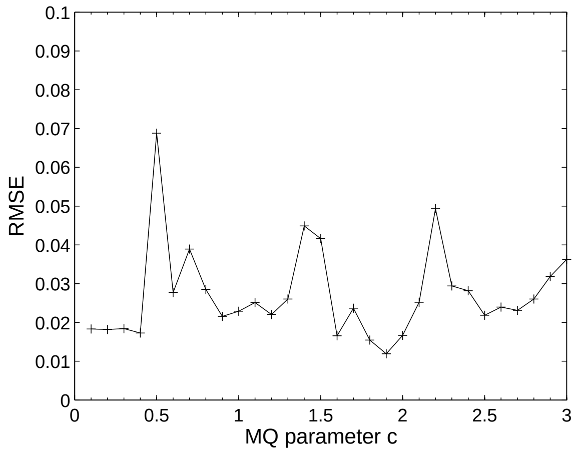

Figure 3: Effect of MQ parameter to numerical results.

is observed at $c = 1.9$ from Fig. 3.

3.3 Inner knot number effect

When using the BKM to solve inhomogeneous problems, inner knots are always considered to improve solution accuracy [Chen and Tanaka (2002)]. On this condition, we examine the effect of inner knot number variation to the numerical results. For inner knot number $M = 0$, $M = 3$, $M = 5$, $M = 10$, Fig. 4 depicts the numerical results versus boundary knot number N when initial vector value $\alpha_i = 0.04$ $(i = 1, 2, \ldots, N+M)$ and MQ parameter $c = 1.9$. It is observed that the largest inner knot number performs the best numerical result when the boundary knot number $N = 33$. Meanwhile, we find that the more inner knot number corresponds with the worse numerical results for boundary knot number $N \geq 81$, while the case without inner knot number performs the highest accuracy and convergence.

4 Conclusions

In this paper, we proposed a combined boundary knot method for solving nonlinear problems which is based on the AEM, the MPS and the BKM. Three aspects having impact to the numerical results are investigated by solving a classical nonlinear boundary value problem on a complex-shaped physical domain.

Among the most important characteristics of the proposed CBKM, we mention several advantages and shortcomings:

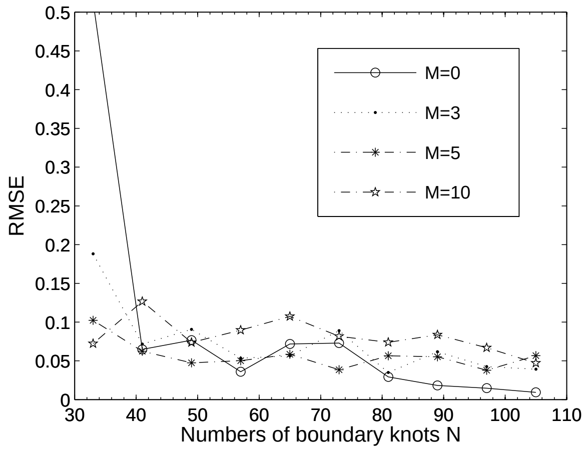

Figure 4: effect of inner knot number to numerical results.

(1) It is an inherently meshless, boundary-only technique since the inner knot is unnecessary.

(2) There is no restriction on the domain geometry as it happens with the BKM.

(3) It is easy to program for a variety of nonlinear problems.

(4) For any nonlinear problems, only the non-singular general solution of the Helmholtz equation is needed.

(5) The numerical solution is sensitive to the MQ parameter and initial value in the 'fsolve' subroutine. Therefore, the optimal choice of the MQ parameter and the initial value remain an open issue for further investigation.

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