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A BEM Approach for Inelastic Analysis of Beam-Foundation Systems under Cyclic Loading

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Abstract: In this paper a Boundary Element Method (BEM) is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on nonlinear inelastic foundation. The beam is subjected to arbitrarily distributed or concentrated vertical cyclic loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to resolve both the plastic part of stress resultants and the foundation reaction along with an efficient iterative process to integrate the inelastic rate equations. The arising boundary value problem is solved employing BEM. Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.

Keywords: inelastic analysis, cyclic loading, beam on foundation, inelastic Winkler model, distributed plasticity, boundary element method

1 Introduction

In engineering practice we often come across the analysis of beams on/in soil medium. The beam-foundation analysis is often required in piles, pile-columns and pile groups embedded in soil medium as well as in beam-columns and railway tracks resting on soil half space. These beam-foundation systems under the action of cyclic loading are usually leaded to the structural's element and/or soil yielding [Allotey and El Naggar (2008); Li et al (2012); Poulos (1981, 1989)]. Moreover, design of beams and engineering structures based on elastic analysis are most likely

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to be extremely conservative not only due to significant difference between initial yield and full plastification in a cross section, but also due to the unaccounted for yet significant reserves of strength that are not mobilized in redundant members until after inelastic redistribution takes place. Thus, material nonlinearity is important for investigating the ultimate strength of a beam that resists bending loading, while distributed plasticity models are acknowledged in the literature [Nukala and White (2004); Teh and Clarke (1999); Saritas and Filippou (2009)] to capture more rigorously material nonlinearities than cross sectional stress resultant approaches [Attalla et al (1994)] or lumped plasticity idealizations [Orbison et al (1982); Ngo-Huu et al (2007)].

During past decades extensive research efforts have been presented concerning *elastic* analysis of beams resting on single or multi-layer *elastic bilateral* foundation models. The majority of the adopted models employed numerical methods such as the finite element method [DasGupta (1974); Zhaohua and Cook (1983); Chiwanga and Valsangkar (1988); Shirm and Giger (1990); Onu (2000); Kim and Shin (2010)] while analytical formulations [Niyogi (1973); Avramidis IE, Morfidis K. (2006)] have also been employed. However, *elastic* analysis of beams on *elastic* foundation, taking into account the realistic *tensionless* character of the subgrade reaction has received limited amount of literature. To begin with, Weitsman (1972) studied the contact area developed between a beam under concentrated load and an elastic half-space. Kaschiev and Mikhajlov (1995) presented a formulation based on the use of Newton's method and the finite element method for beams resting on tensionless Winkler foundation, while for the same problem Zhang and Murphy (2004) presented an analytical/numerical solution making no assumption about either the contact area or the kinematics associated with the transverse deflection of the beam subjected to lateral point load assuming either free or pinned ends. Silveira et al (2008) presented a nonlinear semi-analytical methodology using a Ritz type approach, for the elastic equilibrium and instability analysis of beams, columns and arches resting on a tensionless Winkler elastic foundation, employing Newton-Raphson method together with arc-length iteration procedure in order to solve the nonlinear equations. Zhang (2008) analyzed a beam resting on a tensionless Reissner foundation and demonstrated the improvements of the Reissner foundation model compared with the Winkler one, while Ma et al (2009) used the transfer displacement function method (TDFM) to present the response of an infinite beam resting on a tensionless Pasternak foundation subjected to linearly varying distributed loads. Finally, Sapountzakis and Kampitsis (2010) developed a boundary element method for the nonlinear dynamic analysis of beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation undergoing moderate large deflections under general boundary conditions, taking into account the effects of shear deformation and rotary inertia.

To account for the nonlinear nature of the soil, several researches employed the concept of *elastic* beam on *nonlinear foundation*. In that formulation, the foundation load-displacement relation is assumed to follow a nonlinear law while the beam remains elastic throughout the analysis. Sharma and DasGupta (1975) employed an iteration method using Green's functions for the analysis of uniformly loaded axially constrained Bernoulli beams hinged at both ends assuming an exponential build-up of the foundation reaction with the displacement increment. Beaufait and Hoadley (1980) approximated the nonlinear load-displacement relationship of the Winkle foundation with a bilinear curve and utilized the midpoint difference method to analyze the beam coupled with the weighted averages scheme to estimate the spring stiffness for each iteration, followed by Yankelevsky DZ, Eisenberger M, Adin MA. (1989) who presented an iterative procedure based on the exact stiffness matrix for the beam on Winkler foundation by approximating the load-displacement curve by three to five regions rather than two. Kaliszky and Logo (1994) adopted the extremum principle to analyze a nonlinear elastic beam on nonlinear elastic foundation. Both the beam and the Winkler springs were assumed to follow a bilinear material model while the beam was subdivided into series of rigid bars and the deformation was concentrated in the hinges and spring elements. Sapountzakis and Kampitsis (2011a) studied the nonlinear static analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross-section, partially supported on tensionless three parameter foundation, undergoing moderate large deflections under general boundary conditions, while in Sapountzakis and Kampitsis (2011b) the same authors confirmed the importance of the tensionless nonlinear three-parameter viscoelastic foundation in the nonlinear dynamic analysis of beams under the combined action of arbitrarily distributed or concentrated transverse moving loading.

Contrary to the large amount of research concerning the elastic analysis of beams on either linear or nonlinear elastic foundation, only few studies have taken into account the *inelastic* behavior of both the beam and the foundation. According to this, the beam stress-strain and the foundation load-displacement relations are assumed to follow nonlinear inelastic constitutive laws. Consequently, such beam-foundation models are not commonly used due to the complexity of the problem. Ayoub (2003) presented an inelastic finite element formulation for that is capable of capturing the nonlinear behaviour of both the beam and the foundation. The element is derived from a two-field mixed formulation with independent approximation of forces and displacements and compared with the displacement based formulation. Finally, Mullapudi and Ayoub (2010) expanded the research in inelastic analysis of beams resting on two-parameter foundation where the values for the parameters are derived through an iterative technique that is based on an assumption of plane strain for the soil medium.

In this paper, a Boundary Element Method (BEM) is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on nonlinear inelastic foundation. The beam is subjected to arbitrarily distributed or concentrated vertical cyclic loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to resolve both the plastic part of stress resultants and the foundation reaction along with an efficient iterative process to integrate the inelastic rate equations [Ortiz and Simo (1986)]. The arising boundary value problem is solved employing BEM [Katsikadelis (2002)]. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows

- The formulation is a displacement based one taking into account inelastic redistribution along the beam axis by exploiting material constitutive laws and numerical integration over the cross sections (distributed plasticity approach).
- The cyclic response of the beam-foundation system is thoroughly examined and the influence of the material hardening is investigated.
- The inelasticity of the soil medium is taken into account, employing the nonlinear Winkler foundation model.
- The tensionless character of the foundation is also taken into consideration.
- An incremental - iterative solution strategy is adopted to restore global equilibrium of the beam.
- The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain, while its cross section is an arbitrarily one having at least one axis of symmetry (z-axis).
- To the authors' knowledge, a BEM approach has not yet been used for the solution of the aforementioned problem, while the developed procedure retains most of the advantages of a BEM solution even though domain discretization is required.

Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.

2 Statement of the problem

2.1 Displacements, strains, stresses

Let us consider a prismatic beam of length l (Fig. 1a) of arbitrary constant cross-section having at least one axis of symmetry (z-axis), occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the $\Gamma_j\,(j=1,2,...,K)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1b Cyz is the principal bending coordinate system through the cross section's centroid. The normal stress-strain relationship for the material is assumed to be elastic-plastic-strain hardening with initial modulus of elasticity and yield stress $E$and σ_{Y0}, respectively. The beam is resting on nonlinear inelastic tensionless Winkler type foundation and thus the foundation reaction is expressed as

$$p_f = \begin{cases} k_w w & \text{if } p_f > 0 \\ 0 & \text{if } p_f \leq 0 \end{cases} \tag{1}$$

where $k_w = k_w\,(w, w_y)$ is the Winkler nonlinear inelastic functions depending on the yielding displacement and the current one.

The beam is subjected to the combined action of arbitrarily distributed or concentrated cyclic transverse loading $p_z = p_z\,(x)$ and bending moment $m_y = m_y\,(x)$ acting in the x direction (Fig. 1a). Under the action of the aforementioned loading, the displacement field of the beam is given as

$$\bar{u}(x,z) = u\,(x) + z\theta_y \tag{2a}$$

$$\bar{w}\,(x) = w\,(x) \tag{2b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\,(x)$, $w\,(x)$ are the corresponding components of the centroid C and $\theta_y\,(x,t)$ is the angle of rotation due to bending of the cross-section with respect to its centroid. Employing the strain-displacement relations considering small deflections and adopting the Euler-Bernoulli assumption the following strain components are obtained

$$\varepsilon_{xx} = -z\frac{d^2 w}{dx^2} \tag{3a}$$

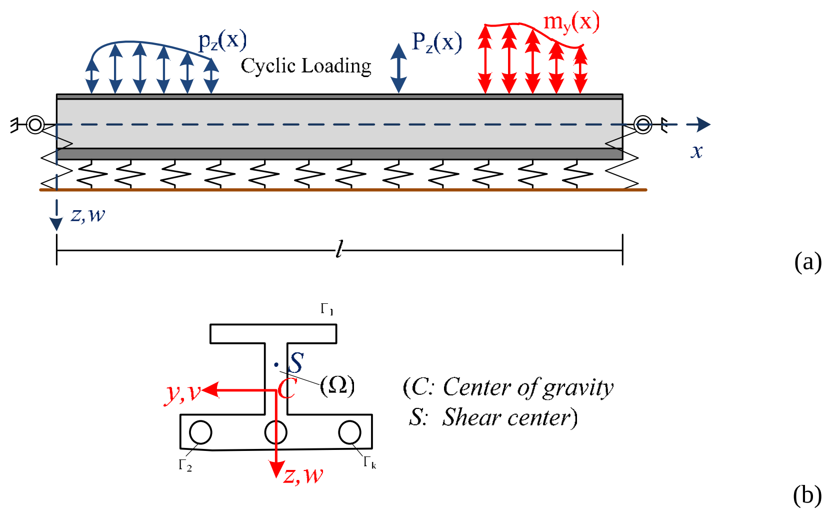


Figure 1: Prismatic beam resting on an inelastic foundation subjected to cyclic bending loading (a) with an arbitrary cross-section having at least one axis of symmetry, occupying the two dimensional region Ω (b).

$$\gamma_{xz} = 0 \Rightarrow \theta_y = -\frac{dw}{dx} \tag{3b}$$

Considering strains to be small, employing the Cauchy stress tensor and assuming an isotropic and homogeneous material without exhibiting any damage during its plastification, the normal stress rate is defined in terms of the corresponding strain one as

$$d\sigma_{xx} = E^* d\varepsilon_{xx}^{el} \tag{4}$$

where $d(\cdot)$ denotes infinitesimal incremental quantities over time (rates), the superscript el denotes the elastic part of the strain component and $E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the plane stress hypothesis is undertaken then $E^* = \frac{E}{1-\nu^2}$ holds [Vlasov (1963)], while E is frequently considered instead of E^* $(E^* \approx E)$ in beam formulations [Vlasov (1963); Armenakas (2006)]. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty in many beam formulations.

As long as the material remains elastic or elastic unloading occurs ($d\varepsilon_{xx} = d\varepsilon_{xx}^{el}$) the stress rate is given with respect to the strain one from eqn. (4). If plastic flow occurs then $d\varepsilon_{xx} = d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}$, where the superscript pl denotes the plastic part of the strain component. A simplified Von Mises yielding criterion is considered ignoring the influence of shear stresses and the yield condition is satisfied when the normal stress is equated with the yield stress of the material, that is

$$f = \sigma_{xx} - \sigma_Y \left(\varepsilon_{eq}^{pl} \right) = 0 \tag{5}$$

where σ_Y is the yield stress of the material and ε_{eq}^{pl} is the equivalent plastic strain, the rate of which is defined in [Crisfield (1991)] and is given as $d\varepsilon_{eq}^{pl} = d\lambda$ ($d\lambda$ is the proportionality factor [Crisfield (1991)]). Moreover, the plastic modulus h is defined as $h = d\sigma_Y / d\varepsilon_{eq}^{pl}$ or $d\sigma_Y = hd\lambda$ and can be estimated from a tension test as $h = E_t E/(E - E_t)$ (Fig. 2). The stress rate is given with respect to the total strain one through eqn. (3) and the strain components as

$$d\sigma_{xx} = Ed\varepsilon_{xx} - Ed\varepsilon_{xx}^{pl} \tag{6}$$

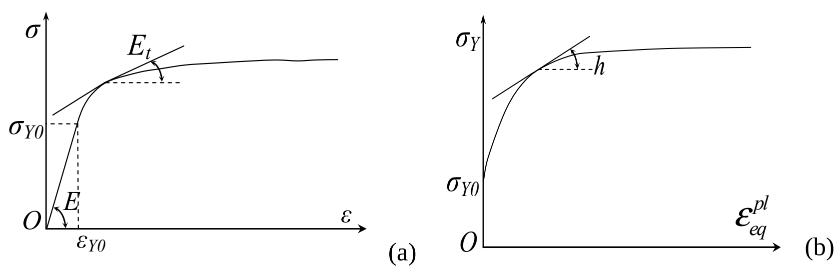


Figure 2: Normal stress - strain (a) and yield stress - equivalent plastic strain (b) relationships.

2.2 Equations of global equilibrium

To establish global equilibrium equations, the principle of virtual work neglecting body forces is employed, that is

$$\int_V (\sigma_{xx} \delta\varepsilon_{xx})\, dV = \int_l \left(p_z \delta w - m_y \delta w' \right) dx - \int_l \left(p_f \right) \delta w dx \tag{7}$$

where the integral quantities represent the strain energy, the external load and foundation reaction work while $\delta(\cdot)$ denotes virtual quantities, V is the volume and l is the length of the beam. The stress resultant corresponding to the internal bending moment of the beam is defined as

$$SM_y = \int_{\Omega} \sigma_{xx} z d\Omega \tag{8}$$

After substituting eqn. (8) into eqn. (7) and conducting some algebraic manipulations, the global equilibrium equations of the beam is obtained as

$$-\frac{d^2 SM_y}{dx^2} + p_f(x) = p_z(x) + \frac{dm_y(x)}{dx} \tag{9}$$

along with its corresponding boundary conditions

$$\alpha_1 \frac{dSM_y}{dx} + \alpha_2 w = \alpha_3 \tag{10a}$$

$$\beta_1 SM_y + \beta_2 \frac{dw}{dx} = \beta_3 \tag{10b}$$

where α_i, β_i $(i = 1,2,3)$ are functions specified at the beam ends. The boundary conditions (10) are the most general ones for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) may be derived from eqns (10) by specifying appropriately the functions α_i and β_i (e.g. for a clamped edge it is $\alpha_2 = \beta_2 = 1$, $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0$).

Since an incremental - iterative approach is adopted for the problem at hand, the incremental version of eqns (9, 10) is firstly written down as

$$-\frac{d^2 \Delta SM_y}{dx^2} + \Delta p_f(x) = \Delta p_z(x) + \frac{d\Delta m_y(x)}{dx} \tag{11}$$

where $\Delta(\cdot)$ denotes incremental quantities (over time), while the incremental stress resultant is given by virtue of eqns (8) and (6) as

$$\Delta SM_y = -EI_y \Delta w'' - \Delta SM_y^{pl} \tag{12}$$

where I_y is the moment of inertia with respect to the principle bending axis y and SM_y^{pl} is the plastic quantity defined as

$$\Delta SM_y^{pl} = E \int_{\Omega} \Delta \varepsilon^{pl} z d\Omega \tag{13}$$

By substituting eqn. (12) in eqn. (11) and forming the incremental version of the boundary conditions (eqns. (10)), the following boundary value problem is obtained

$$EI_y \Delta w'''' + \Delta(p_f) = \Delta p_z(x) + \frac{d\Delta m_y(x)}{dx} - \frac{d^2 \Delta SM_y^{pl}}{dx^2} \text{inside the beam} \tag{14}$$

$$a_1 \frac{d\Delta SM_y}{dx} + a_2 \Delta w = \Delta a_3 \tag{15}$$

$$\beta_1 \Delta SM_y + \beta_2 \frac{d\Delta w}{dx} = \Delta \beta_3 \tag{16}$$

at the beam ends $x = 0, l$.

By dropping the plastic quantities of the above equations, the boundary value problem of the examined problem under elastic conditions is formulated.

3 Integral Representations – Numerical Solution

3.1 Integral representations for the displacement w

According to the precedent analysis, the inelastic problem of beams resting on resting on nonlinear inelastic foundation. reduces to establishing the displacement component $\Delta w(x)$ having continuous derivatives up to the fourth order with respect to x and satisfying the boundary value problem described by the governing differential equation (14) along the beam and the boundary conditions (15-16) at the beam ends $x = 0, l$.

This boundary value problem (eqns (14), (15-16)) is solved employing the BEM [Katsikadelis (2002)], as this is developed in Sapountzakis (2000) for the solution of a fourth order differential equation with constant coefficients. According to this method, let $u(x) = \Delta w(x)$ be the sought solution of the problem. The solution of the fourth order differential equation $d^4u/dx^4 = \Delta w''''$ is given in integral form as [Katsikadelis (2002)]

$$u(\xi) = \int_0^l \frac{d^4u}{dx^4} u^* dx - \left[u^* \frac{d^3u}{dx^3} - \frac{\partial u^*}{\partial x} \frac{d^2u}{dx^2} + \frac{\partial^2 u^*}{\partial x^2} \frac{du}{dx} - \frac{\partial^3 u^*}{\partial x^3} u \right]_0^l \tag{17}$$

where u^* is the fundamental solution given as [Katsikadelis (2002)]

$$u^* = \frac{1}{12} \left(|r|^3 - 3l |r|^2 + 2l^3 \right) \tag{18}$$

with $r = x - \xi$, x, ξ points of the beam. Since EI_y is independent of x, eqn. (17) can be written as

$$\begin{aligned} EI_y u(\xi) &= \int_0^l EI_y \tfrac{d^4u}{dx^4}\Lambda_4(r)dx \\ &- EI_y\left[\Lambda_4(r)\tfrac{d^3u}{dx^3} - \Lambda_3(r)\tfrac{d^2u}{dx^2} + \Lambda_2(r)\tfrac{du}{dx} - \Lambda_1(r)u\right]_0^l \end{aligned} \tag{19}$$

where the kernels $\Lambda_j(r)$ $(j = 1,2,3,4)$ are given as

$$\Lambda_1(r) = \frac{1}{2}sgnr \quad \Lambda_2(r) = \frac{1}{2}(|r| - l) \tag{20a}$$

$$\Lambda_3(r) = \frac{1}{4}|r|(|r| - 2l)\,sgnr \quad \Lambda_4(r) = \frac{1}{12}\left(|r|^3 - 3l|r|^2 + 2l^3\right) \tag{20b}$$

Solving eqn. (14) with respect to $EI_y\Delta w''''$ and substituting the result in eqn. (19), the following integral representation is obtained

$$\begin{aligned} EI_y u(\xi) &= \int_0^l \left(\Delta p_z(x) + \tfrac{d\Delta m_y(x)}{dx} - \tfrac{d^2\Delta SM_y^{pl}}{dx^2} - \Delta(p_f)\right)\Lambda_4(r)dx \\ &- EI_y\left[\Lambda_4(r)\tfrac{d^3u}{dx^3} - \Lambda_3(r)\tfrac{d^2u}{dx^2} + \Lambda_2(r)\tfrac{du}{dx} - \Lambda_1(r)u\right]_0^l \end{aligned} \tag{21}$$

After carrying out several integrations by parts and employing equations (1), eqn. (21) yields

$$\begin{aligned} EI_y u(\xi) = &\int_0^l \Delta p_z(x)\Lambda_4(r)dx - \int_0^l \Delta m_y(x)\Lambda_3(r)dx \\ &- \int_0^l \Delta SM_y^{pl}\Lambda_2(r)dx - \int_0^l \Delta(k_w w)\Lambda_4(r)dx \\ &+ \left[\Lambda_4(r)\frac{d\Delta SM_y}{dx} - \Lambda_3(r)\Delta SM_y + EI_y\Lambda_2(r)\frac{du}{dx} - EI_y\Lambda_1(r)u\right]_0^l \end{aligned} \tag{22}$$

Having in mind that the first derivative of u is contained in eqns (15-16), eqn. (22) is differentiated once with respect to ξ yielding

$$\begin{aligned} EI_y\tfrac{du(\xi)}{d\xi} &= -\int_0^l \Delta p_z(x)\Lambda_3(r)dx + \int_0^l \Delta m_y(x)\Lambda_2(r)dx + \int_0^l \Delta SM_y^{pl}\Lambda_1(r)dx \\ &- \int_0^l \Delta(k_w w)\Lambda_3(r)dx - \left[\Lambda_3(r)\tfrac{d\Delta SM_y}{dx} - \Lambda_2(r)\Delta SM_y + EI_y\Lambda_1(r)\tfrac{du}{dx}\right]_0^l \end{aligned} \tag{23}$$

Observing eqns. (22)-(23), it is deduced that they have been brought into a convenient form to establish a numerical computation of the unknown quantity Δw. Thus, the interval $(0,l)$ is divided into L elements, on each of which Δw is assumed to vary according to a certain law (constant, linear, parabolic etc). The linear element assumption is employed here (Fig. 3) as the numerical implementation is simple and the obtained results are very good. It is worth here noting that this technique does not require either differentiation of shape functions or finite differences application.

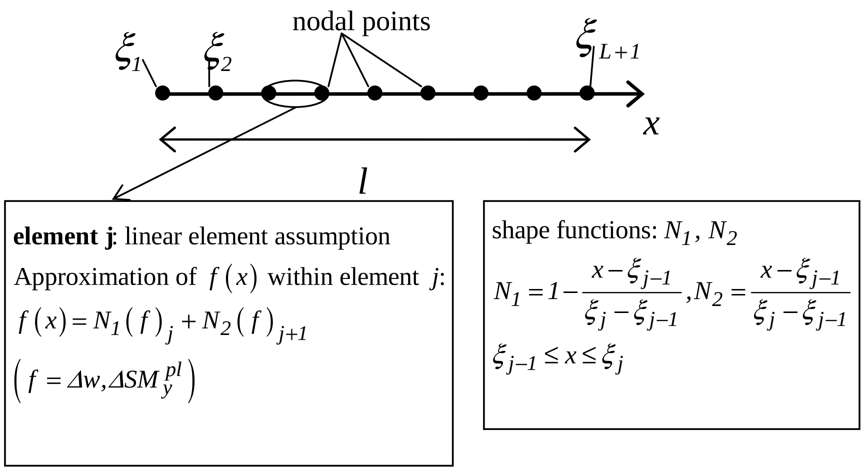


Figure 3: Discretization of the beam interval into linear elements, distribution of the nodal points and approximation of several quantities.

Employing the aforementioned procedure and a collocation technique, a set of $L+1$ algebraic equations is obtained with respect to $L+7$ unknowns, namely the values of $(\Delta w)_i$, $(i=2,3,\ldots,L)$ at the $L-1$ internal nodal points and the boundary values of $(\Delta w)_j$, $\left(\frac{d\Delta w}{dx}\right)_j(\Delta SM_y)_j$, $\left(\frac{d\Delta SM_y}{dx}\right)_j$ $(j=1$ and $L+1)$ at the beam ends $\xi_1=0$, $\xi_{L+1}=l$ (Fig. 3)). Two additional algebraic equations are obtained by applying the integral representation (23) at the beam ends $\xi=0,l$. These $L+3$ equations along with the four boundary conditions (eqns (15-16)) yield a linear system of $L+7$ simultaneous algebraic equations

$$[K]\{\Delta d\}=\{\Delta b_{ext}\}+\left\{\Delta b_{pl}\right\} \tag{24}$$

where $[K]$ is a known generalized stiffness matrix, $\{\Delta d\}$ is a generalized incremental unknown vector given as

$$\{\Delta d\}^T = \Big\{ \; \{(\Delta w)_1 \quad (\Delta w)_2 \quad \ldots \quad (\Delta w)_{L+1} \quad \left(\tfrac{d\Delta w}{dx}\right)_1 \quad \left(\tfrac{d\Delta w}{dx}\right)_{L+1} \quad (\Delta SM_y)_1 \quad (\Delta SM_y)_{L+1} \quad \left(\frac{d\Delta SM_y}{dx}\right)_1 \quad \left(\frac{d\Delta SM_y}{dx}\right)_{L+1} \Big\} \tag{25}$$

while $\{\Delta b_{ext}\}$, $\{\Delta b_{pl}\}$ are known vectors representing all the terms related to the incremental externally applied loading and incremental plastic quantities, respectively.

After solving the system of eqns (24), a post-processing step is required to obtain the derivatives $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ at any nodal point ξ_i $(i = 1,2,...,L+1)$ which is essential to the solution algorithm. The first derivative is obtained from eqn. (23), while the second one from the following integral representation

$$EI_y \frac{d^2u(\xi)}{d\xi^2} = \int_0^l \Delta p_z(x)\Lambda_2(r)dx - \int_0^l \Delta m_y(x)\Lambda_1(r)dx - \Delta SM_y^{pl} - \int_0^l \Delta(k_w w)\Lambda_2(r)dx + \left[\Lambda_2(r)\frac{d\Delta SM_y}{dx} - \Lambda_1(r)\Delta SM_y\right]_0^l \tag{26}$$

which is obtained after differentiating eqn. (22) twice with respect to ξ. The presented integral representations may also be employed to compute $u(x)$ and its derivatives at any interior point of the beam other than ξ_i $(i = 1,2,...,L+1)$.

3.2 Incremental - iterative solution algorithm

Load control over the incremental steps is used and load stations are chosen according to load history and convergence requirements. Incremental stress resultants are decomposed into elastic and plastic part (eqns (12)). They are computed through an iterative procedure since usually, changes between the plastic part of incremental stress resultants of two successive iterations are not negligible. Thus, using the subscript m to denote the load step, the superscript l to denote the iterative cycle and the symbol $\Delta(\cdot)$ to denote incremental quantities, the *l-th* iteration of the *m-th* load step of the incremental - iterative solution algorithm will be described.

Evaluation of the generalized iterative unknown vector $\{\Delta d\}_m^l$ from the solution of the linear system of equations (eqn. (24))

$$[K]\{\Delta d\}_m^l = \{\Delta b_{ext}\}_m + \{\Delta b_{pl}\}_m^{l-1} \tag{27}$$

If $m=1$ and $l=1$, it is assumed that $\left\{\Delta b_{pl}\right\}_1^0=\{0\}$. If $m>1$ and $l=1$, it is assumed that $\left\{\Delta b_{pl}\right\}_m^0=\{0\}$ or $\left\{\Delta b_{pl}\right\}_m^0=\left\{\Delta b_{pl}\right\}_{m-1}^n$, where n is the total number of iterations performed in the previous increment $m-1$.

Evaluation of the incremental unknown derivatives $[\Delta w'(\xi_i)]_m^l$, $[\Delta w''(\xi_i)]_m^l$ ($i=1,2,...,L+1$) by introducing $\{\Delta d\}_m^l$ into eqns (23) and (26), respectively.

Elastic prediction step: For each monitoring station k of the $i-th$ cross section of the beam ($k=1,2,...,N_{dof}$, $i=1,2,...,L+1$): Evaluation of the trial stress components as

$$\left(\sigma_{xx}^{Tr}(\xi_i,z_k)\right)_m^l=\left(\sigma_{xx}(\xi_i,z_k)\right)_m^0+\left(\Delta\sigma_{xx}^{Tr}(\xi_i,z_k)\right)_m^l \tag{28}$$

where the incremental trial stress components are obtained by employing eqns (3), (4) as

$$\left(\Delta\sigma_{xx}^{Tr}(\xi_i,z_k)\right)_m^l=E\left[-\frac{d^2\Delta w(\xi_i)}{dx^2}\right]_m^l(z)_k \tag{29}$$

Perform the yield criterion at each monitoring station k of the $i-th$ cross section of the beam ($k=1,2,...,N_{dof}$, $i=1,2,...,L+1$) employing eqn. (5).

If $f^{\mathrm{Tr}}\leq 0$ then the trial state is the final state and the incremental plastic strain components along with the equivalent plastic strain are updated as

$$\left(\Delta\varepsilon_{xx}^{pl}\right)_m^l=0 \quad \left(\varepsilon_{eq}^{pl}\right)_m^l=\left(\varepsilon_{eq}^{pl}\right)_m^0 \tag{30}$$

If $f^{\mathrm{Tr}}>0$ then plastic flow occurs and return must be made to yield surface (**plastic correction step**). The *plastic flow rule* and the *loading/unloading conditions* can easily be formulated [40, 41].

For each monitoring cross section of the beam i: Evaluation of the plastic quantities $\left[\Delta SM_y^{pl}(\xi_i)\right]_m^l$ ($i=1,2,...,L+1$) by employing a two-dimensional numerical integration scheme to approximate the domain integrals of eqn. (13). In this study, the beam's monitoring cross sections are divided into a number of triangular or quadrilateral cells and standard two-dimensional Gauss quadrature rules are employed in each cell. Thus, the monitoring stations of each cross section coincide with the Gauss points of its cells. If the same number of Gauss points is employed in every cell, then $N_{dof}=N_{cells}\times N_{Gauss}$ holds. Exact patch between adjacent cells is not required due to the combined use of BEM to compute I_y and the fact that only domain integrals are approximated without performing any structural discretization.

Employ the obtained plastic quantities $\left[\Delta SM_y^{pl}(\xi_i)\right]_m^l$ $(i = 1,2,...,L+1)$ to evaluate (a) the vector $\left\{\Delta b_{pl}\right\}_m^l$ representing all the terms of eqn. (22), related to plastic quantities and (b) terms of eqns. (23), (26) related to plastic quantities required to perform step (ii) for the next iteration $l+1$. Apart from elementary computations, the current step requires the computation of line integrals of the form $\int_0^l \Lambda_j(r)\Delta SM_y^{pl}dx$ $(j = 1,2)$ (eqns (22), (23)). A numerical integration scheme must be employed to resolve these integrals since plastic quantities are not known in the whole beam interval $(0,l)$. A semi-analytical scheme has been implemented, according to which ΔSM_y^{pl} vary on an element k $(k = 1,2,...,L)$ (Fig. 2) of the beam interval following the same law that is used to approximate Δw (see section 3.1). This leads to the integration of kernels being products of functions $\Lambda_i(r)$ and two-node linear shape functions, thus it is performed analytically without any difficulty.

Check convergence. Convergence occurs if $\left\| \left[\Delta SM_y^{pl}(\xi_i)\right]_m^l - \left[\Delta SM_y^{pl}(\xi_i)\right]_m^{l-1} \right\|$ is sufficiently small. The iterations continue until the specified accuracy is reached. If convergence is achieved after n iterations then:

For each monitoring station k of the $i-th$ cross section of the beam $(k = 1,2,...,N_{dof}$, $i = 1,2,...,L+1)$: Initialize the stress components along with the equivalent plastic strain for the next increment $m+1$ as

$$(\sigma_{xx})_{m+1}^0 = (\sigma_{xx})_m^n \quad \left(\varepsilon_{eq}^{pl}\right)_{m+1}^0 = \left(\varepsilon_{eq}^{pl}\right)_m^n \tag{31}$$

Resolve (a) the vector $\{\Delta b_{ext}\}_{m+1}$ representing all the terms of eqns (15-16) and (22) related to externally applied loading and (b) terms of eqns (23) and (26) related to externally applied loading required to perform step (ii) for the next increment. Apart from elementary computations, the current step requires the computation of line integrals of the form $\int_0^l \Lambda_i(r)\Delta p_z dx$, $\int_0^l \Lambda_i(r)\Delta m_y dx$ $(i = 1,2,3,4)$. Since the distributions of p_z and m_y are usually prescribed in codes and regulations with simple analytical relations, these integrals are evaluated analytically, demonstrating the efficiency of the developed numerical procedure (e.g. concentrated loads may be treated using the Dirac function, without adhering to any simplifications).

Since convergence is achieved then the foundation reaction is computed employing a simplified Von Mises yielding criterion. The parameters are updated and the process described by steps (i)-(vii) is repeated until the foundation convergence criterion is achieved by using a prescribed tolerance of $tol_{found} = 10^{-10}$.

The increments of the external loading continue till total loading is undertaken or till convergence cannot be satisfied, which means that the last additional increment cannot be undertaken (plastic collapse).

4 Numerical Examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, the accuracy and the range of applications of the developed method.

4.1 Example 1

In the first numerical example a rectangular cross section ($h = 0.60m$, $b = 0.30m$) pinned - fixed beam of length $l = 6.0m$resting on an elastic-plastic Winkler foundation with initial stiffness $k_w = 20MPa$ and yielding force $P_{wY} = 100kN/m$ has been studied (Fig.4a), employing 20 linear longitudinal elements, 400 boundary elements, 72 quadrilateral cells (12 fibers) and a 3×3 Gauss integration scheme for each cell (cross sectional discretization). The beam is subjected to a cyclic uniformly distributed loading acting at $0 \leq x \leq 3.0m$, as presented in Fig. 4a,b. Two material cases have been analyzed, namely an elastic-perfectly plastic with $E = 32318.4MPa$, $\sigma_{Y0} = 20MN/m^2$ and $E_t = 0$ and an elastoplastic-strain hardening with $E_t = 650MPa$.

In Figs 5, 6 the load-displacement curves at the midpoint $w(l/2)$ of the beam are presented for different types of material properties, as compared with a 3-D FEM solution [NX Nastran (2007)] employing 2561 solid elements, ignoring the foundation reaction. Moreover, in Table 1 the normal stress σ_{xx} distribution along the beam's length is presented for different load stages, as compared with the corresponding deformed 3-D FEM contour representation. From this figure and table, a very good agreement between the results is observed verifying the accuracy and applicability of the proposed formulation.

Furthermore, the load-displacement curves at the midpoint $w(l/2)$ of the beam on elastic-plastic Winkler foundation for different types of material properties are depicted in Figs 7, 8, as compared with a FEM solution [NX Nastran (2007)] obtained by employing 2561 solid elements and 81 nonlinear springs following the elastic-plastic law given above. Finally, in Table 2 the maximum beam deflection $w_{\max}$ is presented for different load stages and material properties as compared with those obtained from two FEM models, namely the aforementioned 3-D solid one and a one dimensional model employing 120 beam and spring elements, observing the convergence between the proposed formulation and the solid simulation, as well as the inability of the FEM beam model to capture accurately the systems response.

From these figures and table, the significant influence of the inelastic analysis to the beam-foundation response, as well as the reliability of the proposed method are verified.

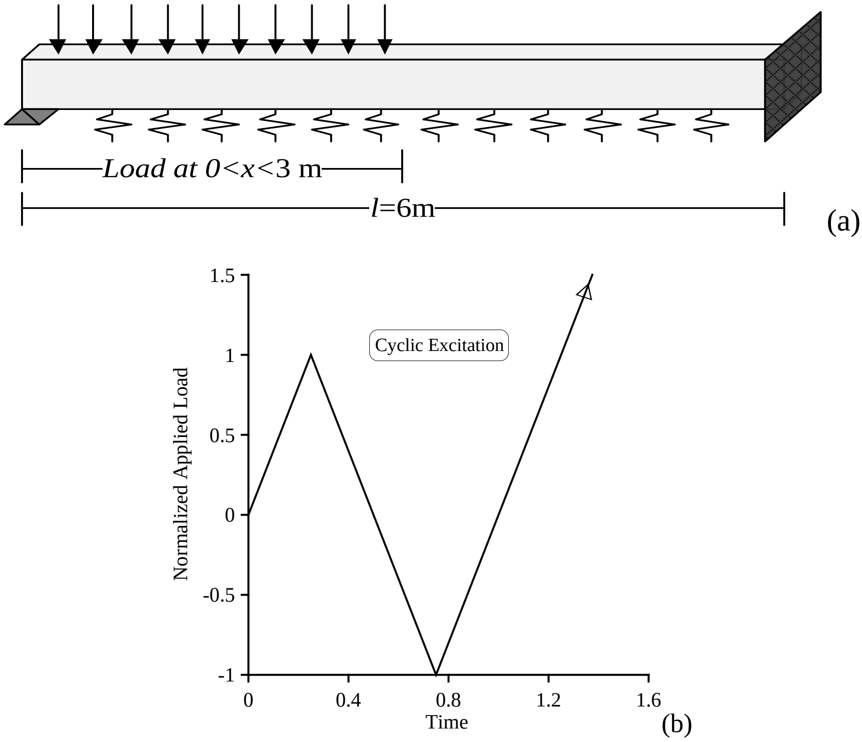


Figure 4: Pinned-fixed beam resting on an elastic-plastic Winkler foundation (a) subjected to a uniformly distributed cyclic loading (b).

4.2 *Example 2*

For comparison reasons the special case of an I-shaped cross section (total height $h = 0.3m$, total width $b = 0.3m$, flange width $t_f = 0.02m$, web width $t_w = 0.01m$) fixed-pinned beam of length $l = 8m$ has been studied, employing 40 linear longitudinal elements, 400 boundary elements, 43 quadrilateral cells (15 fibers) and a 3×3 Gauss integration scheme for each cell (cross sectional discretization). The computational model implemented in the proposed formulation is presented in Fig. 9a. Two material cases have been analyzed, namely an elastic-perfectly plastic one with $E = 213.4GPa$, $\sigma_{Y0} = 285MPa$, $E_t = 0$ and an elastoplastic-strain hardening one with $E_t = 6000MPa$. The beam is subjected to uniformly distributed cyclic

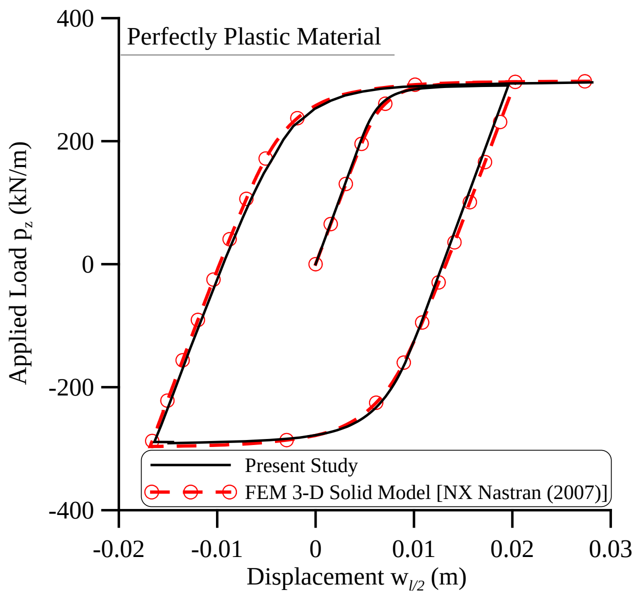


Figure 5: Load – displacement curve at the midpoint of the beam of example 1, in case of elastic-perfectly plastic material.

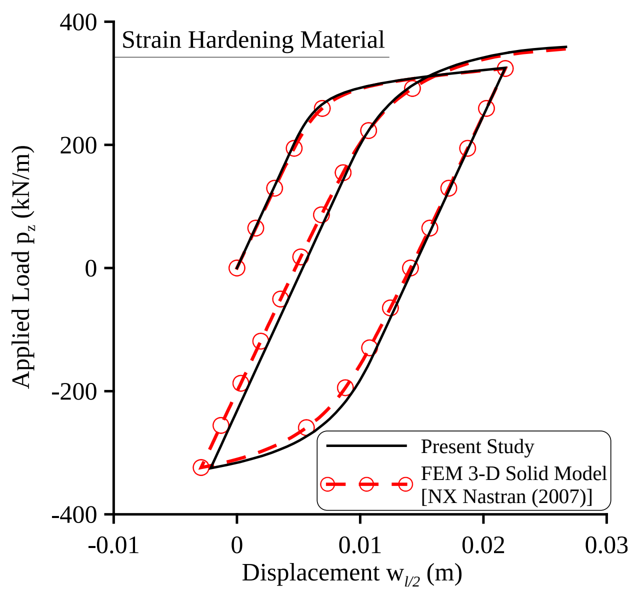
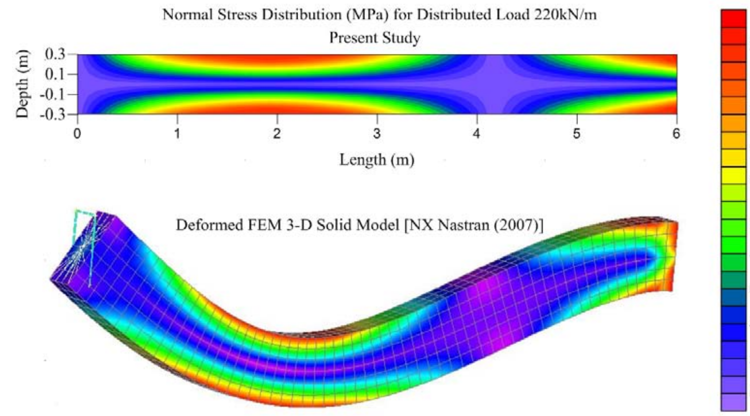
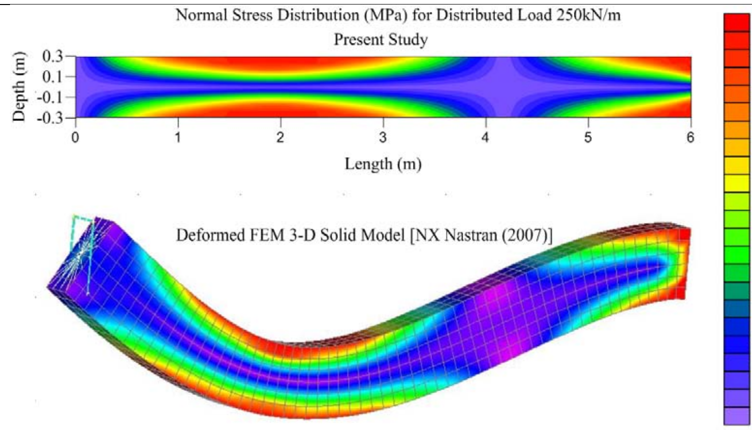


Figure 6: Load – displacement curve at the midpoint of the beam of example 1, in case of elastoplastic-strain hardening material.

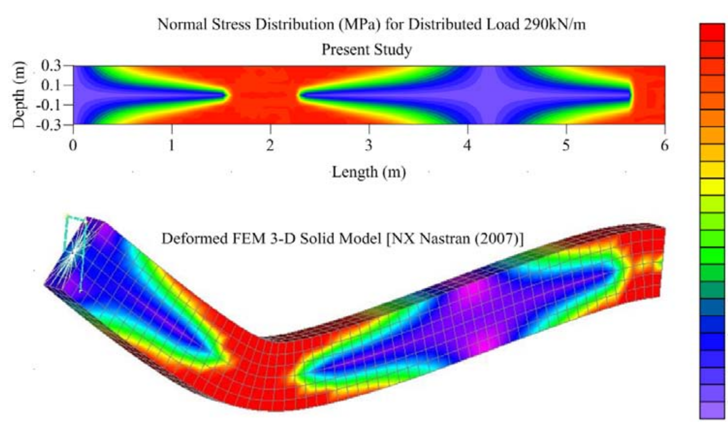
Table 1: Normal stress distribution along the beam's length for different load stages compared with the corresponding deformed 3-D FEM contour representation.



(a)



(b)



(c)

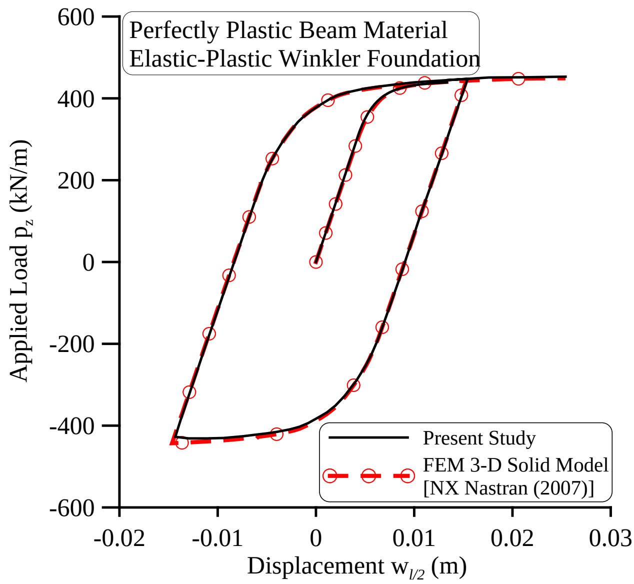


Figure 7: Load – displacement curve at the midpoint of the beam on elastic-plastic Winkler foundation of example 1, in case of elastic-perfectly plastic beam material.

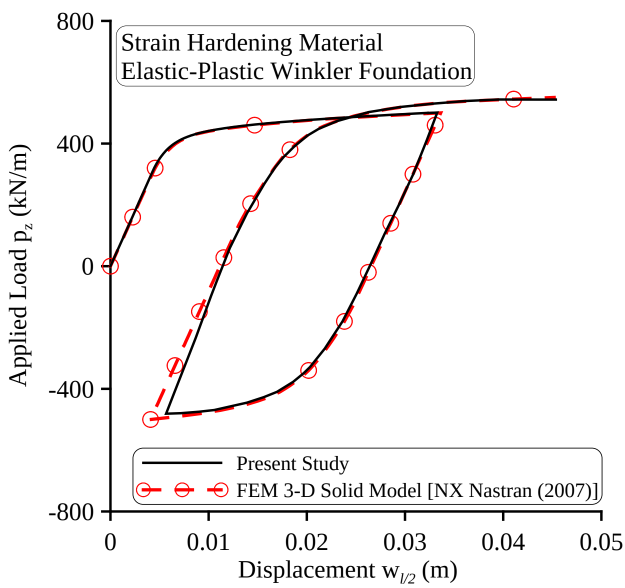


Figure 8: Load – displacement curve at the midpoint of the beam on elastic-plastic Winkler foundation of example 1, in case of elastoplastic-strain hardening beam material.

Table 2: Maximum deflection $w_{\max}$ (cm) of the beam of example 1, for different types of beam and foundation material properties.

Elastic Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650MN/m^2$		
	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam
500	0.987	1.002	1.100	0.980	0.984	1.041
550	1.202	1.213	-	1.158	1.170	1.252
600	1.438	1.468	-	1.364	1.384	1.483

Perfectly Plastic Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650MN/m^2$		
	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam
350	0.567	0.589	0.576	0.566	0.585	0.586
400	0.767	0.780	0.758	0.756	0.769	0.811
440	1.657	1.659	-	1.199	1.215	2.128

Hardening ($k_{wt} = 1.0MPa$) Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650MN/m^2$		
	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam	Present Study	FEM [NX Nastran (2007)] Solid	FEM [NX Nastran (2007)] Beam
400	0.750	0.773	0.810	0.743	0.766	0.789
450	1.663	1.632	-	1.254	1.285	1.938
500	5.689	5.651	-	2.618	2.678	3.876

loading, as presented in Fig.10. The solution obtained from the proposed formulation is compared with a FEM solution [NX Nastran (2007)] obtained employing 2881 quadrilateral shell elements (Fig. 9b).

To demonstrate the convergence of the developed numerical procedure, in Table 3 pairs of applied transverse loading and displacement values at the midpoint of the beam are presented, for both cases of material properties, for three longitudinal discretization schemes. The first loading level of the table corresponds to an elastic behaviour, while the remaining ones refer to inelastic response. Moreover, in this table the ultimate transverse load p_z^u that can be undertaken by the beam (plastic collapse load) is also presented for the aforementioned longitudinal discretization schemes. Moreover, in Figs 11, 12 the load-displacement curves are presented as compared with the aforedescribed FEM solution [NX Nastran (2007)], taking into account or ignoring the material elastoplastic hardening. Excellent agreement between the obtained results and the shell finite element model is observed, illustrating once again the accuracy of the proposed method.

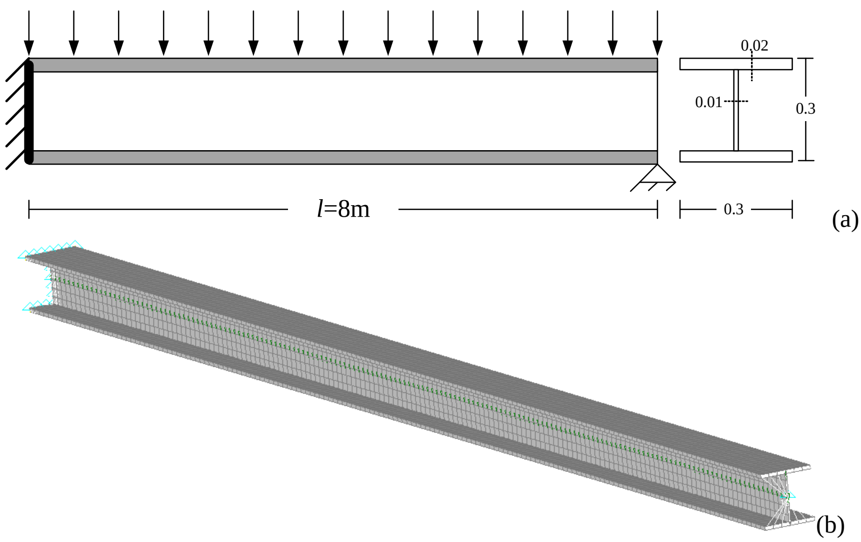


Figure 9: Fixed pinned beam subjected to a uniformly distributed cyclic loading (a) and shell model FEM mesh [NX Nastran (2007)] (b).

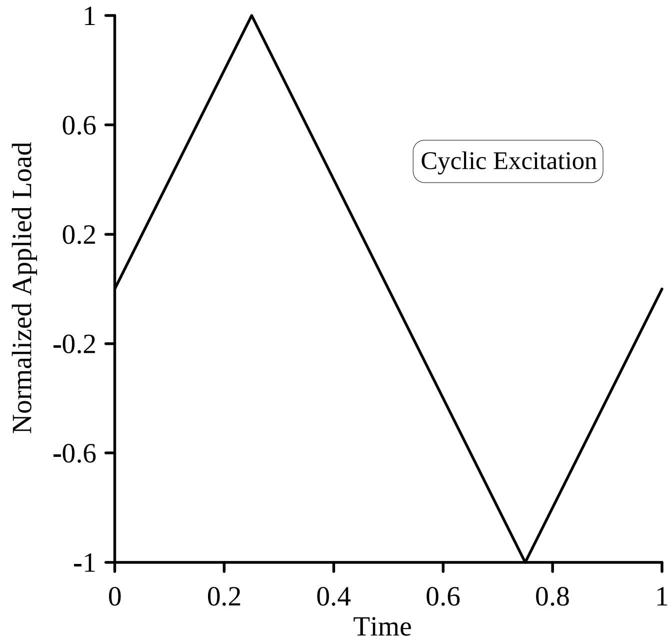


Figure 10: Normalized cyclic excitation of example 2.

4.3 Example 3

As the final numerical example, a mono-symmetric I-shaped cross section of total height $h = 0.3m$, upper/lower flange width $b_f^{top} = 0.3m/b_f^{bot} = 0.4m$, thickness $t_f = 0.02m$, and wed thickness $t_w = 0.01m$, clamped beam ($E = 213400MPa$, $\sigma_{Y0} = 285MPa$) of length $l = 7m$ resting on an inelastic Winkler foundation ($k_w = 25MPa$, $P_{wY} = 100kN/m$, $k_{wt} = 2.5MPa$) has been studied, employing 32 linear longitudinal elements, 400 boundary elements, 43 quadrilateral cells (15 fibers) and a 3×3 Gauss integration scheme for each cell (cross sectional discretization). The beam is subjected to a cyclic concentrated load acting at $x = 2.5m$from the left support.

In Fig. 13 the load-displacement curves at the loading point are presented for different types of beam and soil material properties in case of monotonically increasing concentrate load, verifying the significant influence of the inelastic analysis to the beam-foundation system response and the importance of the subgrade modeling to the beam deflections. Moreover, in Figs 14, 15 the load-displacement curves are presented accounting for or ignoring the beam's and Winkler's spring hardening slope, verifying the importance of the soil nonlinearity to the system's cyclic response.

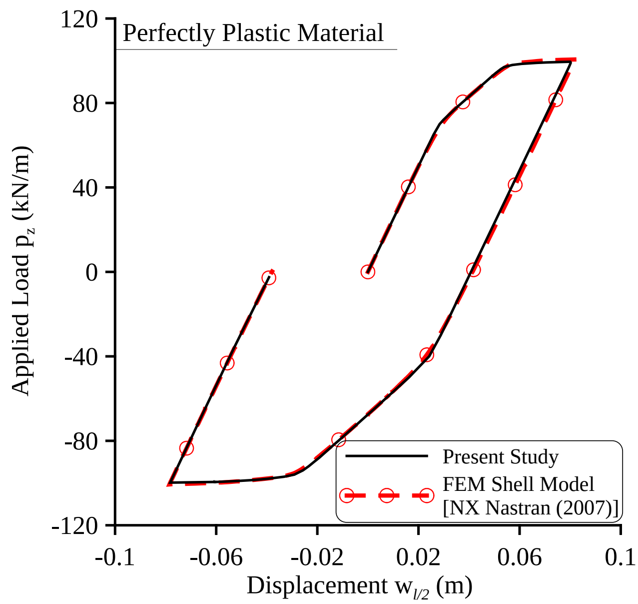


Figure 11: Load – displacement curve at the midpoint of the beam of example 2, in case of elastic-perfectly plastic material.

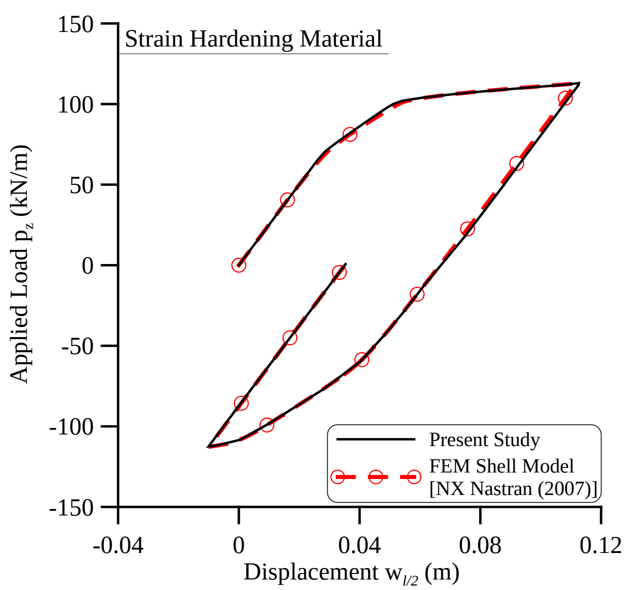


Figure 12: Load – displacement curve at the midpoint of the beam of example 2, in case of elastoplastic-strain hardening material.

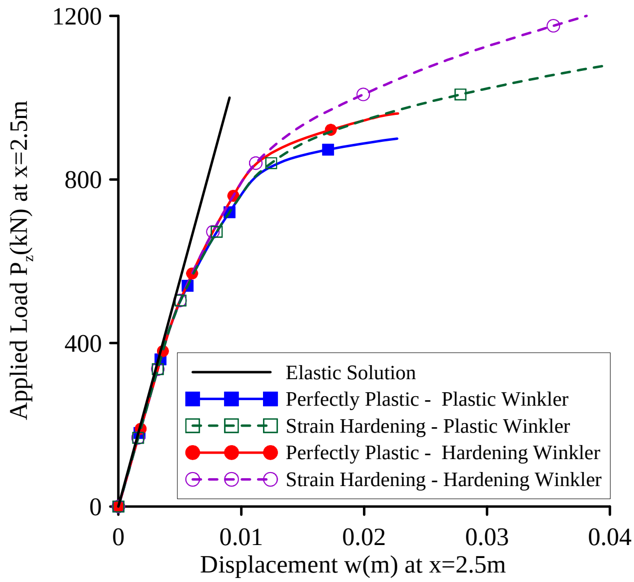


Figure 13: Load – displacement curve at the loading point of the beam of example 3 resting on nonlinear foundation.

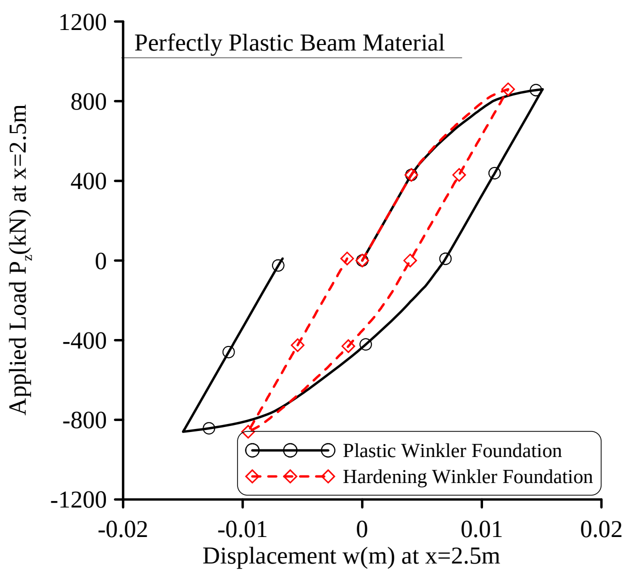


Figure 14: Load – displacement curve at the loading point of the beam of example 3, in case of elastic-perfectly plastic beam material.

Table 3: Applied load versus displacement at $x = l/2$ along with ultimate transverse load p_z^u undertaken by the beam of example 2, for various longitudinal discretization schemes.

	Elastic-perfectly plastic material			Elastoplastic-strain hardening material		
Number of elements	15	30	40	15	30	40
p_z (kN/m)	$w_{l/2} \times 10^{-2}$ (m)					
65	2.597	2.598	2.598	2.597	2.598	2.598
80	3.688	3.719	3.721	3.405	3.518	3.522
90	4.653	4.665	4.671	4.180	4.236	4.288
95	5.020	5.150	5.164	4.598	4.674	4.682
	p_t^u (kN/m)					
	99.8	99.8	100	-		

5 Concluding remarks

In this paper a BEM approach is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on tensionless inelastic foundation. The main conclusions that can be drawn from this investigation are

1. The numerical technique presented in this investigation is well suited for computer aided analysis of prismatic beams of arbitrary simply or multiply connected cross section having at least one axis of symmetry, supported by the most general boundary conditions and subjected to the action of arbitrarily distributed or concentrated vertical loading.

2. The inelastic analysis and the soil nonlinearity are of paramount importance for the cyclic response of the beam-foundation system.

3. Accurate results are obtained using a relatively small number of nodal points across the longitudinal axis.

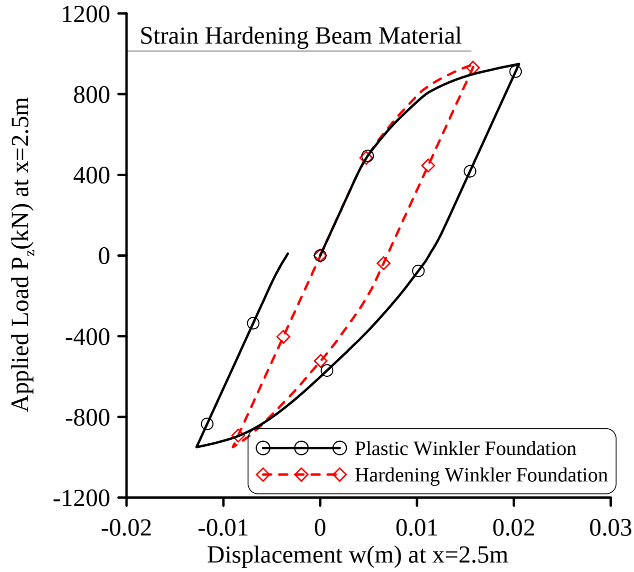


Figure 15: Load – displacement curve at the loading point of the beam of example 3, in case of elastoplastic-strain hardening beam material.

4. A small number of cells (fibers) is required in order to achieve satisfactory convergence.

5. The developed procedure retains most of the advantages of a BEM solution even though domain discretization is required.

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