 CMES, vol.87, no.1, pp.41-54, 2012

Three-Dimensional Unsteady Thermal Stress Analysis by Triple-Reciprocity Boundary Element Method

Yoshihiro Ochiai[1], **Vladimir Sladek**[2] **and Jan Sladek**[2]

Abstract: The conventional boundary element method (BEM) requires a domain integral in unsteady thermal stress analysis with heat generation or an initial temperature distribution. In this paper it is shown that the three-dimensional unsteady thermal stress problem can be solved effectively using the triple-reciprocity boundary element method without internal cells. In this method, the distributions of heat generation and initial temperature are interpolated using integral equations and time-dependent fundamental solutions are used. A new computer program was developed and applied to solving several problems.

Keywords: Thermal Stress, Boundary Element Method, Heat Conduction, Meshless Method

1 Introduction

The unsteady thermal stress problems cannot be solved easily, without using internal cells, by the conventional boundary element method (BEM), in general. Only special cases of ploblems, such as unsteady thermal stress problems with constant heat generation and uniform initial temperature distribution can be solved by the standard BEM without the need for internal cells. When an analysis of thermal stress under arbitrary heat generation or a non-uniform initial temperature distribution within the domain is carried out by the BEM, a domain integral is involved in general [Brebbia et al (1984); Wrobel (2002)]. However, by including the domain integral, the merit of BEM is lost, since the unknowns are not localized on the boundary alone like in pure BEM. Thus, several other methods have been considered. Nowak and Neves proposed a multiple-reciprocity method [Nowak and Neves (1994)]. Tanaka et al. have proposed a dual-reciprocity BEM for transient

[1] Kinki University, Dept. of Mechanical Engineering, 3-4-1 Kowakae, Higashi-Osaka, Osaka, 577-8502 Japan, E-mail: ochiai@mech.kindai.ac.jp
[2] Institute of Construction and Architecture, Slovak Academy of Science, 845 03 Bratislava, Slovakia

heat conduction problems [Tanaka et al (2003)], and V. Sladek and J. Sladek proposed a local boundary integral equation for unsteady heat conduction problems [Sladek and Sladek (2003); Sladek et al (2005)]. However, these methods do not employ a time-dependent fundamental solution, which gives an accurate result. A Laplace transformation can remove the time dependence of the problems, however, it is not suitable under complicated time-dependent boundary conditions. The Laplace transformation method requires internal cells for the initial temperature distribution.

Recently, the efficient treatment of domain integrals has been proposed by the triple-reciprocity BEM or improved multi-reciprocity BEM for steady heat conduction, steady thermal stress and elastoplastic problems [Ochiai and Sekiya (1996, 1995); Ochiai and Kobayashi (1999); Ochiai (2001)]. The triple-reciprocity BEM for two-dimensional heat conduction and thermal stress analysis for an unsteady state has also been proposed [Ochiai et al (2006); Ochiai (2001); Ochiai and Sladek (2004); Ochiai (2003, 2001); Ochiai et al (1996); Ochiai and Kitayama (2009)]. In this paper, the triple-reciprocity BEM is developed for three-dimensional unsteady heat conduction problems. In this method, the heat generation and initial temperature distributions are interpolated using the boundary integral equations. Since the domain integrals are eliminated, no internal cells are required in the present triple-reciprocity method and the time-dependent solution is employed. A new computer program was developed and applied to solving several problems.

2 Theory

Unsteady heat conduction In unsteady heat conduction problems with heat generation $W_1^S(q,t)$, a temperature T is obtained by solving

$$\nabla^2 T + \frac{W_1^S}{\lambda} = \kappa^{-1}\frac{\partial T}{\partial t}, \tag{1}$$

where κ, λ and t are the thermal diffusivity, heat conductivity and time, respectively. Denoting an arbitrary time and the initial temperature by τÓ and $T_1^{0S}(q,0)$, respectively, the boundary integral equation for the temperature in the case of unsteady heat conduction problems is expressed by [Brebbia et al (1984); Wrobel (2002)]

$$cT(P,t) = -\kappa\int_0^t\int_\Gamma [T(Q,\tau)\frac{\partial T_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n}T_1^*(P,Q,t,\tau)]d\Gamma d\tau$$

$$+\kappa\int_0^t\int_\Omega T_1^*(P,q,t,\tau)\frac{W_1^S(q,\tau)}{\lambda}d\Omega d\tau$$

$$+\int_{\Omega} T_1^*(P,q,t,0)T_1^{0S}(q,0)d\Omega, \quad (2)$$

where c=0.5 on the smooth boundary and c=1 in the domain. Γ and Ω_s represent the boundary and the domain, respectively, p and q are respectively an observation point and a loading point, and r is the distance between p and q. The notations p and q are written as P and Q on the boundary, respectively. In the case of three-dimensional problems, the time-dependent fundamental solution $T_1^*(p,q,t,\tau)$ in Eq. (2) for the unsteady temperature analysis problem and its normal derivative are given by [1,2]

$$T_1^*(p,q,t,\tau) = \frac{1}{[4\pi\kappa(t-\tau)]^{3/2}}\exp[-a] \quad (3)$$

$$\frac{\partial T_1^*(p,q,t,\tau)}{\partial n} = \frac{-2r}{\pi^{3/2}[4\kappa(t-\tau)]^{5/2}}\frac{\partial r}{\partial n}\exp(-a) \quad (4)$$

where

$$a = \frac{r^2}{4\kappa(t-\tau)} \quad (5)$$

As shown in Eq. (2), when there is an arbitrary initial temperature or heat generation distribution, a domain integral becomes necessary.

Interpolation An interpolation method for a distribution of heat generation $W_1^S(q,\tau)$ is shown using the boundary integral equations to avoid the use of internal cells. The polyharmonic function $T_1^{[f]}(p,q)$ for the steady state is given by

$$T^{[f]}(p,q) = \frac{r^{2f-3}}{4\pi(2f-2)\,!}, \quad r = |p-q|, \quad (6)$$

with $\nabla^2 T^{[f+1]} = T^{[f]}$ for $(f = 1,2,...)$ and $\nabla^2 T^{[1]}(p,q) = -\delta(p-q)$.

It is appropriate to utilize the following equations for the three-dimensional interpolation [16,17]:

$$\nabla^2 W_1^S(q,\tau) = -W_2^S(q,\tau) \quad (7)$$

$$\nabla^2 W_2^S(q,\tau) = -\sum_{m=1}^{M} W_3^{PA}(q_m,\tau)\delta(q-q_m), \quad (8)$$

where M is the number of internal points for interpolation. Assuming the spatial distribution of $W_2^s(q,\tau)$to be governed by Eq. (8) with point sources, it is known that $W_2^s(q,\tau)$will be divergent at these source points as the particular solution $\sum_{m=1}^{M} T^{[1]}(p,q_m)W_3^{PA}(q_m,\tau)$. Nevertheless, we can evaluate $W_2^s(q,\tau)$ on the boundary. The term W_2^S of Eq. (7) corresponds to the sum of the curvatures $\partial^2 W_1^S/\partial x^2$, $\partial^2 W_1^S/\partial y^2$and $\partial^2 W_1^S/\partial z^2$. The term W_3^{PA} is the unknown strength of a Dirac function distribution. From Eqs. (7) and (8), the following equation can be obtained.

$$\nabla^4 W_1^S(q,\tau) = \sum_{m=1}^{M} W_3^{PA}(q_m,\tau)\delta(q-q_m) \tag{9}$$

This equation corresponds to equation for the deformation of a fictitious thin plate with M point loads. The "deformation" $W_1^S(q,\tau)$ is given, but the "force of the point load" $W_3^{PA}(q,\tau)$ is unknown. $W_3^{PA}(q,\tau)$ is obtained inversely from the "deformation" $W_1^S(q,\tau)$ of the fictitious thin plate. W_2^S corresponds to the moment of the thin plate. The "moment" W_2^S on the boundary is assumed to be 0, which is the same as that in a natural spline. This indicates that the thin plate is simply supported. Similarly, the distribution of the initial temperature can be interpolated as follows.

$$\nabla^2 T_1^{0S}(q,0) = -T_2^{0S}(q,0) \tag{10}$$

$$\nabla^2 T_2^{0S}(q,0) = -\sum_{m=1}^{M} T_3^{0PA}(q_m,0)\delta(q-q_m) \tag{11}$$

Furthermore, the polyharmonic function $T_f^*(p,q,t,\tau)$ in the unsteady heat conduction problem are defined by

$$\nabla^2 T_{f+1}^*(p,q,t,\tau) = T_f^*(p,q,t,\tau), \quad \left(\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) T_1^*(p,q,t,\tau) = -\delta(p-q)\delta(t-\tau). \tag{12}$$

Using Green's theorem twice, and Eqs. (7)-(12), Eq. (2) becomes

$$cT(P,t) = -\kappa \int_0^t \int_\Gamma [T(Q,\tau)\frac{\partial T_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n} T_1^*(P,Q,t,\tau)] d\Gamma d\tau$$

$$+\frac{\kappa}{\lambda}\sum_{f=1}^{2}(-1)^f \int_0^t \int_\Gamma [T_{f+1}^*(P,Q,t,\tau)\frac{\partial W_f^S(Q,\tau)}{\partial n} - \frac{\partial T_{f+1}^*(P,Q,t,\tau)}{\partial n} W_f^S(Q,\tau)] d\Gamma d\tau$$

$$+\frac{\kappa}{\lambda}\sum_{m=1}^{M}\int_0^t W_3^{PA}(q_m,\tau) T_3^*(P,q_m,t,\tau) d\tau$$

$$+\sum_{f=1}^{2}(-1)^f\int_\Gamma [T_{f+1}^*(P,Q,t,0)\frac{\partial T_f^{0S}(Q,0)}{\partial n}-\frac{\partial T_{f+1}^*(P,Q,t,0)}{\partial n}T_f^{0S}(Q,0)]d\Gamma$$

$$+\sum_{m=1}^{M}T_3^{0PA}(q_m,0)T_3^*(P,q_m,t,0). \quad (13)$$

Similarly, starting from the governing equations (7) and (8), we obtain the integral equation constraints for W_1^S and W_2^S [9-11]

$$cW_1^S(P,\tau)=\sum_{f=1}^{2}(-1)^f\int_\Gamma \{T^{[f]}(P,Q)\frac{\partial W_f^S(Q,\tau)}{\partial n}-\frac{\partial T^{[f]}(P,Q)}{\partial n}W_f^S(Q,\tau)\}d\Gamma$$
$$-\sum_{m=1}^{M}T^{[2]}(P,q_m)W_3^{PA}(q_m,\tau) \quad (14)$$

$$cW_2^S(P,\tau)=\int_\Gamma \{T^{[1]}(P,Q)\frac{\partial W_2^S(Q,\tau)}{\partial n}-\frac{\partial T^{[1]}(P,Q)}{\partial n}W_2^S(Q,\tau)\}d\Gamma$$
$$+\sum_{m=1}^{M}T^{[1]}(P,q_m)W_3^{PA}(q_m,\tau) \quad (15)$$

Eventually, from the governing equations (10) and (11) for the initial temperature T_1^{0S} and T_2^{0S}, we obtain

$$cT_1^0(P,0)=\sum_{f=1}^{2}(-1)^f\int_\Gamma \{T^{[f]}(P,Q)\frac{\partial T_f^{0S}(Q,0)}{\partial n}-\frac{\partial T^{[f]}(P,Q)}{\partial n}T_f^{0S}(Q,0)\}d\Gamma$$
$$-\sum_{m=1}^{M}T^{[2]}(P,q_m)T_3^{0PA}(q_m,0) \quad (16)$$

$$cT_2^{0S}(P,0)=\int_\Gamma \{T^{[1]}(P,Q)\frac{\partial T_2^{0S}(Q,0)}{\partial n}-\frac{\partial T^{[1]}(P,Q)}{\partial n}T_2^{0S}(Q,0)\}d\Gamma$$
$$+\sum_{m=1}^{M}T^{[1]}(P,q_m)T_3^{0PA}(q_m,0) \quad (17)$$

Unsteady polyharmonic function The three-dimensional unsteady polyharmonic function $T_{f+1}^*(p,q,t,\tau)$ in Eq. (12) is determined as

$$T_{f+1}^*(p,q,t,\tau)=\int\frac{1}{r^2}\int r^2T_f^*(p,q,t,\tau)drdr, \quad (18)$$

since $\nabla^2 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f(r)\right)$.

Thus, the polyharmonic function $T_f^*(P,q,t,\tau)$ in the unsteady state and its normal derivative are explicitly given by

$$T_2^*(q,p,t,\tau) = \frac{-1}{4\pi^{3/2} r}\gamma(1/2,a), \quad a = r^2/\beta, \quad \beta = 4\kappa(t-\tau) \tag{19}$$

$$T_3^*(q,p,t,\tau) = \frac{-\sqrt{\beta}}{8\pi^{3/2}}\left[\left(\sqrt{a} + \frac{1}{2\sqrt{a}}\right)\gamma(1/2,a) + e^{-a}\right], \tag{20}$$

with γ being the incomplete gamma function defined as $\gamma(\alpha,x) = \int\limits_0^x t^{\alpha-1} e^{-t} dt$.

Unsteady thermal stress Next, in order to obtain the thermal stresses in uncoupled quasi-static thermoelasticity, let us consider the thermoelastic displacement potential $\Phi(P,t)$ for unsteady problems given by [Tanaka et al (2003)]

$$c\Phi(P,t) = -\kappa\int_0^t\int_\Gamma [T(Q,\tau)\frac{\partial \phi_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n}\phi_1^*(P,Q,t,\tau)] d\Gamma d\tau$$

$$+\kappa\int_0^t\int_\Omega \phi_1^*(P,q,t,\tau)\frac{W_1^S(q,\tau)}{\lambda} d\Omega d\tau + \int_\Omega \phi_1^*(P,q,t,0) T_1^{0S}(q,0) d\Omega \tag{21}$$

Denoting Poisson's ratio by ν and the coefficient of linear thermal expansion by α, m_0 is given by m_0=(1+ν)α/(1-ν). Now, let us introduce the high-order function $\phi_f(p,q,t,\tau)$ defined by

$$\phi_f^*(p,q,t,\tau) = m_0 T_{f+1}^*(p,q,t,\tau) \tag{22}$$

Using Green's theorem twice, and Eqs. (7), (8), (10), (11), Eq. (21) becomes

$$c\Phi(P,t) = -\kappa\int_0^t\int_\Gamma [T(Q,\tau)\frac{\partial \phi_1^*(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n}\phi_1^*(P,Q,t,\tau)] d\Gamma d\tau$$

$$+\kappa\lambda^{-1}\sum_{f=1}^{2}(-1)^f\int_0^t\int_\Gamma\left[\frac{\partial W_f^S(Q,\tau)}{\partial n}\phi_{f+1}^*(P,Q,t,\tau) - W_f^S(Q,\tau)\frac{\partial \phi_{f+1}^*(P,Q,t,\tau)}{\partial n}\right] d\Gamma d\tau$$

$$+\kappa\lambda^{-1}\sum_{m=1}^{M}\int_0^t W_3^{PA}(q_m,\tau)\phi_3^*(P,q_m,t,\tau) d\tau$$

$$+\sum_{f=1}^{2}(-1)^f\int_\Gamma\left[\frac{\partial T_f^{0S}(Q,0)}{\partial n}\phi_{f+1}^*(P,Q,t,0) - T_f^{0S}(Q,0)\frac{\partial \phi_{f+1}^*(P,Q,t,0)}{\partial n}\right] d\Gamma$$

$$+\sum_{m=1}^{M} T_3^{0PA}(q_m,0)\phi_3^*(P,q_m,t,0) \tag{23}$$

Using the relationship between the thermoelastic displacement potential $\Phi(P,t)$ and the displacement, the boundary integral representation for the displacement is obtained as Tanaka et al (2003); Ochiai (2001)

$$c_{ij}(P)u_j(P,t) = \int_\Gamma [u_{ij}(P,Q)p_j(Q,t) - p_{ij}(P,Q)u_j(Q,t)]d\Gamma$$

$$+\kappa \int_0^t \int_\Gamma \left[T(Q,\tau)\frac{\partial u_i^{[1]}(P,Q,t,\tau)}{\partial n} \right.$$

$$\left. -\frac{\partial T(Q,\tau)}{\partial n} u_i^{[1]}(P,Q,t,\tau) \right] d\Gamma d\tau + \kappa\lambda^{-1}$$

$$\sum_{f=1}^{2}(-1)^f \int_0^t \int_\Gamma [W_f^S(Q,\tau)\frac{\partial u_i^{[f+1]}(P,Q,t,\tau)}{\partial n} - \frac{\partial W_f^S(Q,\tau)}{\partial n} u_i^{[f+1]}(P,Q,t,\tau)]d\Gamma d\tau$$

$$-\kappa\lambda^{-1}\sum_{m=1}^{M}\int_0^t W_3^{PA}(q_m,\tau)u_i^{[3]}(P,q_m,t,\tau)d\tau$$

$$+\sum_{f=1}^{2}(-1)^f \int_\Gamma [T_f^{0S}(Q,0)\frac{\partial u_i^{[f+1]}(P,Q,t,\tau)}{\partial n} - \frac{\partial T_f^{0S}(Q,0)}{\partial n} u_i^{[f+1]}(P,Q,t,\tau)]d\Gamma$$

$$-\sum_{m=1}^{M} T_3^{0PA}(q_m,0)u_i^{[3]}(P,q_m,t,0) \tag{24}$$

and c_{ij} is the free coefficient. Moreover, u_j and p_j are the $j-$th components of the displacement and surface traction, respectively. Kelvin's solutions, namely, $u_{ij}(p,q)$ and $p_{ij}(p,q)$, are given by

$$u_{ij}(P,Q) = \frac{1}{16\pi(1-\nu)Gr}[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}] \tag{25}$$

$$p_{ij}(P,Q) = \frac{1}{8\pi(1-\nu)Gr^2}\{[(1-2\nu)\delta_{ij}+3r_{,i}r_{,j}]\frac{\partial r}{\partial n} - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\} \tag{26}$$

and n_i is the unit normal component. where ν is Poisson's ratio, and G is the shear modulus. The i-th component of a unit normal vector is denoted by n_i. Moreover, $r_{,i} = \partial r/\partial x_i$.

$$u_i^{[1]}(q,p,t,\tau) = m_0 T_2^*{}_{,i} = \frac{m_0 r_{,i}}{2\pi^{3/2}r^2}\gamma(1.5,a) \tag{27}$$

$$u_i^{[2]}(q,p,t,\tau) = m_0 T_3^*,_i = \frac{m_0 r_{,i}}{8\pi^{3/2}}[-\gamma(0.5,a) + \frac{1}{a}\gamma(1.5,a)] \tag{28}$$

$$u_i^{[3]}(p,q,t,\tau) = \frac{m_0 r_{,i}\, r^2}{32\pi^{3/2}}\left\{[\gamma(2.5,a) - \gamma(1.5,a)]\, a^{-2} - (1+a^{-1})\gamma(0.5,a)\right\} \tag{29}$$

$$\int_{t_{f-1}}^{t_f} u_i^{[1]}(p,q,t,\tau)d\tau = \frac{m_0 r_{,i}}{8\kappa\pi^{3/2}}\left\{\gamma(1/2,z) - \frac{1}{z}\gamma(3/2,z)\right\}\Bigg|_{a_{f-1}}^{a_f}, \quad a_f = \frac{r^2}{4\kappa(t-t_f)} \tag{30}$$

$$\int_{t_{f-1}}^{t_f} u_i^{[2]}(p,q,t,\tau)d\tau = \frac{m_0 r^2 r_{,i}}{32\kappa\pi^{3/2}}\left\{-\frac{1}{2z^2}\gamma(3/2,z) + \left(1+\frac{1}{z}\right)\gamma(1/2,z) + z^{-1/2}e^{-z}\right\}\Bigg|_{a_{f-1}}^{a_f} \tag{31}$$

$$\int_{t_{f-1}}^{t_f} u_i^{[3]}(p,q,t,\tau)d\tau = \frac{m_0 r^4 r_{,i}}{128\kappa\pi^{3/2}}\left\{\left(\frac{2}{9}+\frac{1}{z}+\frac{1}{2z^2}\right)\gamma(1/2,z) + \frac{1}{9}\left(2+\frac{5}{z}\right)z^{-1/2}e^{-z}\right\}\Bigg|_{a_{f-1}}^{a_f} \tag{32}$$

Internal stress In the same manner, internal stress can be obtained.

$$\sigma_{ij}(p,t) = \int_\Gamma [-\sigma_{kij}(p,Q)p_k(Q,t) - S_{kij}(p,Q)u_k(Q,t)]d\Gamma$$

$$+\kappa\int_0^t\int_\Gamma\left[T(Q,\tau)\frac{\partial\sigma_{ij}^{[1]}(P,Q,t,\tau)}{\partial n} - \frac{\partial T(Q,\tau)}{\partial n}\sigma_{ij}^{[1]}(P,Q,t,\tau)\right]d\Gamma d\tau + \kappa\lambda^{-1}$$

$$\sum_{f=1}^{2}(-1)^f\int_0^t\int_\Gamma[W_f^S(Q,\tau)\frac{\partial\sigma_{ij}^{[f+1]}(P,Q,t,\tau)}{\partial n} - \frac{\partial W_f^S(Q,\tau)}{\partial n}\sigma_{ij}^{[f+1]}(P,Q,t,\tau)]d\Gamma d\tau$$

$$-\kappa\lambda^{-1}\sum_{m=1}^{M}\int_0^t W_3^{PA}(q_m,\tau)\sigma_{ij}^{[3]}(P,q_m,t,\tau)d\tau$$

$$+\sum_{f=1}^{2}(-1)^f\int_\Gamma[T_f^{0S}(Q,0)\frac{\partial\sigma_{ij}^{[f+1]}(P,Q,t,0)}{\partial n} - \frac{\partial T_f^{0S}(Q,0)}{\partial n}\sigma_{ij}^{[f+1]}(P,Q,t,0)]d\Gamma$$

$$-\sum_{m=1}^{M} T_3^{0PA}(q_m,0)\sigma_{ij}^{[3]}(P,q_m,t,0) \tag{33}$$

where

$$\sigma_{ij}^{[1]}(p,q,t,\tau) = \frac{Gm_0}{\pi^{3/2}r^3}\left\{\frac{\delta_{ij}}{1-2\nu}[(1+\nu)\gamma(3/2,a)-2\nu\gamma(5/2,a)] - r_{,i}r_{,j}2\gamma(5/2,a)\right\} \tag{34}$$

$$\sigma_{ij}^{[2]}(q,p,t,\tau) = \frac{Gm_0}{4\pi^{3/2}r}\left\{\delta_{ij}\left[\frac{1}{a}\gamma(3/2,a)-\frac{1}{1-2\nu}\gamma(1/2,a)\right] - r_{,i}r_{,j}\left[\frac{3}{a}\gamma(3/2,a)-\gamma(1/2,a)\right]\right\} \tag{35}$$

$$\sigma_{ij}^{[3]}(p,q,t,\tau) = \frac{-Gm_0 r}{32\pi^{3/2}}\left\{\delta_{ij}\left[\frac{2}{1-2\nu}\left((1+2\nu+1/a)\gamma(1/2,a)+\frac{1+2\nu}{\sqrt{a}}e^{-a}\right)-\frac{1}{a^2}\gamma(3/2,a)\right]\right\}$$
$$+ r_{,i}r_{,j}\left[2(1-1/a)\gamma(1/2,a)+\frac{3}{a^2}\gamma(3/2,a)+\frac{2}{\sqrt{a}}e^{-a}\right] \tag{36}$$

3 Numerical Examples

To verify the efficiency of this method, an unsteady thermal stress distribution in a sphere is analyzed. The initial temperature of the sphere is $T_0 = 10°C$, and the temperature on the surface suddenly becomes 0° at the time $t = 0$. It is assumed that the thermal diffusivity is κ =16 mm^{2}s^{-1}and the radius of the sphere is $b = 10$ mm. Figure 1 shows the boundary elements. In this example, internal points are not necessary. Figure 2 shows the radial distributions of the temperature field at several time instants. The solid lines in Fig.2 show the exact solutions, which are given by

$$T(r,t) = \frac{2bT_0}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi r}{b}\exp\left(-\frac{\kappa n^2\pi^2 t}{b^2}\right) \tag{37}$$

Young's modulus E, Poisson's ratio ν and the coefficient of linear thermal expansion α are assumed to be 210 GPa, 0.3 and 11×10^{-6}K^{-1}, respectively. The sphere is not loaded mechanically. Figure 3 shows the numerical and exact results for the radial and circumferential thermal stress distributions.

Next, an unsteady thermal stress distribution in the sphere with the constant heat generation $W_0/\lambda = 10K \cdot mm^{-2}$ is obtained. The initial temperature of the sphere is $T_0 = 0°C$, and the temperature on the surface is $0°C$. The other specifications are the same as in Figs.2 and 3. In this case again, internal nodes are not employed. Figures 4 and 5 show the radial distributions of the temperature and stress fields at several time instants. The solid lines show the exact solutions.

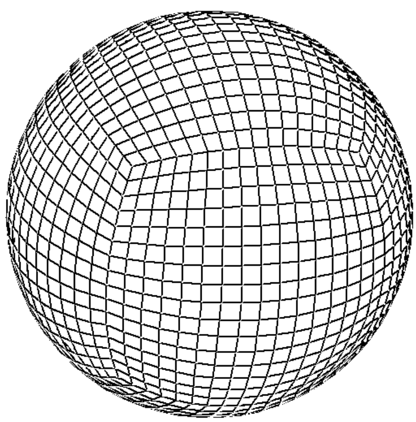

Figure 1: Boundary elements of sphere region

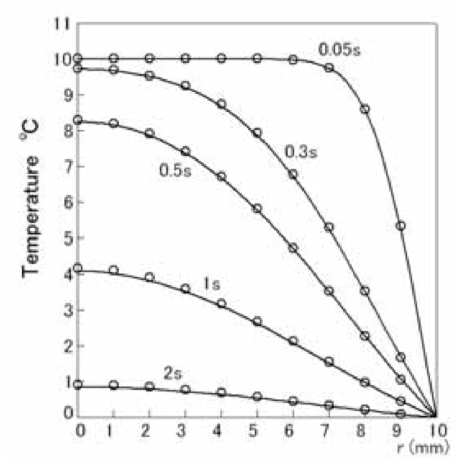

Figure 2: Temperature distributions in sphere

Finally, the unsteady temperature distribution in a solid circular cylinder with an initial temperature distribution is analyzed. The outer diameter is $2b$, the outer temperature is 0 $°C$, and the upper and lower surfaces are adiabatic isolated. Figure 6 shows the boundary elements and internal points. The distribution of initial temperature is given by

$$T(r,0) = T_0 \frac{b^2 - r^2}{b^2}, \tag{38}$$

and step heating is assumed. The thermal diffusivity is $\kappa = 16$ mm^{2}s^{-1}, b=10 mm and $T_0 = 100$ $°C$. Figure 7 shows the exact and numerical results for the temperatures at t=0.2, 0.5, 1 and 2 s obtained by present method. Young's modulus E, Poisson's ratio ν and the coefficient of linear thermal expansion α are assumed to be 210 GPa, 0.3 and 11×10^{-6}K^{-1}, respectively. Figure 8 shows the radial and circumferential thermal stress distributions.

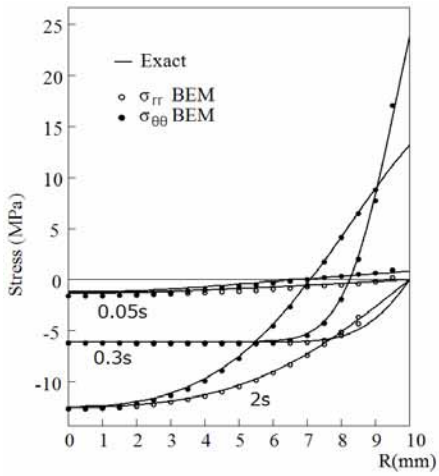

Figure 3: Stress distributions in sphere ($n = 150$)

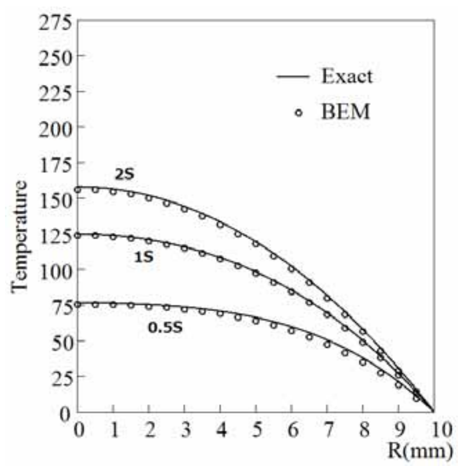

Figure 4: Temperature distributions in sphere with heat generation

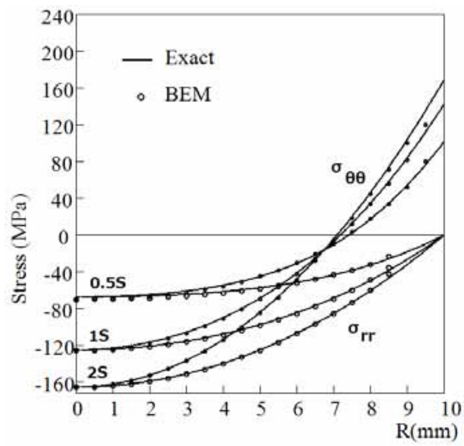

Figure 5: Stress distributions in sphere with heat generation

4 Conclusion

The triple reciprocity boundary element method has been developed for unsteady thermoelastic 3D problems within quasi-static uncoupled thermoelasticity. The well known BEM dimensionality reduction is achieved since the unknowns are localized on the boundary alone. Arbitrary heat sources as well as initial temperature

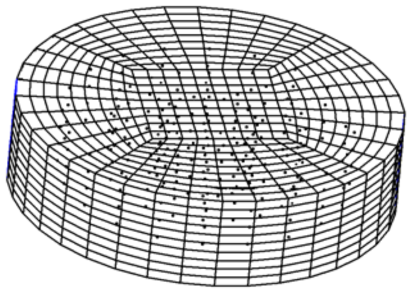

Figure 6: Circular cylinder with initial temperature

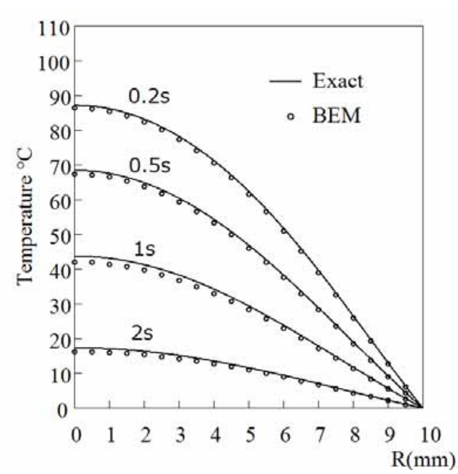

Figure 7: Temperature distributions in cylinder

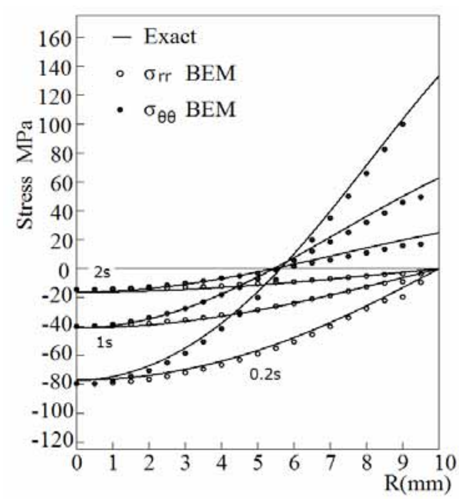

Figure 8: Stress distributions in cylinder

distributions are allowed with prescribing their values at certain interior and boundary points. The triple reciprocity formulation utilizes only low order polyharmonic fundamental solutions. The formulation as well as the developed computer code and the efficiency of the proposed method have been verified in several numerical test examples for which the exact solutions are available.

Acknowledgement: This work was partially supported by the Slovak Research and Development Agency under the contract No. APVV-0032-10.

References

Brebbia, C. A., Telles, J. C. F., Wrobel, L. C. (1984): Boundary Element Techniques: Theory and Applications in Engineering, pp. 47-107, Berlin, Springer-Verlag.

Nowak, A. J., Neves, A. C. (1994): The Multiple Reciprocity Boundary Element Method, Computational Mechanics Publication, Southampton, Boston.

Ochiai, Y., Sekiya, T. (1996): Steady Heat Conduction Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 18, pp. 111-117.

Ochiai, Y., Sekiya, T. (1995): Steady Thermal Stress Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Journal of Thermal Stresses*, Vol. 18, No. 6, pp. 603-620.

Ochiai, Y., Kobayashi, T. (1999): Initial Stress Formulation for Elastoplastic Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 23, pp. 167-173.

Ochiai, Y. (2001): Two-Dimensional Unsteady Heat Conduction Analysis with Heat Generation by Triple-Reciprocity BEM, *International Journal of Numerical Methods in Engineering*, Vol. 51, No. 2, pp. 143-157.

Ochiai, Y., Sladek V., Sladek, J. (2006): Transient Heat Conduction Analysis by Triple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 30, pp. 194-204.

Ochiai, Y. (2001): Two-Dimensional Unsteady Thermal Stress Analysis by Triple-Reciprocity Boundary Element Method, *Journal of Thermal Stresses*, Vol. 24, No. 3, pp. 233-253.

Ochiai Y., Sladek, V. (2004): Numerical Treatment of Domain Integrals without Internal Cells in Three-Dimensional BIEM Formulations, *CMES: Computer Modeling in Engineering & Sciences*, Vol. 6, No. 6, pp. 525-536.

Ochiai, Y. (2003): Multidimensional Numerical Integration for Meshless BEM, *Engineering Analysis with Boundary Elements*, Vol. 27, No. 3, pp. 241-249.

Ochiai Y. (2001): Two-Dimensional Unsteady Heat Conduction Analysis with Heat Generation by Triple-Reciprocity BEM, *International Journal of Numerical Methods for Engineering*, Vol.51, No.2, pp.143-157.

Ochiai, Y., Nishitani, H., Sekiya, T. (1996): Two-dimensional thermal stress analysis under unsteady state by improved boundary element method, *Journal of Thermal Stresses*, Vol.19, pp.107-121.

Ochiai, Y., Kitayama, Y., (2009): Three-dimensional Unsteady Heat Conduction Analysis by Triple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 33, No.6, pp.789-795.

Sharp, S., Crouch, S.L. (1986): Boundary integral method for thermoelasticity problem, *Transaction of the ASME, Journal of Applied Mechanics*, Vol.53, pp.298-302.

Sladek, J., Sladek, V. (2003): Local Boundary Integral Equation Method for Heat Conduction Problem in an Anisotropic Medium, Proceeding of ICCES2003, Chap. 5.

Sladek, V., Sladek, J., Tanaka, M., Zhang, Ch. (2005): Transient heat conduction in anisotropic and functionally graded media by local integral equations, *Engineering Analysis with Boundary Elements*, Vol. 29, pp. 1047-1065.

Tanaka, M., Matsumoto, T., Takakuwa, S. (2003): Dual Reciprocity BEM Based on Time-Stepping Scheme for the Solution of Transient Heat Conduction Problems, Boundary Elements XXV, WIT Press, pp. 299-308.

Wrobel, L. C. (2002): The Boundary Element Method, Volume 1, John Wiley & Sons, West Sussex, pp.97-117.