

A LBIE Method for Solving Gradient Elastostatic Problems

E.J. Sellountos¹, S.V. Tsinopoulos² and D. Polyzos³

Abstract: A Local Boundary Integral Equation (LBIE) method for solving two dimensional problems in gradient elastic materials is presented. The analysis is performed in the context of simple gradient elasticity, the simplest possible case of Mindlin's Form II gradient elastic theory. For simplicity, only smooth boundaries are considered. The gradient elastic fundamental solution and the corresponding boundary integral equation for displacements are used for the derivation of the LBIE representation of the problem. Nodal points are spread over the analyzed domain and the moving least squares (MLS) scheme for the approximation of the interior and boundary variables is employed. Since in gradient elasticity the equilibrium equation is a partial differential equation of forth order, the MLS is ideal for solving these problems since it holds the $C^{(1)}$ continuity property. The companion solution of displacements is explicitly derived and introduced in the LBIEs for zeroing the tractions and double tractions on the local circular boundaries. Two representative numerical examples are presented to illustrate the method, demonstrate its accuracy and assess the gradient effect in the response.

Keywords: LBIE, meshless, gradient elasticity.

1 Introduction

It is well known that classical theory of elasticity fails to describe microstructural and size effects in materials and structures due to the lack of internal length scale parameters in its constitutive equations. This is possible with the use of other enhanced elastic theories where intrinsic parameters correlating the microstructure with the macrostructure are involved in the constitutive equations as well

¹ Department of Mathematics and CEMAT, Instituto Superior Técnico, Technical University of Lisbon, Lisbon, Portugal.

² Department of Mechanical Engineering, Technological and Educational Institute of Patras, GR-26334, Patras, Greece.

³ Department of Mechanical Engineering and Aeronautics, University of Patras, Greece, Institute of Chemical Engineering and High Temperature Process ICETH-FORTH, Rio, Greece.

as in the equilibrium equations of the considered elastic continuum. The most general of these theories are known in the literature as Cosserat elastic theory [Cosserat and Cosserat (1909)], Cosserat theory with constrained rotations or couple stresses theory [Mindlin and Tiersten (1962)], [Koiter (1964)] strain gradient theory [Toupin (1964)], multipolar elastic theory [Green and Rivlin (1964)], higher order strain gradient elastic theory [Mindlin (1964)], [Mindlin (1965)], micromorphic, microstretch and micropolar elastic theories [Eringen (1999)] and non-local elasticity [Eringen (1992)].

A very attractive enhanced elastic theory is the strain gradient elasticity proposed by Mindlin [Mindlin (1964)]. It is known as Form II gradient elasticity and it is a special case of the general gradient theory that Mindlin addressed in the same paper. It is characterized as attractive because i) in the static version only five intrinsic parameters are needed ii) it can be simplified further to the simple gradient elastic theory with only one internal length scale parameter and the most important iii) strains and stresses appearing in the constitutive equations are symmetric as in the classical case. Although elegant, even the simple gradient elastic theory is mathematically much more complex than the classical theory of elasticity and the solution of even very simple boundary value problems is very difficult. An obvious solution is to resort to well-known numerical methods such as the Finite Element Method (FEM) and the Boundary Element Method (BEM).

Undoubtedly the FEM is the most widely used numerical method for solving problems in applied mechanics. Oden et al. [Oden, Rigsby, and Cornett (1970)] were the first who used FEM for solving gradient elastic problems in the framework of Mindlin's first gradient elastic theory [Mindlin (1964)] and, after almost three decades, Shu et al. [Shu, King, and Fleck (1999)] demonstrated a mixed FEM formulation to treat a modified version of Mindlin's theory [Fleck and Hutchinson (1993)]. Since then, two main categories of FEM techniques have appeared in the literature [Papanicolopoulos, Zervos, and Vardoulakis (2010)]. The first deals with displacement FEM formulations and the second with multi-field finite element methodologies. Although simple and efficient, the displacement FEM formulation requires $C^{(1)}$ continuous elements for the interpolation of the displacements. This is due to the presence of higher order gradients of strains in the expressions of potential energy that leads to an equilibrium equation represented by a fourth order partial differential operator in displacements. The use of $C^{(1)}$ elements appears problems dealing with the numerical evaluation of very complicated shape functions, the introduction of many degrees of freedom per elements and inaccuracies coming from the non-isoparametric interpolation of geometry. The alternative multi-field FEM or other $C^{(0)}$ FEM formulations such as mixed formulations [Imatani, Hatada, and Maugin (2005)], [Markolefas, Tsouvalas, and Tsamasphyros (2007)],

Lagrange multipliers [Shu, King, and Fleck (1999)], [Matsushima, Chambon, and Caillerie (2002)], penalty parameters [Zervos (2008)], implicit and explicit formulations [Tenek and Aifantis (2002)], [Askes and Gutierrez (2006)], nonconforming $C^{(0-1)}$ quadrilateral elements [Soh and Chen (2004)], [Zhao, Chen, and Lo (2011)] and continuous-discontinuous finite element approximations [Engel, Garikipati, Hughes, Larson, Mazzei, and Taylor (2002)] are also associated with drawbacks coming from incompatibilities of the different approximated fields.

On the other hand, the BEM is a well-known and powerful numerical tool, successfully used for solving various types of engineering problems [Beskos (1987)], [Beskos (1997)]. Advantages it offers as compared to FEM is the reduction of the dimensionality of the problem by one and the absence of $C^{(1)}$ continuity requirements when gradient elastic problems are dealt with. Tsepoura et al. [Tsepoura, Papargyri-Beskou, and Polyzos (2002)] were the first to use BEM for solving elastostatic problems in the framework of gradient elastic theories. This work was followed by [Tsepoura and Polyzos (2003)], [Papacharalampopoulos, Karlis, Charalambopoulos, and Polyzos (2010)], which are the only studies dealing with two and three dimensional BEM solutions of static gradient elastic and fracture mechanics problems. The works [Karlis, Charalambopoulos, and Polyzos (2010)], [Papacharalampopoulos, Karlis, Charalambopoulos, and Polyzos (2010)] deal with Mindlin's Form II strain gradient elastic theory, while all the other papers implement simplified static versions of Mindlin's theory like the simple strain gradient elasticity and gradient elasticity with surface energy [Vardoulakis, Exadaktylos, and Aifantis (1996)], [Vardoulakis and Sulem (1995)]. The main drawbacks with the aforementioned BEM formulations are the use of two integral equations per node (one for displacements and one for the normal derivative of displacements) and the full populated matrices of the final system of algebraic equations, rendering the solution process time consuming.

Atluri and co-workers proposed the Local Boundary Integral Equation (LBIE) method [Zhu, Zhang, and N.Atluri (1998)] and the Meshless Local Petrov- Galerkin (MLPG) method [Atluri and Zhu (1998)] as alternatives to the BEM and FEM, respectively. Both methods are characterized as "truly meshless" since no background cells are required for the numerical evaluation of the involved integrals. Properly distributed nodal points, without any connectivity requirement, covering the domain of interest as well as the surrounding global boundary are employed instead of any boundary or finite element discretization. All nodal points belong in regular sub-domains (e.g. circles for two-dimensional problems) centered at the corresponding collocation points. The fields at the local and global boundaries as well as in the interior of the subdomains are usually approximated by the Moving Least Squares (MLS) approximation scheme. Owing to regular shapes of the sub-domains, both surface

and volume integrals are easily evaluated. The local nature of the sub-domains leads to a sparse linear system of equations. Both LBIE and MLPG methods are ideal for treating gradient elastic problems since they utilize $C^{(1)}$ continuous MLS interpolation functions. Details on LBIE and MLPG methods one can find in the papers [Sellountos and Polyzos (2003)], [Sladek, Sladek, and Keer (2003)] and in the books [Atluri and Shen (2002)] and [Atluri (2004)].

In the present work, the LBIE method is employed for the solution of 2D problems in gradient elastic materials. The analysis is performed in the context of simple gradient elasticity, the simplest possible special case of Mindlin's Form II gradient elastic theory [Karlis, Tsinopoulos, Polyzos, and Beskos (2007)]. For the sake of simplicity only bodies with smooth boundaries are considered. In contrast to the BEM formulations [Tsepoura, Papargyri-Beskou, and Polyzos (2002)], [Papacharalampopoulos, Karlis, Charalambopoulos, and Polyzos (2010)], only the displacement LBIE is utilized throughout the analyzed domain. This is accomplished with the aid of a companion solution, explicitly derived in Appendix, which zeroes both tractions and double tractions on the boundary of the circular local domains. Thus, the proposed LBIE method has the same philosophy with that of Sellountos and Polyzos [Sellountos and Polyzos (2003)] used for the solution of classical elastic problems. To the authors' best knowledge, this is the first work dealing with meshless LBIE solution of gradient elastic problems. It should be mentioned however, the MLPG methodologies of Tang et al. [Tang, Shen, and Atluri (2003)], and Sun and Liew [Sun and Liew (2008)] for solving problems in the framework of Mindlin's general gradient elastic theory, the LBIE method of Sladek and Sladek [Sladek, Sladek, and Keer (2003)] for treating micropolar elastic problems [Eringen (1999)] and the Element Free Galerkin techniques of Askes and Aifantis [Askes and Aifantis (2002)] and Pamin et al. [Pamin, Askes, and de Borst (2003)] for solving problems in implicit/explicit gradient elasticity and plasticity, respectively. The paper consists of the following five sections: Section 2 presents in brief the simple gradient elastic theory and how it is taken from Mindlin's Form II gradient elasticity. In Section 3 the LBIE of the problem is explicitly derived, while in Section 4 the numerical implementation of the proposed methodology is illustrated. Section 5 provides two numerical examples to demonstrate the accuracy of the method and illustrate the microstructural effects. Finally, Section 6 consists of the conclusions pertaining to this work.

2 Simple Gradient Elastic Theory

Mindlin in the Form II version of his general strain gradient elastic theory [Mindlin (1964)] considered that the potential energy W of an isotropic elastic body with microstructure of volume V is a quadratic form of the strains ε_{ij} and the gradient of

strains, κ_{ijk} i.e.,

$$W = \int_V \left(\frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + \hat{\alpha}_1 \kappa_{iik} \kappa_{kjj} + \hat{\alpha}_2 \kappa_{ijj} \kappa_{ikk} + \hat{\alpha}_3 \kappa_{iik} \kappa_{jjk} + \hat{\alpha}_4 \kappa_{ijk} \kappa_{ijk} + \hat{\alpha}_5 \kappa_{ijk} \kappa_{kji} \right) dV \quad (1)$$

where

$$\varepsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \quad (2)$$

$$\kappa_{ijk} = \partial_i \varepsilon_{jk} = \frac{1}{2} (\partial_i \partial_j u_k + \partial_i \partial_k u_j) = \kappa_{ikj} \quad (3)$$

with ∂_i denoting space differentiation, u_i displacements, λ, μ the Lamé constants and $\hat{\alpha}_1, \dots, \hat{\alpha}_5$ intrinsic microstructural parameters, explicitly defined in [Mindlin (1964)].

Strains ε_{ij} and gradient of strains κ_{ijk} are dual in energy with the Cauchy-like and double stresses, respectively, defined as

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ji} \quad (4)$$

$$\mu_{ijk} = \frac{\partial W}{\partial \kappa_{ijk}} = \mu_{ikj} \quad (5)$$

which implies that

$$\tau_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{ll} \delta_{ij} \quad (6)$$

and

$$\mu_{ijk} = \frac{1}{2} \hat{\alpha}_1 [\kappa_{kll} \delta_{ij} + 2\kappa_{lli} \delta_{jk} + \kappa_{jll} \delta_{ki}] + 2\hat{\alpha}_2 \kappa_{ill} \delta_{jk} + \hat{\alpha}_3 (\kappa_{llk} \delta_{ij} + \kappa_{llj} \delta_{ik}) + 2\hat{\alpha}_4 \kappa_{ijk} + \hat{\alpha}_5 (\kappa_{kij} + \kappa_{jki}) \quad (7)$$

For a static problem, the equilibrium equation of the considered gradient elastic body as well as the possible boundary conditions that establish a well-posed boundary value problem can be determined with the aid of energy variational principle, written as

$$\delta (W - W_1) = 0 \quad (8)$$

where δ denotes variation and W_1 represents the work done by external forces.

By inserting Eq. 1 into Eq. 8 and taking into account the body forces F_k , the following equation of equilibrium is obtained

$$\partial_j (\tau_{jk} - \partial_i \mu_{ijk}) + F_k = 0 \tag{9}$$

accompanied by the classical essential and natural boundary conditions where the displacements u_k and/or tractions p_k have to be defined on the global boundary S of the analyzed domain, the non- classical essential and natural boundary conditions where normal strains $q_k = \partial u_k / \partial n$ and/or double tractions R_k are prescribed on S , and the non- classical boundary condition satisfied only when non-smooth boundaries are dealt with, where the jump traction vector E_k has to be defined at corners and edges. Traction vectors p_k, R_k, E_k are defined as

$$p_k = n_j \tau_{jk} - n_i n_j D \mu_{ijk} - (n_j D_i + n_i D_j) \mu_{ijk} + (n_i n_j D_l n_l - D_j n_i) \mu_{ijk} \tag{10}$$

$$R_k = n_i n_j \mu_{ijk} \tag{11}$$

$$E_k = ||n_i m_j \mu_{ijk}|| \tag{12}$$

where n_i is the unit vector normal to the global boundary S , $D = n_l \partial_l$ and $D_j = (\delta_{jl} - n_j n_l) \partial_l$ is the surface gradient operator.

The non-classical boundary condition Eq. 12 exists only when non-smooth boundaries are considered. Double brackets $||\cdot||$ indicate that the enclosed quantity is the difference between its values taken on the two sides of all corners while m_i is a vector being tangential to the corner line.

Finally, by taking into account relations Eq. 6 and Eq. 7, the equilibrium equation Eq. 9 in terms of displacements is written as

$$\mu \partial_i u_k + (\lambda + \mu) \partial_k \partial_i u_i + [\mu l_2^2 - (\lambda + 2\mu) l_1^2] \partial_j^2 \partial_k \partial_i u_i - \mu l_2^2 \partial_j^2 \partial_i^2 u_k + F_k = 0 \tag{13}$$

where

$$l_1^2 = 2(\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4 + \hat{a}_5) \frac{1}{\lambda + 2\mu} \tag{14}$$

and

$$l_2^2 = (\hat{a}_3 + 2\hat{a}_4 + \hat{a}_5) \frac{1}{2\mu}. \tag{15}$$

The above intrinsic parameters l_1^2, l_2^2 have units of length square (m^2) and represent the effect of the stiffness of the microstructure on the macrostructural behavior of the gradient elastic material. l_1^2 is related to longitudinal deformations while l_2^2 to shear ones. Positive definiteness of the potential energy requires $\mu, \lambda + 2\mu > 0, l_i^2 > 0$.

The simple gradient elastic theory considers that microstructural effects are the same for both longitudinal and shear deformations, i.e. $l_1^2 = l_2^2 = g^2$. The consequence of this simplification is that $\hat{a}_1 = \hat{a}_3 = \hat{a}_5 = 0$ and $\hat{a}_2 = \frac{\lambda}{2}g^2, \hat{a}_4 = \mu g^2$. Thus, the equilibrium equation Eq. 13 and the double stress tensor Eq. 7 obtain the simpler form, respectively

$$(1 - g^2 \partial_j^2) [\mu \partial_i^2 u_k + (\lambda + \mu) \partial_k \partial_i u_i] + F_k = 0 \tag{16}$$

$$\mu_{ijk} = \partial_i (2\mu \varepsilon_{jk} + \lambda \varepsilon_{ll} \delta_{jk}) \tag{17}$$

Equations Eq. 16 in conjunction with Eq. 6, Eq. 17 and the classical and non classical boundary conditions form a well posed boundary value problem, which can be solved with the LBIE method as it is explained in the next section.

3 LBIEs for Gradient Elastic Material

Consider a finite gradient elastic body of volume Ω surrounded by a smooth boundary S . Polyzos et al [Polyzos, Tsepoura, Tsinopoulos, and Beskos (2003)] have proved that for two deformation states of the same gradient elastic body the following reciprocal identity is valid

$$\int_V (F_i^* u_i - F_i u_i^*) dV + \int_S (p_i^* u_i - p_i u_i^*) dS = \int_S \left(R_i \frac{\partial u_i^*}{\partial n} - R_i^* \frac{\partial u_i}{\partial n} \right) dS \tag{18}$$

where F_i represents body forces and p_i, R_i traction and double traction vectors defined by Eq. 10 and Eq. 11, respectively.

Assume as "*" the deformation state, due to the fundamental tensor of the gradient elastic problem, which satisfies the partial differential equation

$$(1 - g^2 \partial_k^2) [\mu \partial_k^2 u_{ij}^*(r) + (\lambda + \mu) \partial_i \partial_k u_{kj}^*(r)] + \delta(r) \delta_{ij} = 0$$

and it has the form [Polyzos, Tsepoura, Tsinopoulos, and Beskos (2003)]

$$\begin{aligned} \tilde{u}_{ij}^* &= \frac{1}{8\pi\mu(1-\nu)} [\Psi(r) \delta_{ij} - X(r) \partial_i r \partial_j r] \\ X(r) &= -1 + \frac{4g^2}{r^2} - 2K_2 \left(\frac{r}{g} \right) \\ \Psi(r) &= -(3-4\nu) \ln r + \frac{2g^2}{r^2} - (3-4\nu) K_0 \left(\frac{r}{g} \right) - K_2 \left(\frac{r}{g} \right) \end{aligned} \tag{19}$$

with $\delta(r)$ being Dirac δ -function, δ_{ij} Kronecker's delta, ν Poisson's ratio, K_0 and K_2 zero and second order modified Bessel functions of second kind, respectively and $r = |\mathbf{x} - \mathbf{y}|$ where \mathbf{x} , \mathbf{y} are the observation and the source point.

By considering zero body forces ($F_i = 0$) and replacing F_i^* by $F_{ij}^* = \delta(r) \delta_{ij}$, identity Eq. 18 leads to an integral representation of the gradient elastic problem of the form

$$\frac{1}{2}u_i(\mathbf{x}) + \int_S \{p_{ij}^*(\mathbf{x}, \mathbf{y})u_j(\mathbf{y}) - u_{ij}^*(\mathbf{x}, \mathbf{y})p_j(\mathbf{y})\} dS_{\mathbf{y}} = \int_S \{q_{ij}^*(\mathbf{x}, \mathbf{y})R_j(\mathbf{y}) - R_{ij}^*(\mathbf{x}, \mathbf{y})q_j(\mathbf{y})\} dS_{\mathbf{y}} \tag{20}$$

where u_{ij}^* is the fundamental solution given by Eq. 19, $q_{ij}^* = \partial u_{ij}^*/\partial n$ and p_{ij}^*, R_{ij}^* fundamental traction and double traction tensors defined by Eq. 10 and Eq. 11, respectively.

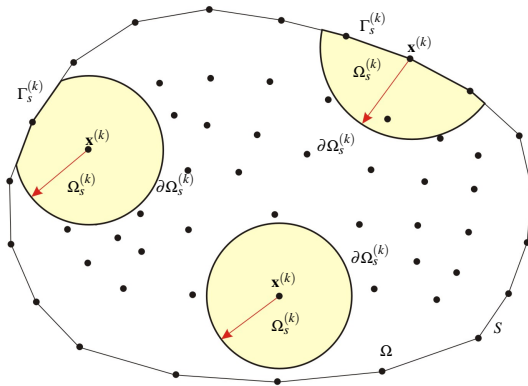


Figure 1: Local domains and boundaries used for the LBIE representation of displacements at point \mathbf{x}^k

Adopting the meshless LBIE methodology of [Sellountos and Polyzos (2003)] both the domain Ω and the global boundary S are covered by randomly distributed points, as shown in Fig 1. Each point \mathbf{x}^k is the center of a circular area called support domain. Taking into account that u_{ij}^* is singular only when $\mathbf{x} \equiv \mathbf{y}$ and applying the reciprocal identity Eq. 18 for the domain between S and $\partial\Omega_s^{(k)}$ (Fig 1), one obtains a local integral representation of displacements at point $\mathbf{x}^{(k)}$ called

LBIE written as

$$u_i(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \left\{ p_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) u_j(\mathbf{y}) - u_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) p_j(\mathbf{y}) \right\} dS_y = \int_{\partial\Omega_s^{(k)}} \left\{ q_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) R_j(\mathbf{y}) - R_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) q_j(\mathbf{y}) \right\} dS_y \quad (21)$$

when the support domain lies entirely in Ω and

$$cu_i(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)} \cup \Gamma_s^{(k)}} \left\{ p_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) u_j(\mathbf{y}) - u_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) p_j(\mathbf{y}) \right\} dS_y = \int_{\partial\Omega_s^{(k)} \cup \Gamma_s^{(k)}} \left\{ q_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) R_j(\mathbf{y}) - R_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) q_j(\mathbf{y}) \right\} dS_y \quad (22)$$

when the support domain intersects the global boundary S .

The constant c is equal to 1 for internal nodes $\mathbf{x}^{(k)} \in \Omega$ and 0.5 for boundary points $\mathbf{x}^{(k)} \in S$ lying on smooth boundary, while $\Gamma_s^{(k)}$ is the intersected part of S .

In order to get rid of tractions and double tractions in integrals defined on $\partial\Omega_s^{(k)}$ the use of companion solution u_{ij}^c is made. It is a regular function of r satisfying the partial differential equation

$$(1 - g^2 \partial_r^2) [\mu \partial_k^2 u_{ij}^c(r) + (\lambda + \mu) \partial_i \partial_k u_{kj}^c(r)] = 0 \quad (23)$$

with the boundary conditions

$$\begin{aligned} u_{ij}^c(r_0) &= u_{ij}^*(r_0) \\ q_{ij}^c(r_0) &= q_{ij}^*(r_0) \end{aligned} \quad (24)$$

where r_0 is the radius of the support domain $\Omega_s^{(k)}$ depicted in Fig 1, u_{ij}^* is the fundamental solution of the gradient elastic problem given by Eq. 20 and $q_{ij}^* = \partial u_{ij}^* / \partial n$, i.e.

$$q_{ij}^* = \frac{1}{8\pi\mu(1-\nu)} \left[\left(\frac{d\Psi}{dr} - \frac{2X}{r} \right) (n_k r_k) \partial r_i \partial r_j + \frac{d\Psi}{dr} n_k r_k \delta_{ij} - \frac{X}{r} (n_i r_j + n_j r_i) \right]$$

The derivation of u_{ij}^c is provided in Appendix. By considering displacements u_i and companion solution u_{ij}^c as the two deformation states of identity Eq. 18, one obtains

$$\int_{\partial\Omega_s^{(k)}} \left\{ p_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) u_j(\mathbf{y}) - u_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) p_j(\mathbf{y}) \right\} dS_y = \int_{\partial\Omega_s^{(k)}} \left\{ q_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) R_j(\mathbf{y}) - R_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) q_j(\mathbf{y}) \right\} dS_y \quad (25)$$

when the support domain lies entirely in Ω and

$$\int_{\partial\Omega_s^{(k)} \cup \Gamma_s^{(k)}} \left\{ p_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) u_j(y) - u_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) p_j(y) \right\} dS_y = \int_{\partial\Omega_s^{(k)} \cup \Gamma_s^{(k)}} \left\{ q_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) R_j(y) - R_{ij}^c(\mathbf{x}^{(k)}, \mathbf{y}) q_j(y) \right\} dS_y \tag{26}$$

when the support domain intersects the global boundary S .

By subtracting Eq. 25 and Eq. 26 from Eq. 21 and Eq. 22, respectively, and applying the boundary conditions Eq. 24, the LBIEs for internal and intersected support domains take the final form

$$u_i(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \hat{p}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) u_j(y) dS_y = \int_{\partial\Omega_s^{(k)}} \hat{R}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) q_j(y) dS_y \tag{27}$$

$$\begin{aligned} cu_i(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \hat{p}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) u_j(y) dS_y + \\ \int_{\Gamma_s^{(k)}} \left\{ \hat{p}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) u_j(y) - \hat{u}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) p_j(y) \right\} dS_y = \\ \int_{\Gamma_s^{(k)}} \left\{ \hat{q}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) R_j(y) - \hat{R}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) q_j(y) \right\} dS_y - \\ \int_{\partial\Omega_s^{(k)}} \hat{R}_{ij}^*(\mathbf{x}^{(k)}, \mathbf{y}) q_j(y) dS_y \end{aligned} \tag{28}$$

where $\hat{a}_{ij} = a_{ij}^* - a_{ij}^c$. Relations Eq. 27 and Eq. 28 provide the displacement LBIE of any internal or boundary point $\mathbf{x}^{(k)}$.

4 Numerical Implementation

Displacements $u_i(\mathbf{x}^k)$ defined at any point \mathbf{x}^k of the analyzed domain are interpolated with the aid of Moving Least Square (MLS) approximation scheme, i.e.

$$u_i(\mathbf{x}^{(k)}) = \sum_{m=1}^n \phi_{ij}(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) \hat{u}_j(\mathbf{x}^{(m)}) \tag{29}$$

where, due to the lack of delta property of the MLS interpolants, $\hat{u}_i(\mathbf{x}^k)$ represent fictitious nodal displacements and not nodal displacements values [Atluri and Zhu (1998)]. However, Gosz and Liu [Gosz and Liu (1996)] explained that for piecewise linear global boundaries and evenly distributed points MLS scheme possesses delta property. Sellountos and Polyzos [Sellountos and Polyzos (2003)] have shown that for uniformly distributed points the delta property of MLS interpolants remains even for curved boundaries. Recently, that result has also been confirmed by the

work of Skouras et al [Skouras, Bourantas, Loukopoulos, and Nikiforidis (2011)]. Thus for uniformly distributed points, $\hat{u}_i(\mathbf{x}^k)$ in Eq. 29 can be replaced by the nodal displacements $u_i(\mathbf{x}^k)$.

The normal derivative of displacements is given by

$$q_i(\mathbf{x}^{(k)}) = N_{ij}^{(1)} \sum_{m=1}^n B_{jl}^{(1)}(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) u_l(\mathbf{x}^{(m)}) \quad (30)$$

with

$$\mathbf{N}^{(1)} = [N_{ij}^{(1)}] = \begin{bmatrix} n_1 & n_2 & 0 & 0 \\ 0 & 0 & n_1 & n_2 \end{bmatrix} \quad (31)$$

$$\mathbf{B}^{(1)} = [B_{jl}^{(1)}] = \begin{bmatrix} \partial_1 \phi & 0 \\ \partial_2 \phi & 0 \\ 0 & \partial_1 \phi \\ 0 & \partial_2 \phi \end{bmatrix}. \quad (32)$$

n_1, n_2 are the two components of the normal vector and ∂_1, ∂_2 derivatives with respect to the coordinate system x_1, x_2 , respectively. Similarly tractions and double tractions are interpolated as

$$p_i(\mathbf{x}^{(k)}) = N_{ij}^{(2)} D_{jl} \sum_{m=1}^n B_{lq}^{(2)}(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) u_q(\mathbf{x}^{(m)}) \quad (33)$$

$$R_i(\mathbf{x}^{(k)}) = N_{ij}^{(2)} D_{jl} N_{lp}^{(3)} \sum_{m=1}^n B_{pq}^{(3)}(\mathbf{x}^{(k)}, \mathbf{x}^{(m)}) u_q(\mathbf{x}^{(m)}) \quad (34)$$

where

$$\mathbf{N}^{(2)} = [N_{ij}^{(2)}] = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} \quad (35)$$

$$\mathbf{N}^{(3)} = [N_{lp}^{(3)}] = \begin{bmatrix} n_1 & n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \\ 0 & n_1 & n_2 & n_1 & n_2 & 0 \end{bmatrix} \quad (36)$$

$$\mathbf{D} = [D_{jl}] = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad (37)$$

$$\mathbf{B}^{(2)} = \left[B_{lq}^{(2)} \right] = \begin{bmatrix} \partial_1 \phi & 0 \\ 0 & \partial_2 \phi \\ \partial_2 \phi & \partial_1 \phi \end{bmatrix} \quad (38)$$

$$\mathbf{B}^{(3)} = \left[B_{pq}^{(3)} \right] = \begin{bmatrix} \partial_1^2 \phi & 0 \\ \partial_1 \partial_2 \phi & 0 \\ \partial_2^2 \phi & 0 \\ 0 & \partial_1^2 \phi \\ 0 & \partial_1 \partial_2 \phi \\ 0 & \partial_2^2 \phi \end{bmatrix} \quad (39)$$

By inserting Eq. 29, Eq. 30, Eq. 33 and Eq. 34 into Eq. 27 and Eq. 28 one obtains in vector form, respectively

$$\begin{aligned} \mathbf{u}(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{p}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \sum_{m=1}^n \phi(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) + \\ \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{R}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(1)}(\mathbf{y}) \cdot \sum_{m=1}^n \mathbf{B}^{(1)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) = \mathbf{0} \end{aligned} \quad (40)$$

$$\begin{aligned} c\mathbf{u}(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{p}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \sum_{m=1}^n \phi(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) + \\ \int_{\Gamma_s^{(k)}} \hat{\mathbf{p}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \sum_{m=1}^n \phi(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) - \\ \int_{\Gamma_s^{(k)}} \hat{\mathbf{u}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(2)} \cdot \mathbf{D} \cdot \sum_{m=1}^n \mathbf{B}^{(2)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) = \\ \int_{\Gamma_s^{(k)}} \hat{\mathbf{q}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \bar{\mathbf{R}}(\mathbf{y}) dS_{\mathbf{y}} - \\ \int_{\Gamma_s^{(k)}} \hat{\mathbf{R}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(1)}(\mathbf{y}) \cdot \sum_{m=1}^n \mathbf{B}^{(1)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) - \\ \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{R}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(1)}(\mathbf{y}) \cdot \sum_{m=1}^n \mathbf{B}^{(1)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) \end{aligned} \quad (41)$$

when double tractions $\mathbf{R} = \bar{\mathbf{R}}$ are prescribed on $\Gamma_s^{(k)}$ and

$$\begin{aligned}
 c\mathbf{u}(\mathbf{x}^{(k)}) + \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{p}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \sum_{m=1}^n \phi(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) + \\
 \int_{\Gamma_s^{(k)}} \hat{\mathbf{p}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \sum_{m=1}^n \phi(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) - \\
 \int_{\Gamma_s^{(k)}} \hat{\mathbf{u}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(2)} \cdot \mathbf{D} \cdot \sum_{m=1}^n \mathbf{B}^{(2)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) = \quad (42) \\
 \int_{\Gamma_s^{(k)}} \hat{\mathbf{q}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(2)}(\mathbf{y}) \cdot \mathbf{D} \cdot \mathbf{N}^{(3)}(\mathbf{y}) \cdot \sum_{m=1}^n \mathbf{B}^{(3)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)}) - \\
 \int_{\Gamma_s^{(k)}} \hat{\mathbf{R}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \bar{\mathbf{q}}(\mathbf{y}) dS_{\mathbf{y}} - \\
 \int_{\partial\Omega_s^{(k)}} \hat{\mathbf{R}}^*(\mathbf{x}^{(k)}, \mathbf{y}) \cdot \mathbf{N}^{(1)}(\mathbf{y}) \cdot \sum_{m=1}^n \mathbf{B}^{(1)}(\mathbf{y}, \mathbf{x}^{(m)}) dS_{\mathbf{y}} \cdot \mathbf{u}(\mathbf{x}^{(m)})
 \end{aligned}$$

when $\mathbf{q} = \bar{\mathbf{q}}$ is prescribed on the boundary $\Gamma_s^{(k)}$.

By collocating equations Eq. 40, Eq. 41 and Eq. 42 at all points, the following system of algebraic equations is obtained

$$\mathbf{K} \cdot \mathbf{u} = \mathbf{f} \quad (43)$$

where vector \mathbf{u} comprises all the unknown internal and boundary nodal values of displacements and \mathbf{f} the known nodal values imposed by the classical and non-classical boundary conditions. Matrix \mathbf{K} contains integrals defined on the surface of support and global boundaries, evaluated numerically with a technique explained in detail in [Sellountos and Polyzos (2003)]. Also, it should be mentioned that \mathbf{K} is sparse and not full populated as in the case of BEM. Finally, by solving Eq. 43 with a typical LU decomposition algorithm, nodal values of displacements are obtained. As soon as displacements are known, normal derivatives, tractions and double tractions are evaluated with the aid of relations Eq. 30, Eq. 33 and Eq. 34, respectively.

5 Numerical Examples

The achieved accuracy of the LBIE method illustrated in the previous two sections is demonstrated with the solution of some representative benchmark problems. The first problem concerns a solid cylinder subjected to an external radial displacement u_0 (classical boundary condition), while the radial deformation vanishes on the surface of the cylinder (non-classical boundary condition). The analytical solution

of this problem is given in [Tsepoura, Tsinopoulos, Polyzos and Beskos (2003)] and has the form

$$u_r = \frac{u_0 [I_0(\alpha/g) + I_2(\alpha/g)]}{2\alpha I_2(\alpha/g)} r - \frac{g u_0}{\alpha I_2(\alpha/g)} I_1(r/g) \tag{44}$$

where u_r represents radial displacements, r is the distance from the center of the cylinder, α is cylinder’s radius and I_n is the modified Bessel function of the first kind and n th order. This problem has been solved numerically by the proposed LBIE method for $u_0 = 0.001m$ and $\alpha = 1m$. The intrinsic parameter has been taken equal to $g = 0.0, 0.1, 0.5$ and 1509 uniformly distributed points have been used for the discretization of the cylinder with their support domain being equal to 0.45m. The obtained results are displayed in Fig 2 and as it is observed they are in very good agreement with the analytical ones. For all the values of g , the maximum error is below 0.5%.

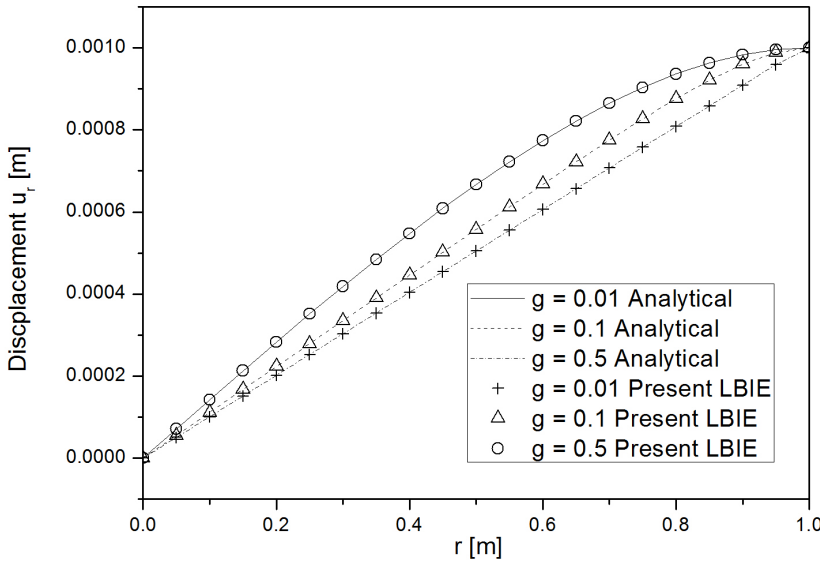


Figure 2: Radial displacement versus radial distance of the solid cylinder of radius $\alpha = 1m$, for $g = 0.0, 0.1, 0.5$. The classical boundary condition is $u_r(\alpha) = u_0$ and the non-classical one $q(\alpha) = \partial u_r / \partial r = 0$.

The same cylinder is subjected to an external pressure P_0 , while the radial double stresses R_r vanish at the boundary. The analytical solution of this problem is also provided in [Tsepoura, Tsinopoulos, Polyzos and Beskos (2003)] and it is

$$u_r = -\frac{P_0(1-2\nu)}{E}r \quad (45)$$

where E, ν stand for Young modulus and Poisson ratio, respectively.

Equation Eq. 45 indicates that the response of the cylinder is as in classical elasticity, i.e. independent of the material characteristic length g^2 . Although the analytical solution of that problem is identical to the classical one, it is a useful benchmark since all the kernels of the considered LBIEs are expressed in terms of double stresses and it is not apparent how do they provide classical solution for different values of g . Assuming again $\alpha = 1m, P_0/E = 1$ and $g = 0.0, 0.1, 0.5$, the radial displacements are evaluated and depicted in Fig (3), as function of r and compared to the corresponding free of g analytical ones. As it is evident from these figures, the agreement between the solutions is very good.

The next problem deals with the tension of a 4m x 4m supported rectangle by a uniform traction $T = 1.4GPa$. The supported and the loaded sides correspond to $y=0$ and $y=4m$, respectively, while the free of stresses sides are symmetrical to y defined by the lines $x = \pm 2m$. The material properties are $\mu=0.7GPa$, Poisson ratio $\nu = 0.3$ and $g = 0.1, 0.5, 0.8$. The considered classical and non-classical boundary conditions are: $p_x = u_y = R_x = R_y = 0$ for the side $y = 0$, $p_x = p_y = R_x = R_y = 0$ for the sides $x = 2m$ and $x = -2m$ and $p_x = q_y = R_x = 0, p_y = T$ for the side $y = 4m$. The analytical solution of the problem has been derived by the authors and it is

$$u_x = -\nu \frac{T}{2\mu}x$$

$$u_y = (1-\nu) \frac{T}{2\mu}y - g \frac{(1-\nu) \frac{T}{2\mu}}{\cosh\left(\frac{h}{g}\right)} \sinh\left(\frac{y}{g}\right) \quad (46)$$

and

$$\epsilon_{xx} = -\nu \frac{T}{2\mu}$$

$$\epsilon_{yy} = (1-\nu) \frac{T}{2\mu} - \frac{(1-\nu) \frac{T}{2\mu}}{\cosh\left(\frac{h}{g}\right)} \cosh\left(\frac{y}{g}\right) \quad (47)$$

$$\epsilon_{xy} = \epsilon_{yx} = 0$$

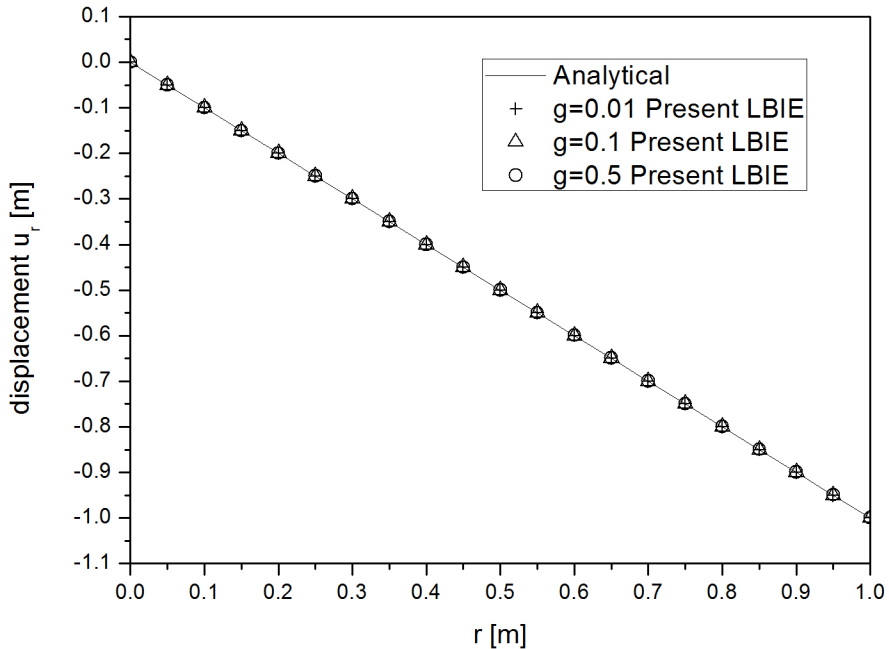


Figure 3: Radial displacement versus radial distance of the solid cylinder of radius $\alpha = 1m$, for $g = 0.0, 0.1, 0.5$. The classical boundary condition is $p_r(\alpha) = P_0$ and the non-classical one $R_r(\alpha) = 0$.

This is a gradient elastic boundary value problem with non-smooth boundary. In order to be solved with the present LBIE methodology, all corners are rounded with arcs of radius $r_e = 0.04$. 485 uniformly distributed points with support domains equal to $r_0 = 0.51$ have been used. Both displacements and strains defined along the axis of symmetry $x = 0$ are calculated and compared to analytical ones in Figs (4) and (5). In both figures the agreement is good, while the gradient effect on the response is apparent. The maximum error for $g=0.8$ is 4.52% for displacements and 4.54% for strains while, the errors become smaller for smaller g . The explanation for that error is the numerical solution of a slightly different problem (rectangle with rounded corners) from that corresponding to analytical solutions Eq. 46 and Eq. 47. The error becomes much smaller by considering Poisson ratio equal to $\nu = 0.0$, which introduces smaller side effects. Indeed, in that case the numerical results

depicted in Figs (6) and (7) appear maximum error 0.42% in both displacements and strains.

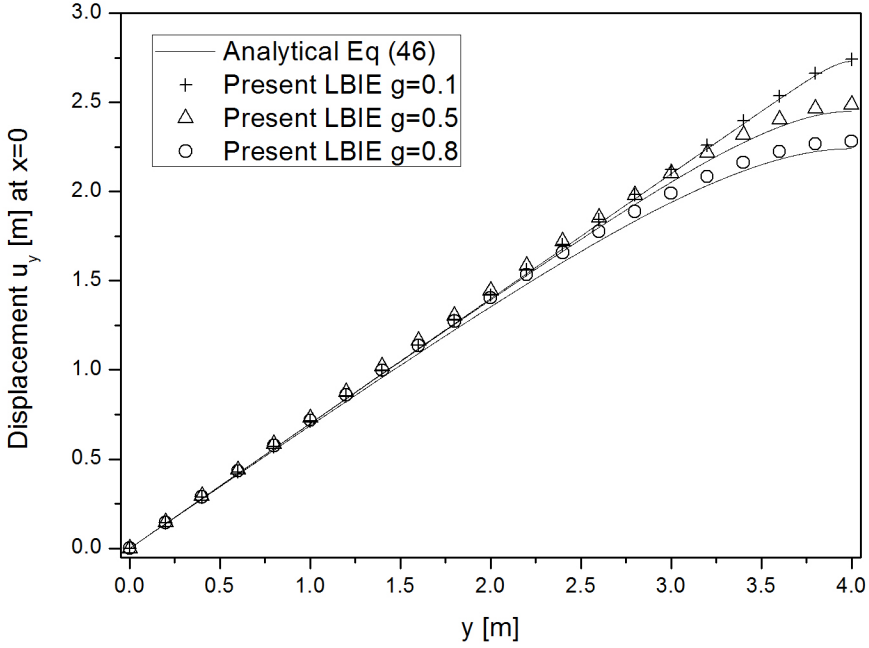


Figure 4: Displacements along the axis of symmetry $x = 0$, for a gradient elastic rectangle ($\mu=0.7\text{GPa}$, $\nu = 0.3$) subjected to a uniform tension T .

Finally, the same problem has been solved with the boundary conditions $p_x = u_y = R_x = R_y = 0$ for the side $y = 0$, $p_x = p_y = R_x = R_y = 0$ for the sides $x = 2m$ and $x = -2m$ and $p_x = R_x = R_y = 0$, $p_y = T$ for the side $y = 4m$. For this problem the analytical solution is identical to the classical elastic one, i.e.

$$\begin{aligned}
 u_x &= -\nu \frac{T}{2\mu} x \\
 u_y &= (1 - \nu) \frac{T}{2\mu} y
 \end{aligned}
 \tag{48}$$

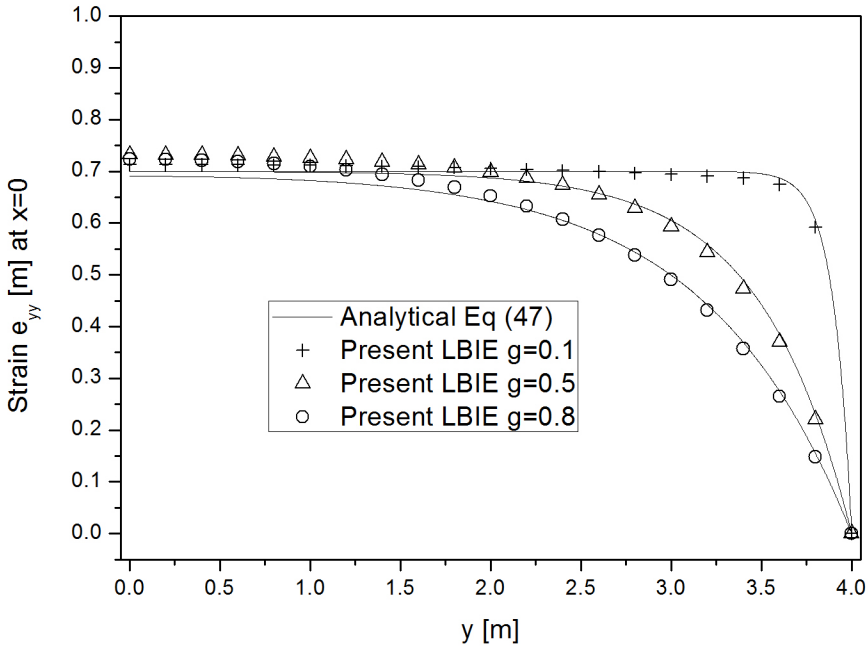


Figure 5: Strains ϵ_{yy} along the axis of symmetry $x = 0$, for the same gradient elastic rectangle of Figure 4.

$$\begin{aligned}\epsilon_{xx} &= -\nu \frac{T}{2\mu} \\ \epsilon_{yy} &= (1 - \nu) \frac{T}{2\mu} \\ \epsilon_{xy} &= \epsilon_{yx} = 0\end{aligned}\tag{49}$$

As in the case of the cylinder, the aim of the present benchmark is twofold: first to assess if the LBIE methodology provides classical solution for different values of g and second to examine if the corners affect the solution of the problem as in Figs (4) and (5). For the numerical solution of the problem, again 485 uniformly distributed points have been used with a support domain being equal to 0.51m. The obtained numerical results are compared to analytical ones and the maximum error in both displacements and strains is 2.1% and 0.17% for Poisson ratios $\nu = 0.3$ and $\nu = 0.0$, respectively.

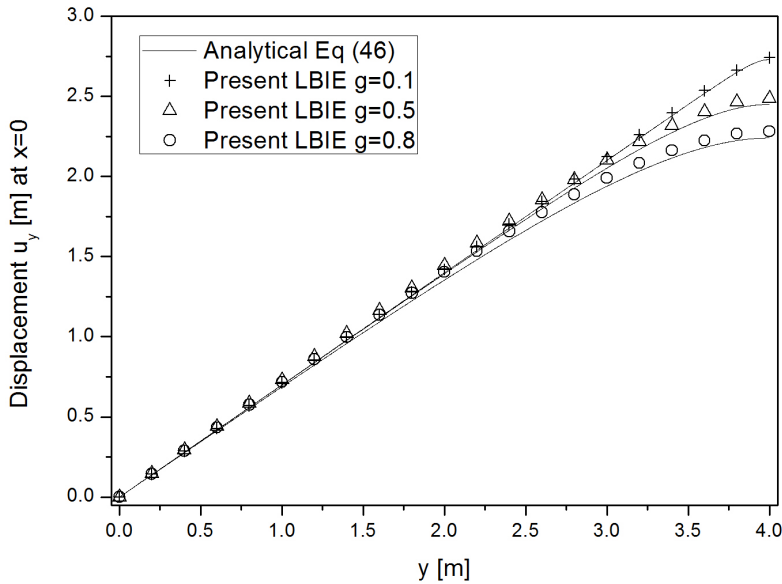


Figure 6: Displacements along the axis of symmetry $x = 0$, for a gradient elastic rectangle ($\mu = 0.7\text{GPa}$, $\nu = 0$) subjected to a uniform tension T .

6 Conclusions

A meshless Local Boundary Integral Equation (LBIE) method for solving two dimensional gradient elastic problems has been proposed. Uniformly distributed points are utilized for the interpolation of the involved fields. Only the LBIE representation of displacements is employed and the Moving Least Square (MLS) approximation scheme is introduced for the meshless representation of displacements throughout the analysed domain. This is possible with the use of a companion solution, explicitly derived in the present work, which zeroes tractions and double tractions on the boundary of support domains. Normal derivatives of displacements, tractions and double tractions on all boundaries are approximated with the derivatives of MLS interpolation functions. The proposed method is an excellent alternative to Finite Element Method (FEM) since it utilizes MLS approximation scheme for the interpolation of the fields avoiding thus elements with $C^{(1)}$ continuity requirements or mixed formulations with many degrees of freedom. Also

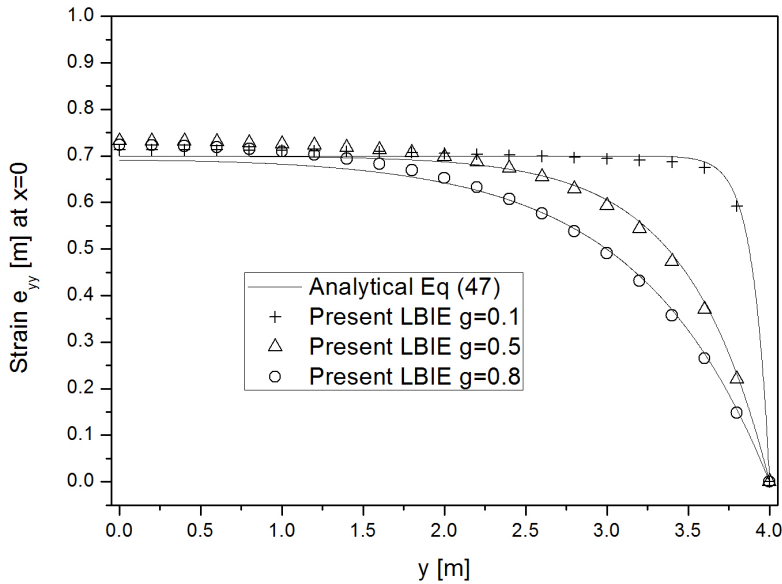


Figure 7: Strains ϵ_{yy} along the axis of symmetry $x = 0$

appears significant advantages over the Boundary Element Method (BEM) because it utilizes for every node only one integral equation and not two as in the case of BEM, and the most important is that it concludes to a final system of algebraic equations which is sparse and not full populated. Two representative numerical examples have been provided to illustrate the method, demonstrate its high accuracy and confirm the effect of the microstructure to macrostructure.

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Appendix A: Gradient elastostatic companion solution

In this appendix the 2D gradient elastostatic companion solution \mathbf{u}^c used for the derivation of LBIEs Eq. 23 and Eq. 24 is explicitly derived. \mathbf{u}^c is a second order tensor and regular function of r satisfying the boundary value problem

$$(1 - g^2 \nabla^2) [(\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u}^c - \mu \nabla \times \nabla \times \mathbf{u}^c] = 0, r \leq r_0 \tag{50}$$

$$\begin{aligned} \mathbf{u}^c(r_0) &= \mathbf{u}^*(r_0) \\ \mathbf{q}^c(r_0) &= \mathbf{q}^*(r_0) \end{aligned} \tag{51}$$

where r_0 is the radius of the support domain, \mathbf{u}^* the fundamental solution of gradient elastic problem given by Eq. 19 and $\mathbf{q}^* = \partial \mathbf{u}^* / \partial n$, i.e.

$$q_{ij}^* = \frac{1}{8\pi\mu} \left[\left(\frac{\partial X}{\partial r} - \frac{2X}{r} \right) (n_k r_k) \partial r_i \partial r_j + \frac{\partial \Psi}{\partial r} n_k r_k \delta_{ij} - \frac{X}{r} (n_i r_j + n_j r_i) \right] \tag{52}$$

Since the partial differential equation Eq. 50 is a multiplication of two partial differential operators, the solution \mathbf{u}^c can be written as

$$\mathbf{u}^c = \mathbf{u}^e + g^2 \mathbf{u}^g \tag{53}$$

where \mathbf{u}^e and \mathbf{u}^c are solutions of the equations, respectively

$$(\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u}^e - \mu \nabla \times \nabla \times \mathbf{u}^e = \mathbf{0} \tag{54}$$

$$(1 - g^2 \nabla^2) \mathbf{u}^g = \mathbf{0} \tag{55}$$

By utilizing cylindrical coordinates and using only the r dependency, one can find that the regular solutions of Eq. 54 and Eq. 55 are

$$u_{ij}^e = \frac{3 - 4\nu}{4\mu(1 - \nu)} C_1 r^2 \partial_{ir} \partial_{jr} - \frac{5 - 4\nu}{8\mu(1 - \nu)} C_1 r^2 \delta_{ij} + \frac{4 - 5\nu}{4\mu(1 - \nu)} C_2 \delta_{ij} \tag{56}$$

$$u_{ij}^g = C_3 I_2(r/g) \partial_{ir} \partial_{jr} + \left[C_4 I_0(r/g) - \frac{1}{2} C_3 I_2(r/g) \right] \delta_{ij} \tag{57}$$

with I_n being the modified Bessel function of the first kind and n th order and C_1, C_2, C_3, C_4 constants to be determined. In view of Eq. 53, Eq. 56 and Eq. 57 the companion solution \mathbf{u}^c is written as

$$u_{ij}^c = \frac{1}{8\pi\mu(1 - \nu)} [\Psi^c(r) - X^c(r) \partial_{ir} \partial_{jr}]$$

$$X^c(r) = -2(3 - 4\nu) C_1 r^2 - 8\pi\mu(1 - \nu) g^2 C_3 I_2(r/g) \tag{58}$$

$$\Psi^c(r) = (5 - 4\nu) C_1 r^2 + 2(4 - 5\nu) C_2 - 4\pi\mu(1 - \nu) g^2 C_3 I_2(r/g) + 8\pi\mu(1 - \nu) g^2 C_4 I_0(r/g)$$

where the coefficients C_1, C_2, C_3, C_4 satisfy the following simple algebraic system of equations taken from the boundary conditions Eq. 51,

$$\frac{1}{8\pi\mu(1 - \nu)} \left(\frac{dX}{dr} - \frac{2X}{r} \right)_{r=r_0} - C_3 \left(g I_2'(r_0/g) - g^2 \frac{2I_2(r_0/g)}{r_0} \right) = 0 \tag{59}$$

$$-\frac{1}{8\pi\mu(1 - \nu)} X(r_0) - \frac{3 - 4\nu}{8\pi\mu(1 - \nu)} C_1 r_0^2 - g^2 C_3 I_2(r_0/g) = 0 \tag{60}$$

$$\frac{1}{8\pi\mu(1 - \nu)} \frac{d\Psi}{dr} \Big|_{r=r_0} - \frac{5 - 4\nu}{4\mu(1 - \nu)} C_1 r_0 + \frac{1}{2} g C_3 I_2'(r_0/g) - g C_4 I_0'(r_0/g) = 0 \tag{61}$$

$$\frac{1}{8\pi\mu(1 - \nu)} \Psi(r_0) + \frac{5 - 4\nu}{4\mu(1 - \nu)} C_1 r_0^2 - \frac{4 - 5\nu}{4\mu(1 - \nu)} C_2 + \frac{1}{2} g^2 C_3 I_2(r_0/g) - g^2 C_4 I_0(r_0/g) = 0 \tag{62}$$

with $I_n'(z)$ meaning dI_n/dz .

