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Numerical Reconstruction of a Space-Dependent Heat Source Term in a Multi-Dimensional Heat Equation

C. Shi[1], **C. Wang**[1] **and T. Wei**[1,2]

Abstract: In this paper, we consider a typical ill-posed inverse heat source problem, that is, we determine a space-dependent heat source term in a multi-dimensional heat equation from a pair of Cauchy data on a part of boundary. By a simple transformation, the inverse heat source problem is changed into a Cauchy problem of a homogenous heat conduction equation. We use the method of fundamental solutions (MFS) coupled with the Tikhonov regularization technique to solve the ill-conditioned linear system of equations resulted from the MFS discretization. The generalized cross-validation rule for determining the regularization parameter is used. Numerical results for four examples in 1D, 2D and 3D cases show that the proposed method is effective and feasible.

Keywords: Inverse heat source , ill-posed problem , the method of fundamental solution , Tikhonov regularization

1 Introduction

In the process of transportation, diffusion and heat conduction of natural materials, the following heat equation is a suitable model:

$$u_t - \Delta u = F(x,t), (x,t) \in \Omega \times (0,T),$$

where u denotes the state variable, Ω is a bounded domain in R^d, and the right hand side f denotes the source term, which depends generally on both space and time. This equation is especially important in some practical physical applications.

Numerical methods on the determination of space-dependent heat source is given in Johansson and Lesnic (2007); Nili Ahmadabadi, Arab, and Maalek Ghaini (2009); Farcas and Lesnic (2006); Yang, Deng, Yu, and Luo (2009); Yan, Yang, and Fu (2009), while the determination of time-dependent heat source is considered in

[1] School of mathematics and statistics, Lanzhou University, Lanzhou 730000, China
[2] Corresponding author.

Yang and Fu (2010); Yang, Dehghan, Yu, and Luo (2011); Battaglia, Cois, Puigsegur, and Oustaloup (2001); Huang and Wang (1999); Farcas and Lesnic (2006); Yan, Fu, and Yang (2008). In some cases the heat source can be written as a separable form $F(x,t) = f(x)g(t)$ and either $f(x)$ or $g(t)$ is unknown. Recovery of one term of a separable heat source where the other term is given are considered in Saitoh, Tuan, and Yamamoto (2003); Neto and Ozisik (1992); Bellassoued and Yamamoto (2011); Geng and Lin (2009); Qian and Li (2011) . A number of regularization methods have been used to handle the ill-posedness of the inverse heat source problem, including the boundary element method Dijkstra, Kakuba, and Mattheij (2011); Denda (2011); Loeffler (2011); Litewka and Sygulski (2010), iterative regularization methods Chen and Yang (2011); George and Kunhanandan (2010); George and Elmahdy (2010); Huang, Wu, and Kim (2010); Anh and Van Chung (2009); Kaltenbacher, Neubauer, and Scherzer (2008) and mollification methods Yin, He, and Yan (2008); Li and Ma (2000); Hào (1996); Mejia (1993); Murio (1992), and linear least squares error method Fernández-Cara and Münch (2011); He, Li, Zhao, and Chen (2011); Bellettini, Caselli, and Mariani (2009) and so on. In Savateev (1995); Su and Silva Neto (2001), the authors recover both parts of an separable source term by using iterative methods.

In this paper we consider a special case that $g(t) = e^{-\lambda t}$ is given and the space-dependent term $f(x)$ is unknown. The similar problem in an unbounded domain is considered in Qian and Li (2011), where the authors used a generalized Tikhonov regularization to get a conditional stability estimate by an additional data at $t = T$, while we use the Cauchy data on a part of boundary. This problem is ill-posed Choulli and Yamamoto (2004). We use a regularized MFS to solve it. The condition number of the linear system of equations from the MFS is huge. To cope with numerical instability, a suitable regularization method must be used. We use the Tikhonov regularization method, and choose the regularization parameter through the generalized cross-validation (GCV) criterion Hansen (1998); Morozov (1984); Engl, Hanke, and Neubauer (1996); Tautenhahn and Hämarik (1999); Golub, Heath, and Wahba (1979).

The method of fundamental solutions was firstly proposed by Kupradze and Aleksidze in Kupradze and Aleksidze (1964). In the last decades, it has been successfully used in solving various linear partial differential equations. Recently, Hon et al. applied the MFS to solve the Cauchy problem of heat equations in one-dimensional case Hon and Wei (2004) and multidimensional case Hon and Wei (2005). The recent development of the MFS on inverse problems can be found in Wei, Chen, and Liu (2012); KoŁodziej and Gorzelańczyk (2012); Boselli, Obrist, and Kleiser (2012); Bin-Mohsin and Lesnic (2012); Karageorghis, Lesnic, and Marin (2012); Abouelsaad, Morsi, and Salama (2011); Kinugawa, Yamamoto, and

Hara (2012).

In the last section, we give some numerical examples including smooth and non-smooth functions up to three dimensional case which show that our method is numerically stable and accurate.

2 Formulation of the problem

Let $\Omega \subset R^d$ be a bounded domain and Γ be a part of boundary $\partial\Omega$. Consider the following inverse heat source problem in Ω, i.e. the temperature satisfies non-homogeneous heat equation

$$u_t = \Delta u + e^{-\lambda t} f(x), \quad x \in \Omega, \ t \in (0,T], \tag{2.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$\frac{\partial u}{\partial n}(x,t) = g(x,t), \quad x \in \Gamma, \ t \in [0,T], \tag{2.3}$$

where $f(x)$ is an unknown heat source term to be determined from an additional measurement data

$$u(x,t) = h(x,t), \quad x \in \Gamma, \ t \in [0,T]. \tag{2.4}$$

We suppose that the given data are consistent, that is, we have

$$h(x,0) = \varphi(x), \quad g(x,0) = \frac{\partial \varphi(x)}{\partial n}, \quad \text{for} x \in \Gamma. \tag{2.5}$$

Take a transformation $v(x,t) = u_t(x,t) + \lambda u(x,t)$, then the problem (2.1)-(2.4) is changed into the following Cauchy problem of homogeneous heat equation

$$v_t(x,t) = \Delta v(x,t), \quad x \in \Omega, \ t \in (0,T], \tag{2.6}$$

$$\frac{\partial v}{\partial n}(x,t) = g_t(x,t) + \lambda g(x,t), \quad x \in \Gamma, \ t \in [0,T], \tag{2.7}$$

$$v(x,t) = h_t(x,t) + \lambda h(x,t), \quad x \in \Gamma, \ t \in [0,T]. \tag{2.8}$$

When v is obtained from the above inverse problem, the unknown heat source term $f(x)$ can be calculated from

$$f(x) = v(x,0) - \lambda \varphi(x) - \Delta\varphi(x). \tag{2.9}$$

We provide the following uniqueness result for problem (2.1)-(2.4).

Theorem 2.1. *The reconstruction of $f(x)$ is unique for problem (2.1)-(2.4).*

Proof. Suppose $g = h = \varphi = 0$ in (2.1)-(2.4), we only need to prove $f = 0$. By the transformation $v(x,t) = u_t(x,t) + \lambda u(x,t)$, we know v satisfies the heat equation (2.6) and $v|_\Gamma = 0$ and $\frac{\partial v}{\partial n}|_\Gamma = 0$. By the uniqueness continuation of heat problem, we know $v = 0$ on $\bar{\Omega} \times [0,T]$. By $u_t(x,t) + \lambda u = 0$, we know $u(x,t) = c(x)e^{\lambda t}$. From $u(x,0) = 0$, we get $u = 0$, further, $f(x) = 0$. □

3 Method of fundamental solutions and Tikhonov regularization

The fundamental solution of the heat equation (2.6) is given by:

$$F(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} H(t), \tag{3.1}$$

where $H(t)$ is the Heaviside function, d is the number of dimensions. Let $\tau > T$ be a constant and denote

$$\phi(x,t) = F(x,t+\tau), \tag{3.2}$$

which is non-singular for all $t > -\tau$. The use of such a parameter is to avoid the singularities .

The approximate solution of problem (2.6)-(2.8) could be given by the following linear combination:

$$v(x,t) = \sum_{j=1}^{N} \omega_j \phi(x - \chi_j, t - \tau_j), \tag{3.3}$$

where $(\chi_j, \tau_j)_{1 \le j \le N}$ are the source points outside $\bar{\Omega} \times [0,T]$, and N is the total number of source points.

Using the boundary conditions (2.8) and (2.7), then we obtain the following linear system of equations:

$$A\omega = b, \tag{3.4}$$

where A is an $N \times N$ square matrix,

$$A = \begin{pmatrix} \phi(x_i - \chi_j, t_i - \tau_j) \\ \frac{\partial \phi}{\partial n}(x_i - \chi_j, t_i - \tau_j) \end{pmatrix}, \tag{3.5}$$

and b is an column vector of length N,

$$b = \begin{pmatrix} h_t(x_i,t_i) + \lambda h(x_i,t_i) \\ g_t(x_i,t_i) + \lambda g(x_i,t_i) \end{pmatrix}, \tag{3.6}$$

where ω_j denotes unknown coefficients to be calculated and (x_i, t_i) are the collocation points on $\Gamma \times [0,T]$.

The linear system (3.4) can not be solved by direct methods, since such an approach would produce a highly unstable solution due to the large condition number of the matrix A which increases explosively as the number of collocation points and τ increase. Several regularization procedures have been developed to solve such ill-conditioned problem. In this paper we adopt the Tikhonov regularization method. The Tikhonov regularized solution ω^α for problem (3.4) is defined as the solution of the following minimization problem:

$$\min_{\omega}\{\|A\omega - b\|^2 + \alpha\|\omega\|^2\}, \tag{3.7}$$

where $\|\cdot\|$ denotes the Euclidean norm and $\alpha > 0$ is called the regularization parameter. The choice of a suitable value of the regularization parameter α is crucial for the accuracy of the final numerical solution and is still undergoing intensive research. There are many methods for the parameter choice, such as the Morozov's discrepancy principle, the L-curve criterion, the cross-validation, and generalized cross-validation Hansen (1998); Morozov (1984); Engl, Hanke, and Neubauer (1996); Tautenhahn and Hämarik (1999); Golub, Heath, and Wahba (1979). In this paper, we employ the generalized cross validation in Engl, Hanke, and Neubauer (1996), i.e. the regularization parameter α is chosen through minimizing the following GCV function:

$$G(\alpha) = \frac{\|A\omega^\alpha - b\|^2}{(\mathrm{tr}(I - AA^I))^2}, \quad \alpha > 0, \tag{3.8}$$

where $A^I = (A^T A + \alpha I)^{-1} A^T$.

Denote the regularized solution of (3.4) by ω^α, then the approximate solution for the problem (2.6)-(2.8) can be written as:

$$v^\alpha(x,t) = \Sigma_{j=1}^{N} \omega_j^\alpha \phi(x - \chi_j, t - \tau_j). \tag{3.9}$$

By (2.9), we get

$$f^\alpha(x) = v^\alpha(x,0) - \lambda\varphi(x) - \Delta\varphi(x). \tag{3.10}$$

4 Numerical experiments

For simplicity, we set $T = 1$ and $\tau = 2$ in the following examples, unless otherwise specified. We use the following formula to generate the noisy data for the initial and boundary data $g(x_i,t_i), h(x_i,t_i)$ and $\varphi(x_i)$

$$g^{\delta}(x_i,t_i) = g(x_i,t_i)(1+\delta \cdot randn(i)), \tag{4.1}$$

$$h^{\delta}(x_i,t_i) = h(x_i,t_i)(1+\delta \cdot randn(i)), \tag{4.2}$$

$$\varphi^{\delta}(x_i) = \varphi(x_i)(1+\delta \cdot randn(i)). \tag{4.3}$$

where $randn(i)$ are random numbers obeying a normal distribution with mean 0 and standard variation 1.

In (3.6) and (3.10), we need to compute the first order derivative of noisy boundary Cauchy data and the second order derivatives of the noisy initial data. We use an improved radial basis functions method similar to the one in Wei and Hon (2007) to obtain the approximations to $g_t(x,t)$ and $h_t(x,t)$ at the collocation points in which the basis functions are chosen as the form $|x-x_i|^7$ and the fourth order polynomial. The Tikhonov regularization and generalized cross validation rule are used to give a stable numerical derivative. The approximations of $\Delta\varphi(x)$ are obtained by calculating the first order derivative to φ and then to $\frac{\partial \varphi}{\partial x_i}$. The calculated numerical derivatives may have oscillations, and we use the robust local regression ("rloess" in MATLAB) to smooth them.

Although there is no convergence result, our numerical results illustrate that the MFS is feasible and effective in our inverse heat source problem.

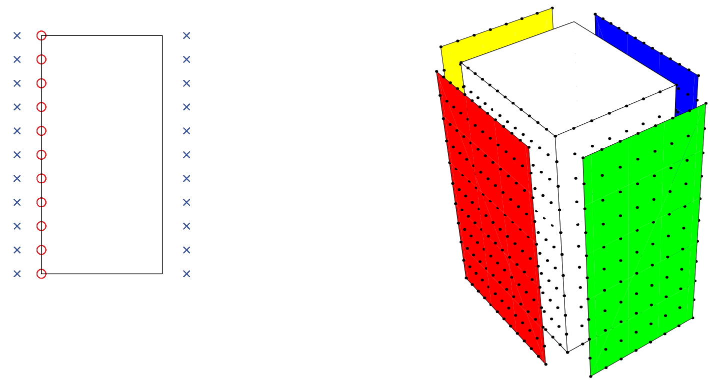
To measure the efficiency of the MFS, we compute the relative root mean square error

$$RES(f) = \frac{\sqrt{\sum_{i=1}^{N_t}(f(\tilde{x}_i) - f^{\alpha}(\tilde{x}_i))^2}}{\sqrt{\sum_{i=1}^{N_t}(f(\tilde{x}_i))^2}}, \tag{4.4}$$

where N_t is the total number of test points in the domain $\bar{\Omega}$ and $\tilde{x}_i$ are test points, $f(\tilde{x}_i)$ and $f^{\alpha}(\tilde{x}_i)$ are the exact and approximate values at the test points.

4.1 One dimensional case

Example 1. In this example, we take $\Omega = (0,1)$ and $\Gamma = \{0\}$, the source points $(-d_x,t_i)$ and $(1+d_x,t_i)$ where $d_x > 0$ is a parameter governing the distance between the source points and the boundary and $t_i = (i-1)T/(k_t-1)$ $(i = 1,2,\cdots,k_t)$ are equidistant times in $[0,T]$. The collocation points are chosen as $(0,t_i)$ $(i = 1,2,\cdots,k_t)$ for the fitting of Cauchy data on the boundary conditions. Figure 1 (a) illustrates the position of these points.



(a) One dimensional case (b) Two dimensional case

Figure 1: Position of source and collocation points.

The exact solution is given by

$$u(x,t) = x^3 - 3x^2 + (6t+2)x - 6t + e^{-t}(\sin \pi x + \cos \pi x), \tag{4.5}$$

$$f(x) = (\pi^2 - 1)(\sin \pi x + \cos \pi x), \tag{4.6}$$

where we use $\lambda = 1$.

The numerical results are shown in Figures 2-4.

The inverse heat source problem investigated in this study is solved by using the uniformly distributed collocation points and source points. The total numbers of collocation points and source points are typically chosen to be 25 and 50, respectively. We add some noises $\delta = 0.005$, 0.01, 0.05 into the exact data. Figure 2 presents the exact and numerical solutions for the heat source $f(x)$ with these noise levels. It can be seen that the results are satisfactory with the noise levels up to $\delta = 0.05$. Hence the MFS, in conjunction with the Tikhonov regularization method, provides stable numerical solutions to the 1-D inverse source problem. We note that the condition number $cond(A)$ of the interpolation matrix A is nearly 10^{20}.

Next, we investigate the convergence of numerical method with respect to the noise level. Figure 3 shows the relative root mean square errors versus various noise levels for Example 1 with a fixed $\tau = 2$, from which we can see that the numerical solution is convergent as the noise level is decay.

In order to investigate the influence of parameters τ and k_t on the accuracy and stability of the numerical solutions for the heat source, in Figure 4 we present *RES* for Example 1 with various parameters k_t for a fixed $\delta = 0.01$ and $\tau = 2$, and various values τ for a fixed $\delta = 0.01, k_t = 25$. It can be seen from Figure 4 that the accuracy of the numerical results is relatively independent on the parameters k_t and τ. This stability of the solutions to the parameters over a fairly large range is a favorable feature of MFS because there is no need to search for optimal values of parameters.

Example 2. In the second example we set $\Omega = (0,\pi)$ and $\Gamma = \{0\}$. The source points are $(-d_x,t_i)$ and $(\pi+d_x,t_i)$ and the collocation points are $(0,t_i)$ similar to Example 1, as depicted in Figure 1 (a).

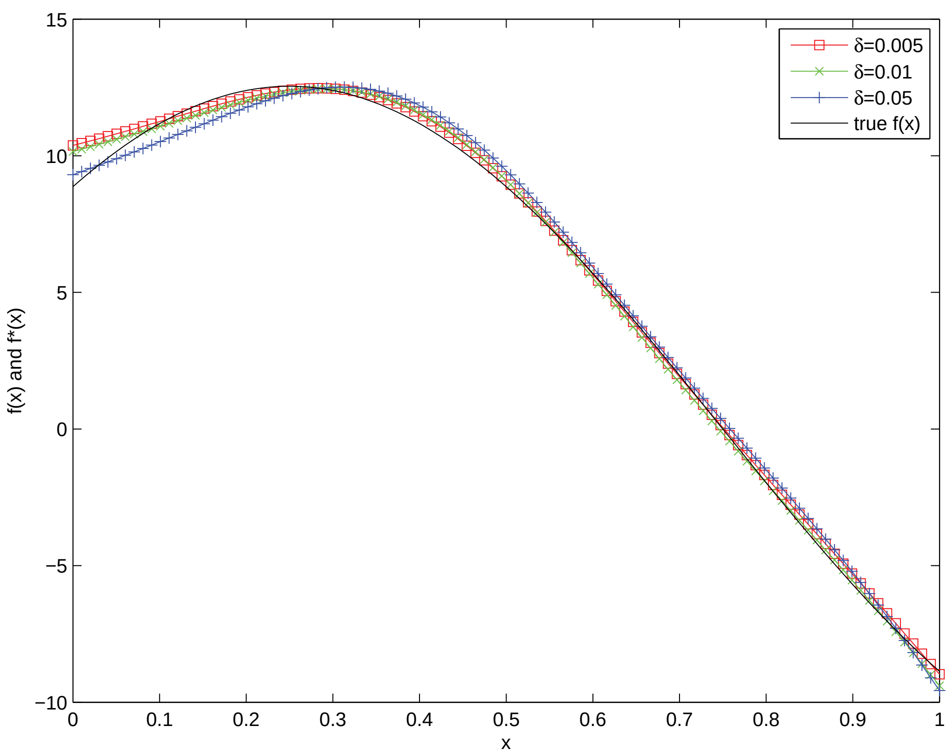


Figure 2: The exact $f(x)$ and its numerical approximations with noise levels $\delta = 0.005, 0.01, 0.05$ for Example 1.

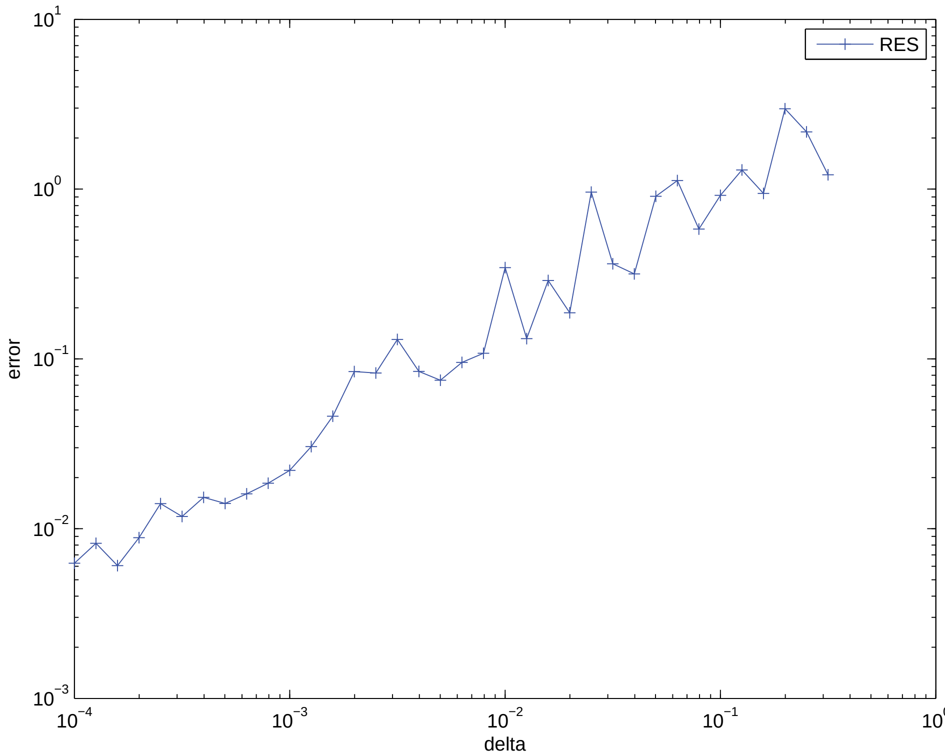


Figure 3: The accuracy of the numerical solutions for Example 1 with respect to the noise levels δ.

The exact solution is given by

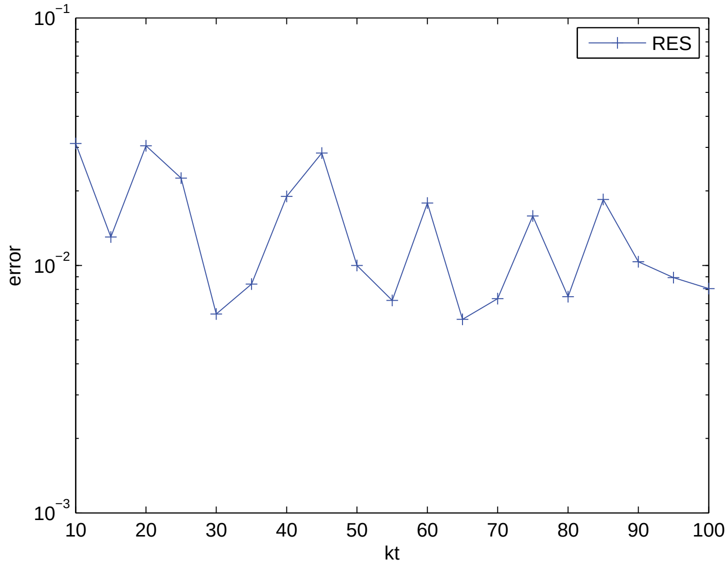
$$u(x,t) = \begin{cases} e^{-t}(2\sin x + (t+1)\cos x - 1), & x < \pi/2, \\ e^{-t}(1 + (t+1)\cos x), & x \geq \pi/2, \end{cases} \tag{4.7}$$

$$f(x) = \begin{cases} 1 + \cos x, & x < \pi/2, \\ -1 + \cos x, & x \geq \pi/2. \end{cases} \tag{4.8}$$

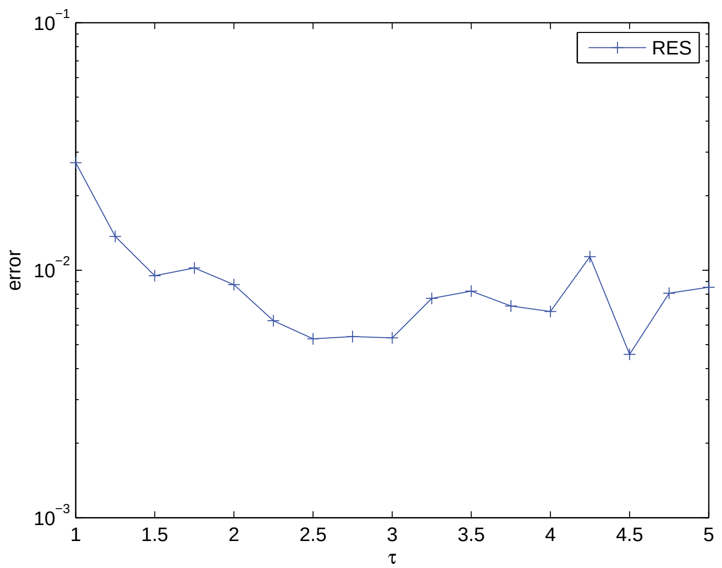
It is easy to prove that the function $u(x,t)$ is a weak solution of equation (2.1) with the initial data

$$u(x,0) = \begin{cases} 2\sin x + \cos x - 1, & x < \pi/2, \\ 1 + \cos x, & x \geq \pi/2, \end{cases} \tag{4.9}$$

and boundary condition $u(0,t) = te^{-t}$, $u(\pi,t) = -te^{-t}$, refer to the definition of weak solution in Evans (1998).



(a) RES versus k_t



(b) RES versus τ

Figure 4: RES of numerical solutions for Example 1 with respect to the parameters k_t or τ .

Numerical results for Example 2 by using various amounts of noises added into the data are presented in Figure 5 and we can see that the numerical results are reasonable.

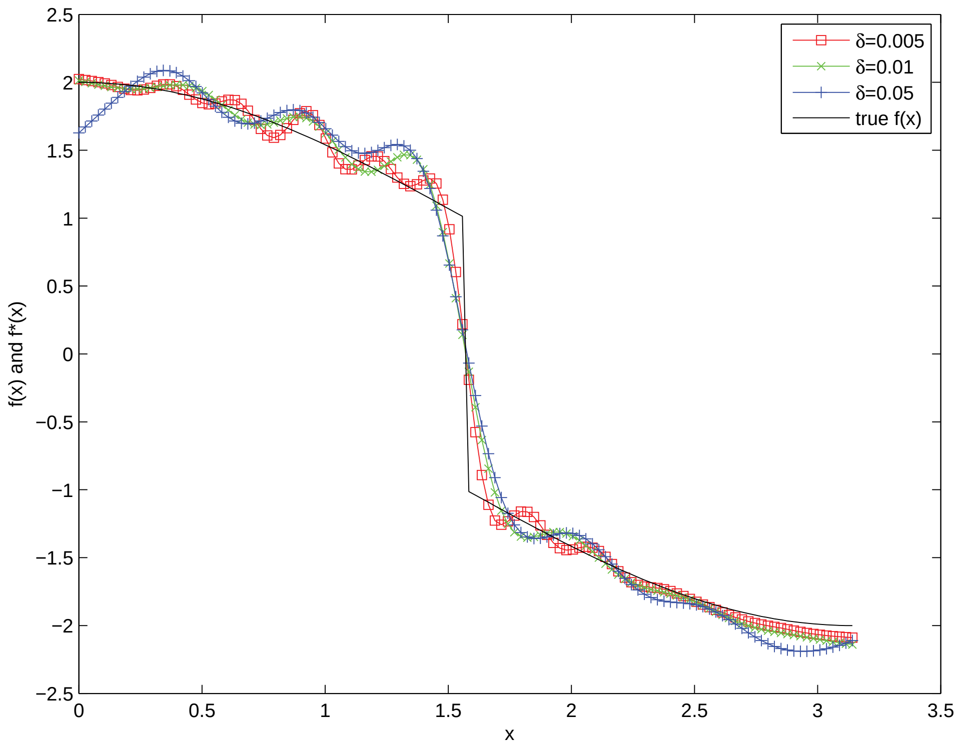


Figure 5: The exact $f(x)$ and its numerical approximations with noise levels $\delta = 0.005, 0.01, 0.05$ for Example 2.

4.2 Two dimensional case

Example 3. The solution domain is taken as $\Omega = \{(x_1,x_2)|0 < x_i < \pi/2, i = 1,2\}$, and $\Gamma = \{0\} \times [0,\pi/2] \bigcup [0,\pi/2] \times \{0\}$. The set of source points can be written as $\mathscr{X} \times \mathscr{T}$, where $\mathscr{X} = \{(-d_{x_1}) \times \mathscr{X}_2\} \bigcup \{(\pi/2 + d_{x_1}) \times \mathscr{X}_2\} \bigcup \{\mathscr{X}_1 \times (-d_{x_2})\} \bigcup \{\mathscr{X}_1 \times (\pi/2 + d_{x_2)})\}$ where $\mathscr{T} = \{t_i = (i-1)T/(k_t-1), i = 1,2,\cdots,k_t\}$ and $\mathscr{X}_1 = \{\chi_i = (i-1)\pi/(2(k_1-1)), i = 1,2,\cdots,k_1\}$ and $\mathscr{X}_2 = \{\chi_i = (i-1)\pi/(2(k_2-1)), i = 1,2,\cdots,k_2\}$ are the sets of equidistant points in $[0,T]$, $[0,\pi/2]$ and $[0,\pi/2]$, respectively and d_{x_i} $(i = 1,2)$ are two parameters. An illustration of the point positions is given in Figure 1(b).

Table 1: RES versus δ for Example 3

δ	0	0.005	0.01	0.05
RES	0.008	0.038	0.189	0.426

We consider an example with the analytical solution

$$u(x,t) = (t+1)e^{-2t}\cos x_1 \sin x_2, \tag{4.10}$$

$$f(x) = \cos x_1 \sin x_2, \tag{4.11}$$

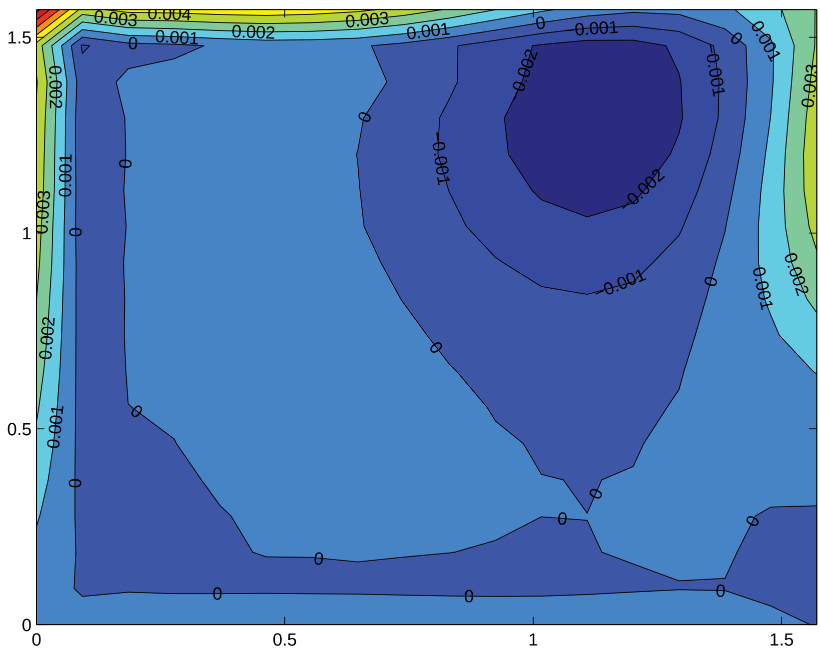
where we use $\lambda = 2$. The numerical results are shown in Figure 6 and Table 1.

In Example 3 the numerical solution is very accurate if using the noise free data and we can see it in Figure 6(a). It can be seen that the numerical results retrieved for the heat source represent good approximations for their analytical values. Furthermore, the numerical heat sources converge towards their corresponding exact solutions as the amount of noise decreases. Hence the MFS, in conjunction with the Tikhonov regularization method, provides stable numerical solutions to the 2-D inverse source problem.

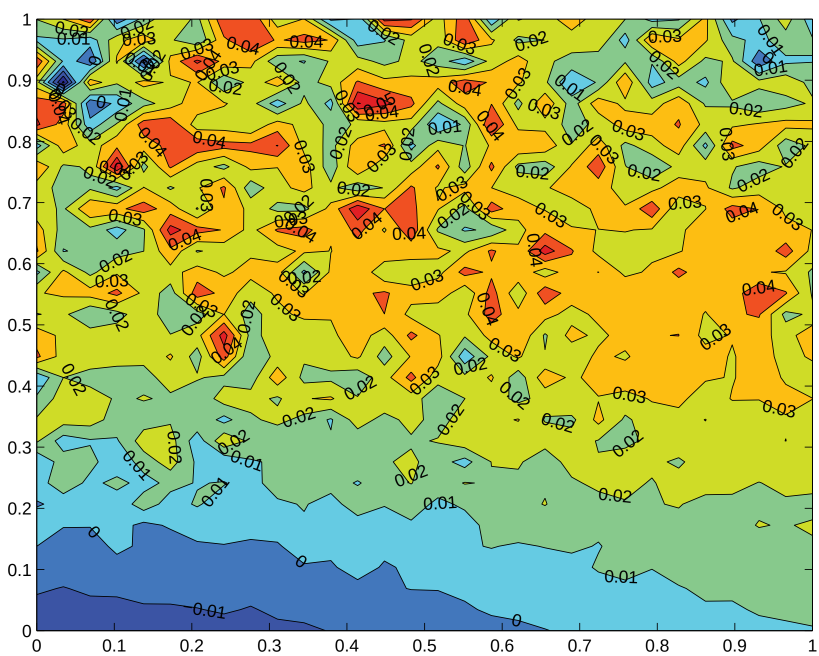
4.3 Three dimensional case

Three-dimensional heat problems are usually not easy to deal with, partly due to the expensive effort in the mesh generation for mesh-dependent techniques and, more importantly, due to the exponential increasing size of the resulting discrete equations. This fact is the so-called curse of dimensionality. The following example is intended to verify numerically the accuracy and efficiency of the present MFS solution for a 3-D problem.

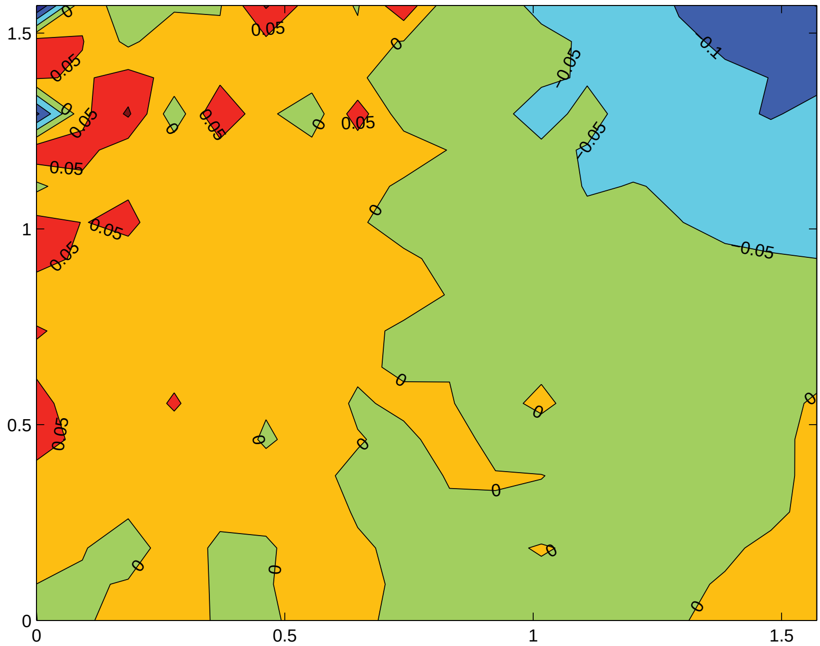
Let $\Omega = \{(x_1,x_2,x_3)|0 < x_i < 1, i = 1,2,3\}$, and $\Gamma = \{x \in \partial\Omega | x_1x_2x_3 = 0\}$. The set of source points can be written as $\mathscr{X} \times \mathscr{T}$, similar to the two-dimensional case, where $\mathscr{X} = (\{-d_{x_1}, 1+d_{x_1}\} \times \mathscr{X}_2 \times \mathscr{X}_3) \bigcup (\mathscr{X}_1 \times \{-d_{x_2}, 1+d_{x_2}\} \times \mathscr{X}_3) \bigcup (\mathscr{X}_1 \times \mathscr{X}_2 \times \{-d_{x_3}, 1+d_{x_3}\})$, and $\mathscr{T} = \{t_i = (i-1)T/(k_t-1), i = 1,2,\cdots,k_t\}$. Here $\mathscr{X}_1 = \{\chi_i = (i-1)/(k_1-1), i = 1,2,\cdots,k_1\}$, $\mathscr{X}_2 = \{\chi_i = (i-1)/(k_2-1), i = 1,2,\cdots,k_2\}$ and $\mathscr{X}_3 = \{\chi_i = (i-1)/(k_3-1), i = 1,2,\cdots,k_3\}$ are sets of equidistant points on $[0,T], [0,1], [0,1], [0,1]$ respectively, and d_{x_i} $(i = 1,2,3)$ are parameters.



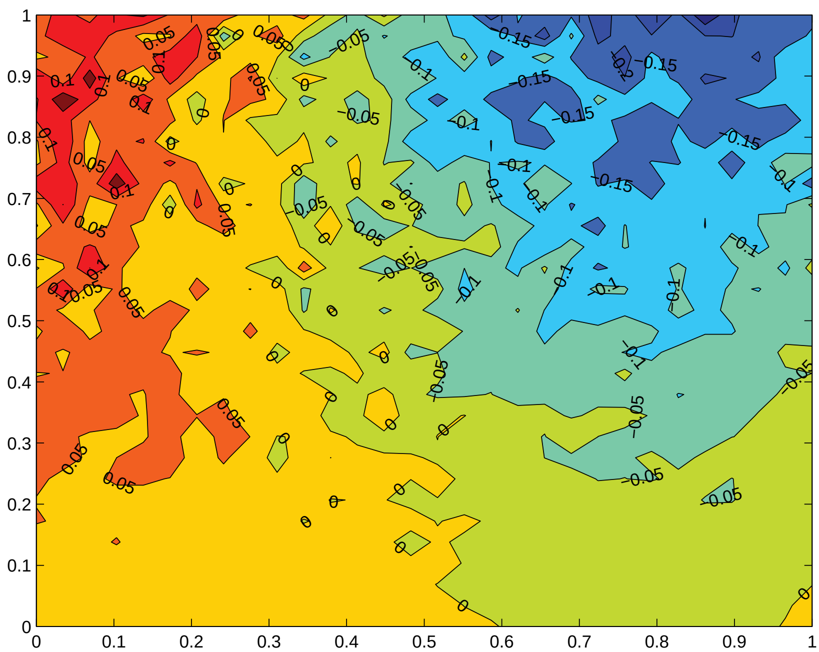
(a) $\delta = 0$



(b) $\delta = 0.005$

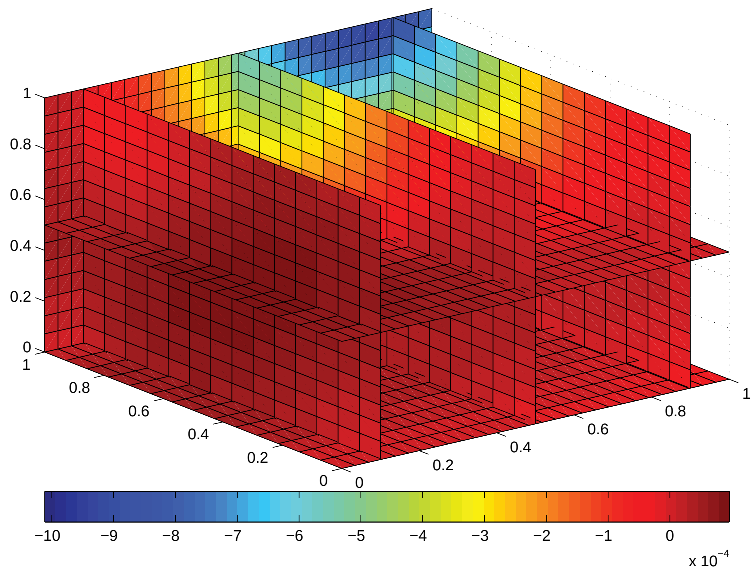


(c) $\delta = 0.01$

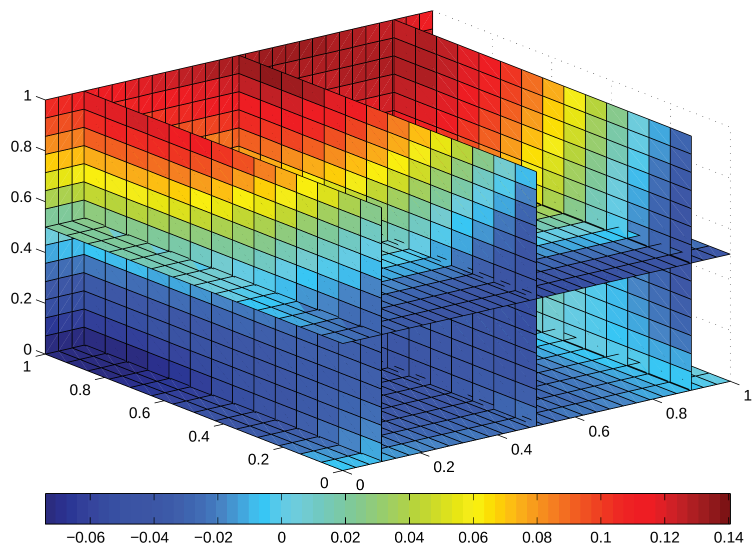


(d) $\delta = 0.05$

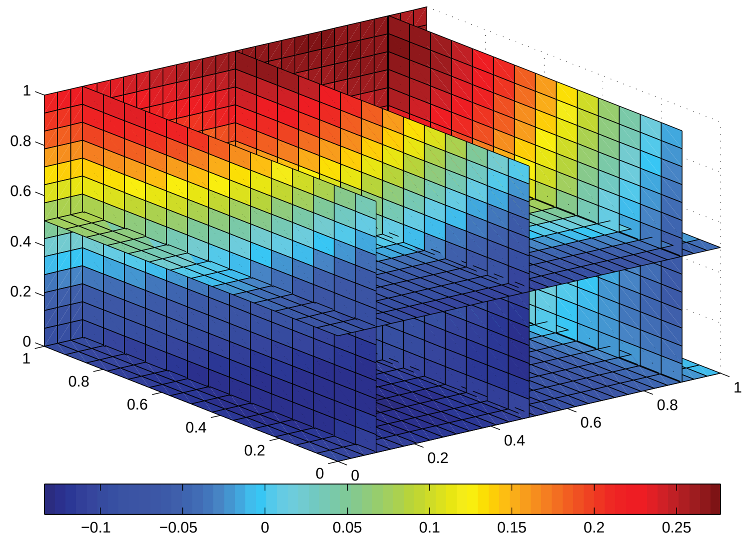
Figure 6: The error distribution for $f(x)$ in Example 3 with various δ.



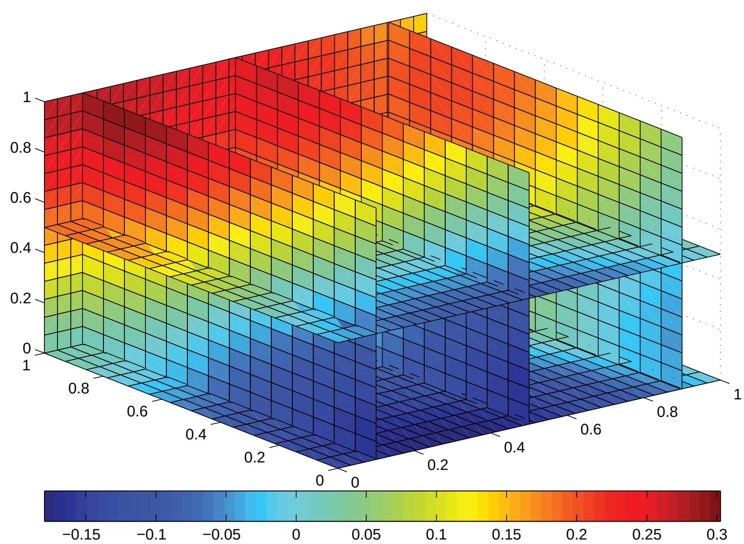
(a) $\delta = 0$



(b) $\delta = 0.001$



(c) $\delta = 0.01$



(d) $\delta = 0.05$

Figure 7: The error distribution for $f(x)$ for Example 4 with various δ.

Table 2: RES versus δ for Example 4

δ	0	0.005	0.01	0.05
RES	0.003	0.031	0.079	0.214

Example 4. Consider an analytical solution

$$u(x,t) = \frac{(x_1^2 + x_2^2 + x_3^2 - 1/\lambda)(1 - e^{-\lambda t}) + 6t}{\lambda} + e^{-\lambda t}(\sin \pi x_1 + \cos \pi x_2 + \sin \pi x_3), \tag{4.12}$$

$$f(x) = x_1^2 + x_2^2 + x_3^2 + \frac{5}{\lambda} + (\pi^2 - 1)(\sin \pi x_1 + \cos \pi x_2 + \sin \pi x_3). \tag{4.13}$$

The numerical results are shown in Figure 7 and Table 2, from which we can see that the computational errors decrease as the noise levels δ decrease. Thus we can conclude that the MFS with the Tikhonov regularization works as well for this 3-D problem as in the previous 1D, 2D cases.

5 Conclusion

In this paper, we investigate a multi-dimensional inverse heat source problem, and determine the space-dependent source term from the Cauchy data on part of boundary and the initial data. We use the method of fundamental solutions combining with the Tikhonov regularization method and GCV choice rule. Four numerical examples in 1D-3D cases show the effective of the proposed method.

Acknowledgement: This work was supported by the NSF of China (10971089) and the Fundamental Research Funds for the Central Universities (lzujbky-2010-k10).

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