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# Upper and Lower Bounds of the Solution for the Superelliptical Plates Problem Using Genetic Algorithms

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**Abstract:** In this article, a new method combining the Mathematical Programming and the Method of Weighted Residual called MP-MWR is presented. Under the validation of maximum principle, and up on the collocation method, the differential equation can be transferred into a bilateral inequality problem. Applying the genetic algorithms helps to find optimal solutions of upper and lower bounds which satisfy the inequalities. Here, the method is verified by analyzing the deflection of superelliptical clamped plate problem. By using this method, the good approximate solution and its error bounds can be obtained effectively and accurately.

**Keywords:** Mathematical Programming, Method of Weighted Residuals, collocation method, superelliptical plate, genetic algorithms.

## 1 Introduction

Until now, solving some nonlinear differential equations of physical engineering problems by either numerical or theoretical methods has still been a challenge to many scholars. For some problems, the analytic exact solutions are impossible to find, and only their approximate solutions can be obtained by some kinds of methods. And the method of weighted residuals (MWR) is one of them.

The method of weighted residuals based on the governing differential equation is a mathematical procedure used to obtain approximate solutions. A good first guess of trial function which sometimes needs experience and intuition is likely to successively improve the approximations. The analytical form of the approximate solution is often more useful and requires less computation time than numerical integration [Finlayson and Scriven (1966)]. Compared to the finite element method and other current methods [Zhang and He (1989)], MWR doesn't rely on the existence of a variational principle for which stationary value need to be sought. Its advantages are program simplicity, shorter computer running time, and less computational error.

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With a rapid development of engineering and technique, a reliable and accuracy solution to the physical problem seems to be guaranteed. Unfortunately, the traditional MWR does not allow the accurate error analysis. For this reason, several techniques came into existence for the treatment of the error bounds of the differential equations. Chen, Lin, and Chen (1997) worked on an error bounds estimate procedure to solve the boundary value problem of differential equations. Moreover, an approximate method using the modern mathematical programming and collocation method to deal with the initial value problems of differential equations was presented by Xing, Li, and Zhu (1997). Appl and Hung (1964) put forward a principle relating to convergent upper and lower bounds in certain continuous boundary value problems.

Genetic algorithms (GAs), a class of probabilistic search algorithms for optimization problems, are first proposed by Goldberg (1989), Davis (1991) and Holland (1992). GAs start with a population of randomly generated candidates and evolve toward better solutions by applying genetic operators, just like the genetic processes occurring in the natural environment. In the last decade, GAs have emerged as a practical and robust search method [Srinivas and Patmaik (1994), Jenkis (1991), Christopher and Donald (1991), Reeves (1993)].

This paper attempts to solve the deflection of the superelliptical plates, using a new double side approach method combing MWR and GAs. These kinds of plates have been broadly used in engineering applications because of the advantage of the curved corners that prevent the stress concentrations, just like some structural and machine elements [Wang, Wang, and Liew (1994)]. However, the numerous studies have been mainly focused on rectangular, circular, and elliptical plates which are the special cases of the superelliptical plates. Considering the lack of contribution in the static behavior of this kind of plate shape, Çeriba?ı, Altay, and Dökmeci (2008) used Galerkin's method to analyze the static behavior of superelliptical clamped plates, and the detailed results were arranged in tabular form. That study was performed for a wide range of superelliptical plates, and the presented results were very useful for practical applications of these sorts of plates.

## 2 Formuation

### *2.1 Formulation of Mathematical programming*

In this study, we investigate the static behavior of the clamped plates by way of the MP-MWR. This kind of method can be used only when the governing equation satisfies the maximum principle [Protter (1967)], the monotonicity of the double Laplace operator can be proved based on the maximum principle of differential equations and this work has been brought up by Zhu (1994). Further, Lee, Chen, and Hung (2002) utilized MP-MWR and applied GAs as the optimization method to find the upper and lower bounds of the solution for an elliptic plate. In that work, the analysis using MP-MWR and GAs was carried out and the presented results were in accordance with the semi-inverse method. Therefore, it indicated that this method worked well with the problem of the elliptic plate, which is a special case of the superelliptical plates.

Now that the governing equation, the double Laplace operator, satisfies the maximum principle, MP-MWR can be applied to this problem.

The governing equation of the plate under uniform loading is given as:

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D} \tag{1}$$

where $w(x, y)$ is the deflection function, $D$ is the flexural rigidity of the plate:

$$D=\frac{Eh^3}{12(1-\nu^2)} \tag{2}$$

Here $E$ is Young's modulus and $\nu$ is the Poisson's ratio of the material of the plate, $h$ is the thickness of the plate, $p$ is a constant related to the surface load.

The superelliptical plates are assumed to be clamped here, therefore, on the boundary $S$, the boundary conditions can be written as:

$$w_S=0 \tag{3}$$

$$\frac{\partial w_S}{\partial n_i}=w_x\times\cos\theta+w_y\times\sin\theta=0 \tag{4}$$

where $n_i$ is the outward normal of the boundary.

Establish an appropriate trial function $Z(x,y,C_j)$ which satisfies the boundary conditions, and then the residuals of this problem can be formulated as:

$$R[Z]=\frac{\partial^4 Z}{\partial x^4}+2\frac{\partial^4 Z}{\partial x^2 \partial y^2}+\frac{\partial^4 Z}{\partial y^4}-\frac{p}{D} \tag{5}$$

$$R_{s1}[Z]=Z \tag{6}$$

$$R_{s2}[Z]=Z_x\times\cos\theta+Z_y\times\sin\theta \tag{7}$$

Because the trial function had been set to satisfy the boundary conditions, so the equation is solved by interior collocation method based on the maximum principle. The problem can be further transferred into a bilateral inequality mathematical programming problem.

The upper bound approximate solution $Z_u(x,y,C_j)$ is to minimize $Z(x_0,y_0,C_j)$ under the conditions:

$$R[Z(x_i,y_i,C_j)] \geq 0 \tag{8}$$

$$R_{S1}[Z(x_i,y_i,C_j)] \geq 0 \tag{9}$$

$$R_{S2}[Z(x_i,y_i,C_j)] \geq 0 \tag{10}$$

On the contrary, the lower bound approximate solution $Z_l(x,y,C_j)$ is to maximize $Z(x_0,y_0,C_j)$ under the conditions:

$$R[Z(x_i,y_i,C_j)] \leq 0 \tag{11}$$

$$R_{S1}[Z(x_i,y_i,C_j)] \leq 0 \tag{12}$$

$$R_{S2}[Z(x_i,y_i,C_j)] \leq 0 \tag{13}$$

where$(x_0,y_0)$ is the target point, $i$ represents the number of collocation points, $j$ stands for the number of undertermined coefficients of the trial function.

### *2.2 Formulation of genetic algorithm*

The method adopted to find the optimal solutions in this article is the genetic algorithms (GAs). The basic procedures of GAs are as follows:

(1) Formulation - Formulate a natural process as the optimization problem;

(2) Initialization - Initialize a population of individuals;

(3) Evaluation - Evaluate the fitness of each individual within the population;

(4) If termination criterion is not satisfied:

(5) Selection- Select individuals for the next population;

(6) Crossover and Mutation- Apply genetic operators to produce new individuals;

(7) Evaluation - Evaluate the new individuals;

(8) Return the best individual.

The flow chart of GAs is shown as Fig. 1.

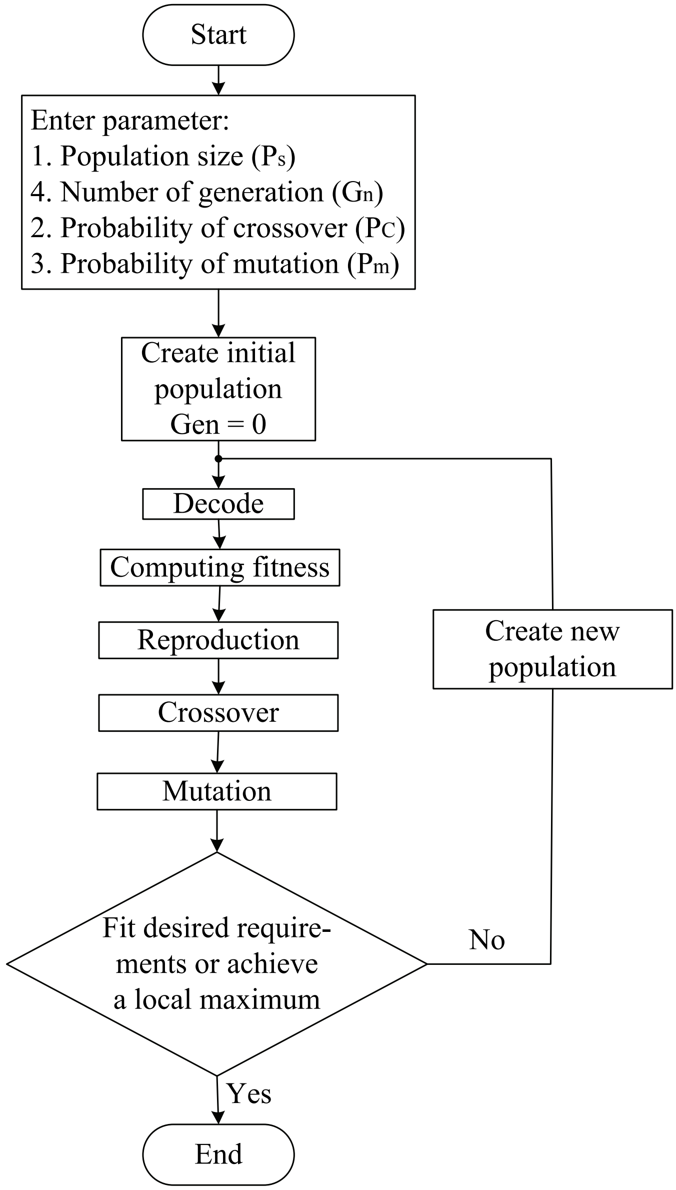


Figure 1: Flow chart of GAs

## 3 Numerical results and discussions

The procedure used to find the upper and lower bounds of the solutions by MP-MWR and GAs is summarized as below:

- Check the problem based on the maximum principle.
- Transfer the problem into a mathematical programming problem using the collocation method in MWR.
- Obtain the optimal solution of the established mathematical programming problem by GAs.

In the Cartesian coordinate, the shape equation of the superelliptical plate can be shown as

$$\frac{x^{2n}}{a^{2n}} + \frac{y^{2n}}{b^{2n}} = 1 \tag{14}$$

where $n$ is the power of the superellipse, $a$ and $b$ are called the semi-diameters of the superellipse, and all of them are positive numbers. Eq. (14) defines a curve ranged in $-a \le x \le a$ and $-b \le y \le b$. Its figure is illustrated in Fig. 2.The power $n$ is chosen as 1, 2, 4, 10 in this study, and we set the values of $a/b$ ratio as 1, 2, 3, 4, 5, 10, 20.

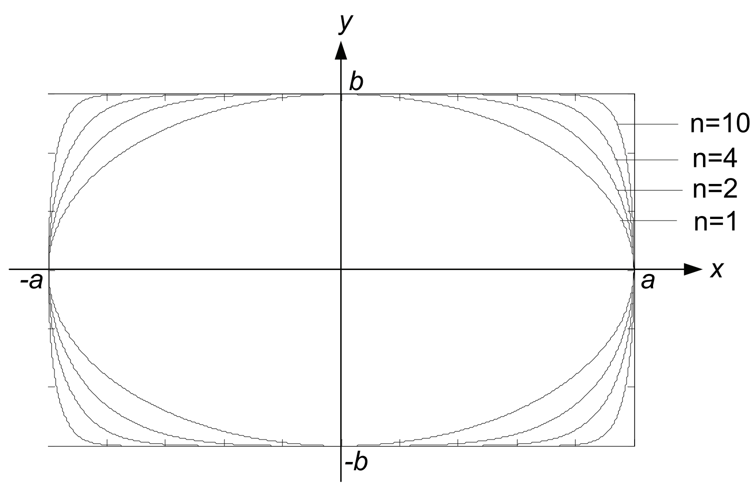


Figure 2: Schematic illustration of the boundary shape of the superelliptical plates

We choose the center of the plate, the location where the maximum reflection occurs, as the index of the objective function. Therefore, the objective of this optimal problem under inequalities is to find the coefficients of the trial function by minimizing $Z(0,0,C_j)$ when

$$R[Z(x_i,y_i,C_j)] \geq 0 \tag{15}$$

$$R_{S1}[Z(x_i,y_i,C_j)] \geq 0 \tag{16}$$

$$R_{S2}[Z(x_i,y_i,C_j)] \geq 0 \tag{17}$$

On the contrary, maximizing $Z(0,0,C_j)$ when

$$R[Z(x_i,y_i,C_j)] \leq 0 \tag{18}$$

$$R_{S1}[Z(x_i,y_i,C_j)] \leq 0 \tag{19}$$

$$R_{S2}[Z(x_i,y_i,C_j)] \leq 0 \tag{20}$$

Once the upper and lower bounds solutions are obtained, then the approximate solution can be determined as $(Z_u+Z_l)/2$. GAs are used to deal with the optimization problem under constraints set in this study, and the parameters are shown in Table 1.

Table 1: The GAs parameters

| Paramters | Value |
|---|---|
| Population size | 100 |
| Number of generations | 1000 |
| Probability of crossover | 0.8 |
| Probability of mutation | 0.01 |
| Selection of refer point | 0 |

In this study, we choose the trial function which satisfies the boundary conditions as:

$$Z(x,y,c) = C_1\left(\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}-1\right)^2 + C_2\left(\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}-1\right)^4 + C_3\left(\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}-1\right)^6$$
$$+ C_4\left(\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}-1\right)^8 + C_5\left(\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}-1\right)^{10} \tag{21}$$

When $n$=1, that is, the case of circular or elliptical boundary shape, the exact solution can be obtained in accordance with the semi-inverse method [Leipholz (1974)]:

$$w(x,y) = \frac{a^4 b^4 p}{8(3a^4 + 2a^2 b^2 + 3b^4)D} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2 \tag{22}$$

However, for other values of $n$, no exact solutions exist, so we solve them by way of the method presented in this article.

Table 2: Deflection at the point (0, 0) of $Z$ for clamped superelliptical plates under uniform loading when $p$=$D$=$b$=1

| $a/b$ | $n$=1 | $n$=2 | $n$=4 | $n$=10 | Rectangle [19] |
|---|---|---|---|---|---|
| 1 | 0.01563<br>0.01563[a] | 0.02009<br>0.01971[b] | 0.02025<br>0.02027[b] | 0.02019<br>0.02017[b] | 0.02016 |
| 2 | 0.03390<br>0.03390[a] | 0.04040<br>0.03973[b] | 0.04044<br>0.04063[b] | 0.04061<br>0.04064[b] | 0.04064 |
| 3 | 0.03835<br>0.03835[a] | 0.04142<br>0.04157[b] | 0.04165<br>0.04198[b] | 0.04175<br>0.04189[b] | — |
| 4 | 0.03985<br>0.03985[a] | 0.04148<br>0.04160[b] | 0.04190<br>0.04174[b] | 0.04182<br>0.04115[b] | — |
| 5 | 0.04052<br>0.04052[a] | 0.04157<br>0.04159[b] | 0.04159<br>0.04172[b] | 0.04170<br>0.04076[b] | — |
| 10 | 0.04139<br>0.04139[a] | 0.04167<br>0.04158[b] | 0.04203<br>0.04212[b] | 0.04201<br>0.04129[b] | — |
| 20 | 0.04160<br>0.04160[a] | 0.04208<br>0.04022[b] | 0.04235<br>0.04230[b] | 0.04243<br>0.04237[b] | — |

[a] results from semi-inverse method [18]
[b] results presented in Ref.[14]

Because the trial function has satisfied the boundary conditions, we adopt interior collocation method to solve the problem. Due to the symmetry of the plate, we need to set the collocation points at the first quadrant only. There are five undetermined coefficients needed to be found and nine uniformly distributed collocation points are arranged in the first quadrant.

It can be observed from Fig. 3∼Fig. 5 that the upper approximate solutions are always distributed on the upper side of the lower approximate solutions in the whole calculation domain, that is, no matter how the shape or size of the plate change, the

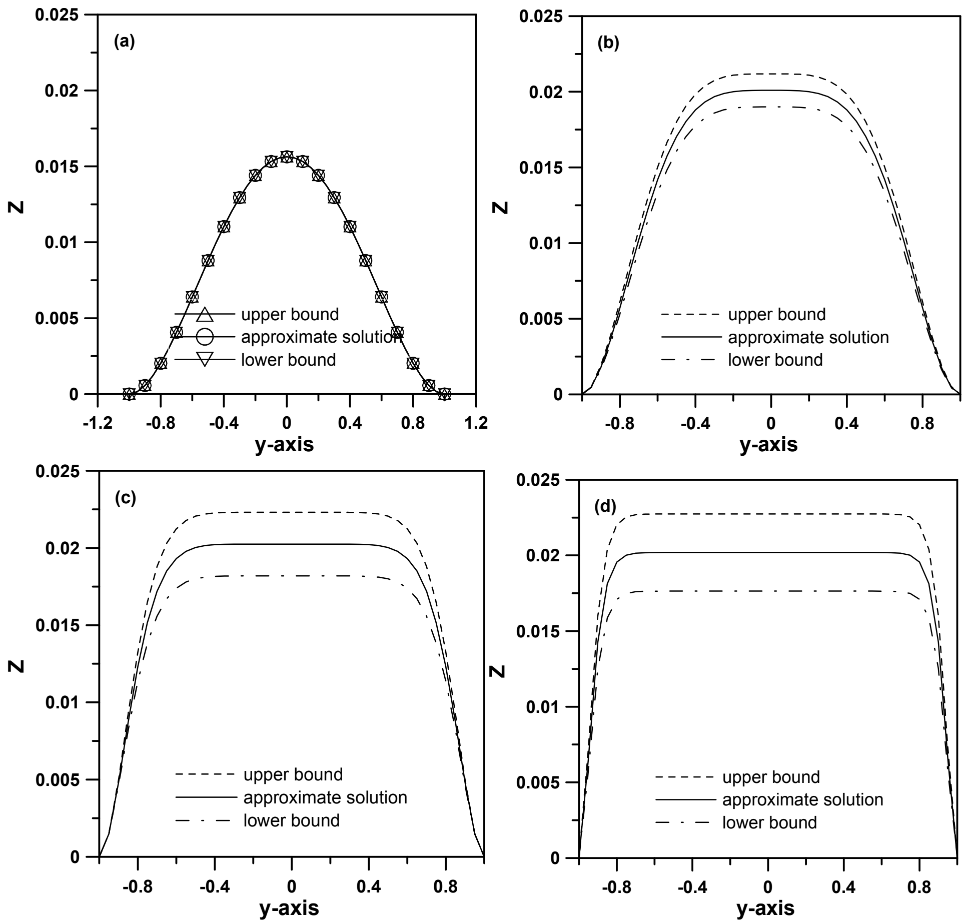


Figure 3: Upper and lower bounds of $Z$ on the y-axis when $a/b$=1. (a) $n$=1 (b) $n$=2 (c) $n$=4 (d) $n$=10

mean approximate solutions always locate between the upper and lower approximate solutions. It always satisfies the requirement for monotonicity.

Table 2 shows the calculated deflection at the center of the plate. It is seen that the first column in Table 2 is precisely the same as the exact solutions from the semi-inverse method. Therefore, it can be concluded that when $n$=1, this method works with high accuracy. Then, compare the results obtained by Timoshenko and Woinowsky-Krieger (1959) for rectangular plates to the results of $n$=10, the case closes to a rectangle, it can be noticed that the results also agree with the

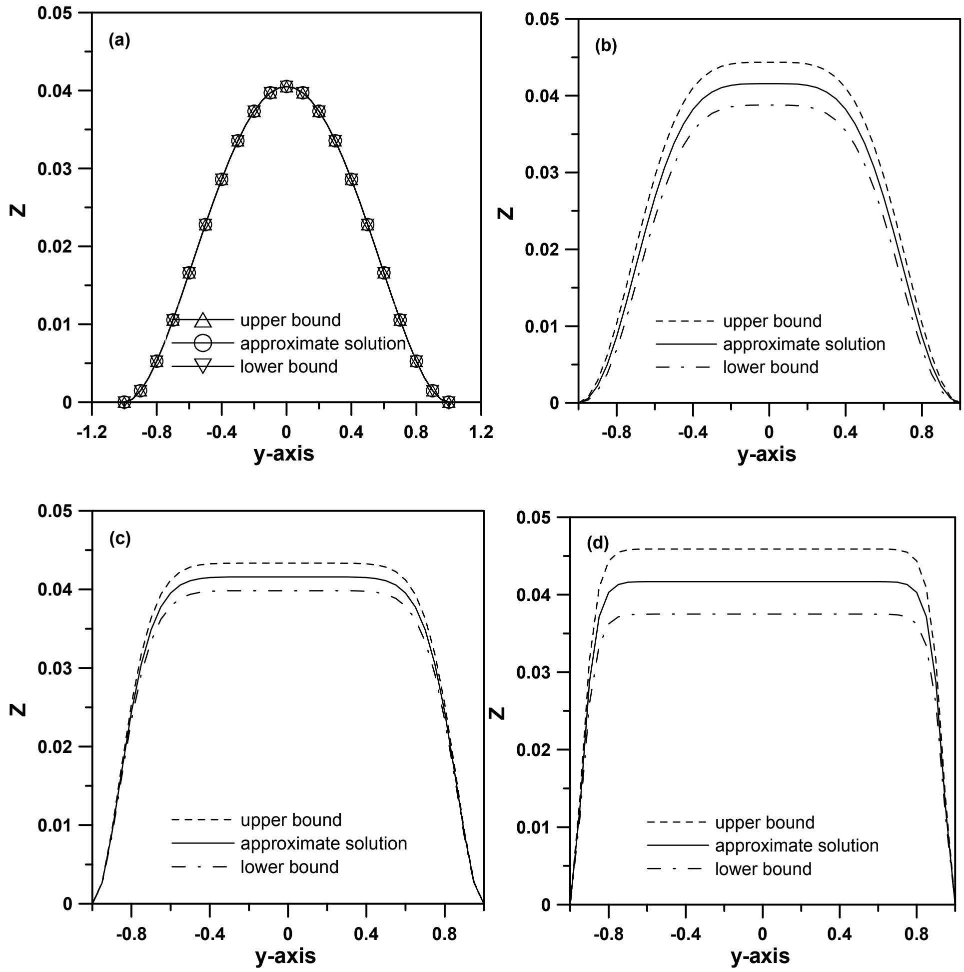


Figure 4: Upper and lower bounds of $Z$ on the y-axis when $a/b$=5. (a) $n$=1 (b) $n$=2 (c) $n$=4 (d) $n$=10

corresponding values. Further, it can be obviously seen that the calculated results agree with those values presented in the work of Çeribaşı, Altay, and Dökmeci (2008) no matter how the values of $n$ changes.

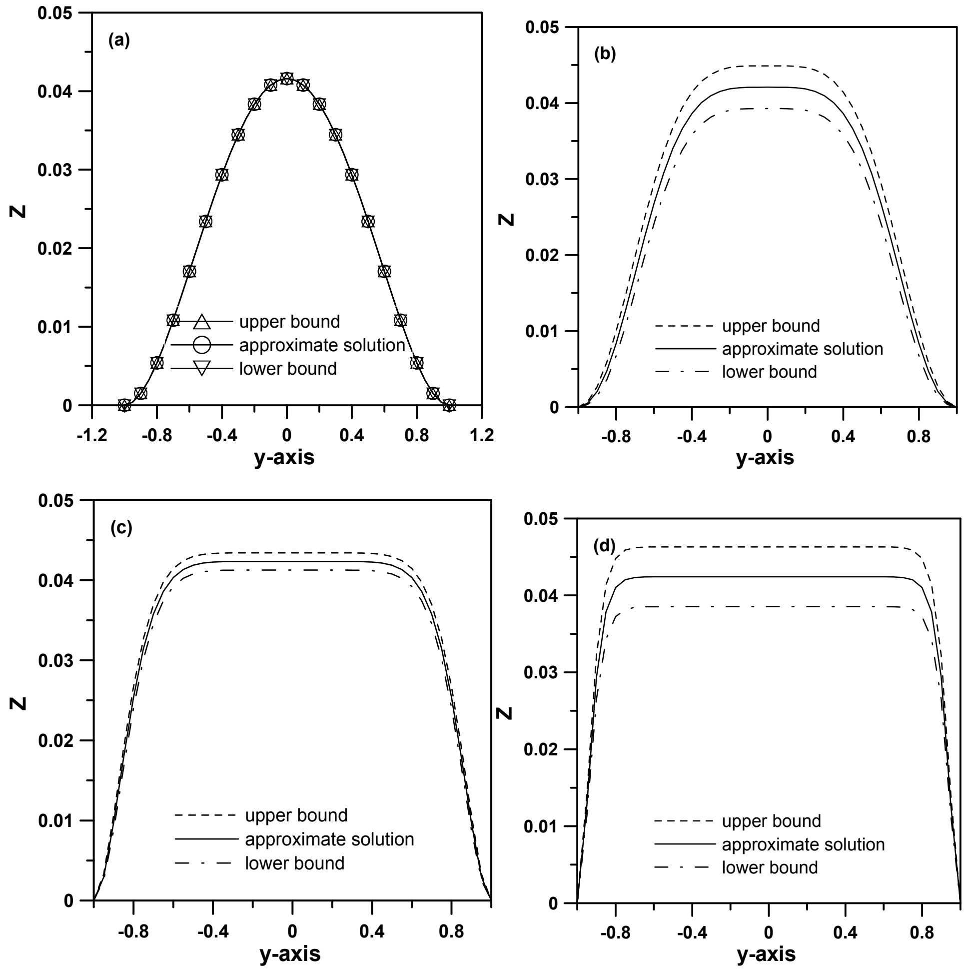


Figure 5: Upper and lower bounds of $Z$ on the y-axis when $a/b$=20. (a) $n$=1 (b) $n$=2 (c) $n$=4 (d) $n$=10

## 4 Conclusion

A new approach has been introduced to analyze the static behavior of the superelliptical plates in this article, and the results presented here were in great agreement with those found by Galerkin's method. Indicate that MP-MWR combining GAs provides another powerful tool for the solution of partial differential equation problems. The advantages of the method can be thus summarized: the memory demand on computer is less than the finite element method; it doesn't need to do the integration so that it requires less calculation load.

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