

## On Determination of a Finite Jacobi Matrix from Two Spectra

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**Abstract:** In this work we study the inverse spectral problem for two spectra of finite order real Jacobi matrices (tri-diagonal matrices). The problem is to reconstruct the matrix using two sets of eigenvalues, one for the original Jacobi matrix and one for the matrix obtained by replacing the first diagonal element of the Jacobi matrix by some another number. The uniqueness and existence results for solution of the inverse problem are established and an explicit procedure of reconstruction of the matrix from the two spectra is given.

**Keywords:** Jacobi matrix, eigenvalue, normalizing numbers, inverse spectral problem.

### 1 Introduction

Let  $J$  be an  $N \times N$  Jacobi matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \quad (1)$$

where for each  $n$ ,  $a_n$  and  $b_n$  are arbitrary real numbers such that  $a_n$  is different from zero:

$$a_n, b_n \in \mathbb{R}, \quad a_n \neq 0. \quad (2)$$

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Define  $\tilde{J}$  to be the Jacobi matrix where all  $a_n$  and  $b_n$  are the same as  $J$ , except  $b_0$  is replaced by  $\tilde{b}_0 \in \mathbb{R}$ ,  $\tilde{b}_0 \neq b_0$ , that is,

$$\tilde{J} = \begin{bmatrix} \tilde{b}_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}. \tag{3}$$

We shall assume for the definiteness that

$$\tilde{b}_0 > b_0. \tag{4}$$

Denote the eigenvalues of the matrices  $J$  and  $\tilde{J}$  by  $\lambda_1, \dots, \lambda_N$  and  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_N$ , respectively. The (finite) sequences  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  are called the *two spectra* of the matrix  $J$ .

The subject of the present paper is the solution of the inverse problem consisting of the following parts:

- (i) Is the matrix  $J$  determined uniquely by its two spectra?
- (ii) To indicate an algorithm for the construction of the matrix  $J$  from its two spectra.
- (iii) To find necessary and sufficient conditions for two given sequences of real numbers  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  to be the two spectra for some matrix of the form (1) with entries from class (2).

The papers Atkinson (1964), De Boor and Golub (1978), Gesztesy and Simon (1997) have been devoted to this problem. However, it appears to us that these papers do not contain a complete solution of the problem. The main result of these papers can be formulated as follows: The two spectra  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  of matrix  $J$  of the form (1) with entries satisfying  $a_n > 0$ ,  $b_n \in \mathbb{R}$  uniquely determine the matrix  $J$  and the number  $\tilde{b}_0$ .

The above papers contain some results on the effective construction of the matrix from two of its spectra. However, these results are of a conditional nature, since it is first assumed that there exists a matrix of the form (1) having the sequences  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  as two of its spectra. In the present paper we shall give a complete solution of the problem.

Another version of the inverse problem for two spectra is to reconstruct the matrix using two sets of eigenvalues, one for the original Jacobi matrix and one for the matrix obtained by deleting the first column and the first row of the Jacobi matrix. Exhaustive results in this direction were obtained in Fu and Hochstadt (1974), Gray and Wilson (1976), Guseinov (1978), Guseinov (2011), Hald (1976), Hochstadt (1967), Hochstadt (1974).

The paper consists, besides this introductory section, of two sections. Section 2 is auxiliary and presents briefly the solution of the inverse problem for finite Jacobi matrices in terms of the eigenvalues and normalizing numbers. Various solutions of this problem are presented in Atkinson (1964), Gesztesy and Simon (1997), Guseinov (2009). In Section 3, we solve our main problem formulated above. At the basis of this solution is the formula

$$\beta_k = \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{b}_0 - b_0} \prod_{j=1, j \neq k}^N \frac{\tilde{\lambda}_j - \lambda_k}{\lambda_j - \lambda_k}, \tag{5}$$

which gives an expression for the normalizing numbers of a finite Jacobi matrix in terms of two of its spectra. Here the difference  $\tilde{b}_0 - b_0$  is expressed by the equation

$$\tilde{b}_0 - b_0 = \sum_{k=1}^N (\tilde{\lambda}_k - \lambda_k). \tag{6}$$

The formulae (5) and (6) also give a conditional solution of the inverse problem in terms of two spectra, because, once we know the numbers  $\{\lambda_k\}_{k=1}^N$  and  $\{\beta_k\}_{k=1}^N$ , we can form the matrix by the prescription given in Section 2. Next we give necessary and sufficient conditions for two sequences of real numbers  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  to be two spectra of a Jacobi matrix of the form (1) with entries in the class (2), i.e. we solve the main problem of this paper. The conditions consist of the following single and simple condition:

$$\lambda_1 < \tilde{\lambda}_1 < \lambda_2 < \tilde{\lambda}_2 < \dots < \lambda_{N-1} < \tilde{\lambda}_{N-1} < \lambda_N < \tilde{\lambda}_N,$$

that is, the numbers  $\lambda_k$  and  $\tilde{\lambda}_k$  interlace.

## 2 Construction of a Jacobi matrix from its eigenvalues and normalizing numbers

In this section we follow the author's paper Guseinov (2009). Given a Jacobi matrix  $J$  of the form (1) with the entries (2), consider the eigenvalue problem  $Jy = \lambda y$  for a column vector  $y = \{y_n\}_{n=0}^{N-1}$ , that is equivalent to the second order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \{0, 1, \dots, N-1\}, \quad a_{-1} = a_{N-1} = 1, \tag{7}$$

for  $\{y_n\}_{n=-1}^N$ , with the boundary conditions

$$y_{-1} = y_N = 0.$$

Denote by  $\{P_n(\lambda)\}_{n=-1}^N$  and  $\{Q_n(\lambda)\}_{n=-1}^N$  the solutions of Eq. (7) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \tag{8}$$

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0. \tag{9}$$

For each  $n \geq 0$ ,  $P_n(\lambda)$  is a polynomial of degree  $n$  and is called a polynomial of first kind and  $Q_n(\lambda)$  is a polynomial of degree  $n - 1$  and is known as a polynomial of second kind. The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{10}$$

holds so that the eigenvalues of the matrix  $J$  coincide with the zeros of the polynomial  $P_N(\lambda)$ . If  $P_N(\lambda_0) = 0$ , then  $\{P_n(\lambda_0)\}_{n=0}^{N-1}$  is an eigenvector of  $J$  corresponding to the eigenvalue  $\lambda_0$ . Any eigenvector of  $J$  corresponding to the eigenvalue  $\lambda_0$  is a constant multiple of  $\{P_n(\lambda_0)\}_{n=0}^{N-1}$ .

As is shown in Guseinov (2009), the equations

$$P_{N-1}(\lambda)Q_N(\lambda) - P_N(\lambda)Q_{N-1}(\lambda) = 1, \tag{11}$$

$$P_{N-1}(\lambda)P'_N(\lambda) - P_N(\lambda)P'_{N-1}(\lambda) = \sum_{n=0}^{N-1} P_n^2(\lambda) \tag{12}$$

hold, where the prime denotes the derivative with respect to  $\lambda$ .

Since the real Jacobi matrix  $J$  of the form (1), (2) is self-adjoint, its eigenvalues are real. Let  $\lambda_0$  be a zero of the polynomial  $P_N(\lambda)$ . The zero  $\lambda_0$  is an eigenvalue of the matrix  $J$  by (10) and hence it is real. Putting  $\lambda = \lambda_0$  in (12) and using  $P_N(\lambda_0) = 0$ , we get

$$P_{N-1}(\lambda_0)P'_N(\lambda_0) = \sum_{n=0}^{N-1} P_n^2(\lambda_0). \tag{13}$$

The right-hand side of (13) is different from zero because the polynomials  $P_n(\lambda)$  have real coefficients and hence are real for real values of  $\lambda$ , and besides  $P_0(\lambda) = 1$ . Therefore,  $P'_N(\lambda_0) \neq 0$ , that is, the root  $\lambda_0$  of the polynomial  $P_N(\lambda)$  is simple. Hence the  $P_N(\lambda)$ , as a polynomial of degree  $N$ , has  $N$  distinct zeros. Thus, any real Jacobi matrix  $J$  of the form (1), (2) has precisely  $N$  real and distinct eigenvalues.

Let  $R(\lambda) = (J - \lambda I)^{-1}$  be the resolvent of the matrix  $J$  (by  $I$  we denote the identity matrix of needed dimension) and  $e_0$  be the  $N$ -dimensional column vector with the components  $1, 0, \dots, 0$ . The rational function

$$w(\lambda) = -\langle R(\lambda)e_0, e_0 \rangle = \langle (\lambda I - J)^{-1}e_0, e_0 \rangle, \tag{14}$$

we call the *resolvent function* of the matrix  $J$ , where  $\langle \cdot, \cdot \rangle$  stands for the standard inner product in  $\mathbb{C}^N$ . This function is known also as the Weyl-Titchmarsh function of  $J$ .

In Guseinov (2009) it is shown that the entries  $R_{nm}(\lambda)$  of the matrix  $R(\lambda) = (J - \lambda I)^{-1}$  (resolvent of  $J$ ) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \leq n \leq m \leq N - 1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \leq m \leq n \leq N - 1, \end{cases}$$

where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}.$$

Therefore according to (14) and using initial conditions (8), (9), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{15}$$

We often will use the following well-known simple useful lemma. We bring it here for easy reference.

**Lemma 1.** *Let  $A(\lambda)$  and  $B(\lambda)$  be polynomials with complex coefficients and  $\deg A < \deg B$ . Next, suppose that  $B(\lambda) = b(\lambda - z_1) \cdots (\lambda - z_N)$ , where  $z_1, \dots, z_N$  are distinct complex numbers and  $b$  is a nonzero complex number. Then there exist uniquely determined complex numbers  $a_1, \dots, a_N$  such that*

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^N \frac{a_k}{\lambda - z_k} \tag{16}$$

for all values of  $\lambda$  different from  $z_1, \dots, z_N$ . The numbers  $a_k$  are given by the equation

$$a_k = \lim_{\lambda \rightarrow z_k} (\lambda - z_k) \frac{A(\lambda)}{B(\lambda)} = \frac{A(z_k)}{B'(z_k)}, \quad k \in \{1, \dots, N\}. \tag{17}$$

**Proof.** For each  $k \in \{1, \dots, N\}$  define the polynomial

$$L_k(\lambda) = b \prod_{j=1, j \neq k}^N (\lambda - z_j) = \frac{B(\lambda)}{\lambda - z_k}$$

of degree  $N - 1$  and set

$$F(\lambda) = A(\lambda) - \sum_{k=1}^N a_k L_k(\lambda),$$

where  $a_k$  is defined by (17). Obviously  $F(\lambda)$  is a polynomial and  $\deg F \leq N - 1$  (recall that  $\deg A < \deg B = N$ ). Since

$$L_k(z_j) = 0 \text{ for } j \neq k, \text{ and } L_k(z_k) = B'(z_k) \neq 0,$$

we have

$$\begin{aligned} F(z_j) &= A(z_j) - \sum_{k=1}^N a_k L_k(z_j) \\ &= A(z_j) - a_j L_j(z_j) = A(z_j) - \frac{A(z_j)}{B'(z_j)} B'(z_j) = 0 \end{aligned}$$

for all  $j \in \{1, \dots, N\}$ . Thus the polynomial  $F(\lambda)$  of degree  $\leq N - 1$  has  $N$  distinct zeros  $z_1, \dots, z_N$ . Then  $F(\lambda) \equiv 0$  and we get

$$\begin{aligned} A(\lambda) &= \sum_{k=1}^N a_k L_k(\lambda) \\ &= \sum_{k=1}^N a_k \frac{B(\lambda)}{\lambda - z_k} = B(\lambda) \sum_{k=1}^N \frac{a_k}{\lambda - z_k}. \end{aligned}$$

This proves (16). Note that the decomposition (16) is unique as for the  $a_k$  in this decomposition the equation (17) necessarily holds.  $\square$

Denote by  $\lambda_1, \dots, \lambda_N$  all the zeros of the polynomial  $P_N(\lambda)$  (which coincide by (10) with the eigenvalues of the matrix  $J$  and which are real and distinct):

$$P_N(\lambda) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_N),$$

where  $c$  is a nonzero constant. Therefore applying Lemma 1 to (15) we can get for the resolvent function  $w(\lambda)$  the following decomposition:

$$w(\lambda) = \sum_{k=1}^N \frac{\beta_k}{\lambda - \lambda_k}, \tag{18}$$

where

$$\beta_k = \frac{Q_N(\lambda_k)}{P'_N(\lambda_k)}. \tag{19}$$

Further, putting  $\lambda = \lambda_k$  in (11) and (12) and taking into account that  $P_N(\lambda_k) = 0$ , we get

$$P_{N-1}(\lambda_k)Q_N(\lambda_k) = 1, \tag{20}$$

$$P_{N-1}(\lambda_k)P'_N(\lambda_k) = \sum_{n=0}^{N-1} P_n^2(\lambda_k), \tag{21}$$

respectively. It follows from (20) that  $Q_N(\lambda_k) \neq 0$  and therefore  $\beta_k \neq 0$ . Comparing (19), (20), and (21), we find that

$$\beta_k = \left\{ \sum_{n=0}^{N-1} P_n^2(\lambda_k) \right\}^{-1}, \tag{22}$$

whence we obtain, in particular, that  $\beta_k > 0$ .

Since  $\{P_n(\lambda_k)\}_{n=0}^{N-1}$  is an eigenvector of the matrix  $J$  corresponding to the eigenvalue  $\lambda_k$ , it is natural, according to the formula (22), to call  $\beta_k$  the *normalizing number* of the matrix  $J$  corresponding to the eigenvalue  $\lambda_k$ .

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k, \beta_k (k = 1, \dots, N)\} \tag{23}$$

of the matrix  $J$  of the form (1), (2) is called the *spectral data* of this matrix.

Determination of the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

Thus, the spectral data consist of the eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl-Titchmarsh function) into partial fractions using the eigenvalues. The resolvent function  $w(\lambda)$  of the matrix  $J$  can be constructed by using equation (15). Another convenient formula for computing the resolvent function is

$$w(\lambda) = -\frac{\det(J_1 - \lambda I)}{\det(J - \lambda I)}, \tag{24}$$

[see Guseinov (2009)] where  $J_1$  is the first truncated matrix (with respect to the matrix  $J$ ) and is obtained by deleting the first column and the first row of the matrix  $J$ .

It follows from (24) that  $\lambda w(\lambda)$  tends to 1 as  $\lambda \rightarrow \infty$ . Therefore multiplying (18) by  $\lambda$  and passing then to the limit as  $\lambda \rightarrow \infty$ , we find

$$\sum_{k=1}^N \beta_k = 1.$$

The *inverse spectral problem* is stated as follows:

- (i) To see if it is possible to reconstruct the matrix  $J$ , given its spectral data (23). If it is possible, to describe the reconstruction procedure.
- (ii) To find the necessary and sufficient conditions for a given collection (23) to be spectral data for some matrix  $J$  of the form (1) with entries belonging to the class (2).

The solution of this problem is well known [see Atkinson (1964), Gesztesy and Simon (1997), Guseinov (2009)] and let us bring here the final result.

Given a collection (23) define the numbers

$$s_l = \sum_{k=1}^N \beta_k \lambda_k^l, \quad l = 0, 1, 2, \dots, \tag{25}$$

and using these numbers introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \dots \tag{26}$$

**Theorem 1.** *Let an arbitrary collection (23) of numbers be given. In order for this collection to be the spectral data for a Jacobi matrix  $J$  of the form (1) with entries belonging to the class (2), it is necessary and sufficient that the following two conditions be satisfied:*

- (i) *The numbers  $\lambda_1, \dots, \lambda_N$  are real and distinct.*
- (ii) *The numbers  $\beta_1, \dots, \beta_N$  are positive and such that  $\beta_1 + \dots + \beta_N = 1$ .*

*Under the conditions (i) and (ii) we have  $D_n > 0$  for  $n \in \{0, 1, \dots, N - 1\}$  and the entries  $a_n$  and  $b_n$  of the matrix  $J$  for which the collection (23) is spectral data, are recovered by the formulae*

$$a_n = \frac{\pm \sqrt{D_{n-1} D_{n+1}}}{D_n}, \quad n \in \{0, 1, \dots, N - 2\}, \quad D_{-1} = 1, \tag{27}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \dots, N - 1\}, \quad \Delta_{-1} = 0, \quad \Delta_0 = s_1, \tag{28}$$

*where  $D_n$  is defined by (26) and (25), and  $\Delta_n$  is the determinant obtained from the determinant  $D_n$  by replacing in  $D_n$  the last column by the column with the components  $s_{n+1}, s_{n+2}, \dots, s_{2n+1}$ .*

It is not difficult to see that for the determinants  $D_n$  defined by (26) and (25) we have  $D_n = 0$  for  $n \geq N$ .

It follows from the above solution of the inverse problem that the matrix (1) is not uniquely restored from the spectral data. This is linked with the fact that the  $a_n$  are determined from (27) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs  $+$  and  $-$ . Namely, let  $\{\sigma_0, \sigma_1, \dots, \sigma_{N-2}\}$  be a given finite sequence, where for each  $n \in \{0, 1, \dots, N-2\}$  the  $\sigma_n$  is  $+$  or  $-$ . We have  $2^{N-1}$  such different sequences. Now to determine  $a_n$  uniquely from (27) for  $n \in \{0, 1, \dots, N-2\}$  we can choose the sign  $\sigma_n$  when extracting the square root. In this way we get precisely  $2^{N-1}$  distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence  $\{\sigma_0, \sigma_1, \dots, \sigma_{N-2}\}$  of signs  $+$  and  $-$ . Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix. In particular, the inverse problem is solvable uniquely in the class of entries  $a_n > 0, b_n \in \mathbb{R}$ .

### 3 Construction of a Jacobi matrix from two of its spectra

Let  $J$  be an  $N \times N$  Jacobi matrix of the form (1) with entries satisfying (2). Define  $\tilde{J}$  to be the Jacobi matrix given by (3), where the number  $\tilde{b}_0$  satisfies (4):  $\tilde{b}_0 > b_0$ . We denote the eigenvalues of the matrices  $J$  and  $\tilde{J}$  by  $\lambda_1 < \dots < \lambda_N$  and  $\tilde{\lambda}_1 < \dots < \tilde{\lambda}_N$ , respectively. We call the collections  $\{\lambda_k (k = 1, \dots, N)\}$  and  $\{\tilde{\lambda}_k (k = 1, \dots, N)\}$  the *two spectra* of the matrix  $J$ .

The inverse problem for two spectra consists in the reconstruction of the matrix  $J$  by two of its spectra.

We will reduce the inverse problem for two spectra to the inverse problem for eigenvalues and normalizing numbers solved above in Section 2.

First let us study some necessary properties of the two spectra of the Jacobi matrix  $J$ .

Let  $P_n(\lambda)$  and  $Q_n(\lambda)$  be the polynomials of the first and second kind for the matrix  $J$ . The similar polynomials for the matrix  $\tilde{J}$  we denote by  $\tilde{P}_n(\lambda)$  and  $\tilde{Q}_n(\lambda)$ . By (10) we have

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda), \tag{29}$$

$$\det(\tilde{J} - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} \tilde{P}_N(\lambda), \tag{30}$$

so that the eigenvalues  $\lambda_1, \dots, \lambda_N$  and  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_N$  of the matrices  $J$  and  $\tilde{J}$  coincide with the zeros of the polynomials  $P_N(\lambda)$  and  $\tilde{P}_N(\lambda)$ , respectively.

The  $P_n(\lambda)$  and  $\tilde{P}_n(\lambda)$  satisfy the same equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \{1, \dots, N-1\}, \quad a_{N-1} = 1, \quad (31)$$

subject to the initial conditions

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \frac{\lambda - b_0}{a_0}; \quad (32)$$

$$\tilde{P}_0(\lambda) = 1, \quad \tilde{P}_1(\lambda) = \frac{\lambda - \tilde{b}_0}{a_0}. \quad (33)$$

The  $Q_n(\lambda)$  also satisfies Eq. (31); besides

$$Q_0(\lambda) = 0, \quad Q_1(\lambda) = \frac{1}{a_0}. \quad (34)$$

Since  $P_n(\lambda)$  and  $\tilde{P}_n(\lambda)$  form, for  $b_0 \neq \tilde{b}_0$ , linearly independent solutions of Eq. (31), the solution  $Q_n(\lambda)$  will be a linear combination of the solutions  $P_n(\lambda)$  and  $\tilde{P}_n(\lambda)$ . Using initial conditions (32), (33), and (34) we find that

$$Q_n(\lambda) = \frac{1}{\tilde{b}_0 - b_0} [P_n(\lambda) - \tilde{P}_n(\lambda)], \quad n \in \{0, 1, \dots, N\}. \quad (35)$$

Replacing  $Q_N(\lambda)$  and  $Q_{N-1}(\lambda)$  in (11) by their expressions from (35), we get

$$P_{N-1}(\lambda)\tilde{P}_N(\lambda) - P_N(\lambda)\tilde{P}_{N-1}(\lambda) = b_0 - \tilde{b}_0. \quad (36)$$

**Lemma 2.** *The matrices  $J$  and  $\tilde{J}$  have no common eigenvalues, that is,  $\lambda_k \neq \tilde{\lambda}_j$  for all values of  $k$  and  $j$ .*

**Proof.** Suppose that  $\lambda$  is an eigenvalue of the matrices  $J$  and  $\tilde{J}$ . Then by (29) and (30) we have  $P_N(\lambda) = \tilde{P}_N(\lambda) = 0$ . But this is impossible by (36) and the condition  $\tilde{b}_0 \neq b_0$ .  $\square$

The following lemma allows us to calculate the difference  $\tilde{b}_0 - b_0$  in terms of the two spectra.

**Lemma 3.** *The equality*

$$\sum_{k=1}^N (\tilde{\lambda}_k - \lambda_k) = \tilde{b}_0 - b_0 \quad (37)$$

holds.

**Proof.** For any matrix  $A = [a_{jk}]_{j,k=1}^N$  the spectral trace of  $A$  coincides with the matrix trace of  $A$  : If  $\mu_1, \dots, \mu_N$  are the eigenvalues of  $A$ , then

$$\sum_{k=1}^N \mu_k = \sum_{k=1}^N a_{kk}.$$

Indeed, this follows from

$$\det(\lambda I - A) = (\lambda - \mu_1) \dots (\lambda - \mu_N)$$

by comparison of the coefficients of  $\lambda^{N-1}$  on the two sides. Therefore we can write

$$\sum_{k=1}^N \tilde{\lambda}_k = \tilde{b}_0 + b_1 + \dots + b_{N-1} \quad \text{and} \quad \sum_{k=1}^N \lambda_k = b_0 + b_1 + \dots + b_{N-1}.$$

Subtracting the last two equalities side by side we arrive at (37).  $\square$

**Lemma 4.** *The eigenvalues of  $J$  and  $\tilde{J}$  interlace: If  $\tilde{b}_0 > b_0$ , then*

$$\lambda_1 < \tilde{\lambda}_1 < \lambda_2 < \tilde{\lambda}_2 < \dots < \lambda_{N-1} < \tilde{\lambda}_{N-1} < \lambda_N < \tilde{\lambda}_N. \tag{38}$$

**Proof.** Replacing  $Q_N(\lambda)$  in (15) by its expression from (35), we get

$$w(\lambda) = \frac{1}{\tilde{b}_0 - b_0} \left[ 1 - \frac{\tilde{P}_N(\lambda)}{P_N(\lambda)} \right].$$

Hence, setting

$$\psi(\lambda) = 1 - (\tilde{b}_0 - b_0)w(\lambda),$$

we obtain

$$\psi(\lambda) = \frac{\tilde{P}_N(\lambda)}{P_N(\lambda)}, \tag{39}$$

and also

$$\psi(\lambda) = 1 - (\tilde{b}_0 - b_0) \sum_{k=1}^N \frac{\beta_k}{\lambda - \lambda_k} \tag{40}$$

by (18). Differentiating (40) we find

$$\psi'(\lambda) = (\tilde{b}_0 - b_0) \sum_{k=1}^N \frac{\beta_k}{(\lambda - \lambda_k)^2}. \tag{41}$$

Inserting (39) in the left side of (41), we get

$$\tilde{P}'_N(\lambda)P_N(\lambda) - \tilde{P}_N(\lambda)P'_N(\lambda) = [P_N(\lambda)]^2(\tilde{b}_0 - b_0) \sum_{k=1}^N \frac{\beta_k}{(\lambda - \lambda_k)^2}.$$

Hence

$$\tilde{P}'_N(\lambda)P_N(\lambda) - \tilde{P}_N(\lambda)P'_N(\lambda) > 0 \quad (-\infty < \lambda < \infty). \tag{42}$$

Recall that  $\lambda_k, \lambda_{k+1}$  are two consecutive zeros of  $P_N(\lambda)$ . Since  $P_N(\lambda)$  has only simple zeros, this implies that  $P'_N(\lambda_k), P'_N(\lambda_{k+1})$  have opposite signs. By (42) we have

$$-\tilde{P}'_N(\lambda_k)P'_N(\lambda_k) > 0, \quad -\tilde{P}'_N(\lambda_{k+1})P'_N(\lambda_{k+1}) > 0,$$

and so  $\tilde{P}_N(\lambda_k), \tilde{P}_N(\lambda_{k+1})$  must also have opposite signs. Consequently  $\tilde{P}_N(\lambda)$  has at least one zero in the interval  $(\lambda_k, \lambda_{k+1})$  for each  $k \in \{1, \dots, N\}$ . Next, since  $P_N(\lambda_N) = 0, P_N(\lambda) \neq 0$  for  $\lambda > \lambda_N$ , and  $P_N(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , it follows that  $P'_N(\lambda_N) > 0$ . Then (42) gives  $\tilde{P}'_N(\lambda_N) < 0$ . Besides we have that  $\tilde{P}_N(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . Therefore  $\tilde{P}_N(\lambda)$  has at least one zero in the interval  $(\lambda_N, \infty)$ . Since there are  $N$  intervals  $(\lambda_1, \lambda_2), \dots, (\lambda_{N-1}, \lambda_N), (\lambda_N, \infty)$  and  $\tilde{P}_N(\lambda)$  is a polynomial of degree  $N$ ,  $\tilde{P}_N(\lambda)$  has a single zero in each of these intervals. Thus (38) is proved.  $\square$

Note that another proof of (38) can be given as follows. Since  $\beta_k > 0$  for all  $k \in \{1, \dots, N\}$ , it follows from (41) that  $\psi'(\lambda) > 0$  for real values of  $\lambda$  if  $\tilde{b}_0 > b_0$ . Therefore  $\psi(\lambda)$  is strictly increasing continuous function on the intervals  $(-\infty, \lambda_1), (\lambda_1, \lambda_2), \dots, (\lambda_{N-1}, \lambda_N), (\lambda_N, \infty)$ . Besides, as follows from (40),

$$\lim_{|\lambda| \rightarrow \infty} \psi(\lambda) = 1, \quad \lim_{\lambda \rightarrow \lambda_k^-} \psi(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \lambda_k^+} \psi(\lambda) = -\infty.$$

Consequently, the function  $\psi(\lambda)$  has no zero in the interval  $(-\infty, \lambda_1)$  and exactly one zero in each of the intervals  $(\lambda_1, \lambda_2), \dots, (\lambda_{N-1}, \lambda_N)$  and  $(\lambda_N, \infty)$ . Since the zeros of the function  $\psi(\lambda)$  coincide with the eigenvalues of  $\tilde{J}$  by (39), the proof is complete.

The following lemma (together with Lemma 3) gives a formula for calculating the normalizing numbers  $\beta_1, \dots, \beta_N$  in terms of the two spectra.

**Lemma 5.** For each  $k \in \{1, \dots, N\}$  the formula

$$\beta_k = \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{b}_0 - b_0} \prod_{j=1, j \neq k}^N \frac{\tilde{\lambda}_j - \lambda_k}{\tilde{\lambda}_j - \lambda_j} \tag{43}$$

holds.

**Proof.** Equating (39) and (40), and taking into account (29), (30), we can write

$$1 - (\tilde{b}_0 - b_0) \sum_{j=1}^N \frac{\beta_j}{\lambda - \lambda_j} = \prod_{j=1}^N \frac{\lambda - \tilde{\lambda}_j}{\lambda - \lambda_j}.$$

Multiply both sides of the last equality by  $\lambda - \lambda_k$  and pass then to the limit as  $\lambda \rightarrow \lambda_k$  to get

$$-(\tilde{b}_0 - b_0)\beta_k = (\lambda_k - \tilde{\lambda}_k) \prod_{j=1, j \neq k}^N \frac{\lambda_k - \tilde{\lambda}_j}{\lambda_k - \lambda_j}.$$

This yields (43).  $\square$

**Theorem 2** (Uniqueness result). *The two spectra  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  of the Jacobi matrix  $J$  of the form (1) in the class*

$$a_n > 0, b_n \in \mathbb{R} \tag{44}$$

*uniquely determine the matrix  $J$  and the number  $\tilde{b}_0 \in \mathbb{R}$  in the matrix  $\tilde{J}$  defined by (3).*

**Proof.** Given the two spectra  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  of the matrix  $J$  we determine uniquely the number (difference)  $\tilde{b}_0 - b_0$  by (37) and then the normalizing numbers  $\beta_k$  ( $k = 1, \dots, N$ ) of the matrix  $J$  by (43). Since the collection of the eigenvalues and normalizing numbers  $\{\lambda_k, \beta_k$  ( $k = 1, \dots, N$ ) $\}$  of the matrix  $J$  determines  $J$  uniquely in the class (44), and the number  $\tilde{b}_0$  is determined uniquely by the equation

$$\tilde{b}_0 = b_0 + \sum_{k=1}^N (\tilde{\lambda}_k - \lambda_k),$$

the proof is complete.  $\square$

The following theorem solves the inverse problem in terms of the two spectra. Its proof given below contains an effective procedure for the construction of the Jacobi matrix from its two spectra.

**Theorem 3.** *In order for given two collections of real numbers  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  to be the spectra of two matrices  $J$  and  $\tilde{J}$ , respectively, of the forms (1) and (3) with the entries in the class (2) and  $\tilde{b}_0 > b_0$ , it is necessary and sufficient that the following inequalities be satisfied:*

$$\lambda_1 < \tilde{\lambda}_1 < \lambda_2 < \tilde{\lambda}_2 < \dots < \lambda_{N-1} < \tilde{\lambda}_{N-1} < \lambda_N < \tilde{\lambda}_N. \tag{45}$$

**Proof.** The necessity of the condition (45) has been proved above in Lemma 4. To prove the sufficiency suppose that two collections of real numbers  $\{\lambda_k\}_{k=1}^N$  and  $\{\tilde{\lambda}_k\}_{k=1}^N$  are given which satisfy the inequalities in (45). We construct  $\beta_k$  ( $k = 1, \dots, N$ ) according to these data by Eq. (43) in which the number (difference)  $\tilde{b}_0 - b_0$  is defined by Eq. (37). It follows from (43), (37), and (45) that  $\beta_k > 0$  ( $k = 1, \dots, N$ ). Let us show that

$$\sum_{k=1}^N \beta_k = 1. \tag{46}$$

To this end we consider the rational function

$$\begin{aligned} T(\lambda) &= \frac{(\lambda - \tilde{\lambda}_1) \cdots (\lambda - \tilde{\lambda}_N)}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_N)} - 1 \\ &= \frac{(\lambda - \tilde{\lambda}_1) \cdots (\lambda - \tilde{\lambda}_N) - (\lambda - \lambda_1) \cdots (\lambda - \lambda_N)}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_N)}. \end{aligned}$$

We have

$$T(\lambda) = -\frac{1}{\lambda} \sum_{j=1}^N (\tilde{\lambda}_j - \lambda_j) + O\left(\frac{1}{|\lambda|^2}\right) \tag{47}$$

as  $|\lambda| \rightarrow \infty$ . On the other hand, applying Lemma 1 to the function  $T(\lambda)$  we can write

$$T(\lambda) = \sum_{k=1}^N \frac{a_k}{\lambda - \lambda_k}, \tag{48}$$

where

$$\begin{aligned} a_k &= \lim_{\lambda \rightarrow \lambda_k} (\lambda - \lambda_k) T(\lambda) \\ &= (\lambda_k - \tilde{\lambda}_k) \prod_{j=1, j \neq k}^N \frac{\lambda_k - \tilde{\lambda}_j}{\lambda_k - \lambda_j} = -\beta_k \sum_{j=1}^N (\tilde{\lambda}_j - \lambda_j); \end{aligned}$$

in the last equality we have used the constuction of  $\beta_k$ . Equating (47) and (48), we have

$$-\frac{1}{\lambda} \sum_{j=1}^N (\tilde{\lambda}_j - \lambda_j) + O\left(\frac{1}{|\lambda|^2}\right) = -\left\{ \sum_{j=1}^N (\tilde{\lambda}_j - \lambda_j) \right\} \sum_{k=1}^N \frac{\beta_k}{\lambda - \lambda_k}.$$

Multiplying here both sides by  $\lambda$  and passing then to the limit as  $|\lambda| \rightarrow \infty$  we arrive at (46).

Consequently, the collection  $\{\lambda_k, \beta_k (k = 1, \dots, N)\}$  satisfies the conditions of Theorem 1 and hence there exist a Jacobi matrix  $J$  of the form (1) with entries from the class (2) such that the  $\lambda_k (k = 1, \dots, N)$  are the eigenvalues and the  $\beta_k (k = 1, \dots, N)$  are the corresponding normalizing numbers for  $J$ . Having the matrix  $J$ , in particular, its entry  $b_0$ , we construct the number  $\tilde{b}_0$  by

$$\tilde{b}_0 = b_0 + \sum_{k=1}^N (\tilde{\lambda}_k - \lambda_k) \tag{49}$$

and then the matrix  $\tilde{J}$  by (3) according to the matrix  $J$  and (49). It remains to show that  $\{\tilde{\lambda}_k\}_{k=1}^N$  is the spectrum of the constructed matrix  $\tilde{J}$ . Denote the eigenvalues of  $\tilde{J}$  by  $\mu_1 < \dots < \mu_N$ . We have to show that  $\mu_k = \tilde{\lambda}_k (k = 1, \dots, N)$ .

By the direct spectral problem we have (Lemma 5)

$$\beta_k = \frac{\mu_k - \lambda_k}{\tilde{b}_0 - b_0} \prod_{j=1, j \neq k}^N \frac{\mu_j - \lambda_k}{\lambda_j - \lambda_k}.$$

On the other hand, by our construction of  $\beta_k$ ,

$$\beta_k = \frac{\tilde{\lambda}_k - \lambda_k}{\tilde{b}_0 - b_0} \prod_{j=1, j \neq k}^N \frac{\tilde{\lambda}_j - \lambda_k}{\lambda_j - \lambda_k}.$$

Equating the last two equations we obtain

$$\prod_{j=1}^N (\mu_j - \lambda_k) = \prod_{j=1}^N (\tilde{\lambda}_j - \lambda_k), \quad k = 1, \dots, N.$$

This means that the polynomials  $(\mu_1 - \lambda) \cdots (\mu_N - \lambda)$  and  $(\tilde{\lambda}_1 - \lambda) \cdots (\tilde{\lambda}_N - \lambda)$  of degree  $N$  with the same leading coefficients coincide at  $N$  distinct points  $\lambda_1, \dots, \lambda_N$ . Then these polynomials coincide identically and hence  $\mu_k = \tilde{\lambda}_k (k = 1, \dots, N)$ . The proof is complete.  $\square$

#### 4 Conclusion

In this paper, an inverse spectral problem for two spectra of finite order real Jacobi matrices has been solved. First the inverse spectral problem with respect to the spectral data has been discussed and recalled how to reconstruct the real Jacobi matrices from the spectral data. The spectral data consist of the eigenvalues and

associated normalizing numbers derived by decomposing the resolvent function (Weyl-Titchmarsh function) of the Jacobi matrix into partial fractions using the eigenvalues. Then the the inverse problem for two spectra has been reduced to the inverse problem for spectral data. The uniqueness and existence results for solution of the inverse problem with respect to two spectra have been established and an explicit procedure of reconstruction of the matrix from the two spectra has been given.

The Jacobi matrices appear in a variety of applications. A distinguishing feature of the Jacobi matrices from others is that they are related to certain three-term recursion equations (second order linear difference equations). Therefore these matrices can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. Many versions of the inverse spectral problem for Jacobi matrices form analogs of problems of inverse Sturm-Liouville theory [Levitan and Gasymov (1964)], in which a coefficient-function or “potential” in a second-order differential equation is to be recovered, given either the spectral function, or alternatively given two sets of eigenvalues corresponding to two given boundary conditions at one end, the boundary condition at the other end being fixed. Spectral and inverse spectral theory for Jacobi matrices plays a fundamental role in the investigation of completely integrable nonlinear lattices, in particular, in the investigation of the Toda lattices [Teschl (2000)].

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