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Natural Boundary Element Method for Bending Problem of Infinite Plate with a Circular Opening under the Boundary Loads

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Abstract: Based on the complex functions theory in elastic mechanics, the bending deflection formula expressed by the complex Fourier series is derived for the infinite plate with a circular opening at first, then the boundary conditions of the circular opening are expanded in Fourier Series, and the unknown coefficients of the Fourier series are determined by comparing coefficients method. By means of the convolution of the complex Fourier series and some basic formulas in the generalized functions theory, the natural boundary integral formula or the analytical deflection formulas expressed by the boundary displacement or loads are developed for the infinite plates with a circular opening under the three common boundary conditions of the circular opening. These analytical formulas can be directly used to solve the bending problems of the infinite plates with a circular opening under the conditions of the clapped edge, simply supported edge and free edge. At last, some examples of using these analytical formulas indicate that under simple boundary conditions we can easily obtain the analytical solutions for the bending problem of the infinite plate with a circular opening, while for the bending problems with some complicated boundary conditions we can get their numerical solutions by these developed formulas.

Keywords: infinite plate with a circular opening, bending deflection, complex Fourier series; generalized functions; natural boundary element method

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1 Introduction

The bending problem of infinite plate with circular opening is usually solved by the complex function method combined with the Cauchy integral [Huang (1992); Qu (2000); Xu(1990)]. But, the calculation of Cauchy integral is tedious, and for the different problems some different methods or technique must be used in order to calculate the Cauchy integral, so the bending problem of infinite plate with openings is not easily solved directly by a uniform analytical formula.

In this paper the Cauchy integral is avoided, and by the natural boundary element method (NBEM) the bending deflection formula of the infinite plate with a circular opening under the boundary loads is derived. The natural boundary element method is a branch of boundary element methods, based on a Green functional method, a complex variable method, or a method using a Fourier series to induce a Dirichlet boundary value problem as a differential equation into Poisson integration formula of the studied area or to induce Neumann boundary value problem of differential equation into a strong singular boundary integral equation [Yu (1993)]. The natural boundary element method has been widely used to solve the plane problems [Dong (2006) and Chen (2006)], bending problems [Li, Dong and Zhao (2011)] and engineering problems [Peng et al. (2009); Pan (2008)]. In addition, Liu and Yu (2008), Yu and Du (2003), Zhao et al., (2000), have studied some coupling methods between NBEM and finite element methods.

2 The bending deflection formula expressed by complex Fourier series for infinite plates with a Circular opening under the boundary loads

The bending problem under boundary loads for infinite plates with opening can be attributed to the boundary value problems of homogeneous bi-harmonic equation, namely, the deflection function u satisfies:

$$\Delta^2 u = 0 \tag{1}$$

where Δ is Laplacian operator. The solution of Eq. (1) is also called the bi-harmonic function. According to the complex variable method [Xu (1990)] of the bending problem for infinite plates with openings, the bi-harmonic function u can be expressed as

$$u = \Re[\bar{z}\varphi_1(z) + \theta_1(z)] \tag{2}$$

where $\varphi_1(z)$ and $\theta_1(z)$ are analytic functions, and when the principal vector and principal moment of the loads on the opening edge are zero, there are:

$$\varphi_1(z) = Bz + \sum_{n=0}^{\infty} a_n z^{-n} \tag{3}$$

$$\theta_1'(z) = \psi_1(z) = (B_1 + iC_1)z + \sum_{n=0}^{\infty} b_n z^{-n} \tag{4}$$

where $a_0 = 0$, and B, B_1, C_1 can be expressed by the loads M_x^∞, M_y^∞, M_{xy}^∞ at the infinite point, that is:

$$B = -\frac{M_y^\infty + M_x^\infty}{4D(1+\mu)}, \quad B_1 = \frac{M_y^\infty - M_x^\infty}{2D(1-\mu)}, \quad C_1 = \frac{2M_{xy}^\infty}{2D(1-\mu)} \tag{5}$$

where μ is poisson ratio, and D is bending rigidity of the plate.
By integration of Eq.(4) we have:

$$\theta_1(z) \ = \frac{B_1 + iC_1}{2} z^2 + b_0 z + b_1 \ln z + \sum_{n=1}^{\infty} (-\frac{1}{n} b_{n+1} z^{-n}) + B' + iC' \tag{6}$$

where B', C' are constants of integration.
Take Eqs.(3) and (6) into Eq. (2), we have:

$$u = \Re\left[\bar{z}(Bz + \sum_{n=1}^{\infty} a_n z^{-n}) + \frac{B_1 + iC_1}{2} z^2 + b_0 z + b_1 \ln z + B' + iC' + \sum_{n=1}^{\infty} (-\frac{1}{n} b_{n+1}) z^{-n}\right] \tag{7}$$

Without loss of generality, the radius of the circular opening for the infinite plate is set as unit one. Let $z = re^{i\theta}$, $r > 1$. Take z into Eq. (7), we have

$$\begin{aligned} u = u(r,\theta) = Br^2 + \frac{B_1}{2} r^2 \cos 2\theta - \frac{C_1}{2} r^2 \sin 2\theta + b_0 r \cos\theta + b_1 \ln r + B' \\ + \sum_{n=1}^{\infty} a_n r^{-n+1} \cos(n+1)\theta \ + \sum_{n=1}^{\infty} (-\frac{1}{n} b_{n+1}) r^{-n} \cos n\theta, \ r > 1 \end{aligned} \tag{8}$$

Considering that $\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta})$, the series in Eq. (8) can be further expanded into complex Fourier series:

$$\sum_{n=1}^{\infty} a_n r^{-n+1} \cos(n+1)\theta + \sum_{n=1}^{\infty} (-\frac{1}{n} b_{n+1}) r^{-n} \cos n\theta = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} [\frac{a_{|n|-1}}{2} r^2 + (-\frac{1}{2|n|} b_{|n|+1})] r^{-|n|} e^{in\theta}$$

Substituting the above equation into Eq. (8), we can get the complex Fourier series form of deflection function as follows:

$$u(r,\theta)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}[\frac{a_{|n|-1}}{2}r^2+(-\frac{b_{|n|+1}}{2|n|})]r^{-|n|}e^{in\theta}+Br^2+\frac{B_1}{2}r^2\cos 2\theta-\frac{C_1}{2}r^2\sin 2\theta$$
$$+b_0 r\cos\theta+b_1\ln r+B',\ r>1 \quad (9)$$

3 Analytical formula of bending deflection for infinite plate corresponding to the three common boundary conditions of the opening

Bending problems for infinite plate with opening have the following three types of boundary conditions of the opening:

$$(a)\ u|_\Gamma=u_0,\ \frac{\partial u}{\partial n}|_\Gamma=u_n.\quad (b)\ u|_\Gamma=u_0,\ Mu=m.\quad (c)\ Tu=t,\ Mu=m.$$

where Γ is the opening boundary of the plate, and differential boundary value operators Tu, Mu are respectively:

$$\begin{cases} Tu &= \left[\frac{\partial}{\partial r}\Delta u+(1-\mu)\frac{1}{r\partial\theta}(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}u-\frac{1}{r^2}\frac{\partial}{\partial\theta}u)\right]_\Gamma=\frac{V_r}{D}, \\ Mu &= \left[\mu\Delta\mathrm{u}+(1\text{ - }\mu)\frac{\partial^2}{\partial \mathrm{r}^2}u\right]_\Gamma=-\frac{M_r}{D} \end{cases} \quad (10)$$

where V_r, M_r are respectively the radial distribution of shear force and bending moment on the opening boundary. Next, analytical formula for bending deflection corresponding to the above three types of the boundary conditions will be derived respectively.

3.1 The natural boundary integral formula of bending deflection under the first boundary condition

The first boundary condition is $u|_\Gamma=u_0(\theta)$, $\frac{\partial u}{\partial n}|_\Gamma=u_n(\theta)$. According to Eq.(9), we have

$$u_0(\theta)=u(1,\theta)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(\frac{1}{2}a_{|n|-1}-\frac{1}{2|n|}b_{1+|n|})\cdot e^{in\theta}$$
$$+B+b_0\cos\theta+B'+\frac{B_1}{2}\cos 2\theta-\frac{C_1}{2}\sin 2\theta \quad (11)$$

$$u_n(\theta)=-\left.\frac{\partial u}{\partial r}\right|_{r=1}=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}[(|n|-2)\frac{a_{|n|-1}}{2}-\frac{b_{1+|n|}}{2})]\cdot e^{in\theta}-2B$$
$$-b_0\cos\theta-b_1-B_1\cos 2\theta+C_1\sin 2\theta \quad (12)$$

Suppose

$$u_0(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \quad c_{-n} = \bar{c}_n \tag{13}$$

$$u_n(\theta) = \sum_{-\infty}^{\infty} d_n e^{in\theta}, \quad d_{-n} = \bar{d}_n \tag{14}$$

Comparing the coefficients on the right side of Eqs.(11) and (13) and of Eqs.(12) and(14), respectively, by simplification we have

$$\begin{cases} a_{|n|-1} = -B_1 + \frac{C_1}{i} + 2c_2 - d_2, & n = 2 \\ a_{|n|-1} = -B_1 - \frac{C_1}{i} + 2c_{-2} - d_{-2}, & n = -2 \\ a_{|n|-1} = |n|\, c_n - d_n,\ n \neq 0, & \pm 1,\ \pm 2 \end{cases} \tag{15}$$

$$\begin{cases} b_{|n|+1} = -2B - d_0, & n = 0 \\ b_{|n|+1} = -(c_1 + d_1), & n = 1 \\ b_{|n|+1} = -(c_{-1} + d_{-1}), & n = -1 \\ b_{|n|+1} = -B_1 + \frac{C_1}{i} - 2d_2, & n = 2 \\ b_{|n|+1} = -B_1 + \frac{C_1}{i} - 2d_{-2}, & n = -2 \\ b_{|n|+1} = |n| \cdot [(|n|-2)c_n - d_n], & n \neq 0,\ \pm 1,\ \pm 2 \\ b_0 = c_1 - d_1 \end{cases} \tag{16}$$

Substituting a_n, b_n into Eq. (9), where parts of the series become

$$\begin{aligned} &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} [\frac{a_{|n|-1}}{2} r^2 + (-\frac{b_{|n|+1}}{2\,|n|})] r^{-|n|} e^{in\theta} \\ &= \sum_{-\infty}^{\infty} \left\{ \frac{1}{2}(r^2-1)\,|n|\, c_n + c_n - \frac{1}{2}(r^2-1) d_n \right\} r^{-|n|} e^{in\theta} \\ &-\frac{1}{2}(1-r^2) d_0 - c_0 + (\frac{1}{2} r^{-2} - 1) B_1 \cos 2\theta + (1 - \frac{1}{2r^2}) C_1 \sin 2\theta - (c_1 - d_1) r \cos\theta, \end{aligned} \tag{17}$$

According to the convolution of complex Fourier series

$$\sum_{n=-\infty}^{\infty} |n| r^{-|n|} e^{in\theta} * \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \sum_{n=-\infty}^{\infty} \left(2\pi |n| r^{-|n|}\right) c_n e^{in\theta}$$

and the basic formula in generalized functions, when $r > 1$

$$\sum_{n=-\infty}^{\infty} r^{-|n|} e^{in\theta} = \frac{r^2-1}{1+r^2-2r\cos\theta} \tag{18}$$

$$\sum_{n=-\infty}^{\infty} |n|\, r^{-|n|} e^{in\theta} = \frac{2r^3\cos\theta - 4r^2 + 2r\cos\theta}{(1+r^2-2r\cos\theta)^2} \tag{19}$$

Then, the series in Eq.(17) can be further simplified as

$$\sum_{-\infty}^{\infty} \left\{ \frac{1}{2}(r^2-1)\,|n|\, c_n + c_n - \frac{1}{2}(r^2-1) d_n \right\} r^{-|n|} e^{in\theta}$$
$$= \int_0^{2\pi} \left\{ \frac{(r^2-1)^2[r\cos(\theta-\theta')-1]}{2\pi[1+r^2-2r\cos(\theta-\theta')]^2} u_0(\theta') - \frac{(r^2-1)^2 u_n(\theta')}{4\pi[1+r^2-2r\cos(\theta-\theta')]} \right\} \mathrm{d}\theta' \tag{20}$$

For convenience, let

$$f(r,\theta) = \int_0^{2\pi} \left\{ \frac{(r^2-1)^2[r\cos(\theta-\theta')-1]}{2\pi[1+r^2-2r\cos(\theta-\theta')]^2} u_0(\theta') - \frac{(r^2-1)^2 u_n(\theta')}{4\pi[1+r^2-2r\cos(\theta-\theta')]} \right\} \mathrm{d}\theta' \tag{21}$$

Substituting Eq.(20) into Eq.(17) and then the results into Eq.(9), the natural boundary integral formula of the bending deflection under the first boundary condition is obtained as follows:

$$u(r,\theta) = \frac{1}{2}(B_1\cos 2\theta - C_1 \sin 2\theta)(r^{-2}+r^2-2) - (2B+d_0)[\ln r - \frac{(r^2-1)}{2}] + f(r,\theta) \tag{22}$$

where $d_0 = \frac{1}{2\pi}\int\limits_0^{2\pi} u_n(\theta) d\theta$. B, B_1 and C_1 are determined by Eq.(5), and $f(r,\theta)$ is determined by Eq.(21).

In particular, when the opening boundary is clamped supported, we have

$$u_0(\theta) = 0, \quad u_n(\theta) = 0, \quad f(r,\theta) = 0 \tag{23}$$

Then, according to Eqs.(13) and (14), we can obtain $c_n = 0$ and $d_n = 0$. Taking these known conditions into Eq. (22), we have

$$u(r,\theta) = \frac{1}{2}(B_1\cos 2\theta - C_1 \sin 2\theta)(r^{-2} + r^2 - 2)\ - 2B\ln r + (r^2-1)B, \quad r>1 \tag{24}$$

Eq.(24) is the analytical bending deflection formula of infinite plate with a unit circular opening under the clapped supported inner edge and boundary loads.

3.2 The boundary integral formula of bending deflection under the second boundary condition

The second boundary condition is $u|_\Gamma = u_0(\theta),\ Mu = m$. According to Eqs.(9) and (10), we have

$$\begin{aligned} Mu &= \left[\mu\Delta u + (1-\mu)\frac{\partial^2 u}{\partial r^2}\right]_{r=1} \\ &= \sum_{\substack{-\infty\\ n\neq 0}}^{\infty} \left[(1-\mu)n^2 - (3+\mu)\,|n| + 2(1+\mu)\right]\frac{a_{|n|-1}}{2}e^{in\theta} + 2B(1+\mu) - b_1(1-\mu) \\ &\quad - \sum_{\substack{-\infty\\ n\neq 0}}^{\infty}(1-\mu)(1+|n|)\frac{b_{|n|+1}}{2}e^{in\theta} + (1-\mu)(B_1\cos 2\theta - C_1\sin 2\theta) \end{aligned} \tag{25}$$

Suppose

$$Mu = \sum_{-\infty}^{\infty} g_n e^{in\theta}, \quad \bar{g}_{-n} = g_n \tag{26}$$

and $u_0(\theta) = \sum\limits_{-\infty}^{\infty} c_n e^{in\theta}$, $\bar{c}_{-n} = c_n$ is the same as Eq.(13).

Comparing the coefficients on the right side of Eqs.(11) and (13) and of Eqs.(25) and (26), respectively, we have

$$\begin{cases} n=0: & c_0 = B + B'\ , \\ n=\pm 1: & c_1 = c_{-1} = \frac{b_0}{2} - \frac{b_2}{2}\ , \\ n=\pm 2: & c_{\pm 2} = \frac{B_1}{4} + \frac{a_1}{2} - \frac{b_3}{4} \mp \frac{C_1}{4i}, \\ n\neq 0,\ \pm 1,\ \pm 2: & c_n = \frac{a_{|n|-1}}{2} - \frac{b_{1+|n|}}{2|n|}, \end{cases} \tag{27}$$

$$\begin{cases} n=0: & g_0 = 2B(1+\mu) - b_1(1-\mu) \\ n=\pm 1: & g_1 = g_{-1} = -(1-\mu)b_2 - \frac{\mu}{2}b_0 \\ n=\pm 2: & g_{\pm 2} = -2\mu a_1 - \frac{3(1-\mu)}{2}b_3 + \frac{(1-\mu)}{2}B_1 \mp \frac{(1-\mu)}{2i}C_1 \\ n\neq 0,\ \pm 1,\ \pm 2: & g_n = \left[(1-\mu)n^2 - (3+\mu)|n| + 2(1+\mu)\right]\frac{a_{|n|-1}}{2} \\ & \quad -(1-\mu)(1+|n|)\frac{b_{1+|n|}}{2} \end{cases} \tag{28}$$

Similar to the derivation process in section 2.1, the coefficients $a_{|n|-1}$ and $b_{|n|+1}$ can be solved from the Eqs.(27) and (28) and expressed by c_n and g_n, substituting $a_{|n|-1}$ and $b_{|n|+1}$ into Eq. (9), then the analytical formula of bending deflection corresponding to the second boundary condition of the unit circular opening is developed as follows:

$$\begin{aligned} u(r,\theta) = & \\ \sum_{n=-\infty}^{\infty} & \left\{ \frac{(1-\mu)(n^2+|n|)c_n - g_n}{4|n| - 2(1+\mu)} r^2 - \frac{\left[(1-\mu)n^2 - (3+\mu)|n| + 2(1+\mu)\right]c_n - g_n}{4|n| - 2(1+\mu)} \right\} \\ & \cdot r^{-|n|} e^{in\theta} + (B_1\cos 2\theta - C_1 \sin 2\theta)\left(\frac{r^2}{2} - \frac{1+\mu}{2(3-\mu)} r^{-2} - \frac{1-\mu}{3-\mu}\right) \\ & + b_1 \ln r - \frac{1-\mu}{2(1+\mu)} b_1 (1-r^2),\ r>1, \end{aligned} \tag{29}$$

where B, B_1 and C_1 are determined by Eq.(5), and

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta)\cdot e^{-in\theta} d\theta,\ \ g_n = \frac{1}{2\pi}\int_0^{2\pi} Mu\cdot e^{-in\theta} d\theta,\ \ b_1 = \frac{2B(1+\mu)-g_0}{1-\mu} \tag{30}$$

In particular, for the infinite plate with a unit circular opening simply supported and $u_0(\theta)=0$, $Mu=0$, then from Eqs.(13) and (26), we get $c_n=0$ and $g_n=0$. Substituting them into Eq.(29), we have

$$\begin{aligned} u(r,\theta) = & (B_1\cos 2\theta - C_1\sin 2\theta)\left[\frac{r^2}{2} - \frac{1+\mu}{2(3-\mu)} r^{-2} - \frac{1-\mu}{3-\mu}\right] \\ & + \frac{2(1+\mu)}{1-\mu} B \ln r - B(1-r^2),\ r>1 \end{aligned} \tag{31}$$

The Eq.(31) is the analytical formula of bending deflection for the infinite plate under the boundary loads with a unit circular opening which is simply supported and has no bending moment.

3.3 The analytical formula of bending deflection under the third boundary condition of opening

The third boundary condition is$Tu = t,\ Mu = m.$ According to Eqs.(9) and (10), we have

$$\begin{aligned} Tu &= \left[\frac{\partial \Delta u}{\partial r} + (1-\mu)\frac{\partial}{r\partial\theta}\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}u - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)\right]_{r=1} \\ &= 2(1-\mu)(C_1\sin 2\theta - B_1\cos 2\theta) \\ &\quad + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty}\left\{\left[(1-\mu)\left|n\right|^3 + (3+\mu)n^2 - 4\left|n\right|\right]\frac{a_{|n|-1}}{2} - (1-\mu)(n^2+|n|)\frac{b_{|n|+1}}{2}\right\}e^{in\theta} \end{aligned} \tag{32}$$

Suppose

$$Tu = \sum_{-\infty}^{\infty} h_n e^{in\theta}, \quad h_{-n} = \bar{h}_n \tag{33}$$

and $Mu = \sum\limits_{-\infty}^{\infty} g_n e^{in\theta}$, $g_{-n} = \bar{g}_n$ is the same as Eq.(26).

Similarly, comparing the coefficients on the right side of Eqs.(32) and (33) and of Eqs.(25) and(26), we obtain

$$g_n = \begin{cases} 2B(1+\mu) - b_1(1-\mu), & n = 0 \\ -2\mu a_1 - \frac{3(1-\mu)}{2}b_3 + \frac{(1-\mu)}{2}(B_1 - \frac{C_1}{i}), & n = 2 \\ -2\mu a_1 - \frac{3(1-\mu)}{2}b_3 + \frac{(1-\mu)}{2}(B_1 + \frac{C_1}{i}), & n = -2 \\ \left[(1-\mu)n^2 - (3+\mu)\left|n\right| + 2(1+\mu)\right]\frac{a_{|n|-1}}{2} - (1-\mu)(1+|n|)\frac{b_{1+|n|}}{2}, & n \neq 0,\ \pm 2 \end{cases} \tag{34}$$

$$h_n = \begin{cases} (6-2\mu)a_1 - 3(1-\mu)b_3 + (1-\mu)(\frac{C_1}{i} - B_1), & n = 2 \\ (6-2\mu)a_1 - 3(1-\mu)b_3 - (1-\mu)(\frac{C_1}{i} + B_1), & n = -2 \\ \frac{(1-\mu)|n|^3 + (3+\mu)n^2 - 4|n|}{2}a_{|n|-1} - (1-\mu)(n^2+|n|)\frac{b_{1+|n|}}{2}, & n \neq 0,\ \pm 2 \end{cases} \tag{35}$$

Thus, $a_{|n|-1}$, $b_{|n|+1}$ expressed by g_n and h_n can be solved from the above equations, substituting $a_{|n|-1}$, $b_{|n|+1}$ into Eq.(9), we have

$$u(r,\theta)=\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\frac{|n|\,g_n-h_n}{(6+2\mu)(|n|-n^2)}r^{2-|n|}e^{in\theta}-$$
$$\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\frac{\left[(1-\mu)n^2-(3+\mu)\,|n|+2(1+\mu)\right]h_n-\left[(1-\mu)\,|n|^3+(3+\mu)n^2-4\,|n|\right]g_n}{(6+2\mu)(1-\mu)(n^4-n^2)}r^{-|n|}e^{in\theta}+$$
$$(B_1\cos 2\theta-C_1\sin 2\theta)[\frac{1-\mu}{3+\mu}(1-\frac{r^{-2}}{2})+\frac{r^2}{2}]+Br^2+b_1\ln r+B',\ r>1 \tag{36}$$

where

$$g_n=\frac{1}{2\pi}\int_0^{2\pi}Mu\cdot e^{-in\theta}d\theta,\quad h_n=\frac{1}{2\pi}\int_0^{2\pi}Tu\cdot e^{-in\theta}d\theta,\quad b_1=\frac{2B(1+\mu)-g_0}{1-\mu} \tag{37}$$

Eq.(36) is the analytical formula of bending deflection for infinite plate under the third opening boundary condition.

In particular, for the plate with a circular opening of free edge, that is$Tu=t=0$, $Mu=m=0$, so from Eqs.(37), we can get$g_n=0$ and $h_n=0$. Furthermore, from Eq.(36), we have

$$u(r,\theta)=(B_1\cos 2\theta-C_1\sin 2\theta)[\frac{1-\mu}{3+\mu}(1-\frac{r^{-2}}{2})+\frac{r^2}{2}]\ +Br^2+\frac{2(1+\mu)}{1-\mu}B\ln r+B' \tag{38}$$

where B, B_1 and C_1 is determined by Eq.(5), and B' is the displacement of rigid body. Eq.(38) is the analytical formula of bending deflection for the infinite plate under the boundary loads with a unit circular opening of free edge.

4 Examples

4.1 Example 1

As shown in Fig.1, an infinite plate with a circular opening is loaded with a uniform moment M_0 along the x direction of the outer boundary.

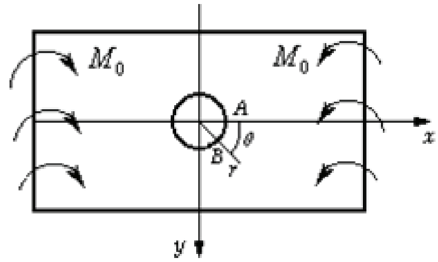

Figure 1: An infinite plate with a uniform boundary bending moment M_0

According to Fig.1, we have

$$M_x^\infty = M_0, \quad M_y^\infty = M_{xy}^\infty = 0 \tag{39}$$

From Eq.(5), we get

$$B_1 = \frac{-M_0}{2D(1-\mu)}, \quad B = -\frac{M_0}{4D(1+\mu)}, \quad C_1 = 0 \tag{40}$$

Substituting Eqs.(39) and (40) into Eq.(38), we obtain

$$u(r,\theta) = -\frac{M_0}{2D(1-\mu)}\cos 2\theta\left[\frac{1-\mu}{3+\mu}(1-\frac{r^{-2}}{2})+\frac{r^2}{2}\right] - \frac{M_0}{4D(1+\mu)}r^2 - \frac{M_0}{2(1-\mu)D}\ln r + B' \tag{41}$$

The internal forces on the cross section of the plate can be solved by deflection function $u(r,\theta)$, and the answer is the same as the result by complex function method [Qu (2000)].

4.2 Example 2

As shown in Fig.2, an infinite plate with a circular clapped opening, its outer boundary is free and a uniform distribution moment M_0 is loaded on the inner boundary.

In this example, the outer boundary at the infinite point is free, so from Eq.(5), we have

$$B = B_1 = C_1 = 0$$

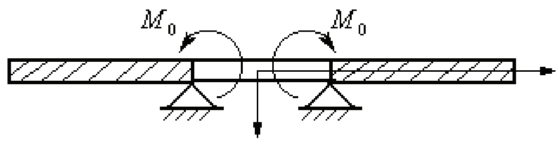

Figure 2: An infinite plate with a uniform distribution bending moment M_0 on its inner boundary

Also, the inner boundary is simply supported and its deflection$u_0(\theta) = 0$, its distribution bending moment$M_r = M_0$. Thus, from Eq.(10), we have $Mu = -\frac{M_0}{D}$.
Taking the above results into the deflection formula of the second boundary condition Eq.(29), considering that

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta)\cdot e^{-in\theta}d\theta = 0$$

$$g_n = \frac{1}{2\pi}\int_0^{2\pi} Mu\cdot e^{-in\theta}d\theta = 0,\ n \neq 0$$

$$g_0 = \frac{1}{2\pi}\int_0^{2\pi} Mud\theta = -\frac{M_0}{D}$$

$$b_1 = \frac{2B(1+\mu) - g_0}{1-\mu} = \frac{M_0}{D(1-\mu)}$$

We have

$$u(r,\theta) = \frac{M_0}{D(1-\mu)}\ln r, \quad r > 1 \tag{42}$$

Eq.(42) is the same as the bending answer of a circular plate for which its inner radius is unit one and outer radius tends to infinity[Xu(1990].

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