On the Determination of the Singular Sturm-Liouville Operator from Two Spectra

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Abstract: In this paper an inverse problem by two given spectrum for a second-order differential operator with coulomb singularity of the type $\frac{A}{x}$ in zero point (here A is constant), is studied. It is well known that two spectrum $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely determine the potential function $q(x)$ in the singular Sturm-Liouville equation defined on interval $(0, \pi]$. The aim of this paper is to prove the generalized degeneracy of the kernel $K(x,t)$. In particular, we obtain a new proof of the Hochstadt's theorem concerning the structure of the difference $\widetilde{q}(x) - q(x)$.

Keywords: Coulomb Potential, Spectrum, Singular Sturm-Liouville Operator.

1   Introduction

Learning about the motion of electrons moving under Coulomb potential is of significance in quantum theory. Solving these types of problems provides us to find energy levels not only hydrogen atom but also single valance electron atoms such as sodium.

For hydrogen atom, Coulomb potential is given by $U = \frac{-e^2}{r}$, where r is the radius of the nucleus, e is electronic charge. Accordingly we use time dependent Schrödinger equation;

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x,y,z)\Psi, \quad \int\limits_{R^3} |\Psi|^2\, dxdydz = 1, \tag{1.1}$$

where Ψ is the wave function, $\hbar$ is Planck's constant and m is the mass of electron (see, e.g. [Blokhintsev (1949); Fock (1932)]). If we make the necessary transfor-

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mation, then we can get a Sturm-Liouville equation with Coulomb potential

$$-y'' + \left[\frac{A}{x} + q(x)\right] y = \lambda y$$

where λ is a parameter which corresponds to the energy.

Now let us consider two singular Sturm-Liouville problems

$$-y'' + \left[\frac{A}{x} + q(x)\right] y = \lambda y, \ \lambda = s^2, 0 < x \le \pi, \tag{1.2}$$

$$y(0) = 0, \tag{1.3}$$

$$y'(\pi) - Hy(\pi) = 0, \tag{1.4}$$

and

$$-y'' + \left[\frac{A}{x} + \widetilde{q}(x)\right] y = \lambda y, \ \lambda = s^2, 0 < x \le \pi, \tag{1.5}$$

$$y(0) = 0,$$

$$y'(\pi) - \widetilde{H}y(\pi) = 0, \tag{1.6}$$

where $q(x), \widetilde{q}(x) \in C[0, \pi]$, A and H reel constants and $\frac{y(x)}{x} \in C[0, \pi]$.

Denote the spectrum of this first problem by $\{\lambda_n\}_{n=0}^{\infty}$ and the spectrum of the second by $\left\{\widetilde{\lambda}_n\right\}_{n=0}^{\infty}$. Next we denote by $\varphi(x, \lambda)$ the solution of (1.2) and by $\widetilde{\varphi}(x, \lambda)$ the solution of (1.5) satisfying the initial condition (1.3).

It is well known that there exists a function $K(x, t)$ such that

$$\widetilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \int_0^x K(x, t)\varphi(t, \lambda)dt. \tag{1.7}$$

The function $K(x, t)$ satisfies the equation

$$\frac{\partial^2 K}{\partial x^2} - \left(\frac{A}{x} + \widetilde{q}(x)\right) K = \frac{\partial^2 K}{\partial t^2} - \left(\frac{A}{t} + q(t)\right) K, \tag{1.8}$$

and the conditions;

$$K(x, x) = \frac{1}{2} \int_0^x (\widetilde{q}(r) - q(r)) dr, \tag{1.9}$$

$$K(x, 0) = 0. \tag{1.10}$$

This problem can be solved by using the Riemann method (see, e.g. [Courant and Hilbert (1953); Volk (1953)]).

We set

$$c_n = \int_0^\pi \varphi^2(x, \lambda_n)dx, \quad \widetilde{c}_n = \int_0^\pi \widetilde{\varphi}^2(x, \widetilde{\lambda}_n)dx, \tag{1.11}$$

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{c_n}, \quad \widetilde{\rho}(\lambda) = \sum_{\widetilde{\lambda}_n < \lambda} \frac{1}{\widetilde{c}_n}. \tag{1.12}$$

The function $\rho(\lambda)$ $(\widetilde{\rho}(\lambda))$ is called the spectral function of problem (1.2)-1.4 [(1.5), (1.6)]. Problem (1.2)-(1.4) is regarded as an unperturbed problem, while (1.5), (1.6) is considered as a perturbation of (1.2)-(1.4).

It is known fact that the knowledge of two spectra for given singular Sturm-Liouville equation makes it possible to recover its spectral function, i.e., to find the numbers $\{c_n\}$ (see [Amirov,Cakmak and Gulyaz (2006)]). More exactly, suppose that, in addition to the spectrum of problem (1.2)-(1.4), we also know the spectrum $\{\mu_n\}$ of the problem

$$\begin{aligned}
-y'' + \left[\frac{A}{x} + q(x)\right]y &= \lambda y, \; \lambda = s^2, 0 < x \le \pi, \\
y(0) &= 0, \\
y'(\pi) - H_1 y(\pi) &= 0, \quad H_1 \ne H.
\end{aligned} \tag{1.13}$$

Knowing $\{\lambda_n\}$ and $\{\mu_n\}$,we can calculate the numbers $\{c_n\}$. Similarly, for (1.5), if, in addition to $\left\{\widetilde{\lambda}_n\right\}$, we also know the spectrum $\{\widetilde{\mu}_n\}$ determined by the boundary conditions

$$y(0) = 0,$$

$$y'(\pi) - \widetilde{H}_1 y(\pi) = 0, \quad \widetilde{H}_1 \ne \widetilde{H} \tag{1.14}$$

then it follows that we can determine the numbers $\{\widetilde{c}_n\}$.

It is also shown that

$$s_n = \sqrt{\lambda_n} = n + \frac{1}{2} + \frac{A}{2\pi} \frac{\ln(n + \frac{1}{2})}{(n + \frac{1}{2})} + \frac{c_0}{(n + \frac{1}{2})} + O(\frac{\ln n}{n^2}), \tag{1.15}$$

$$\|\varphi_n\|^2 = \int_0^\pi \varphi^2(x, \lambda_n)dx = \frac{\pi}{2} + \frac{A\pi^2}{4} \frac{1}{(n + \frac{1}{2})} + O(\frac{\ln n}{n^2}), \tag{1.16}$$

where

$$c_0 = \frac{1}{\pi} \left(AM_1 - H + \frac{A \ln \pi}{2} + \frac{1}{2} \int_0^\pi q(t)dt \right),$$

$$\beta(x) = AM_1 + \frac{1}{2} \int_0^x q(t)dt,$$

$$M_1 = M + \frac{\sin 2}{2}, \; M = \int_0^1 \frac{\sin^2 \xi}{\xi} d\xi,$$

(see [Amirov, Cakmak and Gulyaz (2006)]).

Theorem 1.1 *Consider the operator*

$$Ly \equiv -y'' + \left[\frac{A}{x} + q(x) \right] y, \tag{1.17}$$

subject to the boundary conditions

$$y(0) = 0, \tag{1.18}$$
$$y'(\pi) - Hy(\pi) = 0, \tag{1.19}$$

where $q(x)$ is square integrable on $(0, \pi]$. Let $\{\lambda_n\}$ be the spectrum of L subject to (1.18) and (1.19).

If (1.19) is replaced by the new boundary condition

$$y'(\pi) - H_1 y(\pi) = 0 \tag{1.20}$$

then a new operator and a new spectrum, say $\{\mu_n\}$, result.

Now consider the second operator

$$\widetilde{L}y \equiv -y'' + \left[\frac{A}{x} + \tilde{q}(x) \right] y \tag{1.21}$$

where $\tilde{q}$ is square integrable on $(0, \pi]$. Suppose that $\widetilde{L}$ has the spectrum $\left\{ \tilde{\lambda}_n \right\}$ with $\left\{ \tilde{\lambda}_n \right\} = \{\lambda_n\}$ for all n under the boundary conditions (1.18) and

$$y'(\pi) - \widetilde{H}y(\pi) = 0, \tag{1.22}$$

$\widetilde{L}$ with the boundary conditions (1.18) and

$$y'(\pi) - \widetilde{H}_1 y(\pi) = 0 \tag{1.23}$$

is assumed to have the spectrum $\{\widetilde{\mu}_n\}$. Assumed that H, $H_1 \neq H$, $\widetilde{H}$ and $\widetilde{H}_1 \neq \widetilde{H}$ are real numbers that are not infinite.

Denote by Λ_0 the finite index set for which $\widetilde{\mu}_n \neq \mu_n$. Under the above assumptions, it follows that the kernel $K(x,t)$ is degenerate in the extended sense:

$$K(x,t) = \sum_{\Lambda_0} c_n \widetilde{\varphi}_n(x) \varphi_n(t) \tag{1.24}$$

where φ_n and $\widetilde{\varphi}_n$ are suitable solutions of (1.2) and (1.5).

Proof. It follows from (1.7) that

$$\widetilde{\varphi}'(x,\lambda) = \varphi'(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \frac{\partial K}{\partial x}\varphi(t,\lambda)dt \tag{1.25}$$

and

$$\widetilde{\varphi}'(x,\lambda) - \widetilde{H}\widetilde{\varphi}(x,\lambda)$$
$$= \varphi'(x,\lambda) - \widetilde{H}\varphi(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \left(\frac{\partial K}{\partial x} - \widetilde{H}K\right)\varphi(t,\lambda)dt \tag{1.26}$$

Substituting $x = \pi$ and $\lambda = \lambda_n$ into the last equation and using the boundary conditions (1.19) and (1.22), we obtain

$$\left(H - \widetilde{H}\right)\varphi(\pi,\lambda_n) + K(\pi,\pi)\varphi(\pi,\lambda_n) + \int_0^\pi \left(\frac{\partial K}{\partial x} - \widetilde{H}K\right)_{x=\pi}\varphi(t,\lambda_n)dt = 0. \tag{1.27}$$

As $n \to \infty$ and $\varphi(\pi,\lambda_n) \to (-1)^n$, the integral on the right-hand side tends to zero, (see [Amirov, Cakmak and Gulyaz (2006)]). Therefore, from (1.27) we get

$$K(\pi,\pi) = \widetilde{H} - H \tag{1.28}$$

$$\int_0^\pi \left(\frac{\partial K}{\partial x} - \widetilde{H}K\right)_{x=\pi}\varphi(t,\lambda_n)dt = 0, \ n = 0, 1, ... \tag{1.29}$$

Since the systems of functions $\varphi(t,\lambda_n)$ is complete, it follows from the last equation that

$$\left(\frac{\partial K}{\partial x} - \widetilde{H}K\right)_{x=\pi} = 0, \ 0 < t \leq \pi \tag{1.30}$$

We now use the condition imposed on the second mentioned spectrum. Using (1.7) again, we obtain

$$\widetilde{\varphi}'(x,\lambda) - \widetilde{H}_1 \widetilde{\varphi}(x,\lambda)$$
$$= \varphi'(x,\lambda) - \widetilde{H}_1 \varphi(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \left(\frac{\partial K}{\partial x} - \widetilde{H}_1 K \right) \varphi(t,\lambda)dt \quad (1.31)$$

Setting $x = \pi$ and $\lambda = \mu_n$ $(n \in \Lambda)$ and using (1.20) and (1.23), we get

$$\int_0^\pi \left(\frac{\partial K}{\partial x} - \widetilde{H}_1 K \right)_{x=\pi} \varphi(t,\mu_n)dt + \left(H_1 - \widetilde{H}_1 \right) \varphi(\pi,\mu_n) + K(\pi,\pi)\varphi(\pi,\mu_n) = 0$$
$$(1.32)$$

In the last equation as $n \to \infty$, the left-hand side tends to zero and $\varphi(\pi,\mu_n) \to (-1)^n$. Therefore

$$K(\pi,\pi) = \widetilde{H}_1 - H_1 \tag{1.33}$$

$$\int_0^\pi \left(\frac{\partial K}{\partial x} - \widetilde{H}_1 K \right)_{x=\pi} \varphi(t,\mu_n)dt = 0, \; n \in \Lambda \tag{1.34}$$

Comparing (1.28) and (1.33), we obtain $\widetilde{H} - H = \widetilde{H}_1 - H_1$. For $n \in \Lambda_0$, relation (1.26) for $(x = \pi$ and $\lambda = \mu_n)$ yields

$$\int_0^\pi \left(\frac{\partial K}{\partial x} - \widetilde{H}_1 K \right)_{x=\pi} \varphi(t,\mu_n)dt = \widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi,\mu_n). \tag{1.35}$$

It follows from (1.34) and (1.35) that

$$\left(\frac{\partial K}{\partial x} - \widetilde{H}_1 K \right)_{x=\pi} = \sum_{\Lambda_0} \frac{\widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2} \varphi(t,\mu_n), \; 0 < t \le \pi. \tag{1.36}$$

We derive from (1.30) and (1.36) the following equations:

$$K(\pi,t) = \frac{1}{\widetilde{H} - \widetilde{H}_1} \sum_{\Lambda_0} \frac{\widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2} \varphi(t,\mu_n) \tag{1.37}$$

$$\left(\frac{\partial K(x,t)}{\partial x} \right)_{x=\pi} = -\frac{\widetilde{H}}{\widetilde{H} - \widetilde{H}_1} \sum_{\Lambda_0} \frac{\widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2} \varphi(t,\mu_n), 0 < t \le \pi. \tag{1.38}$$

The function $K(x,t)$ satisfies (1.8). Therefore, it follows from the initial conditions (1.37) and (1.38) that, in triangle I (see Figure 1), we have

$$K(x,t) = \frac{1}{\widetilde{H} - \widetilde{H}_1} \sum_{\Lambda_0} \frac{\widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2} \left[\widetilde{c}(x,\mu_n) - \widetilde{H}\widetilde{t}(x,\mu_n) \right] \varphi(t,\mu_n) \tag{1.39}$$

where $\widetilde{c}(x,\lambda)$ and $\widetilde{t}(x,\lambda)$ are the solutions of (1.5) satisfying the initial conditions

$$\widetilde{c}(\pi,\lambda) = \widetilde{t}'(\pi,\lambda) = 1, \ \ \widetilde{t}(\pi,\lambda) = \widetilde{c}'(\pi,\lambda) = 0. \tag{1.40}$$

The function $K(x,t)$ and sum (1.39) satisfy (1.10); therefore, they coincide in
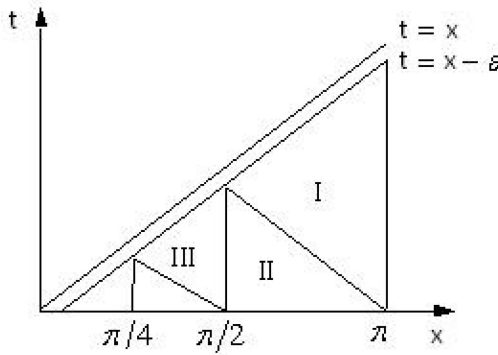

Figure 1:

triangle II; consequently, they coincide in triangle III because solutions of (1.8) satisfy the same initial conditions on the line $x = \frac{\pi}{2}$, etc., i.e., $K(x,t)$ is expressed by (1.39) throughout the triangle $0 < x \le t \le \pi$ (see, e.g. [Levitan (1978); Panakhov (1987)]).

Hence, we obtain Hochstadt's result in a somewhat more general formulation, (see [Hochstadt (1973)]). ∎

Theorem 1.2 *If the spectra and $\{\lambda_n\}$ and $\left\{\widetilde{\lambda}_n\right\}$ coincide and $\{\mu_n\}$ and $\{\widetilde{\mu}_n\}$ differ in a finite number of their terms , i.e., $\widetilde{\mu}_n = \mu_n$ for $n \in \Lambda$, then*

$$\widetilde{q}(x) - q(x) = \sum_{\Lambda_0} \widetilde{c}_n \frac{d}{dx} \left(\widetilde{\varphi}_n \cdot \varphi_n\right), \tag{1.41}$$

where φ_n and $\widetilde{\varphi}_n$ are suitable solutions of (1.2) and (1.5).

Proof. We obtain from (1.9) the equation

$$\widetilde{q}(x) - q(x) = 2\frac{dK(x,x)}{dx} \tag{1.42}$$

Differentiating (1.39) and setting $t = x$, we obtain

$$\tilde{q}(x) - q(x) =$$
$$\frac{2}{\widetilde{\widetilde{H}} - \widetilde{H}_1} \sum_{\Lambda_0} \frac{\widetilde{\varphi}'(\pi, \mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi, \mu_n)}{\|\varphi(t, \mu_n)\|^2} \frac{d}{dx} \left\{ \left[\widetilde{c}(x, \mu_n) - \widetilde{H} \widetilde{t}(x, \mu_n) \right] \varphi(x, \mu_n) \right\} \quad (1.43)$$

Consequently,

$$\tilde{q}(x) - q(x) = \sum_{\Lambda_0} \widetilde{c}_n \frac{d}{dx} \left(\widetilde{\varphi}_n . \varphi_n \right), \quad (1.44)$$

where $\widetilde{c}(x, \mu_n) - \widetilde{H} \widetilde{t}(x, \mu_n) = \widetilde{\varphi}_n$, $\varphi(x, \mu_n) = \varphi_n(x, \mu_n)$, and

$$\widetilde{c}_n = \frac{2 \left[\widetilde{\varphi}'(\pi, \mu_n) - \widetilde{H}_1 \widetilde{\varphi}(\pi, \mu_n) \right]}{\left(\widetilde{\widetilde{H}} - \widetilde{H}_1 \right) \|\varphi(t, \mu_n)\|^2}. \quad (1.45)$$

This completes the proof of Theorem 1.2 . We note that similar problem was investigated in, (see [Hochstadt (1973)]). ■

Theorem 1.3 *If the spectra and $\{\lambda_n\}$ and $\left\{ \widetilde{\lambda}_n \right\}$ coincide and $\{\mu_n\}$ and $\{\widetilde{\mu}_n\}$ differ by a finite number of their terms, then the integral equation*

$$K(x,t) + \int_0^x K(x, \xi) F(\xi, t) d\xi + F(x,t) = 0 \quad \text{for } 0 < t \leq x \leq \pi \quad (1.46)$$

is degenerate in the extended sense. In (1.46),

$$F(x,t) = \int_0^\infty \varphi(x, \lambda) \varphi(t, \lambda) d_\lambda \left\{ \widetilde{\rho}(\lambda) - \rho(\lambda) \right\} \quad (1.47)$$

$$= \sum_{n=0}^\infty \left\{ \frac{1}{\widetilde{c}_n} \varphi(x, \widetilde{\lambda}_n) \varphi(t, \widetilde{\lambda}_n) - \frac{1}{c_n} \varphi(x, \lambda_n) \varphi(t, \lambda_n) \right\}.$$

Proof. From (1.46) we obtain, for $x = \pi$

$$F(\pi, t) = -K(\pi, t) - \int_0^\pi K(\pi, \xi) F(\xi, t) d\xi \quad (1.48)$$

Substituting in this equation, in place of $K(\pi,t)$, the expansion (1.37) and, in place of $F(\xi,t)$, the expansion (1.47) $\left(\text{in which } \lambda_n \text{ and } \widetilde{\lambda}_n \text{ are replaced by } \mu_n \text{ and } \widetilde{\mu}_n\right)$, we obtain, upon using the orthogonality of the functions $\varphi(t,\mu_n)$ the equality

$$F(\pi,t) = -\int_0^\pi \sum_{\Lambda_0} \alpha_n \varphi(\xi,\mu_n) \sum_{\Lambda_0} \frac{1}{\widetilde{c}_k} \varphi(\xi,\widetilde{\mu}_k)\varphi(t,\widetilde{\mu}_k)d\xi \tag{1.49}$$

where

$$\alpha_n = \frac{\widetilde{\varphi}'(\pi,\mu_n) - \widetilde{H}_1\widetilde{\varphi}(\pi,\mu_n)}{\left(\widetilde{H} - \widetilde{H}_1\right)c_n} \tag{1.50}$$

$$c_n = \|\varphi(t,\mu_n)\|^2, \; \widetilde{c}_k = \|\widetilde{\varphi}(t,\widetilde{\mu}_k)\|^2. \tag{1.51}$$

Next, from (1.2) we easily obtain, for $k < N$

$$-\varphi''(\xi,\mu_n) + \left[\frac{A}{\xi} + q(\xi)\right]\varphi(\xi,\mu_n) = \mu_n\varphi(\xi,\mu_n) \tag{1.52}$$

$$-\varphi''(\xi,\widetilde{\mu}_k) + \left[\frac{A}{\xi} + q(\xi)\right]\varphi(\xi,\widetilde{\mu}_k) = \widetilde{\mu}_k\varphi(\xi,\widetilde{\mu}_k) \tag{1.53}$$

from these equations we obtain

$$-\varphi''(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k) + \varphi''(\xi,\widetilde{\mu}_k)\varphi(\xi,\mu_n) = (\mu_n - \widetilde{\mu}_k)\varphi(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k) \tag{1.54}$$

and

$$\int_0^\pi \frac{d}{d\xi}\left(\varphi'(\xi,\widetilde{\mu}_k)\varphi(\xi,\mu_n) - \varphi'(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k)\right)d\xi$$

$$= \int_0^\pi (\mu_n - \widetilde{\mu}_k)\varphi(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k)d\xi$$

$$\left[\varphi'(\xi,\widetilde{\mu}_k)\varphi(\xi,\mu_n) - \varphi'(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k)\right]_0^\pi = \int_0^\pi (\mu_n - \widetilde{\mu}_k)\varphi(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k)d\xi$$

$$\left[\varphi'(\pi,\widetilde{\mu}_k)\varphi(\pi,\mu_n) - \varphi'(\pi,\mu_n)\varphi(\pi,\widetilde{\mu}_k)\right] = (\mu_n - \widetilde{\mu}_k)\int_0^\pi \varphi(\xi,\mu_n)\varphi(\xi,\widetilde{\mu}_k)d\xi$$

$$\int_0^\pi \varphi(\xi,\mu_n)\,\varphi(\xi,\widetilde{\mu}_k)\,d\xi = \frac{\varphi'(\pi,\widetilde{\mu}_k)\,\varphi(\pi,\mu_n) - \varphi'(\pi,\mu_n)\,\varphi(\pi,\widetilde{\mu}_k)}{(\mu_n - \widetilde{\mu}_k)}, \quad (\mu_n \neq \widetilde{\mu}_k).$$

Therefore

$$F(\pi,t) = \sum_{\Lambda_0} \alpha_n \varphi(t,\widetilde{\mu}_n) \tag{1.55}$$

where

$$\alpha_n = \frac{1}{\widetilde{c}_n}\left[\varphi(\pi,\widetilde{\mu}_n)\sum_{\Lambda_0} a_k \frac{\varphi'(\pi,\mu_k)}{\mu_k - \widetilde{\mu}_n} - \varphi'(\pi,\widetilde{\mu}_n)\sum_{\Lambda_0} \alpha_k \frac{\varphi(\pi,\mu_n)}{\mu_k - \widetilde{\mu}_n}\right]. \tag{1.56}$$

To calculate $F_x(\pi,t)$ we differentiate (1.46) with respect to x,we then obtain

$$K_x(x,t) + \int_0^x K_x(x,\xi)F(\xi,t)d\xi + F_x(x,t) + K(x,x)F(x,t) = 0 \tag{1.57}$$

Putting $x = \pi$ here and replacing $K(\pi,\pi)$ by (1.33), $F(\pi,t)$ by (1.55) and $K_x(x,t)$ by (1.38), we find that

$$F_x(\pi,t) = \sum_{\Lambda_0} b_n \varphi(t,\widetilde{\mu}_n) \tag{1.58}$$

where the b_n are constants which we shall not write out.

From (1.47), $F(x,t)$ satisfies the equation

$$\frac{\partial^2 F}{\partial x^2} - \left(\frac{A}{x} + \widetilde{q}(x)\right)F = \frac{\partial^2 F}{\partial t^2} - \left(\frac{A}{t} + q(t)\right)F. \tag{1.59}$$

Therefore, from the boundary conditions (1.55) and (1.58), we find that in the triangle I (see Figure I)

$$F(x,t) = \sum_{\Lambda_0} [\alpha_n c(x,\widetilde{\mu}_n) + b_n t(x,\widetilde{\mu}_n)]\,\varphi(t,\widetilde{\mu}_n) \tag{1.60}$$

where $c(x,\lambda)$ and $t(x,\lambda)$ are the solutions of (1.5) satisfying the boundary conditions

$$c(\pi,\lambda) = s'(\pi,\lambda) = 1, \quad c'(\pi,\lambda) = s(\pi,\lambda) = 0. \tag{1.61}$$

It is also evident from (1.47) that $F(x,t)$ satisfies the boundary condition.

$$F(x,\xi)_{\xi=0} = 0 \tag{1.62}$$

This boundary condition is satisfied, obviously, by the sum (1.60). Therefore (1.60) is valid in the triangle II, etc., i.e., the kernel $F(x,t)$ is degenerate in the extended sense, which is what we wished to prove, (see [Levitan and Sargsyan (1970); Panakhov and Yilmazer (2006)]). ■

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