# T-Trefftz Voronoi Cell Finite Elements with Elastic/Rigid Inclusions or Voids for Micromechanical Analysis of Composite and Porous Materials 

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#### Abstract

In this paper, we develop T-Trefftz Voronoi Cell Finite Elements (VCF-EM-TTs) for micromechanical modeling of composite and porous materials. In addition to a homogenous matrix in each polygon-shaped element, three types of arbitrarily-shaped heterogeneities are considered in each element: an elastic inclusion, a rigid inclusion, or a void. In all of these three cases, an inter-element compatible displacement field is assumed along the element outer-boundary, and interior displacement fields in the matrix as well as in the inclusion are independently assumed as T-Trefftz trial functions. Characteristic lengths are used for each element to scale the T-Trefftz trial functions, in order to avoid solving systems of ill-conditioned equations. Two approaches for developing element stiffness matrices are used. The differences between these two approaches are that, the compatibility between the independently assumed fields at the outer- as well as the innerboundary, are enforced alternatively, by Lagrange multipliers in multi-field boundary variational principles, or by collocation at a finite number of preselected points. Following a previous paper of the authors, these elements are denoted as VCFEM-TT-BVP and VCFEM-TT-C respectively. Several two dimensional problems are solved using these elements, and the results are compared to analytical solutions and that of VCFEM-HS-PCE developed by [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)]. Computational results demonstrate that VCFEM-TTs developed in this study are much more efficient than VCFEM-HS-PCE developed by Ghosh, et al., because domain integrations are avoided in VCFEM-TTs. In addition, the accuracy of stress fields computed by VCFEM-HS-PCE by [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)] seem to be very poor as compared to analytical solutions, because the polynomial Airy stress function is highly incomplete for problems in a doubly-connected domain (as in this case, when a inclusion or a void is present in the element). However, the results of VCFEM-TTs


[^0]developed in the present paper are very accurate, because, the compete T-Trefftz trial functions derived from positive and negative power complex potentials are able to model the singular nature of these stress concentration problems. Finally, out of these two methods, VCFEM-TT-C is very simple, efficient, and does not suffer from LBB conditions. Because it is almost impossible to satisfy LBB conditions a priori, we consider VCFEM-TT-C to be very useful for ground-breaking studies in micromechanical modeling of composite and porous materials.

Keywords: T-Trefftz, VCFEM, matrix, inclusion, void, variational principle, collocation, LBB conditions, completeness, efficiency

## 1 Introduction

Primal finite elements, which involve displacement-type nodal shape functions, are widely accepted and applied in computer modeling of physical problems. This is because of their simplicity, efficiency, stability and established convergence. However, the disadvantages of these elements are also well-known, such as unsatisfactory performance in problems which involve constraints (shear/membrane/incompressibility locking), low convergence rate for problems which are of singular nature (stress concentration problems/ fracture mechanics problems), difficulty to satisfy higher-order continuity requirements (plates and shells), sensitivity to mesh distortion, etc. Carefully formulated hybrid/mixed finite elements based on multifield assumptions, on the other hand, can mitigate or even resolve such problems. Therefore, since their early development in 1960s, different types of hybrid/mixed finite element methods have demonstrated their advantages in various problems.
To mention some of the successful development/applications of hybrid/mixed elements, [Atluri (1975)] developed a set of general variational principles by modifying the Hu-Washizu principle, and used them to develop various hybrid/mixed models in linear elasticity, including the hybrid stress element, the hybrid strain element, the hybrid displacement element, etc. Some of these models were also extended to develop finite elements with drilling degrees of freedoms in [Iura and Atluri (1992); Cazzani and Atluri (1993)], and for geometrical as well as material nonlinear problems in [Atluri (1980)]. [Tong, Pian and Lasry (1973); Atluri, Kobayashi, and Nakagaki (1975)] developed hybrid displacement elements for modeling cracks with very coarse meshes. [Bratianu and Atluri (1983); Ying and Atluri (1983)] developed mixed finite elements for modeling Stokes flows, which eliminate incompressibility locking without resolving to selective reduced-order integrations. [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)] developed Voronoi cell finite elements (VCFEM) and applied them to multi-scale analysis of structures composed of heterogeneous materials. [Jirousek and Teodorescu (1982); Jirousek
and Guex (1986)] developed hybrid Trefftz elements for two-dimensional solid mechanical problems and plate bending problems. [Cai, Paik and Atluri (2009 a,b); Cai, Paik and Atluri (2010); Cai, Paik and Atluri (2010); Zhu, Cai, Paik and Atluri (2010)] developed locking-free hybrid/mixed finite elements for modeling large rotation deformations of beams/rods/plates/shells considering von-Karman type of nonlinearity in co-rotational frames.
However, in spite of their widely recognized advantages, there are essentially two major drawbacks that have been limiting the engineering applications of hybrid/mixed finite elements. One is the increased computational burden caused by matrix inversion for each and every element, and the need to generate at least two other different element matrices through integrations over the element domain, in the process of developing the element stiffness matrix. The other is the questionable stability of finite element solutions. Matrix inversion is difficult to avoid as long as multi-field variational principles are used for element derivation. Regarding stability, [Babuska (1973); Brezzi (1974)] analyzed the existence, uniqueness, stability and convergence of problems with Lagrange multipliers and established the socalled LBB conditions. Inability to satisfy the LBB conditions in general would plague the solvability and stability of hybrid/mixed finite element equations. [Rubinstein, Punch and Atluri (1983); Punch and Atluri (1984); Xue, Karlovitz and Atluri (1985)] used sophisticated group theory to develop guidelines for selecting independent fields which will satisfy the LBB conditions, under the condition that the element is undistorted. For an arbitrarily distorted element, to the best of the authors' knowledge, there is no rational way of satisfying LBB conditions a priori.
By noticing that all the previous hybrid/mixed models suffer from LBB conditions because multi-field variational principles use Lagrange multipliers to enforce constraints, [Dong and Atluri (2011)] presented a simple approach to avoid LBB conditions when developing hybrid/mixed elements. The essential idea was to enforce the compatibility between independently assumed fields, using collocation or the least squares method, instead of using Lagrange multipliers in multi-field variational principles. This approach was therefore used in [Dong and Atluri (2011a); Dong and Atluri (2011b)] to develop simple, stable, and efficient assumed strain or T-Trefftz four-node elements with/without drilling degrees of freedoms, as well as Voronoi Cell Finite Elements based on Radial Basis Functions (VCFEM-RBF) and Voronoi Cell Finite Elements based on T-Trefftz basis functions (VCFEM-TTs) for micromechanical modeling of heterogeneous materials.
In this paper, we extend VCFEM-TTs developed in [Dong and Atluri (2011b)] so that an elastic/rigid inclusion or void can be considered to be present in each Voronoi Cell Finite Element. For each element, in addition to assuming an interelement compatible displacement field along the element outer-boundary, inde-
pendent displacement fields in the matrix material as well as in the inclusion are assumed as characteristic-length-scaled T-Trefftz trial functions. Two approaches are used alternatively to develop finite element equations. The first approach uses multi-field boundary variational principles to enforce all the conditions in a variational sense. On the other hand, the second approach uses collocation method to relate independently assumed displacement fields to nodal displacements, and develop finite element equations based on a primitive-field boundary variational principle. We denote these two classes of elements as VCFEM-TT-BVP and VCFEM-TT-C. By numerical examples, it is clearly shown that both of these two classes of elements are much more accurate and efficient than VCFEM-HS-PCE, which are developed by [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)]. Compared to VCFEM-TTs developed in this study, VCFEM-HS-PCE not only has stability issues, but also gives very poor solutions of stress distribution in the element, simply because the polynomial Airy stress function is highly incomplete for problems in a doubly-connected domain. Among the many VCFEM-TTs developed in this paper, because VCFEM-TT-C is simple, efficient, and so not suffer from LBB conditions, we consider this class of elements to be very useful for micromechanical modeling of composite and porous materials.
The rest of this paper is organized as follows: in section 2, we introduce the characteristic-length-scaled T-Trefftz trial functions as independently assumed displacement fields; in section 3, we develop VCFEM-TT-BVP using multi-field boundary variational principles; in section 4, we develop VCFEM-TT-C using collocation and a primitive-field boundary varitional principle; in section 5, we make some comments on advantages of VCFEM-TTs compared to VCFEM-HS-PCE developed by Ghosh and his coworkers; in section 6, we compare the performance of different elements through numerical examples; in section 7, we complete this paper with some concluding remarks.

## 2 Independent Displacement Fields: T-Trefftz Trial Functions Scaled by Characteristic Lengths

Consider a linear elastic solid undergoing infinitesimal elasto-static deformation. Cartesian coordinates $x_{i}$ identify material particles in the solid. $\sigma_{i j}, \varepsilon_{i j}, u_{i}$ are Cartesian components of the stress tensor, strain tensor and displacement vector respectively. $\bar{f}_{i}, \bar{u}_{i}, \bar{t}_{i}$ are Cartesian components of the prescribed body force, boundary displacement and boundary traction vector. $S_{u}, S_{t}$ are displacement boundary and traction boundary of the domain $\Omega$. We use ()$_{, i}$ to denote differentiation with respect to $x_{i}$. The equations of linear \& angular momentum balance, constitutive
equations, compatibility equations, and boundary conditions can be written as:
$\sigma_{i j, j}+\bar{f}_{i}=0$ in $\Omega$
$\sigma_{i j}=\sigma_{j i}$ in $\Omega$
$\sigma_{i j}=E_{i j k l} \varepsilon_{k l}\left(\right.$ or $\left.\varepsilon_{i j}=C_{i j k l} \sigma_{k l}\right)$ in $\Omega$ for a linear elastic solid
$\varepsilon_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right) \equiv u_{(i, j)}$ in $\Omega$
$n_{j} \sigma_{i j}=\bar{t}_{i}$ at $S_{t}$
$u_{i}=\bar{u}_{i}$ at $\mathrm{S}_{u}$
Consider that the domain $\Omega$ is discretized into elements $\Omega^{e}$ with element boundary $\partial \Omega^{e}$, each element boundary can be divided into $S_{u}^{e}, S_{t}^{e}, \rho^{e}$, which are intersections of $\partial \Omega^{e}$ with $S_{u}, S_{t}$ and other element boundaries respectively. For elements developed in this study, an inclusion or void $\Omega_{c}^{e}$ is present inside each element, which satisfy $\Omega_{c}^{e} \subset \Omega^{e}, \partial \Omega_{c}^{e} \cap \partial \Omega^{e}=\emptyset$, see Fig. 1. We denote the matrix material in each element as $\Omega_{m}^{e}$, such that $\Omega_{m}^{e}=\Omega^{e}-\Omega_{c}^{e}$.
When an elastic inclusion is considered, we denote the displacement field in $\Omega_{m}^{e}$ and $\Omega_{c}^{e}$ as $u_{i}^{m}$ and $u_{i}^{c}$, the strain and stress fields corresponding to which are $\varepsilon_{i j}^{m}, \sigma_{i j}^{m}$ and $\varepsilon_{i j}^{c}, \sigma_{i j}^{c}$ respectively. We also denote the displacement field along $\partial \Omega^{e}$ as $\tilde{u}_{i}^{m}$, which is inter-element compatible. Then, in addition to $u_{i}^{c}, \varepsilon_{i j}^{c}, \sigma_{i j}^{c}$ satisfying (1)(2)(3)(4) in each $\Omega_{c}^{e}, u_{i}^{m}, \varepsilon_{i j}^{m}, \sigma_{i j}^{m}$ satisfying (1)(2)(3)(4) in each $\Omega_{m}^{e}$, satisfying (5)(6) at $S_{u}^{e}, S_{t}^{e}$, displacement continuity and traction reciprocity conditions at each $\rho^{e}$ should be considered:
$u_{i}^{m}=\tilde{u}_{i}^{m}$ at $\partial \Omega^{e}$
$\left(n_{j} \sigma_{i j}^{m}\right)^{+}+\left(n_{j} \sigma_{i j}^{m}\right)^{-}=0$ at $\rho^{e}$
Displacement continuity and traction reciprocity conditions at $\partial \Omega_{c}^{e}$ should also be considered:
$u_{i}^{m}=u_{i}^{c}$ at $\partial \Omega_{c}^{e}$
$-n_{j} \sigma_{i j}^{m}+n_{j} \sigma_{i j}^{c}=0$ at $\partial \Omega_{c}^{e}$
where $n_{j}$ is the unit outer-normal vector at $\partial \Omega_{c}^{e}$.
When a rigid inclusion is considered, because only rigid-body displacement is allowed for the inclusion, there is no need to assume $u_{i}^{c}$. The following conditions need to be satisfied at $\partial \Omega_{c}^{e}$ :
$u_{i}^{m}($ non - rigid - body $)=0$ at $\partial \Omega_{c}^{e}$

$$
\begin{align*}
\int_{\partial \Omega_{c}^{e}} n_{j} \sigma_{i j}^{m} d S & =0  \tag{12}\\
\int_{\partial \Omega_{c}^{e}} e_{g h i} x_{h} n_{j} \sigma_{i j}^{m} d S & =0
\end{align*}
$$

which means, the non-rigid-body displacements vanish at $\partial \Omega_{c}^{e}$, and the resultant force and moment on $\partial \Omega_{c}^{e}$ are zero.
When a void is to be considered, for VCFEM-TT-BVP, $\tilde{u}_{i}^{c}$ is assumed only along $\partial \Omega_{c}^{e}$, and the following displacement continuity and traction free conditions are to be satisfied:
$u_{i}^{m}=\tilde{u}_{i}^{c}$ at $\partial \Omega_{c}^{e}$
$n_{j} \sigma_{i j}^{m}=0$ at $\partial \Omega_{c}^{e}$
For VCFEM-TT-C, on the other hand, there is no need to assume such a boundary field $\tilde{u}_{i}^{c}$, and only condition (14) needs to be satisfied. Details of the difference on assumed displacement fields are explained in section 3 and section 4. Assumed fields of VCFEM-TT-BVP and VCFEM-TT-C with three cases of heterogeneities are summarized in Fig. 2 and Fig. 3.
It should be noted that, for a priori equilibrated displacement fields, condition (12) is a necessary condition of (10) or (14). Hence, for problems with elastic inclusion or voids, condition (12) is satisfied as long as conditions (10) or (14) are satisfied.
In T-Trefftz elements derived in this study, $u_{i}^{m}, u_{i}^{c}$ should satisfy (1)(2)(3)(4) as well as (12) a priori, $\tilde{u}_{i}^{m}$ should satisfy (5) a priori, while other afore-mentioned conditions are satisfied using boundary variational principles or using the collocation method.
For plane stress or plane strain problems where body force are negligible, T-Trefftz trial functions in the matrix $\Omega_{m}^{e}$, which satisfy (1)(2)(3)(4) a priori, can be generated by two complex potentials $\phi_{m}\left(z_{e}\right)$ and $\chi_{m}^{\prime}\left(z_{e}\right)$, see [Muskhelishvil (1954)]:
$u_{1}^{m}+i u_{2}^{m}=\left[\kappa_{m} \phi_{m}\left(z_{e}\right)-z_{e} \overline{\phi_{m}^{\prime}\left(z_{e}\right)}-\overline{\chi_{m}^{\prime}\left(z_{e}\right)}\right] / 2 G_{m}$ in $\Omega_{m}^{e}$
In (15), $z_{e}=\left(x_{1}+i x_{2}\right)-\left(x_{1}^{e}+i x_{2}^{e}\right)$ with $i=\sqrt{-1} . S^{e}:\left(x_{1}^{e}, x_{2}^{e}\right)$ is the Trefftz source point for element $e . G_{m}$ and $\kappa_{m}$ are defined as:

$$
\begin{align*}
\kappa_{m} & = \begin{cases}3-4 v_{m} & \text { for plane strain problems } \\
\left(3-v_{m}\right) /\left(1+v_{m}\right) & \text { for plane stress problems }\end{cases}  \tag{16}\\
\mathrm{G}_{m} & =\frac{E_{m}}{2\left(1+v_{m}\right)}
\end{align*}
$$

where $E_{m}, v_{m}$ are the Young's modulus and Poisson ratio of the matrix .
It should be noted that, $\phi_{m}\left(z_{e}\right)$ and $\chi_{m}^{\prime}\left(z_{e}\right)$ need to be constructed in such a way that the trial functions are relatively complete for the specific domain of interest. Because $\Omega_{m}^{e}$ is a doubly-connected domain, one locates the source point $S^{e}$ inside $\Omega_{c}^{e}$, and $\phi_{m}\left(z_{e}\right), \chi_{m}^{\prime}\left(z_{e}\right)$ are assumed in terms of both positive and negative power series, as well as a logarithmic function, see [Yeih, Liu, Kuo and Atluri (2010)] for further discussion:
$\phi_{m}\left(z_{e}\right)=(i A+B) \ln z_{e}+\sum_{n=1}^{\infty}\left(i \alpha_{n}^{1}+\alpha_{n}^{2}\right) z_{e}^{n}+\sum_{n=-1}^{-\infty}\left(i \alpha_{n}^{1}+\alpha_{n}^{2}\right) z_{e}^{n}$
$\chi_{m}^{\prime}\left(z_{e}\right)=\kappa(i A-B) \ln z_{e}+\sum_{n=0}^{\infty}\left(i \alpha_{n}^{3}+\alpha_{n}^{4}\right) z_{e}^{n}+\sum_{n=-1}^{-\infty}\left(i \alpha_{n}^{3}+\alpha_{n}^{4}\right) z_{e}^{n}$


Figure 1: A VCFEM with an arbitrarily-shaped inclusion or void

As discussed in [Muskhelishvil (1954)], $A, B, \alpha_{-1}^{3}$ are determined by the resultant force and moment on $\partial \Omega_{c}^{e}$. In all of these three cases, the resultant forces and moment on $\partial \Omega_{c}^{e}$ are zero as stated in condition (12), thus we prescribe the value of these parameters to be zero:
$A=B=\alpha_{-1}^{3}=0$
When an elastic inclusion is considered, $u_{i}^{c}$ can be assumed using complex potentials $\phi_{c}\left(z_{e}\right)$ and $\chi_{c}\left(z_{e}\right)$ :
$\left.u_{1}^{c}+i u_{2}^{c}=\left[\kappa_{c} \phi_{c}\left(z_{e}\right)-z_{e} \overline{\phi_{c}^{\prime}\left(z_{e}\right.}\right)-\overline{\chi_{c}^{\prime}\left(z_{e}\right)}\right] / 2 G_{c}$ in $\Omega_{c}^{e}$
where $\kappa_{c}, G_{c}$ are defined in a similar fashion to that of $\kappa_{m}, G_{m}$ in (16).
Because $\Omega_{c}^{e}$ is a simply connected domain, $\phi_{c}\left(z_{e}\right)$ and $\chi_{c}^{\prime}\left(z_{e}\right)$ are assumed in terms of positive power series only:
$\phi_{c}\left(z_{e}\right)=\sum_{n=1}^{\infty}\left(i \beta_{n}^{1}+\beta_{n}^{2}\right) z_{e}^{n}$
$\chi_{c}^{\prime}\left(z_{e}\right)=\sum_{n=0}^{\infty}\left(i \beta_{n}^{3}+\beta_{n}^{4}\right) z_{e}^{n}$
When a rigid inclusion is considered, $u_{i}^{c}$ does need to be assumed. Assuming $u_{i}^{m}$ is good enough in order to develop finite element equations.
When a void is considered, for VCFEM-TT-BVP, $\tilde{u}_{i}^{c}$ is assumed only at $\partial \Omega_{c}$. In this case, because $\kappa_{c}, G_{c}$ is not well defined, instead of (19), it is reasonable to assume $\tilde{u}_{i}^{c}$ as:

$$
\begin{align*}
& \tilde{u}_{1}^{c}+i \tilde{u}_{2}^{c}=\left[\kappa_{m} \tilde{\phi}_{c}\left(z_{e}\right)-z_{e} \overline{\phi_{c}^{\prime}\left(z_{e}\right)}-\overline{\tilde{\chi}_{c}^{\prime}\left(z_{e}\right)}\right] / 2 G_{m} \text { at } \partial \Omega_{c}^{e} \\
& \tilde{\phi}_{c}\left(z_{e}\right)=\sum_{n=1}^{\infty}\left(i \gamma_{n}^{1}+\gamma_{n}^{2}\right) z_{e}^{n}  \tag{21}\\
& \tilde{\chi}_{c}^{\prime}\left(z_{e}\right)=\sum_{n=0}^{\infty}\left(i \gamma_{n}^{3}+\gamma_{n}^{4}\right) z_{e}^{n}
\end{align*}
$$

Actually, because $\tilde{u}_{i}^{c}$ is only defined on the boundary of $\Omega_{c}^{e}$, it is not necessary to assume it in terms of complex potentials. It can also be assumed in terms of Fourier series of polar coordinate $\theta_{e}$. In this study, we adopt the assumption of (21).
Now that the displacement fields are defined, undetermined parameters $\alpha_{n}^{k}, \beta_{n}^{k}, \gamma_{n}^{k}$ can be related to nodal displacements of the element using either multi-field boundary variational principles or using collocation method. However, similar to what is frequently encountered in T-Trefftz methods, a system of ill-conditioned equations is to be solved in order to establish such a relation. This is because of the exponential growth of the term $z^{n}$ with respect to the order $n$. [Liu (2007a, 2007b)] introduced the concept of characteristic length to scale the T-Trefftz trial functions for Laplace equations, and it was later extended to solve general ill-conditioned linear algebra equations in [Liu, Yeih and Atluri (2009)]. Since this method successfully resolved the ill-conditioned nature of Trefftz method for Laplace equations, it is also applied in this study, in the context of plane stress/strain solid mechanics.
For each element with source point $S^{e}:\left(x_{1}^{e}, x_{2}^{e}\right)$, two characteristic lengths $R_{m k}$ and $R_{m p}$ are defined for $u_{i}^{m}$, the displacement field in the matrix. $R_{m k}$ is equal to the minimum distance between the source point $S^{e}$ and any point in $\Omega_{m}^{e}$, therefore
$\left|\left(\frac{z_{e}}{R_{m k}}\right)^{n}\right|$ is confined between 0 and 1 for any negative $n$. A characteristic length $R_{m p}$ is also defined, which is equal to the maximum distance between the source point $S_{k}$ and any point in $\Omega_{m}^{e}$, therefore $\left|\left(\frac{z_{e}}{R_{m p}}\right)^{n}\right|$ is confined between 0 and 1 for any positive $n$. Complex potentials $\phi_{m}\left(z_{e}\right)$ and $\chi_{m}^{\prime}\left(z_{e}\right)$ are thereafter scaled as:

$$
\begin{align*}
\phi_{m}\left(z_{e}\right) & =\sum_{n=1}^{N}\left(i \alpha_{n}^{1}+\alpha_{n}^{2}\right)\left(\frac{z_{e}}{R_{m p}}\right)^{n}+\sum_{n=-1}^{-M}\left(i \alpha_{n}^{1}+\alpha_{n}^{2}\right)\left(\frac{z_{e}}{R_{k p}}\right)^{n} \\
\chi_{m}^{\prime}\left(z_{e}\right) & =\sum_{n=0}^{N}\left(i \alpha_{n}^{3}+\alpha_{n}^{4}\right)\left(\frac{z_{e}}{R_{m p}}\right)^{n}+\sum_{n=-2}^{-M}\left(i \alpha_{n}^{3}+\alpha_{n}^{4}\right)\left(\frac{z_{e}}{R_{k p}}\right)^{n}+\alpha_{-1}^{4}\left(\frac{z_{e}}{R_{k p}}\right)^{-1} \tag{22}
\end{align*}
$$

It should be noted that, in (22), the upper and lower bound of order $n$ are considered as $N, M$ for numerical implementation. Also, compared to (17), the three modes corresponding to $A, B, \alpha_{-1}^{3}$ are eliminated beforehand, according to (18).
For $u_{i}^{c}$, the displacement field in the inclusion, or $\tilde{u}_{i}^{c}$, the displacement field along $\partial \Omega_{c}^{e}$ when a void is to be considered, a characteristic length $R_{c p}$ is also considered. $R_{c p}$ is equal to the maximum distance between the source point $S^{e}$ and any point in $\Omega_{c}^{e}$, therefore $\left|\left(\frac{z_{e}}{R_{c p}}\right)^{n}\right|$ is confined between 0 and 1 for any positive $n$.
Complex potentials $\phi_{c}\left(z_{e}\right)$ and $\chi_{c}^{\prime}\left(z_{e}\right)$ are thereafter scaled as:

$$
\begin{align*}
\phi_{c}\left(z_{e}\right) & =\sum_{n=1}^{L}\left(i \beta_{n}^{1}+\beta_{n}^{2}\right)\left(\frac{z_{e}}{R_{c p}}\right)^{n} \\
\chi_{c}^{\prime}\left(z_{e}\right) & =\sum_{n=0}^{L}\left(i \beta_{n}^{3}+\beta_{n}^{4}\right)\left(\frac{z_{e}}{R_{c p}}\right)^{n} \tag{23}
\end{align*}
$$

Complex potentials $\tilde{\phi}_{c}\left(z_{e}\right)$ and $\tilde{\chi}_{c}^{\prime}\left(z_{e}\right)$ are scaled in a similar fashion:

$$
\begin{align*}
& \tilde{\phi}_{c}\left(z_{e}\right)=\sum_{n=1}^{P}\left(i \gamma_{n}^{1}+\gamma_{n}^{2}\right)\left(\frac{z_{e}}{R_{c p}}\right)^{n}  \tag{24}\\
& \tilde{\chi}_{c}^{\prime}\left(z_{e}\right)=\sum_{n=0}^{P}\left(i \gamma_{n}^{3}+\gamma_{n}^{4}\right)\left(\frac{z_{e}}{R_{c p}}\right)^{n}
\end{align*}
$$

Similarly, the upper bound of order $n$ is taken as $L$ and $P$ respectively for numerical implementation. A relatively larger number of modes should be used in order to accurately model the stress distribution in the element. In this study, we use $M=$ $N=L=P=8$.

As will be shown in numerical examples of section 6 , by using $R_{m k}, R_{m p}, R_{c p}$ to scale the T-Trefftz trial functions, we successfully avoid solving systems of illconditioned equations. Without using $R_{m k}, R_{m p}, R_{c p}$, it is almost impossible to develop stiffness matrices based these displacement assumptions.
It should also be noted that, $\alpha_{0}^{3}, \alpha_{0}^{4}, \alpha_{1}^{1}, \beta_{0}^{3}, \beta_{0}^{4}, \beta_{1}^{1}$, and $\gamma_{0}^{3}, \gamma_{0}^{4}, \gamma_{1}^{1}$ correspond to the three rigid-body modes for two-dimensional problems. These three modes should be eliminated beforehand for VCFEM-TT-BVP, but should be preserved for VCFEM-TT-C. All other modes are independent, non-rigid-body modes.
Moreover, the displacement assumptions considered in this section are all invariant with respect to change of coordinate systems. Therefore, the element stiffness matrices developed from these displacement assumptions are expected to be invariant. Now that the displacement filed, $\tilde{u}_{i}, u_{i}^{m}, u_{i}^{c}$ (or $\tilde{u}_{i}^{c}$ ) are independently assumed, element stiffness matrices can be developed using methods in the next two sections.
(a)

(c)
(b)


Figure 2: Assumed fields for VCFEM-TT-BVP: (a) with an elastic inclusion; (b) with a rigid inclusion; (c) with a void

## 3 T-Trefftz VCFEMs Using Multi-Field Boundary Variational Principles

In this section, T-Trefftz VCFEMs are developed using multi-field boundary variational principles.
An inter-element compatible displacement field $\tilde{u}_{i}^{m}$ is assumed at $\partial \Omega^{e} . \tilde{u}_{i}^{m}$ can be assumed to be linear, quadratic, or of higher-order on each edge of the element, depending on the number of nodes on each edge. In this study, we simply assume $\tilde{u}_{i}^{m}$ to be linear on each edge. Using matrix and vector notation, we have:
$\tilde{\mathbf{u}}_{m}=\tilde{\mathbf{N}}_{m} \mathbf{q}$ at $\partial \Omega^{e}$

The displacement field in the matrix $u_{i}^{m}$ is derived from complex potentials assumed in (22). We have the displacement field in $\Omega_{m}^{e}$ and its corresponding traction field $t_{i}^{m}$ at $\partial \Omega_{m}^{e}, \partial \Omega_{c}^{e}$ as:

$$
\begin{align*}
\mathbf{u}_{m} & =\mathbf{N}_{m} \alpha \text { in } \Omega_{m}^{e}  \tag{26}\\
\mathbf{t}_{m} & =\mathbf{R}_{m} \alpha \text { at } \partial \Omega_{m}^{e}, \partial \Omega_{c}^{e}
\end{align*}
$$

When an elastic inclusion is to be considered, the displacement field in the inclusion, $u_{i}^{c}$ is derived from complex potentials assumed in (23). Similarly, we have:

$$
\begin{align*}
\mathbf{u}_{c} & =\mathbf{N}_{c} \alpha \text { in } \Omega_{c}^{e} \\
\mathbf{t}_{c} & =\mathbf{R}_{c} \alpha \text { at } \partial \Omega_{c}^{e} \tag{27}
\end{align*}
$$

It should be noted that, the traction fields in (26)(27) are as follows:

1. at $\partial \Omega^{e}, t_{i}^{m}=n_{j} E_{i j k l}^{m} u_{(k, l)}^{m}$, where $n_{j}$ is the unit outer-normal vector at $\partial \Omega^{e}$;
2. at $\partial \Omega_{c}^{e}, t_{i}^{m}=-n_{j} E_{i j k l}^{m} u_{(k, l)}^{m}, t_{i}^{c}=n_{j} E_{i j k l}^{c} u_{(k, l)}^{c}$, where $n_{j}$ is the unit out-normal vector of $\partial \Omega_{c}^{e}$.
Therefore, finite element equations can be derived using the following three-field boundary variational principle:

$$
\begin{align*}
\pi_{1}\left(\tilde{u}_{i}^{m}, u_{i}^{m}, u_{i}^{c}\right) & =\sum_{e}\left\{-\int_{\partial \Omega^{e}+\partial \Omega_{c}^{e}} \frac{1}{2} t_{i}^{m} u_{i}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\}  \tag{28}\\
& +\sum_{e}\left\{\int_{\partial \Omega_{c}^{e}} t_{i}^{m} u_{i}^{c} d S+\int_{\partial \Omega_{c}^{e}} \frac{1}{2} t_{i}^{c} u_{i}^{c} d S\right\}
\end{align*}
$$

which leads to Euler-Lagrange equations:

$$
\begin{align*}
& t_{i}^{m}=\bar{t}_{i} \text { at } \mathrm{S}_{t}^{e} \\
& u_{i}^{m}=\tilde{u}_{i} \text { at } \partial \Omega^{e} \\
& t_{i}^{m+}+t_{i}^{m-}=0 \text { at } \rho^{e}  \tag{29}\\
& u_{i}^{m}=u_{i}^{c} \text { at } \partial \Omega_{c}^{e} \\
& t_{i}^{m}+t_{i}^{c}=0 \text { at } \partial \Omega_{c}^{e}
\end{align*}
$$

Substitute (25), (26), (27) into (28), we obtain:

$$
\begin{align*}
\delta \pi_{1}(\mathbf{q}, \boldsymbol{\alpha}, \boldsymbol{\beta}) & =0 \\
& =\delta \sum_{e}\left(-\frac{1}{2} \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}+\boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha \beta} \boldsymbol{\beta}+\frac{1}{2} \boldsymbol{\beta}^{T} \mathbf{H}_{\beta \beta} \boldsymbol{\beta}-\mathbf{q}^{T} \mathbf{Q}\right) \\
& =\sum_{e}\left(-\delta \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\delta \boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}+\delta \mathbf{q}^{T} \mathbf{G}_{\alpha q}^{T} \boldsymbol{\alpha}-\delta \mathbf{q}^{T} \mathbf{Q}\right) \\
& +\sum_{e}\left(\delta \boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha \beta} \boldsymbol{\beta}+\delta \boldsymbol{\beta}^{T} \mathbf{G}_{\alpha \beta}^{T} \boldsymbol{\alpha}+\delta \boldsymbol{\beta}^{T} \mathbf{H}_{\beta \beta} \boldsymbol{\beta}\right) \\
\mathbf{G}_{\alpha \beta} & =\int_{\partial \Omega_{c}^{e}} \mathbf{R}_{m}^{T} \mathbf{N}_{c} d S \\
\mathbf{G}_{\alpha q} & =\int_{\partial \Omega^{e}} \mathbf{R}_{m}^{T} \tilde{\mathbf{N}}_{m} d S \\
\mathbf{H}_{\alpha \alpha} & =\int_{\partial \Omega^{e}+\partial \Omega_{c}^{e}} \mathbf{R}_{m}^{T} \mathbf{N}_{m} d S \\
\mathbf{H}_{\beta \beta} & =\int_{\partial \Omega_{c}^{e}} \mathbf{R}_{c}^{T} \mathbf{N}_{c} d S \\
\mathbf{Q} & =\int_{S_{t}^{e}} \tilde{\mathbf{N}}_{m}^{T} \overline{\mathbf{t}} d S \tag{30}
\end{align*}
$$

This leads to finite element equations:
$\delta\left\{\begin{array}{c}\mathbf{q} \\ \boldsymbol{\beta}\end{array}\right\}^{T}\left[\begin{array}{cc}\mathbf{G}_{\alpha q}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha q} & \mathbf{G}_{\alpha q}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha \beta} \\ \mathbf{G}_{\alpha \beta}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha q} & \mathbf{G}_{\alpha \beta}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha \beta}+\mathbf{H}_{\beta \beta}\end{array}\right]\left\{\begin{array}{l}\mathbf{q} \\ \boldsymbol{\beta}\end{array}\right\}=\delta\left\{\begin{array}{c}\mathbf{q} \\ \boldsymbol{\beta}\end{array}\right\}^{T}\left\{\begin{array}{c}\mathbf{Q} \\ \mathbf{0}\end{array}\right\}$
Since $\delta \boldsymbol{\beta}$ is arbitrary, (31) can be further simplified so that $\mathbf{q}$ is the only unknown, by static-condensation.
When the inclusion is rigid, $u_{i}^{c}$ does not need to be assumed, and we use the following variational principle:
$\pi_{2}\left(\tilde{u}_{i}^{m}, u_{i}^{m}\right)=\sum_{e}\left\{-\int_{\partial \Omega^{e}+\partial \Omega_{c}^{e}} \frac{1}{2} t_{i}^{m} u_{i}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\}$
which leads to Euler-Lagrange equations:
$t_{i}^{m}=\bar{t}_{i}$ at $\mathrm{S}_{t}^{e}$
$u_{i}^{m}=\tilde{u}_{i}$ at $\partial \Omega^{e}$
$t_{i}^{m+}+t_{i}^{m-}=0$ at $\rho^{e}$
$u_{i}^{m}($ non - rigid $-\operatorname{body})=0$ at $\partial \Omega_{c}^{e}$
We should point out that, $u_{i}^{m}$ is made to produce zero resultant force and moment at $\partial \Omega_{c}$, because of (18). This is to say, the following conditions are satisfied a priori:

$$
\begin{align*}
\int_{\partial \Omega_{c}^{e}} t_{i}^{m} d S & =0 \\
\int_{\partial \Omega_{c}^{e}} e_{i j k} x_{j} t_{k}^{m} d S & =0 \tag{34}
\end{align*}
$$

And only because Eq. (34) is satisfied a priori, stationarity of (32) leads to rigidbody $u_{i}^{m}$ at $\partial \Omega_{c}^{e}$ instead of vanishing $u_{i}^{m}$ at $\partial \Omega_{c}^{e}$.
Substituting (25)(26) into variational principle (32), we obtain:

$$
\begin{align*}
\delta \pi_{2}(\mathbf{q}, \alpha) & =0 \\
& =\delta \sum_{e}\left(-\frac{1}{2} \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}-\mathbf{q}^{T} \mathbf{Q}\right)  \tag{35}\\
& =\sum_{e}\left(-\delta \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\delta \boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}+\delta \mathbf{q}^{T} \mathbf{G}_{\alpha q}^{T} \boldsymbol{\alpha}-\delta \mathbf{q}^{T} \mathbf{Q}\right)
\end{align*}
$$

And corresponding finite element equations are:

$$
\begin{equation*}
\sum_{e}\left(\delta \mathbf{q}^{T} \mathbf{G}_{\alpha q}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha q} \mathbf{q}-\delta \mathbf{q}^{T} \mathbf{Q}\right)=0 \tag{36}
\end{equation*}
$$

When the element include a void instead of an elastic/rigid inclusion, $\tilde{u}_{i}^{c}$ is merely assumed at $\partial \Omega_{c}^{e}$, from (21). We have:
$\tilde{\mathbf{u}}_{c}=\tilde{\mathbf{N}}_{c} \gamma$ at $\partial \Omega_{c}^{e}$
We use the following variational principle:

$$
\begin{align*}
\pi_{3}\left(\tilde{u}_{i}^{m}, u_{i}^{m}, \tilde{u}_{i}^{c}\right) & =\sum_{e}\left\{-\int_{\partial \Omega^{e}+\partial \Omega_{c}^{e}} \frac{1}{2} t_{i}^{m} u_{i}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\}  \tag{38}\\
& +\sum_{e} \int_{\partial \Omega_{c}^{e}} t_{i}^{m} \tilde{u}_{i}^{c} d S
\end{align*}
$$

which leads to Euler-Lagrange equations:
$t_{i}^{m}=\bar{t}_{i}$ at $\mathrm{S}_{t}^{e}$
$u_{i}^{m}=\tilde{u}_{i}$ at $\partial \Omega^{e}$
$t_{i}^{m+}+t_{i}^{m-}=0$ at $\rho^{e}$
$u_{i}^{m}=\tilde{u}_{i}^{c}$ at $\partial \Omega_{c}^{e}$
$t_{i}^{m}=0$ at $\partial \Omega_{c}^{e}$
Substituting (25)(26)(37) into variational principle (38), we have:

$$
\begin{align*}
\delta \pi_{3}(\mathbf{q}, \alpha, \gamma) & =0 \\
& =\delta \sum_{e}\left(-\frac{1}{2} \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}+\boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha \gamma} \gamma-\mathbf{q}^{T} \mathbf{Q}\right) \\
& =\sum_{e}\left(-\delta \boldsymbol{\alpha}^{T} \mathbf{H}_{\alpha \alpha} \boldsymbol{\alpha}+\delta \boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha q} \mathbf{q}+\delta \mathbf{q}^{T} \mathbf{G}_{\alpha q}^{T} \boldsymbol{\alpha}-\delta \mathbf{q}^{T} \mathbf{Q}\right)  \tag{40}\\
& +\sum_{e}\left(\delta \boldsymbol{\alpha}^{T} \mathbf{G}_{\alpha \gamma} \gamma+\delta \gamma^{T} \mathbf{G}_{\alpha \gamma}^{T} \boldsymbol{\alpha}\right) \\
\mathbf{G}_{\alpha \gamma} & =\int_{\partial \Omega_{c}^{e}} \mathbf{R}_{m}^{T} \tilde{\mathbf{N}}_{c} d S
\end{align*}
$$

And corresponding finite element equations are:
$\delta\left\{\begin{array}{l}\mathbf{q} \\ \gamma\end{array}\right\}^{T}\left[\begin{array}{ll}\mathbf{G}_{\alpha q}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha q} & \mathbf{G}_{\alpha q}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha \gamma} \\ \mathbf{G}_{\alpha \gamma}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha q} & \mathbf{G}_{\alpha \gamma}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha \gamma}\end{array}\right]\left\{\begin{array}{l}\mathbf{q} \\ \gamma\end{array}\right\}=\delta\left\{\begin{array}{l}\mathbf{q} \\ \gamma\end{array}\right\}^{T}\left\{\begin{array}{c}\mathbf{Q} \\ \mathbf{0}\end{array}\right\}$
Similarly, equation (41) can be further simplified by static-condensation.
In this section, we have developed T-Trefftz VCFEMs with elastic/rigid inclusions or voids, the finite element equations of which are (31), (36) and (41) respectively. Because these VCFEMs are all developed using multi-field boundary variational principles, we denote this class of elements as: VCFEM-TT-BVP.
VCFEM-TT-BVP is expected to be much more efficient and accurate than VCFEMs developed in [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)], which use Airy stress functions and the modified principle of complementary energy. However, the development of stiffness matrices of VCFEM-TT-BVP still seems to be somehow complicated. In additions, Lagrange multipliers involved in (28), (32), (38) render VCFEM-TT-BVP to suffer from LBB conditions. LBB conditions are almost impossible to be satisfied a priori, and failure to satisfy LBB conditions make finite element solutions unstable, sometimes even not unique. Hence, in next section, we develop another class of T-Trefftz VCFEMs, which are very simple and do not involve LBB conditions.


Figure 3: Assumed fields for VCFEM-TT-C: (a) with an elastic inclusion; (b) with a rigid inclusion; (c) with a void

## 4 T-Trefftz VCFEMs Using Collocation and a Primitive Field Boundary Variational Principle

In section 3, VCFEM-TT-BVP is developed using multi-field variational principles. In these variational principles, compatibility between $u_{i}^{m}$ and $\tilde{u}_{i}^{m}$ at $\partial \Omega^{e}$, compatibility between $u_{i}^{m}$ and $u_{i}^{c}$ ( or $\tilde{u}_{i}^{c}$ ) at $\partial \Omega_{c}^{e}$, and traction reciprocity conditions ( or traction free conditions ) at $\partial \Omega_{c}^{e}$ are all enforced by Lagrange multipliers in a variational sense. Using Lagrange multipliers renders finite elements rather complicated and plagued by LBB conditions. [Dong and Atluri (2011); Dong and Atluri (2011b)] proposed to use the collocation method instead of using Lagrange multipliers to enforce these conditions, leading to very simple finite element formulations without involving LBB conditions. In this section, we develop VCFEM-TTs using this method, and denote them as VCFEM-TT-C.
A finite number of collocation points are selected along $\partial \Omega^{e}$ and $\partial \Omega_{c}^{e}$, denoted as $x_{i}^{m p} \in \partial \Omega^{e}, p=1,2 \ldots$, and $x_{i}^{c q} \in \partial \Omega_{c}^{e}, q=1,2 \ldots$
Collocations are carried out in the following manner:

1. When an elastic inclusion is considered, $u_{i}^{m}, u_{i}^{c}$ and their corresponding tractions $t_{i}^{m}, t_{i}^{c}$ are assumed as in (26)(27), and $\tilde{u}_{i}^{m}$ as in (25). Therefore, we enforce the following conditions at corresponding collocation points:

$$
\begin{align*}
& 2 G_{m} u_{i}^{m}\left(x_{j}^{m p}, \boldsymbol{\alpha}\right)=2 G_{m} \tilde{u}_{i}\left(x_{j}^{m p}, \mathbf{q}\right) \quad x_{j}^{m p} \in \partial \Omega^{e} \\
& 2 G_{m} u_{i}^{m}\left(x_{j}^{c q}, \boldsymbol{\alpha}\right)-2 G_{m} u_{i}^{c}\left(x_{j}^{c q}, \boldsymbol{\beta}\right)=0 \quad x_{j}^{c q} \in \partial \Omega_{c}^{e}  \tag{42}\\
& R_{m k} t_{i}^{m}\left(x_{j}^{c q}, \boldsymbol{\alpha}\right)+R_{m k} t_{i}^{( }\left(x_{j}^{c q}, \boldsymbol{\beta}\right)=0 \quad x_{j}^{c q} \in \partial \Omega_{c}^{e}
\end{align*}
$$

$2 G_{m}$ and $R_{m k}$ are used as the weights of collocation equations for displacements and tractions receptively. By using these weight functions, collocation equations of displacements and tractions are considered to be of relatively equal order.
By selecting enough number of collocation points, and solving (42) in a leastsquare sense, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are related to $\mathbf{q}$ in the following way:

$$
\begin{align*}
\boldsymbol{\alpha} & =\mathbf{C}_{\alpha q}^{1} \mathbf{q} \\
\boldsymbol{\beta} & =\mathbf{C}_{\beta q}^{1} \mathbf{q} \tag{43}
\end{align*}
$$

2. When the inclusion is rigid, there is no need to assume $u_{i}^{c}$, and the following collocations are considered:

$$
\begin{align*}
& 2 G_{m} u_{i}^{m}\left(x_{j}^{m p}, \boldsymbol{\alpha}\right)=2 G_{m} \tilde{u}_{i}\left(x_{j}^{m p}, \mathbf{q}\right) \quad x_{j}^{m p} \in \partial \Omega^{e}  \tag{44}\\
& 2 G_{m} u_{i}^{m}\left(\text { non - rigid-body }, x_{j}^{c q}, \boldsymbol{\alpha}\right)=0 \quad x_{j}^{c q} \in \partial \Omega_{c}^{e}
\end{align*}
$$

According to section 2, it is very easy to isolate the rigid-body part and non-rigidbody part of $u_{i}^{m} . u_{i}^{m}$ (rigid - body) is the displacement field derived from complex potentials with $\alpha_{0}^{3}, \alpha_{0}^{4}, \alpha_{1}^{1}$ as coefficients. $u_{i}^{m}$ (non-rigid - body) is therefore the other part of $u_{i}^{m}$. By solving (44), we obtain:

$$
\begin{equation*}
\alpha=\mathbf{C}_{\alpha q}^{2} \mathbf{q} \tag{45}
\end{equation*}
$$

3. When a void is considered instead of a inclusion, there is no need to assume $u_{i}^{c}$ or $\tilde{u}_{i}^{c}$. This is different from VCFEM-TT-BVP. For VCFEM-TT-BVP, $\tilde{u}_{i}^{c}$ is assumed on the boundary of the void for in order to be used as Lagrange multipliers to enforce traction free condition at $\partial \Omega_{c}^{e}$. Since here, we directly enforce displacement continuity and traction free condition in a strong form at collocation points, there is no need to assume a boundary displacement field $\tilde{u}_{i}^{c}$. The following conditions are enforced:

$$
\begin{align*}
& G_{m} u_{i}^{m}\left(x_{j}^{m p}, \boldsymbol{\alpha}\right)=G_{m} \tilde{u}_{i}\left(x_{j}^{m p}, \mathbf{q}\right) \quad x_{j}^{m p} \in \partial \Omega^{e} \\
& R_{m k} t_{i}^{m}\left(x_{j}^{c q}, \boldsymbol{\alpha}\right)=0 \quad x_{j}^{c q} \in \partial \Omega_{c}^{e} \tag{46}
\end{align*}
$$

By solving (46), we have:
$\alpha=\mathbf{C}_{\alpha q}^{3} \mathbf{q}$
Now that the interior displacement field is related to nodal displacements, finite element equations can be derived from the following primitive-field boundary variational principle:
$\pi_{4}\left(u_{i}\right)=\sum_{e}\left\{\int_{\partial \Omega^{e}} \frac{1}{2} t_{i} u_{i} d \Omega-\int_{S_{t}^{e}} \bar{t}_{i} u_{i} d S\right\}$
which leads to Euler-Lagrange equations:

$$
\begin{align*}
& t_{i}=\bar{t}_{i} \text { at } S_{t m} \\
& t^{+}+t^{-}=0 \text { at } \rho_{m} \tag{49}
\end{align*}
$$

Substitute corresponding displacement fields into (48), we obtain finite element equations:

$$
\begin{align*}
& \sum_{m}\left(\delta \mathbf{q}^{T} \mathbf{C}_{\alpha q}{ }^{s T} \mathbf{M}_{\alpha \alpha} \mathbf{C}_{\alpha q}^{s} \mathbf{q}-\delta \mathbf{q}^{T} \mathbf{Q}\right)=0, s=1,2 \text { or } 3  \tag{50}\\
& \mathbf{M}_{\alpha \alpha}=\int_{\partial \Omega^{e}} \mathbf{R}_{m}^{T} \mathbf{N}_{m} d S
\end{align*}
$$

When $s$ is equal to 1, 2, and $3, \mathbf{C}_{\alpha q}^{s}{ }^{T} \mathbf{M}_{\alpha \alpha} \mathbf{C}_{\alpha q}^{s}$ is the stiffness matrix for VCFEM with an elastic inclusion, a rigid inclusion, and a void respectively.
It can be seen that the element stiffness matrix (50) is much simpler than that for VCFEM-TT-BVP. And because integration of only one matrix $\mathbf{M}_{\alpha \alpha}$ along the outer boundary is needed, VCFEM-TT-C is expected to be computationally more efficient than VCFEM-TT-BVP. Finally, as explained previously, VCFEM-TT-C does not suffer from LBB conditions, which is a tremendous advantage of VCFEM-TTC over VCFEM-TT-BVP, as well as VCFEM-HS-PCE developed by Ghosh and his coworkers.

## 5 Comparison to VCFEMs Using the Hybrid Stress Approach

[Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)] proposed the idea of discretizing the solution domain using Dirichlet tessellation, and developing corresponding VCFEMs with inclusions/voids to solve problems of micromechanics of materials. Because this class of VCFEMs is based on a similar theoretical foundation to that of the hybrid-stress elements developed in [Pian (1964)], using the modified principle of complementary energy, we denote VCFEMs developed by Ghosh
(a)


Figure 4: Assumed fields for VCFEM-TT-C: (a) with an elastic inclusion; (b) with a rigid inclusion; (c) with a void
and his coworkers as VCFEM-HS-PCE. In this section, we review the development of VCFEM-HS-PCE for elements with inclusions/voids, and make comments on some obvious advantages of VCFEM-TTs over VCFEM-HS-PCE.
The assumed fields of VCFEM-HS-PCE are shown in Fig. 4. The displacement field $\tilde{u}_{i}^{m}$ on the outer-boundary $\partial \Omega^{e}$ is assumed to be linear on each edge. The innerboundary $\partial \Omega_{c}^{e}$ is approximated using line segments, and the displacement field $\tilde{u}_{i}^{c}$ is assumed to be linear along each segment. Using matrix and vector notation, we have:

$$
\begin{align*}
\tilde{\mathbf{u}}_{m} & =\tilde{\mathbf{N}}_{m} \mathbf{q}_{m} \text { at } \partial \Omega^{e} \\
\tilde{\mathbf{u}}_{c} & =\tilde{\mathbf{N}}_{c} \mathbf{q}_{c} \text { at } \partial \Omega_{c}^{e} \tag{51}
\end{align*}
$$

When an elastic inclusion is present in the element, independent stress field $\sigma_{i j}^{m}$ in the matrix $\Omega_{m}^{e}$, stress field $\sigma_{i j}^{c}$ in the inclusion $\Omega_{c}^{e}$ are derived from polynomial Airy stress function. For example, the first 12 modes are:

$$
\left[\begin{array}{c}
\sigma_{11}: \\
\sigma_{22}: \\
\sigma_{12}:
\end{array}\right]\left[\begin{array}{cccccccccccc}
0 & 0 & 1 & 0 & 0 & x_{1} & x_{2} & 0 & 0 & x_{1}^{2} & 2 x_{1} x_{2} & x_{2}^{2} \\
1 & 0 & 0 & x_{1} & x_{2} & 0 & 0 & x_{1}^{2} & 2 x_{1} x_{2} & x_{2}^{2} & 0 & 0 \\
0 & 1 & 0 & 0 & -x_{1} & -x_{2} & 0 & 0 & -x_{1}^{2} & -2 x_{1} x_{2} & -x_{2}^{2} & 0
\end{array}\right]
$$

It should be noted that, these stress modes should also be scaled by characteristic lengths in a similar manner to that of VCFEM-TTs as explained in section 2.
The stress fields in the domain and their corresponding tractions at the boundary can be written as:

$$
\begin{align*}
\sigma_{m} & =\mathbf{P}_{m} \boldsymbol{\alpha} \text { in } \Omega_{m}^{e} \\
\mathbf{t}_{m} & =\mathbf{R}_{m} \boldsymbol{\alpha} \text { at } \partial \Omega^{e}, \partial \Omega_{c}^{e}  \tag{52}\\
\sigma_{c} & =\mathbf{S}_{c} \boldsymbol{\beta} \text { in } \Omega_{c}^{e} \\
\mathbf{t}_{m} & =\mathbf{P}_{c} \boldsymbol{\beta} \text { at } \partial \Omega_{c}^{e}
\end{align*}
$$

And the following variational principle is used:

$$
\begin{align*}
& \pi_{5}\left(\tilde{u}_{i}^{m}, \tilde{u}_{i}^{c}, \sigma_{i j}^{m}, \sigma_{i j}^{c}\right) \\
& =\sum_{e}\left\{-\int_{\Omega_{m}^{e}} \frac{1}{2} \sigma_{i j}^{m} S_{i j k l}^{m} \sigma_{i j k l}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S+\int_{\partial \Omega_{c}^{e}} t_{i}^{m} \tilde{u}_{i}^{c} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\}  \tag{53}\\
& +\sum_{e}\left\{-\int_{\Omega_{c}^{e}} \frac{1}{2} \sigma_{i j}^{c} S_{i j k l}^{c} \sigma_{i j k l}^{c} d S+\int_{\partial \Omega_{c}^{e}} t_{i}^{c} \tilde{u}_{i}^{c} d S\right\}
\end{align*}
$$

Substituting (51)(52) into (53), we obtain finite element equations:

$$
\delta\left\{\begin{array}{l}
\mathbf{q}_{m}  \tag{54}\\
\mathbf{q}_{c}
\end{array}\right\}^{T}\left[\begin{array}{cc}
\mathbf{G}_{\alpha m}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha m} & \mathbf{G}_{\alpha m}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha c} \\
\mathbf{G}_{\alpha c}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha m} & \mathbf{G}_{\alpha c}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha c}+\mathbf{G}_{\beta c}^{T} \mathbf{H}_{\beta \beta}^{-1} \mathbf{G}_{\beta c}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{q}_{m} \\
\mathbf{q}_{c}
\end{array}\right\}=\delta\left\{\begin{array}{l}
\mathbf{q}_{m} \\
\mathbf{q}_{c}
\end{array}\right\}^{T}\left\{\begin{array}{c}
\mathbf{Q} \\
\mathbf{0}
\end{array}\right\}
$$

where

$$
\begin{align*}
\mathbf{G}_{\alpha m} & =\int_{\partial \Omega^{e}} \mathbf{R}_{m}^{T} \tilde{\mathbf{N}}_{m} d S \\
\mathbf{G}_{\alpha c} & =\int_{\partial \Omega_{c}^{e}} \mathbf{R}_{m}^{T} \tilde{\mathbf{N}}_{c} d S \\
\mathbf{G}_{\beta c} & =\int_{\partial \Omega_{c}^{e}} \mathbf{R}_{c}^{T} \tilde{\mathbf{N}}_{c} d S \\
\mathbf{H}_{\alpha \alpha} & =\int_{\Omega_{m}^{e}} \mathbf{P}_{m}^{T} \mathbf{S}_{m} \mathbf{P}_{m} d \Omega  \tag{55}\\
\mathbf{H}_{\beta \beta} & =\int_{\Omega_{c}^{e}} \mathbf{P}_{c}^{T} \mathbf{S}_{c} \mathbf{P}_{c} d \Omega \\
\mathbf{Q} & =\int_{S_{t}^{e}} \tilde{\mathbf{N}}_{m}^{T} \overline{\mathbf{t}} d S
\end{align*}
$$

When a rigid inclusion is present in the element, as far as the knowledge of the authors, finite element equations have not been developed by Ghosh or his coworkers.

However, the following variational principle can be used:

$$
\begin{equation*}
\pi_{6}\left(\tilde{u}_{i}^{m}, \sigma_{i j}^{m}\right)=\sum_{e}\left\{-\int_{\Omega_{m}^{e}} \frac{1}{2} \sigma_{i j}^{m} S_{i j k l}^{m} \sigma_{i j k l}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\} \tag{56}
\end{equation*}
$$

which leads to finite element equations:

$$
\begin{equation*}
\sum_{e}\left(\delta \mathbf{q}_{m}^{T} \mathbf{G}_{\alpha m}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha m} \mathbf{q}_{m}-\delta \mathbf{q}_{m}^{T} \mathbf{Q}\right)=0 \tag{57}
\end{equation*}
$$

When a void is present, we consider the following variational principle:

$$
\begin{align*}
& \pi_{7}\left(\tilde{u}_{i}^{m}, \tilde{u}_{i}^{c}, \sigma_{i j}^{m}\right) \\
& =\sum_{e}\left\{-\int_{\Omega_{m}^{e}} \frac{1}{2} \sigma_{i j}^{m} S_{i j k l}^{m} \sigma_{i j k l}^{m} d S+\int_{\partial \Omega_{m}^{e}} t_{i}^{m} \tilde{u}_{i}^{m} d S+\int_{\partial \Omega_{c}^{e}} t_{i}^{m} \tilde{u}_{i}^{c} d S-\int_{S_{t}^{e}} \bar{t}_{i} \tilde{u}_{i}^{m} d S\right\} \tag{58}
\end{align*}
$$

And finite element equations are obtained as:

$$
\delta\left\{\begin{array}{c}
\mathbf{q}_{m}  \tag{59}\\
\mathbf{q}_{c}
\end{array}\right\}^{T}\left[\begin{array}{cc}
\mathbf{G}_{\alpha m}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha m} & \mathbf{G}_{\alpha m}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha c} \\
\mathbf{G}_{\alpha c}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha m} & \mathbf{G}_{\alpha c}^{T} \mathbf{H}_{\alpha \alpha}^{-1} \mathbf{G}_{\alpha c}
\end{array}\right]\left\{\begin{array}{l}
\mathbf{q}_{m} \\
\mathbf{q}_{c}
\end{array}\right\}=\delta\left\{\begin{array}{l}
\mathbf{q}_{m} \\
\mathbf{q}_{c}
\end{array}\right\}^{T}\left\{\begin{array}{c}
\mathbf{Q} \\
\mathbf{0}
\end{array}\right\}
$$

Although nothing is wrong with variational principles (53)(56) or (58), VCFEM-HS-PCE has several serious disadvantages compared to VCFEM-TTs developed in section 3 and section 4.
Firstly, $\sigma_{i j}^{m}, \sigma_{i j}^{c}$ are highly incomplete, no matter whether an inclusion or a void is present in the element. Therefore, computed stress field by VCFEM-HS-PCE is highly inaccurate, as shown in section 6. [Moorthy and Ghosh (1996)] observed this phenomenon for elements with voids, and made the following explanation: "This is because the stress functions are required to meet the zero traction condition at the void boundary and undergo very large gradients near the interface." Such an explanation is misleading and incorrect. The poor solution of VCFEM-HS-PCE is due to the fact that polynomial Airy stress function is highly incomplete for problems in a doubly-connected domain. In order to improve the performance of VCFEM-HS-PCE, biharmonic functions which are complete in a doubly connected domain can be used as Airy stress functions to construct independently assumed stress fields, instead of using the complicated and misleading approach in [Moorthy and Ghosh (1996)].
Secondly, in order to develop the element stiffness matrix of VCFEM-HS-PCE, domain integration is needed in addition to boundary integrations. Domain integration necessitates dividing the domain of element into triangles/quadrangles. In addition, for elements with inclusions/voids, a large number of stress modes are necessary.

Therefore, high-order domain integration increases the computational burden significantly. For this reason, VCFEM-TTs are computationally much more efficient than VCFEM-TT-CEVP.

In addition, the Lagrange multipliers involved in variational principles (53)(56)(58) make VCFEM-HS-PCE plagued by LBB conditions. Because the incompleteness of Airy stress functions, it is very likely that assumed stress fields are not able to satisfy LBB conditions even if a large number stress modes are used. Inability of satisfying LBB conditions leads to rank-deficiency of the stiffness matrix and the instability of finite element solutions, as observed in [Moorthy and Ghosh (1996)] for porous media. On the other hand, due to the completeness of assumed displacement fields in VCFEM-TT-BVP, rank-deficiency rarely happen as long as enough number of T-Trefftz basis functions are used. Furthermore, because the development of VCFEM-TT-C does not involve Lagrange multipliers, VCFEM-TT-C does not suffer from LBB conditions. This is a significant advantage of VCFEM-TT-C over the other two classes of VCFEMs.
Finally, for VCFEM-HS-PCE, approximating the inner-boundary $\partial \Omega_{c}^{e}$ with line segments invokes further errors of solutions. Assuming linear displacement field along each segment also makes the final stiffness matrix rank-deficient. Three zero energy modes need to be suppressed, which are the modes where the inner boundary is undergoing rigid-body displacements while the outer-boundary stays still. Therefore, the rigid-body displacements of the inner boundary should be eliminated beforehand, or constrained to be the same of the rigid-body displacements of the outer-boundary, using Lagrange multipliers or penalty method, leading to complicated finite element formulations.

## 6 Numerical Examples

We compare the performances of different VCFEMs by conducting numerical experiments. All codes are programed using MATLAB in a 64 -bit WINDOWS operating system, and executed on a PC computer equipped with Intel Q8300 2.5GHZ CPU, and 8G system memory.
Firstly, we illustrate the reason why we use characteristic lengths to scale the TTrefftz trial functions. A pentagonal element with nodal coordinates $(-100,-100)$, $(100,-100),(100,100),(0,250),(-100,-100)$ is used, see Fig. 5. Plane stress case is considered. An elliptical inclusion/void is present in the element. Material properties of the matrix are $E_{m}=1, v_{m}=0.25$. Three kinds of heterogeneities are considered: an elastic inclusion with $E_{c}=2, v_{c}=0.3$, a rigid inclusion, and a void. Stiffness matrices of VCFEM-TT-C are computed, with and without using characteristic lengths to scale T-Trefftz trial functions. Condition numbers of the coeffi-


Figure 5: A pentagonal element with elliptical inclusion/void used for condition number test, eigenvalue test, CPU time comparison and patch test
cient matrix of equations (42)(44)(46) are shown in Tab. 1. We can clearly see that by scaling the T-Trefftz functions using characteristic lengths defined in section 2, the resulting systems of equations have significantly smaller condition number. Although not shown here, scaling T-Trefftz trial functions using characteristic lengths also has similar effect on VCFEM-TT-BVP as well as VCFEM-HS-PCE. In the following examples, the characteristic length as defined in section 2 is always used.

Table 1: Condition number of coefficient matrices of equations (42)(44)(46) used to relate $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to $\mathbf{q}$ with/without using characteristic lengths to scale T-Trefftz Trial functions for the element shown in Fig. 5

|  | Elastic Inclusion |  | Rigid Inclusion |  | Void |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Characteristic <br> Length | Scaled | Not <br> scaled | Scaled | Not <br> scaled | Scaled | Not <br> scaled |
| Condition number | $1.0 \times 10^{4}$ | $1.7 \times 10^{33}$ | $1.5 \times 10^{3}$ | $2.3 \times 10^{33}$ | $1.1 \times 10^{4}$ | $4.5 \times 10^{34}$ |

Using the same element shown in Fig. 5, we compute the eigenvalues of element stiffness matrices of different VCFEMs. This is conducted in the original and rotated global Cartesian coordinate system. Experimental results are shown in Tab. 2-4.
As can clearly be seen, these elements are stable and invariant for this regular el-
ement, because additional zero energy modes do not exist, and eigenvalues do not vary with respect to change of coordinate systems. However, this does not mean that LBB conditions are satisfied by VCFEM-HS-PCE or VCFEM-TT-BVP for an arbitrary element.
In addition, we can see that for elements with elastic/rigid inclusions, eigenvalues of all these three classes of VCFEMs are have no major differences. This indicates that these elements will give somehow similar solutions of nodal displacements. On the other hand, when a void is present, the eigenvalues of VCFEM-HS-PCE are obviously lower than that of VCFEM-TT-BVP and VCFEM-TT-C. This indicates that the nodal solutions of VCFEM-HS-PCE will be quite different from that of VCFEM-TTs when a void is present. This expectation is confirmed in some following examples. However, as shown later, no matter an elastic/rigid inclusion or a void is considered, computed stress field by VCFEM-HS-PCE is always poor, compared to highly accurate solutions of VCFEM-TTs.

Table 2: Eigenvalues of stiffness matrices of different VCFEMs when an elastic inclusion is considered

| Eigenvalues <br> Rotation $=0^{\circ} \& 45^{\circ}$ | VCFEM-TT-BVP | VCFEM-TT-C | VCFEM-HS-PCE <br> [Ghosh et al.] |
| :---: | :---: | :---: | :---: |
| 1 | 1.8027 | 1.8078 | 1.7926 |
| 2 | 0.9228 | 0.9240 | 0.9185 |
| 3 | 0.7522 | 0.7539 | 0.7485 |
| 4 | 0.6367 | 0.6428 | 0.6335 |
| 5 | 0.6027 | 0.6081 | 0.6008 |
| 6 | 0.4836 | 0.4846 | 0.4828 |
| 7 | 0.2139 | 0.2222 | 0.2120 |
| 8 | 0.0000 | 0.0000 | 0.0000 |
| 9 | 0.0000 | 0.0000 | 0.0000 |
| 10 | 0.0000 | 0.0000 | 0.0000 |

We also compare the CPU time required for computing stiffness matrix of the element in Fig. 5, using different VCFEMs. The CPU time required for each element is normalized to that of VCFEM-HS-PCE, and is listed in Tab. 5. Only the case of an elastic inclusion is considered, but results of other cases also follow a similar pattern. As can be seen, VCFEM-TT-C is computationally the most efficient. And VCFEM-HS-PCE is computationally the most expensive. As explained in previous sections, this is due to: VCFEM-HS-PCE necessitates domain integrations as well as boundary integrations; VCFEM-TT-BVP needs integrations over the outer- as

Table 3: Eigenvalues of stiffness matrices of different VCFEMs when a rigid inclusion is considered

| Eigenvalues <br> Rotation $=0^{\circ} \& 45^{\circ}$ | VCFEM-TT-BVP | VCFEM-TT-C | VCFEM-HS-PCE <br> [Ghosh et al.] |
| :---: | :---: | :---: | :---: |
| 1 | 1.9510 | 1.9376 | 1.9005 |
| 2 | 1.0143 | 1.0089 | 0.9839 |
| 3 | 0.8205 | 0.8034 | 0.8060 |
| 4 | 0.6694 | 0.6710 | 0.6590 |
| 5 | 0.6299 | 0.6171 | 0.6196 |
| 6 | 0.4994 | 0.4967 | 0.4969 |
| 7 | 0.2192 | 0.2259 | 0.2140 |
| 8 | 0.0000 | 0.0000 | 0.0000 |
| 9 | 0.0000 | 0.0000 | 0.0000 |
| 10 | 0.0000 | 0.0000 | 0.0000 |

Table 4: Eigenvalues of stiffness matrices of different VCFEMs when a void is considered

| Eigenvalues <br> Rotation $=0^{\circ} \& 45^{\circ}$ | VCFEM-TT-BVP | VCFEM-TT-C | VCFEM-HS-PCE <br> [Ghosh et al.] |
| :---: | :---: | :---: | :---: |
| 1 | 1.3263 | 1.4147 | 1.1600 |
| 2 | 0.7086 | 0.7860 | 0.6152 |
| 3 | 0.6057 | 0.6169 | 0.4982 |
| 4 | 0.4594 | 0.5312 | 0.3648 |
| 5 | 0.3596 | 0.4603 | 0.2383 |
| 6 | 0.3242 | 0.3976 | 0.2013 |
| 7 | 0.1835 | 0.2096 | 0.1238 |
| 8 | 0.0000 | 0.0000 | 0.0000 |
| 9 | 0.0000 | 0.0000 | 0.0000 |
| 10 | 0.0000 | 0.0000 | 0.0000 |

well as the inner boundary; for VCFEM-TT-C, integrations merely over the outer boundary are needed.
We also conduct the one-element patch test. The element shown in Fig. 5 is considered. The materials of the matrix and the inclusion are the same, with material properties $E=1, v=0.25$. A uniform traction is applied to the upper edges. The vertical displacement of node 1 and 2 are prescribed to that of the exact solution. The horizontal displacement of node 1 is also prescribed to that of the exact solu-

Table 5: CPU time required for computing the stiffness matrix of different VCFEMs for the element in Fig. 5 with an elastic inclusion considered

| Normalized <br> CPU Time | VCFEM-TT-BVP | VCFEM-TT-C | VCFEM-HS-PCE <br> [Ghosh et al.] |
| :---: | :---: | :---: | :---: |
|  | 0.59 | 0.34 | 1.00 |

tion. The exact solution is that of the uniform tension problem:
$u_{x}=-\frac{P v}{E} x$
$u_{y}=\frac{P}{E} y$
The error is defined as follows:
Error $=\frac{\left\|\mathbf{q}-\mathbf{q}^{\text {exact }}\right\|}{\left\|\mathbf{q}^{\text {exact }}\right\|}$
where $\mathbf{q}$ and $\mathbf{q}^{\text {exact }}$ are the computed and exact nodal displacement vector of the element. And $|||\mid$ represents the 2-norm. Experimental results are shown in Tab. 6. As can be seen, VCFEM-HS and VCFEM-TT-BVP can pass the patch test with errors equal or less than an order of $10^{-8}$. Although the error for VCFEM-TT-C is much larger, but still in an order of $10^{-4}$. We consider the performance of all VCFEMs to be satisfactory in this one-element patch test.

Table 6: Performances of different VCFEMs in patch test

| Error | VCFEM-TT-BVP | VCFEM-TT-C | VCFEM-HS-PCE [Ghosh et al.] |
| :---: | :---: | :---: | :---: |
|  | $5.4 \times 10^{-8}$ | $6.6 \times 10^{-4}$ | $6.0 \times 10^{-10}$ |

In order to evaluate the overall performances of different VCFEMs to model problems with inclusions or voids, we consider the following problem. An infinite plate with a circular elastic/rigid inclusion or hole is subject to under remote tension $P$. The raidus of the circular inclusion/hole is $R$. Exact solution of this problem can be found in [Muskhelishvil (1954)]. When an elastic inclusion is considered, the displacement field in the matrix is:
$u_{r}^{m}=\frac{P}{8 G_{m} r}\left\{\left(\kappa_{m}-1\right) r^{2}+2 \gamma_{m} R^{2}+\left[\beta_{m}\left(\kappa_{m}+1\right) R^{2}+2 r^{2}+\frac{2 \delta_{m} R^{4}}{r^{2}}\right] \cos 2 \theta\right\}$
$u_{\theta}^{m}=-\frac{P}{8 G_{m} r}\left\{\beta_{m}\left(\kappa_{m}+1\right) R^{2}+2 \gamma_{m}-\frac{2 \delta_{m} R^{4}}{r^{2}}\right\} \sin 2 \theta$


Figure 6: An infinite plate with a circular elastic/rigid inclusion or hole under remote tension

The displacement field is the inclusion is:

$$
\begin{align*}
& u_{r}^{c}=\frac{\operatorname{Pr}}{8 G_{c}}\left\{\beta_{c}\left(\kappa_{c}-1\right)+\left[\frac{\gamma_{c}\left(\kappa_{c}-3\right)}{R^{2}} r^{2}+2 \delta_{c}\right] \cos 2 \theta\right\}  \tag{63}\\
& u_{\theta}^{c}=\frac{\operatorname{Pr}}{8 G_{c}}\left\{\frac{\gamma_{c}\left(\kappa_{c}+3\right)}{R^{2}} r^{2}-2 \delta_{c}\right\} \sin 2 \theta
\end{align*}
$$

where

$$
\begin{align*}
\beta_{m} & =-\frac{2\left(G_{c}-G_{m}\right)}{G_{m}+G_{c} \kappa_{m}} \\
\gamma_{m} & =\frac{G_{m}\left(\kappa_{c}-1\right)-G_{c}\left(\kappa_{m}-1\right)}{2 G_{c}+G_{m}\left(\kappa_{c}-1\right)} \\
\delta_{m} & =\frac{G_{c}-G_{m}}{G_{m}+G_{c} \kappa_{m}}  \tag{64}\\
\beta_{c} & =\frac{G_{c}\left(\kappa_{m}+1\right)}{2 G_{c}+G_{m}\left(\kappa_{c}-1\right)} \\
\gamma_{c} & =0 \\
\delta_{c} & =\frac{G_{c}\left(\kappa_{m}+1\right)}{G_{m}+G_{c} \kappa_{m}}
\end{align*}
$$

When we consider the limit cases of $G_{c}=\infty$ or $G_{c}=0$, the displacement field in the matrix as shown in (62) will be the that of the case where an rigid inclusion or a void is present in this infinite plate.

A plane stress case is considered. The material properties of the matrix are $E_{m}=$ $1, v_{m}=0.25$. When an elastic inclusion is considered, the material properties of the inclusion are $E_{c}=2, v_{c}=0.3$. The magnitude of the remote tension $P$ is equal to 1 . The radius of the inclusion/void is 0.1 . For numerical implementation, the infinite plate is truncated to a finite square plate. The length of each side of the truncated plate is equal to2. For all three cases with an elastic/rigid inclusion or a void, only one element is used, as shown in Fig. 6. Traction boundary conditions are applied to the outer-boundary of the element. The horizontal displacement of node 1 and node 4 , and the vertical displacement of node 4 are prescribed to that of the exact solution.
Computed horizontal and vertical displacement of node $2\left(u_{2}, v_{2}\right)$ are shown in Tab. 7. This is consistent with our previous analysis of element eigenvalues. When a elastic/rigd inclusion is considered, the numerical error given by VCFEM-HSPCE on nodal displacements is still somehow acceptable. However, when a void is considered, VCFEM-HS-PCE of Ghosh et al. gives meaningless solutions. On the other hand, the accuracies of VCFEM-TT-BVP and VCFEM-TT-C are always satisfactory.

Table 7: Computed horizontal and vertical displacements of node 2, for the problem shown in Fig. 6

|  | Elastic Inclusion |  | Rigid Inclusion |  | Void |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $u_{2}$ | $v_{2}$ | $u_{2}$ | $v_{2}$ | $u_{2}$ | $v_{2}$ |
| VCFEM-TT-BVP | 0.9947 | 0.2480 | 0.9866 | 0.2446 | 1.0268 | 0.2634 |
| VCFEM-TT-C | 0.9947 | 0.2480 | 0.9866 | 0.2446 | 1.0268 | 0.2635 |
| VCFEM-HS-PCE <br> [Ghosh et al.] | 1.0589 | 0.1855 | 1.0567 | 0.1828 | 2.0713 | 0.9333 |
| Exact Solution | 0.9986 | 0.2501 | 0.9964 | 0.2502 | 1.0069 | 0.2506 |

We also compare the computed $\sigma_{11}$ along axis $x_{2}, \sigma_{22}$ along axis $x_{1}$, to that of the exact solution. As shown in Fig. 7-9, no matter an elastic inclusion, a rigid inclusion, or a void is considered, computed stresses by VCFEM-HS-PCE of Ghosh et al. is far different from the exact solution. In contrast, VCFEM-TTs always give very accurate computed stresses, even though only one element is used. For this reason, we consider VCFEM-TTs to be much more suitable than VCFEM-HSPCE for multi-scale modeling of heterogeneous materials, where stress field in the macro- as well as in the micro- scale are both of high importance.
In the last example, we determine the homogenized elastic material properties of composite or porous media using different VCFEMs. Aluminum matrix is consid-


Figure 7: Computed $\sigma_{11}$ along axis $x_{2}, \sigma_{22}$ along axis $x_{1}$ for the problem in Fig. 6, with an elastic inclusion


Figure 8: Computed $\sigma_{11}$ along axis $x_{2}, \sigma_{22}$ along axis $x_{1}$ for the problem in Fig. 6, with a rigid inclusion


Figure 9: Computed $\sigma_{11}$ along axis $x_{2}, \sigma_{22}$ along axis $x_{1}$ for the problem in Fig. 6, with a void


Figure 10: Three types of RVEs: (a) a unit-cell model with a circular inclusion/void, (b) 25 VCFEMs with random circular inclusions/voids, (c) 25 VCFEMs with random elliptical inclusions/voids
ered with $E_{m}=68.3 \mathrm{GPa}, v_{m}=0.3$. Three types of heterogeneities are considered: with Boron inclusions, with rigid inclusions, and with voids. The material properties of Boron inclusions are: $E_{c}=379.3 \mathrm{GPa}, v_{c}=0.1$. The volume fraction of inclusions/voids are $20 \%$.
As shown in Fig. 10, three types representative volume element (RVE) are used to determine homogenized material properties: (a) a unit-cell model with a circular inclusion/void, (b) 25 VCFEMs with random circular inclusions/voids, (c) 25 VCFEMs with random elliptical inclusions/voids. For each RVE, a uniform ten-
sion is applied in the vertical direction. Periodic boundary conditions are applied, which constrains each edge to stay in a straight line after deformation. Plane stress problem is considered. The homogenized in-plane elastic modulus is determined by dividing the tensile force with the extension in the vertical direction. And the homogenized Poisson's ratio is computed as the ratio of the contraction in the horizontal direction and the extension in the vertical direction. Numerical results using different elements are listed in Tab. 8-10.
As can be seen, when Boron inclusions or rigid inclusions are considered, homogenized Young's modulus and Poisson's ratio are very close using different VCFEMs. But when voids are present instead of inclusions, VCFEM-HS-PCE of Ghosh et al. gives significantly different results from that of VCFEM-TT-BVP and VCFEM-TTC. This observation consistent with that of the plate with inclusion/void problem as shown in Fig. 6. When an elastic/rigid inclusion is considered, VCFEM-HS-PCE can give acceptable nodal displacements. However, the displacement field solution of VCFEM-HS-PCE is meaningless. Because VCFEM-HS-PCE also always gives very poor results of computed stress field, we consider that VCFEM-TTs are much more suitable for micromechanical modeling of composite and porous materials.

Table 8: Computed material properties using RVEs in Fig. 9, with Boron inclusions

| RVE | Element Type | Young's Modulus (GPa) | Poisson's Ratio |
| :---: | :---: | :---: | :---: |
|  | VCFEM-TT-BVP | 88.20 | 0.2745 |
|  | VCFEM-TT-C | 88.17 | 0.2745 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 87.88 | 0.2733 |
| $(b)$ | VCFEM-TT-BVP | 88.05 | 0.2739 |
|  | VCFEM-TT-C | 89.88 | 0.2783 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 86.69 | 0.2807 |
| $(c)$ | VCFEM-TT-BVP | 89.01 | 0.2812 |
|  | VCFEM-TT-C | 88.13 | 0.2852 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 87.24 | 0.2819 |

## 7 Conclusions

T-Trefftz Voroni Cell Finite Elements (VCFEM-TTs) with elastic/rigid inclusions or voids are developed in this study, for micromechanical modeling of composite and porous materials. For each element, in addition to assuming an interelement compatible displacement field along the element outer-boundary, independent displacement fields in the matrix as well as in the inclusion are assumed as characteristic-length-scaled T-Trefftz trial functions. Two approaches are used

Table 9: Computed material properties using RVEs in Fig. 9, with rigid inclusions

| $R V E$ | Element Type | Young's Modulus (GPa) | Poisson's Ratio |
| :---: | :---: | :---: | :---: |
|  | VCFEM-TT-BVP | 97.31 | 0.2827 |
|  | VCFEM-TT-C | 97.22 | 0.2832 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 97.96 | 0.2753 |
| $(b)$ | VCFEM-TT-BVP | 99.31 | 0.2847 |
|  | VCFEM-TT-C | 96.72 | 0.2907 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 95.18 | 0.2904 |
| $(c)$ | VCFEM-TT-BVP | 99.41 | 0.2981 |
|  | VCFEM-TT-C | 95.03 | 0.3241 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 96.53 | 0.2935 |

Table 10: Computed material properties using RVEs in Fig. 9, with void

| RVE | Element Type | Young's Modulus (GPa) | Poisson's Ratio |
| :---: | :---: | :---: | :---: |
|  | VCFEM-TT-BVP | 42.23 | 0.2656 |
|  | VCFEM-TT-C | 42.17 | 0.2669 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 36.56 | 0.3750 |
| $(b)$ | VCFEM-TT-BVP | 41.91 | 0.2822 |
|  | VCFEM-TT-C | 41.89 | 0.2804 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 32.43 | 0.4325 |
| $(c)$ | VCFEM-TT-BVP | 40.99 | 0.2976 |
|  | VCFEM-TT-C | 42.56 | 0.2715 |
|  | VCFEM-HS-PCE [Ghosh et al.] | 31.91 | 0.4342 |

alternatively to develop element stiffness matrices. VCFEM-TT-BVP uses multifield boundary variational principles to enforce all the conditions in a variational sense. On the other hand, VCFEM-TT-C uses collocation method to relate independently assumed displacement fields to nodal displacements, and develop finite element equations based on a primitive-field boundary variational principle.
Through numerical examples, it can be clearly seen both of these two classes of elements are much better than VCFEM-HS-PCE, which are the elements developed by [Ghosh and Mallett (1994); Ghosh, Lee and Moorthy (1995)]. In contrast to the high accuracy of VCFEM-TTs, VCFEM-HS-PCE of Ghosh et al. always gives very poor solutions of stress distribution in the element, simply because the polynomial Airy stress function is highly incomplete for problems in a doublyconnected domain. VCFEM-TTs are also computationally much more efficient than VCFEM-HS-PCE of Ghosh et al., because domain integrations are avoided.

Among VCFEM-TTs, because VCFEM-TT-C is simple, the most efficient, and do not suffer from LBB conditions, we consider this class of elements to be very useful for micromechanical modeling of composite and porous materials.
Although the present work is conducted in the context of two-dimensional linear elastic solid mechanics, extension to three-dimensional problems and geometrical as well as material nonlinear problems is quite straight-forward. This will be reserved for future study.

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