

## On application of the Stochastic Finite Volume Method in Navier-Stokes problems

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**Abstract:** The main aim of this article is numerical solution of the fully coupled Navier-Stokes equations with Gaussian random parameters. It is provided thanks to the specially adopted Finite Volume Method, modified using the generalized stochastic perturbation technique. This Stochastic Finite Volume Method is applied to model 3D problem with uncertainty in liquid viscosity and a coefficient of the heat conduction, separately. Probabilistic moments and characteristics of up to the fourth order are determined with the use of the Response Function Method realized numerically via the polynomial interpolation. Although mathematical formulation of the SFVM has been proposed in addition to the problems including single random variable, it is possible to extend it towards multi-parametric cases, also for the correlated random variables. Inclusion of the boundary conditions in both global and local sense is quite straightforward but needs some prior experimental basic statistics.

**Keywords:** Finite Volume Method, stochastic perturbation technique, Navier-Stokes equations, symbolic computing

### 1 Introduction

Computational solution of the fully coupled Navier-Stokes equations is still really challenging problem, especially when defined in terms of random or, especially, stochastic coefficients. One may find an application of the Monte-Carlo simulation technique or some polynomial chaos expansions, but they minor points are the enormously large total time of simulation or unavailability of higher than the second order output statistics [Ghanem and Spanos (2002), Kleiber and Hien (1992), Xiu (2007)]. This is the reason to apply, quite independently from the numerical discrete strategy, the generalized stochastic perturbation technique, where previous limitation to small input random dispersions is eliminated through an application of

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any order expansions [Kamiński (2010), Kamiński and Carey (2005), Kleiber and Hien (1992)]. On the other hand, instead of a time consuming implementation of the Direct Differentiation Method (DDM), the Response Function Method (RFM) is preferred, so that instead of up to the  $n$ th order coupled Navier-Stokes equations we solve for polynomial approximations of the state functions relating the PVT solution with the input random variable(s). This approximation is proposed here in a local sense – the response functions for velocities, pressures and temperatures may be different in each discrete point of the computational domain. This idea is connected here with the classical deterministic formulation of the Finite Volume Method (FVM). As it is known, this method (FVM) is one of the available computer methods for evaluation of the partial differential equations as a system of the algebraic equations [Schäfer (2006)] that can be stochastically extended in a similar way. A very useful property of the FVM is that the balance principles, which are the basis for the mathematical modeling of continuum mechanical problems, per definition, are also fulfilled for the discrete equations. A starting point for the FVM is a decomposition of the problem domain into both regular and irregular control volumes (CVs), where each CV is represented by its midpoint only. This is the main difference to the Finite Element Method (FEM), where the equilibrium equations are formed and solved in the nodal points of the mesh only, which are located in the corners (and midpoints for higher order approximations) of each finite element. Let us note that, quite similarly to the FEM [9], there are (a) regular and irregular points of the mesh (grid), quite similar to those applied in the Finite Difference Methods, (b) triangular, quadrilateral as well as the polygonal grids, both structured and unstructured in plane FVM discretizations [Bailey and Cross (1995)], as well as (c) - some volumetric divisions using cubes or various polyhedra, for instance, where the mass center of such a 3D sub-domain may be treated as the discrete point in the FVM grid. Nowadays, the Finite Volume Method has well documented applications in classical linear elasticity [Bijelonja et al. (2006), Onate et al. (1994)] also for complex domains [Demirdzic and Muzafarjija (1994)], various plate types bending analysis [Wheel (1997)], thermo-elasto-plasticity [Demirdzic and Martinovic (1993)], even elasto-visco-plasticity [Taylor et al. (1995)] and for the dynamic fracture problems solutions [Ivankovic (1994)]. Special importance applications of the FVM are of course in flow problems, where is used on unstructured grids for the compressible flows [Cueto-Felgueroso et al. (2007)], with moving meshes [Demirdzic and Muzafarjija (1995)], also for lubrication processes modeling [Durany et al. (2006)].

Computational analysis is provided in a hybrid way here – the FVM freeware code OpenFVM is engaged to solve all N-S problems necessary to build up the response functions. The internal symbolic polynomial interpolation of the system MAPLE

accompanied with the perturbation-based formulas implemented in this program lead to the final probabilistic moments of the fluid state. Numerical visualization is carried out in the freeware FEPlot used before for the FEM and FDM output files and procedures. Computational illustration deals with the compressible fluid flow in a cubic domain and this flow proceeds with two Gaussian input random variables – heat conductivity coefficient and, separately, fluid viscosity. We compute twice up to fourth order probabilistic characteristics of the PVT solution to validate an importance of both physical parameters. Although those input parameters are state-independent, further extension of the proposed SFVM towards numerical modeling of nonlinear, i.e. temperature-dependent, systems will be also possible.

## 2 The Generalized perturbation method

Let us consider the random variable  $b$  being some sigma algebra defined uniquely by  $(\Omega, F, P)$ , where  $\Omega$  denotes the set of possible realizations of  $b$ ,  $F$  is some real valued function, while  $P$  is the probability measure. The expected value of  $b$  is defined as

$$E(b) = \int_{\Omega} b dP \quad (1)$$

assuming that the right hand side integral exists. Let us further define the probability density function (PDF) of the variable  $b$  by  $p_b(x)$ , so that the above expectation can be rewritten as

$$E(b) = \int_{-\infty}^{+\infty} x p_b(x) dx \quad (2)$$

assuming no additional truncation on this variable. Further, one can define the central probabilistic moment of the  $m^{th}$  order for the variable  $b$  as

$$\mu_m(b) = \int_{-\infty}^{+\infty} (b - E[b])^m p_b(x) dx. \quad (3)$$

Let us note that  $b$  represents further some physical parameters of the system as well as their state functions like temperatures, pressures or fluid velocities and usually has arbitrarily chosen Gaussian distribution truncated according to its physical meaning.

As it is known [Kamiński (2010), Kamiński and Carey (2005), Kleiber and Hien (1992)], the basic idea of the stochastic perturbation approach follows the classical

perturbation expansion idea and is based on approximation of all input variables, and the state functions of the problem via the truncated Taylor series about their spatial expectations in terms of a parameter  $\varepsilon > 0$ . For example, in the case of the heat conductivity  $k$ , the  $n$ th order truncated expressions are written as

$$k = k^0 + \frac{\partial k}{\partial b} \Delta b + \frac{1}{2} \varepsilon^2 \frac{\partial^2 k}{\partial b^2} (\Delta b)^2 + \dots + \frac{1}{n!} \varepsilon^n \frac{\partial^n k}{\partial b^n} (\Delta b)^n, \tag{4}$$

where the  $n$ th - order variation is expressed as

$$\varepsilon^n (\Delta b)^n = \varepsilon^n (b - b^0)^n. \tag{5}$$

The symbol  $(.)^0$  represents the value of the function  $(.)$  taken at the expectations  $b^0$ . Then, the fluid velocity may be expanded as

$$v_i = k^0 + \varepsilon \frac{\partial v_i}{\partial b} \Delta b + \frac{1}{2} \varepsilon^2 \frac{\partial^2 v_i}{\partial b^2} (\Delta b)^2 + \dots + \frac{1}{n!} \varepsilon^n \frac{\partial^n v_i}{\partial b^n} (\Delta b)^n. \tag{6}$$

Traditionally, the stochastic perturbation approach to all the physical problems is entered by the respective perturbed equations of the  $0^{th}$ ,  $1^{st}$  and successively higher orders being a modification of the relevant variational integral formulation. It is well known from the SFEM formulations that the system of linear algebraic equations, which is the basis of the model, i.e.

$$K_{\alpha\beta}(b) T_{\beta}(b) = Q_{\alpha}(b), \quad \alpha, \beta = 1, \dots, N, \tag{7}$$

where  $K_{\alpha\beta}(b)$  is the heat conductivity matrix,  $T_{\beta}(b)$  denotes the discrete temperatures vector in the system, while  $Q_{\alpha}(b)$  is the heat flux vector, may be transformed into the following systems of recursive equations:

$$\begin{cases} K_{\alpha\beta}^0 T_{\beta}^0 = Q_{\alpha}^0 \\ (\dots) \\ \sum_{k=0}^n \binom{n}{k} \frac{\partial^k K_{\alpha\beta}}{\partial b^k} \frac{\partial^{(n-k)} T_{\beta}}{\partial b^{(n-k)}} = \frac{\partial^n Q_{\alpha}}{\partial b^n} \end{cases} \tag{8}$$

In order to calculate the expected values and higher order probabilistic moments of displacements, strains and stresses functions, the same Taylor expansion is employed to the definitions of probabilistic moments calculated for any random state variables assuming their continuous character. Therefore, the most important first two probabilistic moments of these functions are derived from the definition; the expectation equals

$$E [T(b)] = T^0(b) + \frac{1}{2} \varepsilon^2 \frac{\partial^2 T}{\partial b^2} \mu_2(b) + \frac{1}{4!} \varepsilon^4 \frac{\partial^4 T}{\partial b^4} \mu_4(b) + \frac{1}{6!} \varepsilon^6 \frac{\partial^6 T}{\partial b^6} \mu_6(b) + \dots + \frac{1}{(2m)!} \varepsilon^{2m} \frac{\partial^{2m} T}{\partial b^{2m}} \mu_{2m}(b), \tag{9}$$

where the odd order terms are equal to zero for the Gaussian random deviates. In the case of a single Gaussian input random variable  $b$ , the generalized expansion given in Eq. (4) simplifies of course to Eq. (9) with  $\mu_{2m}$  being the central  $2m^{th}$  probabilistic moment. Using this extension of the random output, a desired efficiency of the expected values can be achieved by the appropriate choice of  $m$  and  $\epsilon$  corresponding to the input probability density function type, probabilistic moment interrelations, acceptable error of the computations, etc.; this choice can be made by the comparative studies with Monte-Carlo simulations or theoretical results obtained by direct symbolic integration. A treatment similar to that from above leads to the following result for the variance of any state function

$$\begin{aligned} Var(T) = & \left(\frac{\partial T}{\partial b}\right)^2 \mu_2(b) + \left(\frac{1}{4}\left(\frac{\partial^2 T}{\partial b^2}\right)^2 + \frac{2}{3!}\frac{\partial T}{\partial b}\frac{\partial^3 T}{\partial b^3}\right) \mu_4(b) + \\ & + \left(\left(\frac{1}{3!}\right)^2\left(\frac{\partial^3 T}{\partial b^3}\right)^2 + \frac{1}{4!}\frac{\partial^4 T}{\partial b^4}\frac{\partial^2 T}{\partial b^2} + \frac{2}{5!}\frac{\partial^5 T}{\partial b^5}\frac{\partial T}{\partial b}\right) \mu_6(b). \end{aligned} \tag{10}$$

Consecutively, the  $m^{th}$  order probabilistic moment for the structural response function in the  $10^{th}$  order stochastic Taylor expansion is introduced as

$$\begin{aligned} \mu_m(T(b)) = & \int_{-\infty}^{+\infty} \left(T^0(b) + \sum_{i=1}^n \frac{\epsilon^i}{i!} \Delta b^i \frac{\partial^i T(b)}{\partial b^i} - E[T(b)]\right)^m p(b) db = \\ \cong & \int_{-\infty}^{+\infty} \left(\epsilon \Delta b \frac{\partial T(b)}{\partial b} + \epsilon^2 \frac{(\Delta b)^2}{2!} \frac{\partial^2 T(b)}{\partial b^2} + \dots + \frac{\partial^{10} T(b)}{\partial b^{10}} \epsilon^{10} \frac{(\Delta b)^{10}}{10!}\right)^m p(b) db. \end{aligned} \tag{11}$$

It is necessary to point out that this methodology is valid for a single random variable with any probability density function; further simplifications may be obtained by a specification of this PDF. This methodology will essentially change in the case of random field as well as of two and more correlated random variables.

### 3 Navier-Stokes equations

The system of basic equilibrium equations to be extended towards stochastic analysis and to be solved numerically can be written with boundary conditions as follows:

$$\rho \left( \frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) = \sigma_{i,j} + f_i^B, \tag{12}$$

$$v_{i,i} = 0, \tag{13}$$

$$\sigma_{ij} = -p \delta_{ij} + 2\mu \epsilon_{ij}, \tag{14}$$

$$\rho c \left( \frac{\partial \Theta}{\partial t} + \Theta_{,i} v_j \right) = (k \Theta_{,i})_{,i} + q^B, \tag{15}$$

where

$$\epsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{16}$$

State variables in Eqs. (12-16) show successively the velocities and pressures in the analyzed liquid, strain tensor, stress tensor as well as the temperature distribution. Let us assume the following boundary conditions consecutively for the velocity

$$v_i = \bar{v}_i; \quad x \in \partial\Omega_v, \tag{17}$$

the stress

$$\sigma_{ij}n_j = \hat{f}_i; \quad x \in \partial\Omega_\sigma, \tag{18}$$

for the temperature

$$\Theta = \bar{\Theta}; x \in \partial\Omega_\Theta \tag{19}$$

and for the heat flux

$$k \frac{\partial \Theta}{\partial t} = \bar{q}; x \in \partial\Omega_q. \tag{20}$$

For the numerical solution of differential equations above we apply variational formulation, where the equations are numerically integrated over the given volume  $\Omega$ . This operation allows to obtain the starting equations of thermodynamic equilibrium in the following notation:

$$\int_{\Omega} \delta v_i \rho (\dot{v}_i + v_{i,j} v_j) d\Omega + \int_{\Omega} \delta \epsilon_{ij} (2\mu \epsilon_{ij} - p \delta_{ij}) d\Omega = \int_{\Omega} \delta v_i f_i^B d\Omega + \int_{\partial\Omega} \delta v_i \bar{f}_i d(\partial\Omega_\sigma). \tag{21}$$

$$\int_{\Omega} \delta p v_{i,i} d\Omega = 0, \tag{22}$$

$$\int_{\Omega} \delta \Theta \rho c_p (\dot{\Theta} + \Theta_{,i} v_i) d\Omega + \int_{\Omega} k \delta \Theta_{,i} \Theta_{,i} d\Omega = \int_{\Omega} \delta \Theta q^B d\Omega + \int_{\partial\Omega} \delta \bar{\Theta} \bar{q} d(\partial\Omega). \tag{23}$$

Eqs. (21-23) are consecutively transformed using the perturbation method and discretized by the FVM scheme for the numerical solution of the problem coupled with random parameters.

## 4 The Stochastic Finite Volume Method

### 4.1 FVM discretization

We present for a simplicity the FVM basics for the heat conduction problem reduced from the heat transfer [Carslaw and Jaeger (1986)] in Eq. (15) together with the relevant boundary conditions. Let us consider further some continuous temperature variations  $\delta T(x_i)$  defined in the interior of the region  $\Omega$  and vanishing on  $\partial\Omega_T$ . The variational formulation of this problem is proposed as

$$\int_{\Omega} (\rho c \dot{T} \delta T + k_{ij} T_{,j} \delta T_{,i} - g \delta T) d\Omega - \int_{\partial\Omega_q} \bar{q} \delta T d(\partial\Omega) = 0; x_i \in \Omega; \tau \in [0, \infty). \quad (24)$$

Assuming simple heat conduction problem, where  $\dot{T} = 0$  for  $\tau \in [0, \infty)$  and dropping off the internal heat generation one obtains

$$\int_{\Omega} k_{ij} T_{,j} \delta T_{,i} d\Omega - \int_{\partial\Omega_q} \bar{q} \delta T d(\partial\Omega) = 0; x_i \in \Omega; \tau \in [0, \infty). \quad (25)$$

The Finite Volume Method discretization of this equation proceeds using the complete partition of the entire computational domain into the finite set of the non-overlapping control volumes. Then, considering the homogeneous domain, Eq. (27) takes the following form:

$$\sum_{\beta=1}^{n_f} \int_{S_{\beta}} k \nabla T ds = \sum_{\beta=1}^{n_B} \int_{S_{\beta}} \hat{q} ds. \quad (26)$$

The left hand side integration over the entire computational domain is replaced with the surface integrals over the external surfaces of all finite volumes. Further, the following discretization is used with respect to the continuous temperature field:

$$T(r) = T_{\alpha} + (\nabla T)_{\alpha} (r - r_{\alpha}), \quad (27)$$

where  $r_{\alpha}$  is the position vector of the  $\alpha$ th finite volume center (see Fig. 1).

Further, there holds

$$(\nabla T)_{\alpha} = D^{-1} \cdot c, \quad (28)$$

for

$$D = \sum_{\beta=1}^{n_f} (r_{\beta} - r_{\alpha})^T (r_{\beta} - r_{\alpha}), \quad (29)$$

and

$$c = \sum_{\beta=1}^{n_f} (r_{\beta} - r_{\alpha})^T (T_{\beta} - T_{\alpha}). \tag{30}$$

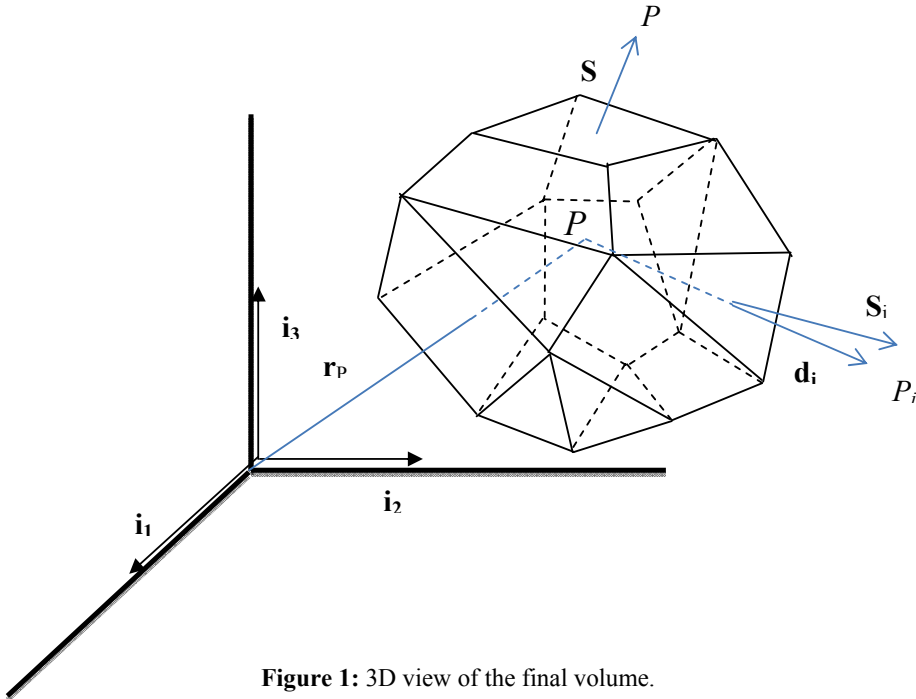


Figure 1: 3D view of the final volume.

One may easily employ Eq. (41) to recalculate the temperatures at the finite volume faces necessary to provide the RHS boundary integrals. There holds

$$T_{\beta} = \frac{1}{2} (T_{\alpha} + T_{\beta}) + \frac{1}{2} \left\{ (\nabla T)_{\alpha} (r_{\beta} - r_{\alpha}) + (\nabla T)_{\beta} (\tilde{r}_{\beta} - r_{\beta}) \right\}, \tag{31}$$

where  $\tilde{r}_{\beta}$  is the position vector of the surface center  $\beta$ . So that, for each finite volume one obtains

$$K_{\gamma\alpha} T_{\alpha} - \sum_{\beta=1}^{n_i} K_{\gamma\beta} T_{\beta} = Q_{\gamma}, \tag{32}$$

where the following notation has been used:

$$K_{\gamma\beta} = k \frac{s_{\beta} \cdot s_{\beta}}{d_{\beta} \cdot s_{\beta}}, \tag{33}$$



$$K_{\gamma\alpha} = \sum_{\beta=1}^{n_f} K_{\gamma\beta} \quad (34)$$

and  $Q_\alpha$  includes the RHS of Eq. (28). Assemblage of the global thermal equilibrium proceeds by rewriting of Eq. (34) for all  $N$  finite volumes and it yields

$$K_{\gamma\alpha} T_\alpha = Q_\gamma, \quad \gamma, \alpha = 1, \dots, N. \quad (35)$$

where the system matrix  $K_{\gamma\alpha}$  is analogous to the heat conductivity matrix used widely in the FEM.

#### 4.2 Stochastic perturbation-based FVM

The set of Eqs. (21-23) is extended next using the generalized perturbation probabilistic second moment technique presented in the next section. Next, equating terms of equal orders in the resulting expressions, the zeroth, first and second-order equations for the laminar flow considered are obtained as

- 0th order ( $\epsilon^0$  terms, one partial differential equation):

$$\begin{aligned} \int_{\Omega} \delta v_i \rho^0 \left( \dot{v}_i^0 + v_{i,j}^0 v_j^0 \right) d\Omega + \int_{\Omega} \delta \epsilon_{ij} \left( 2\mu^0 \epsilon_{ij}^0 - p^0 \delta_{ij} \right) d\Omega = \\ = \int_{\Omega} \delta v_i (f_i^B)^0 d\Omega + \int_{\partial\Omega_\sigma} \delta v_i \left( \tilde{f}_i \right)^0 d(\partial\Omega) \end{aligned}, \quad (36)$$

$$\int_{\Omega} \delta p v_{i,i}^0 d\Omega = 0, \quad (37)$$

$$\begin{aligned} \int_{\Omega} \delta \theta \rho^0 c_p^0 \left( \dot{\theta}^0 + \theta_{,i}^0 v_i^0 \right) d\Omega + \int_{\Omega} k^0 \delta \theta_{,i} \theta_{,i}^0 d\Omega = \\ = \int_{\Omega} \delta \theta (q^B)^0 d\Omega + \int_{\partial\Omega_q} \delta \hat{\theta} (\tilde{q})^0 d(\partial\Omega); \end{aligned} \quad (38)$$

- first order equations:

$$\begin{aligned} \int_{\Omega} \delta v_i \left( \frac{\partial \rho}{\partial b} v_i^0 + \rho^0 \frac{\partial v_i}{\partial b} + \frac{\partial \rho}{\partial b} v_{i,j}^0 v_j^0 + \rho^0 \frac{\partial v_{i,j}}{\partial b} v_j^0 + \rho^0 v_{i,j}^0 \frac{\partial v_j}{\partial b} \right) d\Omega + \\ + \int_{\Omega} \delta \epsilon_{ij} \left( 2 \frac{\partial \mu}{\partial b} \epsilon_{ij}^0 + 2\mu^0 \frac{\partial \epsilon_{ij}}{\partial b} - \frac{\partial p}{\partial b} \delta_{ij} \right) d\Omega = \int_{\Omega} \delta v_i \frac{\partial f_i^B}{\partial b} d\Omega + \int_{\partial\Omega_\sigma} \delta v_i \frac{\partial \tilde{f}_i}{\partial b} d(\partial\Omega) \end{aligned} \quad (39)$$

$$\int_{\Omega} \delta p \frac{\partial v_{i,i}}{\partial b} d\Omega = 0, \quad (40)$$

$$\begin{aligned} & \int_{\Omega} \delta \theta \left( \frac{\partial \rho}{\partial b} c_p^0 \dot{\theta}^0 + \rho^0 \frac{\partial c_p}{\partial b} \dot{\theta}^0 + \rho^0 c_p^0 \frac{\partial \dot{\theta}}{\partial b} + \frac{\partial \rho}{\partial b} c_p^0 \theta_{,i}^0 v_i^0 + \rho^0 \frac{\partial c_p}{\partial b} \theta_{,i}^0 v_i^0 \right) d\Omega + \\ & + \int_{\Omega} \delta \theta \left( \rho^0 c_p^0 \frac{\partial \theta_{,i}}{\partial b} v_i^0 + \rho^0 c_p^0 \theta_{,i}^0 \frac{\partial v_i}{\partial b} \right) d\Omega \tag{41} \\ & + \int_{\Omega} \delta \theta_{,i} \left( \frac{\partial \rho}{\partial b} \theta_{,i}^0 + k^0 \frac{\partial \theta_{,i}}{\partial b} \right) d\Omega = \int_{\Omega} \delta \theta \frac{\partial q^B}{\partial b} d\Omega + \int_{\partial \Omega_q} \delta \hat{\theta} \frac{\partial \bar{q}}{\partial b} d(\partial \Omega) \end{aligned}$$

- second order equations (where higher than the second order partial derivatives of >material parameters with respect to random variable b are postponed):

$$\begin{aligned} & \int_{\Omega} \delta v_i \rho^0 \left( \frac{\partial^2 \dot{v}_i}{\partial b^2} + \frac{\partial^2 v_{i,j}}{\partial b^2} v_j^0 + v_{i,j}^0 \frac{\partial^2 v_j}{\partial b^2} \right) d\Omega + 2 \int_{\Omega} \delta \varepsilon_{ij} \mu^0 \frac{\partial^2 \varepsilon_{ij}}{\partial b^2} d\Omega = \\ & - \int_{\Omega} \delta \varepsilon_{ij} \left( 4 \frac{\partial \mu}{\partial b} \frac{\partial \varepsilon_{ij}}{\partial b} - \frac{\partial^2 \mu}{\partial b^2} \delta_{ij} \right) d\Omega + \int_{\Omega} \delta v_i \frac{\partial^2 f_i^B}{\partial b^2} d\Omega + \int_{\partial \Omega_{\sigma}} \delta v_i \frac{\partial^2 \bar{f}_i}{\partial b^2} d(\partial \Omega) \tag{42} \\ & - 2 \int_{\Omega} \delta v_i \left( \frac{\partial \rho}{\partial b} \frac{\partial \dot{v}_i}{\partial b} + \frac{\partial \rho}{\partial b} \frac{\partial v_{i,j}}{\partial b} v_j^0 + \rho^0 \frac{\partial v_{i,j}}{\partial b} \frac{\partial v_j}{\partial b} + \frac{\partial \rho}{\partial b} v_{i,j}^0 \frac{\partial v_j}{\partial b} \right) d\Omega \end{aligned}$$

$$\int_{\Omega} \delta p \frac{\partial^2 v_{i,i}}{\partial b^2} d\Omega = 0, \tag{43}$$

$$\begin{aligned} & \int_{\Omega} \delta \theta \rho^0 c_p^0 \left( \frac{\partial^2 \dot{\theta}}{\partial b^2} + \frac{\partial^2 \theta_{,i}}{\partial b^2} v_i^0 + \theta_{,i}^0 \frac{\partial^2 v_i}{\partial b^2} \right) d\Omega + \int_{\Omega} \delta \theta_{,i} k^0 \frac{\partial^2 \theta_{,i}}{\partial b^2} d\Omega = \\ & = \int_{\Omega} \delta \theta \frac{\partial^2 q^B}{\partial b^2} d\Omega + \int_{\partial \Omega_q} \delta \hat{\theta} \frac{\partial^2 \bar{q}}{\partial b^2} d(\partial \Omega) - 2 \int_{\Omega} \delta \theta \left( \frac{\partial \rho}{\partial b} c_p^0 \frac{\partial \theta_{,i}}{\partial b} v_i^0 + \frac{\partial \rho}{\partial b} c_p^0 \theta_{,i}^0 \frac{\partial v_i}{\partial b} \right) d\Omega + \\ & - 2 \int_{\Omega} \delta \theta \left( \rho^0 \frac{\partial c_p}{\partial b} \frac{\partial \theta_{,i}}{\partial b} v_i^0 + \rho^0 \frac{\partial c_p}{\partial b} \theta_{,i}^0 \frac{\partial v_i}{\partial b} + \rho^0 c_p^0 \frac{\partial \theta_{,i}}{\partial b} \frac{\partial v_i}{\partial b} \right) d\Omega - 2 \int_{\Omega} \delta \theta_{,i} \frac{\partial k}{\partial b} \frac{\partial \theta_{,i}}{\partial b} d\Omega \tag{44} \end{aligned}$$

Let us observe that this formulation can be generalized to the *n*th order perturbation and then, the following coupled equations are obtained thanks to the chain differentiation rule:

$$\begin{aligned} & \int_{\Omega} \delta v_i \left( \sum_{k=0}^n \binom{n}{k} \frac{\partial^k \rho}{\partial b^k} \frac{\partial^{n-k} \dot{v}_i}{\partial b^{n-k}} + \sum_{k=0}^n \binom{n}{k} \frac{\partial^k \rho}{\partial b^k} \left( \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{\partial^m v_{i,j}}{\partial b^m} \frac{\partial^{n-k-m} v_j}{\partial b^{n-k-m}} \right) \right) d\Omega \\ & + 2 \int_{\Omega} \delta \varepsilon_{ij} \left( \sum_{k=0}^n \binom{n}{k} \frac{\partial^k \mu}{\partial b^k} \frac{\partial^{n-k} \varepsilon_{ij}}{\partial b^{n-k}} - \frac{\partial^n \mu}{\partial b^n} \delta_{ij} \right) d\Omega \\ & = \int_{\Omega} \delta v_i \frac{\partial^n f_i^B}{\partial b^n} d\Omega + \int_{\partial \Omega_{\sigma}} \delta v_i \frac{\partial^n \bar{f}_i}{\partial b^n} d(\partial \Omega) \tag{45} \end{aligned}$$

$$\int_{\Omega} \delta p \frac{\partial^n v_{i,i}}{\partial b^n} d\Omega = 0, \tag{46}$$

$$\begin{aligned}
 & \int_{\Omega} \delta \theta \left( \sum_{k=0}^n \binom{n}{k} \left( \sum_{m=0}^k \binom{k}{m} \frac{\partial^k \rho}{\partial b^k} \frac{\partial^{k-m} c_p}{\partial b^{k-m}} \right) \frac{\partial^{n-k} \theta}{\partial b^{n-k}} \right) d\Omega + \\
 & + \int_{\Omega} \delta \theta \left( \sum_{k=0}^n \binom{n}{k} \left( \sum_{m=0}^k \binom{k}{m} \frac{\partial^k \rho}{\partial b^k} \frac{\partial^{k-m} c_p}{\partial b^{k-m}} \right) \left( \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{\partial^l \theta_i}{\partial b^l} \frac{\partial^{n-k-l} v_i}{\partial b^{n-k-l}} \right) \right) d\Omega + \\
 & + \int_{\Omega} \delta \theta_i \sum_{k=0}^n \binom{n}{k} \frac{\partial^k k_i}{\partial b^k} \frac{\partial^{n-k} \theta_i}{\partial b^{n-k}} d\Omega = \int_{\Omega} \delta \theta \frac{\partial^n q^B}{\partial b^n} d\Omega + \int_{\partial \Omega_q} \delta \hat{\theta} \frac{\partial^n \tilde{q}}{\partial b^n} d(\partial \Omega)
 \end{aligned} \tag{47}$$

Because of the great complexity of such a solution, the second order perturbation approach was usually preferred, where computing the zeroth order velocity, pressure and temperature functions from Eqs. (36-38), next, their first order approximations using Eqs. (39-41) and, finally, the second order terms from Eqs. (42-44), the first two probabilistic moments of these functions are derived.

The key feature of this approach is to determine numerically the coefficients  $D_{\beta m}$  for each node of the initial FVM mesh and each power of the polynomial representations of the nodal temperatures with respect to the random input.

Now the goal would be to compute up to  $n$ th order velocities, pressures and temperatures and calculate their first four probabilistic moments and coefficients. Since unknown computational error resulting from up to  $n$ th order equilibrium equations and the direct access to the source code we propose the Response Function Method where a polynomial approximation of the state function (like temperatures below) in a given sub-volume center with respect to the input random variable  $b$  is proposed in the following form [Kamiński (2011)]:

$$T_{\beta} = D_{\beta m}^T b^m, \quad m = 0, \dots, n - 1; \beta = 1, \dots, N. \tag{48}$$

so that there holds

$$\theta(x_i) = N_{\beta}(x_i) T_{\beta} = N_{\beta}(x_i) D_{\beta m}^T b^m; \quad i = 1, 2; \alpha = 1, 2, \dots, N, m = 0, \dots, n - 1; \tag{49}$$

Therefore, the temperature gradients are similarly determined as

$$\theta_{,j} = N_{\beta,j} T_{\beta} = N_{\beta,j} D_{\beta m}^T b^m, \quad i = 1, 2, m = 0, \dots, n - 1. \tag{50}$$

Analogously, the pressures and velocities are represented as

$$p_{\beta} = D_{\beta m}^p b^m, \quad v_{\beta} = D_{\beta m}^v b^m, \quad m = 0, \dots, n - 1; \beta = 1, \dots, N. \tag{51}$$

Generally, the approximating polynomial orders for the PVT solution components do not need to be exactly the same – this choice is affected mainly by the uncertainty source and physical interpretation in the given problem.

Recovery of the local approximations for the state functions proceeds from several trial solutions using the classical FVM around the mean value of the random input parameters. So that typical discretization is enriched with the new index  $\alpha=1,\dots,N$ , where  $N$  denotes the total number of approximating points (the same as the number of sub-volumes having each  $n$  surfaces). Then, Eqn (36) is discretized in each finite volume  $l$  as

$$\begin{aligned} & \left( \frac{\Delta \rho^{(\alpha)} v^{(\alpha)}}{\Delta t} \right)_l + \frac{1}{V_l} \sum_{j=1}^n \rho_j^{(\alpha)} v_j^{(\alpha)} v_j^{(\alpha)} A_j - \frac{1}{V_l} \sum_{j=1}^n \mu_j^{(\alpha)} \nabla v_j^{(\alpha)} A_j \\ & = \left( \nabla v^{(\alpha)} \right)_l \nabla \mu_l^{(\alpha)} - \left( \nabla p^{(\alpha)} \right)_l + \rho_l^{(\alpha)} g^{(\alpha)} \end{aligned} \tag{52}$$

where  $V_l$  is the volume of the  $l$ th sub-volume and  $A_j$  is the area vector of the face  $j$ . The pressure gradient in  $x_i$  direction is calculated here using the Gauss integration scheme as

$$\nabla p_l^{(\alpha)}(x_i) = \frac{1}{V_l} \sum_{j=1}^n p_j^{(\alpha)} A_j n_j(x_i) \tag{53}$$

and, analogously, for velocity

$$\nabla v_l^{(\alpha)} = \frac{1}{V_l} \sum_{j=1}^n v_j^{(\alpha)} A_j \tag{54}$$

where central differencing scheme is applied to determine the given value at the cell face center. The area vectors remain constant during the response polynomials recovery as far as the domain geometry is defined deterministically. Adopting the following definitions in Eqn (53):

$$\begin{cases} K_l^{v(\alpha)} = \frac{\rho_l^{(\alpha)}}{\Delta t} + \frac{1}{V_l} \sum_{j=1}^n \left\{ (1 - \beta) \rho_j^{(\alpha)} v_j^{(\alpha)} A_j + \mu_j^{(\alpha)} \frac{A_j}{|d_j|} \right\} \\ \bar{K}_{lj}^{v(\alpha)} = \frac{1}{V_l} \left( \beta \rho_j^{(\alpha)} v_j^{(\alpha)} A_j - \mu_j^{(\alpha)} \frac{A_j}{|d_j|} \right) \\ Q_l^{v(\alpha)} = \frac{\rho_l^{(\alpha)} v_l^{(\alpha)}(t-1)}{\Delta t} - \frac{1}{V_l} \sum_{j=1}^n p_j^{(\alpha)} A_j n_j - \rho_l^{(\alpha)} g^{(\alpha)} \\ \quad + \left( \nabla v_l^{(\alpha)}(t-1) \right) \left( \nabla \mu_l^{(\alpha)}(t-1) \right) \end{cases} \tag{55}$$

We obtain finally for the  $l$ th finite volume the algebraic equations system

$$K_l^{v(\alpha)} v_l^{(\alpha)}(t) + \sum_{j=1}^n \bar{K}_{lj}^{v(\alpha)} v_{lj}^{(\alpha)}(t) = Q_l^{v(\alpha)} \tag{56}$$

So that the global momentum equation in the RFM-based SFVM yields

$$\sum_{l=1}^N K_l^{v(\alpha)} v_l^{(\alpha)}(t) + \sum_{l=1}^N \sum_{j=1}^n \bar{K}_{lj}^{v(\alpha)} \bar{v}_{lj}^{(\alpha)}(t) = \sum_{l=1}^N Q_l^{v(\alpha)} \quad (57)$$

The coefficient  $\beta$  is the interpolation factor connecting the given finite volume and its particular face  $m$  as

$$v_{ml} = v_m \beta + v_l (1 - \beta) \quad (58)$$

We discretize similarly the continuity equation on the subvolume level as

$$\sum_{j=1}^n v_j^{(\alpha)} A_j = 0 \quad (59)$$

Analogous considerations as before lead us to the following matrix equation at the discrete level:

$$K_l^{p(\alpha)} p_l^{(\alpha)}(t) + \sum_{j=1}^n \bar{K}_{lj}^{p(\alpha)} \bar{p}_{lj}^{(\alpha)}(t) = Q_l^{p(\alpha)} \quad (60)$$

having the global form

$$\sum_{l=1}^N K_l^{p(\alpha)} p_l^{(\alpha)}(t) + \sum_{l=1}^N \sum_{j=1}^n \bar{K}_{lj}^{p(\alpha)} \bar{p}_{lj}^{(\alpha)}(t) = \sum_{l=1}^N Q_l^{p(\alpha)} \quad (61)$$

Finally, the SFVM discretization of the heat transfer equation is provided as

$$K_l^{\theta(\alpha)} \theta_l^{(\alpha)}(t) + \sum_{j=1}^n \bar{K}_{lj}^{\theta(\alpha)} \bar{\theta}_{lj}^{(\alpha)}(t) = Q_l^{\theta(\alpha)} \quad (62)$$

where it is assumed that

$$\left\{ \begin{aligned} K_l^{\theta(\alpha)} &= \frac{\rho_l^{(\alpha)} c_l^{(\alpha)}}{\Delta t} + v_{li}^{(\alpha)} \frac{1}{V_l} \rho_l^{(\alpha)} c_l^{(\alpha)} \sum_{j=1}^n \left\{ (1 - \beta) A_{lj} n_{lji} + k_l^{(\alpha)} \frac{A_j}{|d_j|} \right\} \\ \bar{K}_{lj}^{\theta(\alpha)} &= v_{li}^{(\alpha)} \frac{1}{V_l} \rho_l^{(\alpha)} c_l^{(\alpha)} \beta A_{lj} n_{jli}, \quad i = 1, 2, 3 \\ Q_l^{\theta(\alpha)} &= \frac{\rho_l^{(\alpha)} c_l^{(\alpha)}}{\Delta t} \theta_l^{(\alpha)}(t - 1) + \phi_l^{(\alpha)} \end{aligned} \right. \quad (63)$$

and  $\phi_l^{(\alpha)}$  is the viscous dissipation in the  $l$ th finite volume. The global heat transfer equation for the SFVM yields

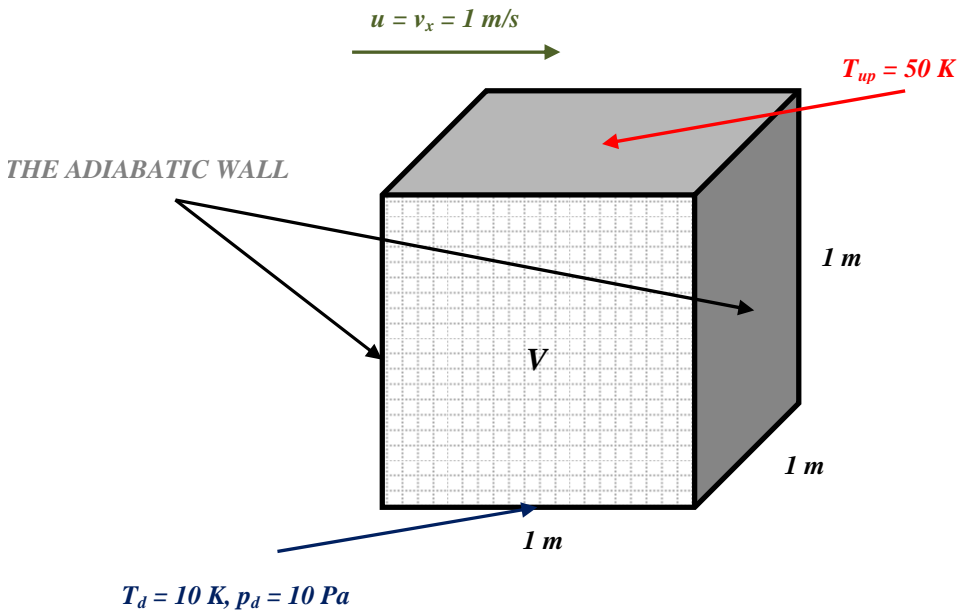
$$\sum_{l=1}^N K_l^{\theta(\alpha)} \theta_l^{(\alpha)}(t) + \sum_{l=1}^N \sum_{j=1}^n \bar{K}_{lj}^{\theta(\alpha)} \bar{\theta}_{lj}^{(\alpha)}(t) = \sum_{l=1}^N Q_l^{\theta(\alpha)} \quad (64)$$

Simultaneous solution to the system of Eqns. (57,62,64) enables for the polynomial approximation of the pressures, temperatures and velocities in a fluid and final algebraic derivation of their probabilistic characteristics provided numerically in the next section.

## 5 Computational analysis

### 5.1 Random viscosity modeling

Let us consider a cube of unit dimensions divided into 400 equal and also cubic finite volumes containing a fluid with the following physical parameters – density  $\rho = 1 \text{ kg/m}^3$ , specific heat  $c = 100 \text{ J/kg} \cdot \text{K}$ , coefficient of thermal conductivity  $k = 10 \text{ W/m} \cdot \text{K}$ , zero compressibility  $\beta = 0.0 \text{ m}^2/\text{N}$  and viscosity being the input Gaussian random parameter. It has the expected value equal to  $E[\mu] = 10^{-1} \text{ Pa} \cdot \text{s}$  and coefficient of variation having the value  $\alpha(\mu) = 0.15$ . The imposed boundary conditions for this cube are shown schematically on the Fig. 2 – the problem is restricted to 2D analysis to make easier final visualization of the resulting state functions and their probabilistic characteristics.



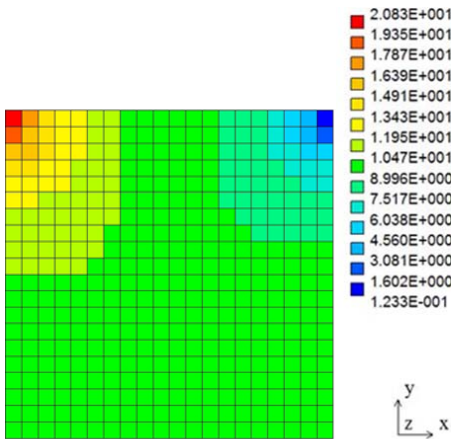
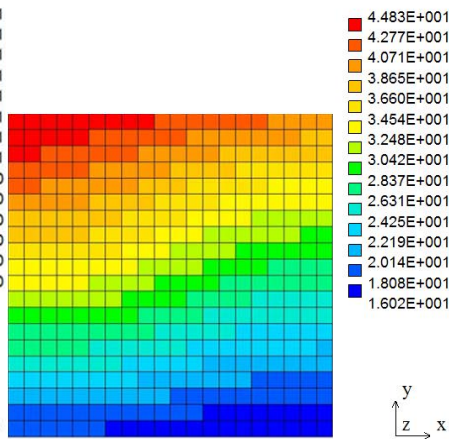
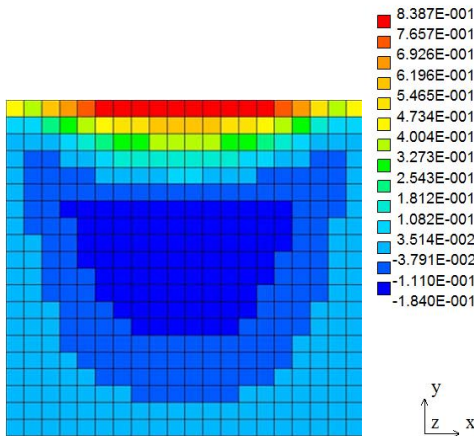
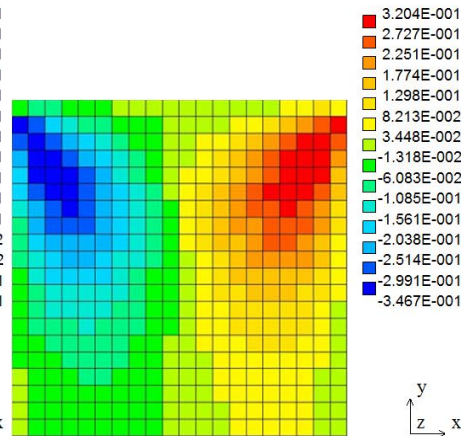
**Figure 2:** Boundary conditions for the test cube.

Computational analysis has been performed in three different computer systems –

(a) OpenFVM, where deterministic problems with varying random parameters have been solved consecutively, (b) symbolic environment of the mathematical package MAPLE, where the local response functions were recovered on the basis of previous models and where probabilistic moments were programmed and derived as well as (c) by using the internet available freeware FEPlot 3.1, where spatial distribution of the resulting probabilistic characteristics were provided. The response functions were obtained through 11-point FVM trials in OpenFVM, where random viscosity was uniformly modified within the interval  $\mu = [5 \cdot 10^{-2}, 15 \cdot 10^{-2}] \text{ Pa} \cdot \text{s}$ . Spatial distributions of the expected values, coefficients of variation, skewness and kurtosis for the pressure  $p$ , velocities in this flow ( $u, v$ ) and, finally, temperature  $T$  in this domain are shown in Figs. 3-6 (from  $a$  to  $d$ , correspondingly to the state variable). The time increment has been chosen as  $\Delta t = 0.10$  sec and the computations have been stopped at  $t_k = 10$  secs. It makes probabilistic analysis 11 times longer plus additional time to transfer and process all the data in the system MAPLE and finally preprocess them for the FEPlot needs. The non-stationary analysis of N-S equations in deterministic version costs more than 2 hours on the same processor, so that the Monte-Carlo analysis (with about  $10^5$  random trials) in this case would demand the very massive parallel computing process.

The expected values given in Figs. 3 have spatial distributions and the particular values almost the same as their deterministic counterparts reaching extremum values at the upper edge of the examined domain. As it was expected, the viscosity coefficient  $\mu$  variations caused a significant change in the distribution of velocity  $u$  inside the cube test and also in pressure distribution, whereas the effect on temperature seems to be negligibly small. The largest coefficients of variation (CoV, see Figs. 4) are noticed for the horizontal velocity components, than – for the vertical one, whereas random dispersion of the pressure and temperature generally have secondary importance here. Location of the absolute maximum of these coefficients almost perfectly coincide with the minimum values of the corresponding expectations and they are usually larger than the input CoV of the fluid viscosity. The most regular spatial distribution of this probabilistic parameter is detected for the temperature field, however the values are practically negligible here, which is expected since the randomness propagates into it according to the coupling with the fluid transport equation only.

Higher order statistics (cf. Figs. 5 and 6) form quite irregular patterns in the domain analyzed, where dominating part of the PVT solution remains Gaussian (according to the skewness and kurtosis). However, maximum values of both coefficients may be even many times more than the values adequate to the normal probability density function and appear in the exceptional cases only. A detailed comparison of Fig. 5a and 6a and the remaining pairs of these probabilistic characteristics shows

Figure 3a: Expected value  $E[p]$ .Figure 3b: Expected value  $E[T]$ .Figure 3c: Expected value  $E[u]$ .Figure 3d: Expected value  $E[v]$ .

clearly that the patterns of skewness and kurtosis for the specific components of the PVT solution almost strictly coincide. So that the larger deviations from the values typical Gaussian distribution in both cases have the same location in this domain.

## 5.2 Random heat conductivity coefficient

Quite a similar analysis was conducted using constant viscosity value  $\mu = 10^{-2} Pa \cdot s$ , given parameters  $\rho$ ,  $c$ ,  $\beta$  and the coefficient of thermal conductivity taken as the input Gaussian random variable with the same coefficient of variation and the expectation equal to  $E[k] = 10 W/m \cdot K$ ; analogously as in Sec. 5.1 we provide 11



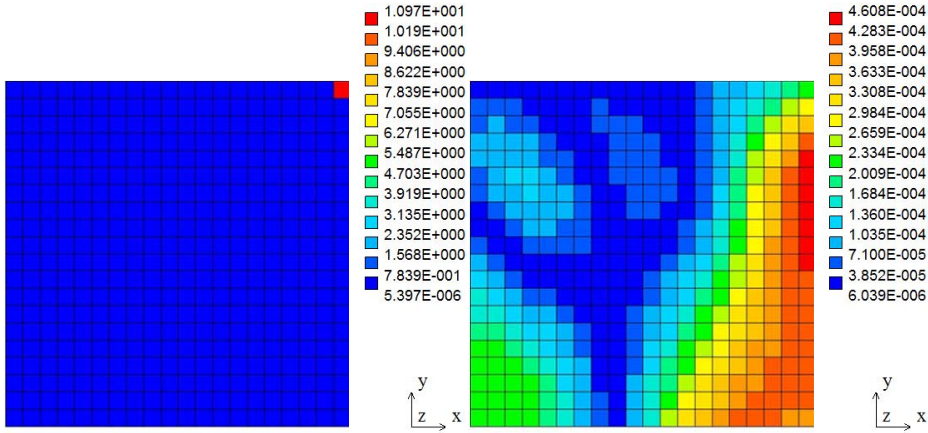
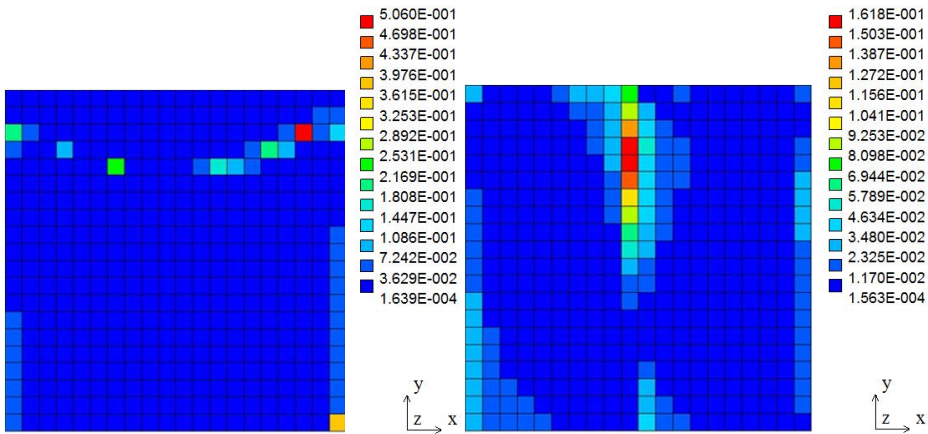
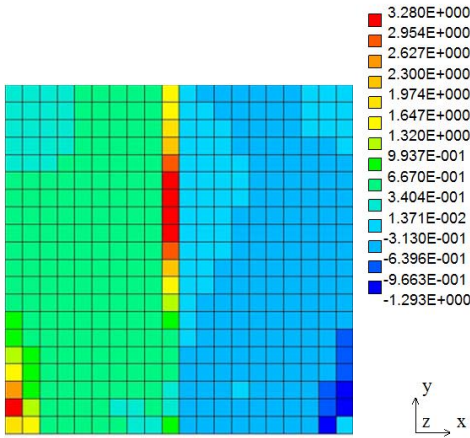
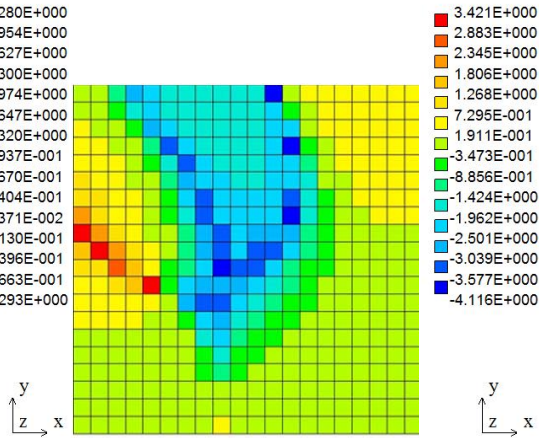
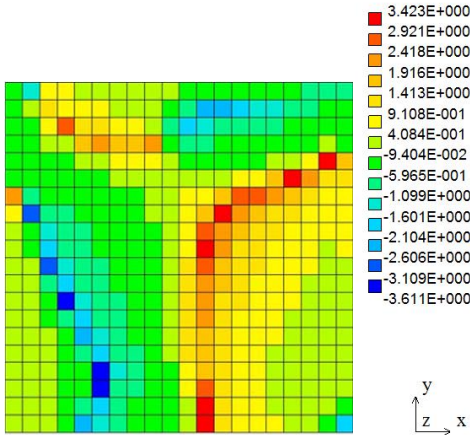
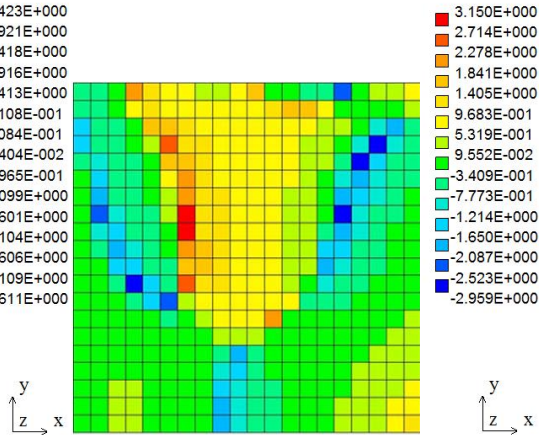


Figure 4a: Coefficient of variation  $\alpha(p)$ . Figure 4b: Coefficient of variation  $\alpha(T)$ .



deterministic solutions with  $k \in [5...15] \text{ W/m}\cdot\text{K}$ . The results obtained (expectations, coefficients of variation, skewness and kurtosis) for the temperature  $T$ , flow velocity  $v_x \equiv u$ ,  $v_y \equiv v$  and pressure  $p$  show illustrations 7a-7d and 8-10 below. Assumed variations of the thermal conductivity coefficient  $k$  cause a significant reduction in the temperature difference  $\Delta T = T_{up} - T_d$  (see Fig. 2). The other physical quantities –  $u$ ,  $v$ ,  $p$  are totally independent from this parameter and that is why the detailed statistical analysis and visualization was conducted only for temperature  $T$ .

As it is typical for the stochastic perturbation-based methods, the expectations of

Figure 5a: Skewness  $\beta(p)$ .Figure 5b: Skewness  $\beta(T)$ .Figure 5c: Skewness  $\beta(u)$ .Figure 5d: Skewness  $\beta(v)$ .

the state functions computed at the mean values of various probabilistic parameters are exactly the same – one may compare Fig. 3a against Fig. 7a etc. Some small exceptions at the minimum values follow rather the discrepancies of the deterministic computer technique itself. Contrary to the previous cases now all higher order characteristics – CoV, skewness and kurtosis all have the very regular spatial distributions without any local large gradients and outstanding extremum values. Since that the extremum values within those coefficients all coincide with each other and, further, the uncertainty at the output temperature field is significantly smaller than the input CoV of the heat conductivity. Particular values of skewness and kurtosis show that the state function of the temperature is very distant from the Gaussian

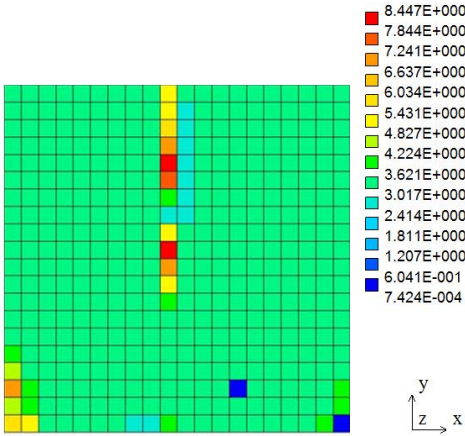


Figure 6a: Kurtosis  $\kappa(p)$ .

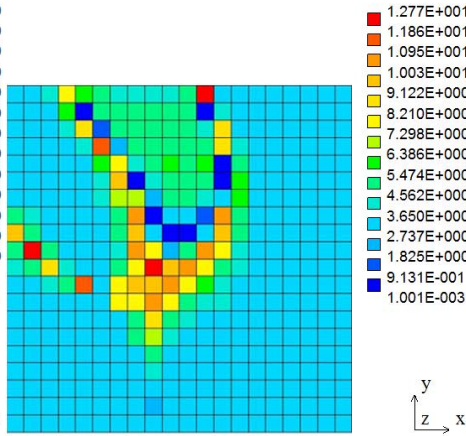


Figure 6b: Kurtosis  $\kappa(T)$ .

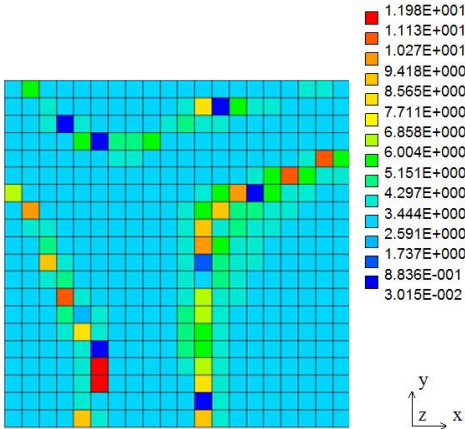


Figure 6c: Kurtosis  $\kappa(u)$ .

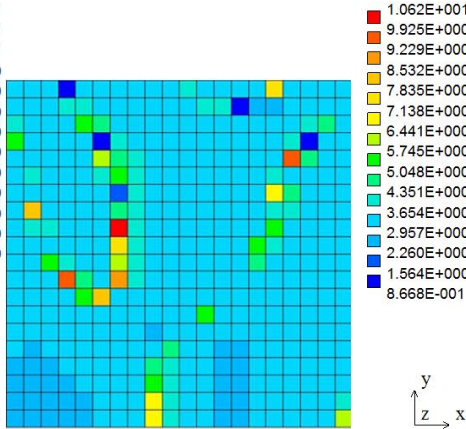
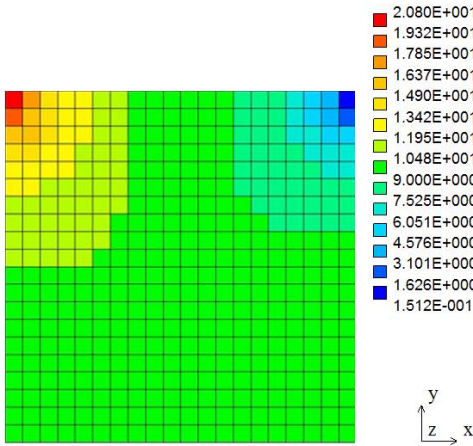
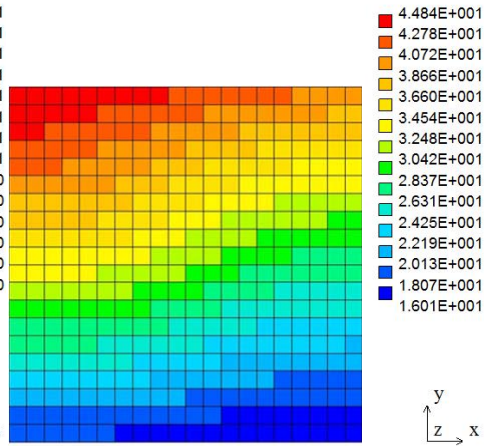
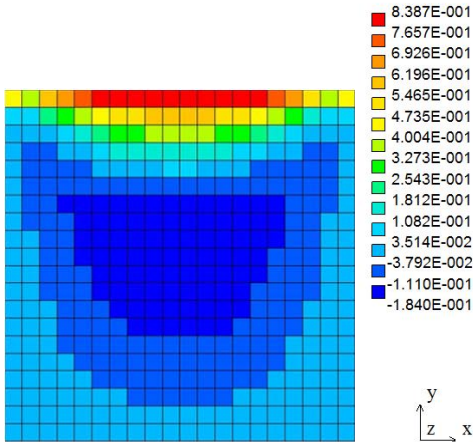
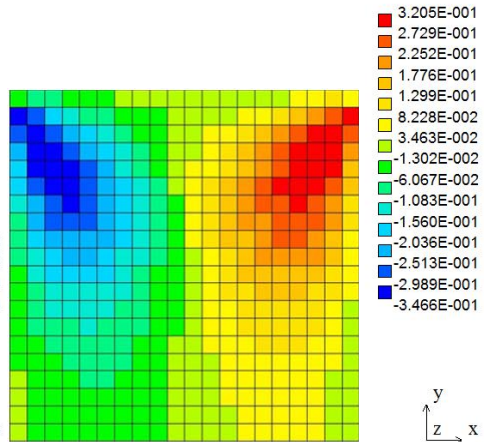


Figure 6d: Kurtosis  $\kappa(v)$ .

random field, so that the first two probabilistic moments information is not sufficient to characterize it uniquely.

### 6 Concluding remarks

[1] The Stochastic perturbation-based Finite Volume Method proposed in this paper in conjunction with the discrete Response Function Methods seems to be an efficient alternative to both Monte-Carlo simulation technique (according to time consumption closer to the deterministic origin) and stochastic polynomial chaos expansions (since enable for higher than the second order statistics analysis). Thanks

Figure 7a: Expected value  $E[p]$ .Figure 7b: Expected value  $E[T]$ .Figure 7c: Expected value  $E[u]$ .Figure 7d: Expected value  $E[v]$ .

to the usage of the RFM technique a numerical error (of unknown amount) inherent in the Direct Differentiation Method version of the perturbation-based SFEM and resulting from the solution to the increasing order hierarchical equations. Now, the proposed technique deficiency in this context is the approximation error typical for the Least Square Method smoothening of the local response function since further partial differentiation at the expected values and probabilistic equations for the moments and coefficients are free from any computational error.

[2] The method presented enables for a randomization of the deterministic FVM models with a single Gaussian input random variable, however it can be relatively

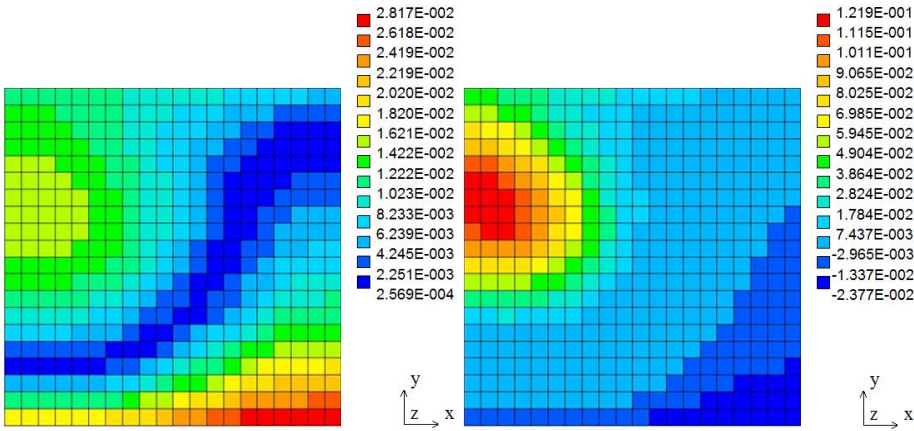


Figure 8: Coefficient of variation  $\alpha(T)$ .

Figure 9: Skewness  $\beta(T)$ .

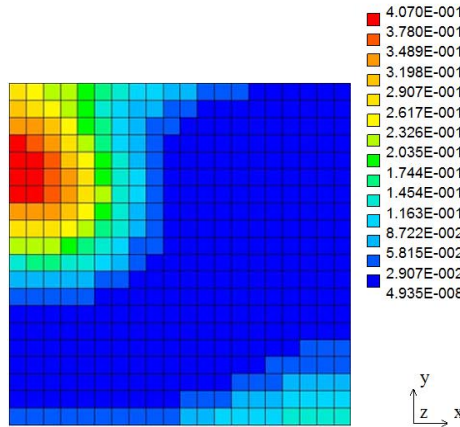


Figure 10: Kurtosis  $\kappa(T)$ .

easy extended towards multi-parametric problems with both uncorrelated and correlated uncertainty sources, but then the additional cross-correlations needs to be given and inserted into the equations for all probabilistic moments of the state parameters. A randomization of the non-linear problems with the state-dependent physical parameters of the fluids does not seem to be impossible but randomness propagation step-by-step in such a computational analysis may make higher order statistics extremely large. It can be concluded using the numerical results given above, where some skewness and kurtosis maxima, many times larger than those corresponding to the Gaussian PDFs, are detected at the minimum values of the expectations.

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