 CMES, vol.78, no.2, pp.77-94, 2011

Acoustic Design Shape and Topology Sensitivity Formulations Based on Adjoint Method and BEM

T. Matsumoto[1], T. Yamada[1], T. Takahashi[1], C.J. Zheng[2] and S. Harada[1]

Abstract: Shape design and topology sensitivity formulations for acoustic problems based on adjoint method and the boundary element method are presented and are applied to shape sensitivity analysis and topology optimization of acoustic field. The objective function is assumed to consist only of boundary integrals and quantities defined at certain number of discrete points. The adjoint field is defined so that the sensitivity of the objective function does not include the unknown sensitivity coefficients of the sound pressures and particle velocities on the boundary and in the domain. Since the final sensitivity expression does not have the sensitivity coefficients of the sound pressure and particle velocity on the boundary, BEM analyses only for the primary acoustic field and the adjoint field are needed to calculate the sensitivities of the objective function. The derived formulations are applied to shape sensitivity analyses and a topology optimization of a sound scatterer placed in an infinite space. The level-set method is utilized to control the shape of the domain in the iterative process of obtaining the optimum shape of the scatterer.

Keywords: Acoustics sensitivity analysis, adjoint method, topology optimization, level-set method.

1 Introduction

Due to the development of fast computation algorithms [Rokhlin (1985)], BEM can be considered as a strong analysis tool for shape optimization problems that requires re-meshing in the shape modification process. Shape optimization problems are usually solved by minimizing an appropriately defined objective function. In order to calculate the value of objective function and its sensitivities with respect to design variables, we can use boundary element method (BEM) based on either the direct differentiation method [Matsumoto, Tanaka, Miyagawa, and Ishii (1993); Matsumoto, Tanaka, and Yamada (1995); Koo (1997); Zheng, Matsumoto, Taka-

[1] Nagoya University, Nagoya, Japan.
[2] USTC, Hefei, China.

hashi, and Chen (2011)] or the adjoint variable method [Haug, Choi, and Komkov (1986)]. Direct differentiation method uses a boundary integral equation for the sensitivity coefficients of the boundary quantities. Hence, when the number of design variables are large, we have to repeat calculations of the sensitivities of the boundary quantities for all the design variables. On the other hand, because the adjoint method defines a system that can eliminate the unknown sensitivities of the quantities on the boundary and in the domain, we have to solve only the original acoustic problem and the adjoint problem when calculating the sensitivities of the objective function.

In this study, an adjoint method approach is shown for shape and topology optimization of acoustic field. The objective function is assumed to be defined with the sound pressure and the particle velocity on the boundary and with quantities at a finite number of internal points. The boundary element method is used for the analysis of the original acoustic problem and the corresponding adjoint problem. The adjoint problem and the sensitivity expression are derived for a typical form of objective function that is appropriate for using BEM. The derived topological sensitivity is used in the topology optimization procedure [Yamada, Izui, Nishiwaki, and Takezawa (2010)] based on the geometric modeling using level-set function [Wang, Lim, Khoo, and Wang (2007)]. Some numerical examples are shown to demonstrate the effectiveness of the present approach.

2 Boundary integral equations

The governing differential equation for the propagation of time-harmonic acoustic waves in a homogeneous and isotropic acoustic medium is the following Helmholtz equation:

$$\nabla^2 p(x) + k^2 p(x) + s(x) = 0 \tag{1}$$

where $p(x)$ is the sound pressure at point x, ∇^2 is the Laplace operator, $k = 2\pi f/C$ is the wavenumber with is the frequency f and sound speed C, and $f(x)$ is sound source.

The boundary conditions are

$$p(x) = \bar{p}(x) \quad \text{on } \Gamma_p \tag{2}$$

$$q(x) = \frac{\partial p}{\partial n}(x) = -\mathrm{i}\omega\rho\, v(x) = \bar{q}(x) \quad \text{on } \Gamma_q \tag{3}$$

where n denotes the outward normal direction, i the imaginary unit, ω the circular frequency, ρ the density of the medium, and v the particle velocity. An over-scribed bar ($\bar{}$) indicates that the value is given on the boundary.

The integral representation of the solution to the Helmholtz equation is given by

$$p(x)+\int_{\Gamma} q^*(x,y)p(y)\,d\Gamma(y) = \int_{\Gamma} p^*(x,y)q(y)\,d\Gamma(y) + \int_{\Omega} p^*(x,y)s(y)\,d\Omega(y) \tag{4}$$

where x is the collocation point, y is the source point, Γ is the boundary of the acoustic field, Ω is the domain, and $p^*(x,y)$ is the fundamental solution, for 2-D problems given by

$$p^*(x,y) = -\frac{\mathrm{i}}{4}H_0^{(2)}(kr) \tag{5}$$

with $r=|x-y|$, and for 3-D given by

$$p^*(x,y) = \frac{e^{\mathrm{i}kr}}{4\pi r} \tag{6}$$

where $H_0^{(2)}$ is the Hankel function of the second kind of order 0. Also, $q^*(x,y)$ is the normal derivative of $p^*(x,y)$.

Although Eq.(4) has a domain integral originated from the sound source distribution $s(x)$, it results in a summation of the values of the fundamental solution at some discrete points in the field when the sound source is the summation of concentrated sources as

$$s(x) = \sum_{\alpha} I^{\alpha}\delta(x-z^{\alpha}) \tag{7}$$

where α counts for concentrated sound sources, I^{α} is the intensity of the sound source at z^{α}, and $\delta(x-z^{\alpha})$ is Dirac's delta function. In this case, the domain integral in Eq.(4) becomes as

$$\int_{\Omega} p^*(x,y)s(y)\,d\Omega(y) = \int_{\Omega} p^*(x,y)\sum_{\alpha} I^{\alpha}\delta(y-z^{\alpha})\,d\Omega(y) = \sum_{\alpha} I^{\alpha} p^*(x,z^{\alpha}) \tag{8}$$

Also, the gradients of the sound pressure at an internal points can be calculated by using the following representation:

$$\begin{aligned} p_{,i}(x)+\int_{\Gamma}\frac{\partial q^*(x,y)}{\partial x_i}p(y)\,d\Gamma(y) &= \int_{\Gamma}\frac{\partial p^*(x,y)}{\partial x_i}q(y)\,d\Gamma(y) \\ &\quad + \int_{\Omega}\frac{\partial p^*(x,y)}{\partial x_i}s(y)\,d\Omega(y) \end{aligned} \tag{9}$$

The boundary integral equation is obtained by taking the limit of point x to the boundary Γ, as follows:

$$C(x)p(x)+\fint_{\Gamma} q^*(x,y)p(y)\,d\Gamma(y) = \int_{\Gamma} p^*(x,y)q(y)\,d\Gamma(y) + \int_{\Omega} p^*(x,y)s(y)\,d\Omega(y) \tag{10}$$

where $C(x)$ is a constant that becomes 1/2 when x lies on a smooth part of the boundary. The integral symbol $⨍$ denotes that the integral is evaluated in the sense of Cauchy's principal value.

Also, an additional boundary integral equation is combined with Eq.(10) for exterior problems to avoid computation errors occurring at fictitious eigen-frequencies. It is obtained as the normal derivative of Eq.(10) at x, as follows:

$$C(x)q(x)+ ⨎_{\Gamma} \tilde{q}^*(x,y)p(y)\,d\Gamma(y) = ⨍_{\Gamma} \tilde{p}^*(x,y)q(y)\,d\Gamma(y) + \int_{\Omega} \tilde{p}^*(x,y)s(y)\,d\Omega(y) \tag{11}$$

where $(\tilde{\ }) = \partial(\,)/\partial n(x)$, and the integral symbol $⨎$ denotes that the integral is evaluated in the sense of finite part of divergent integral.

For exterior acoustic problems, a linear combination of Eqs.(10) and (11) is used [Burton and Miller (1971)] so that the solutions do not suffer from errors at the fictitious eigen frequencies of the corresponding interior problem.

In particular, when x lies on a smooth, and flat part of the boundary of a three-dimensional field, Cauchy's principal value and the finite part of the singular integrals of Eqs.(10) and (11) can be evaluated analytically, and we have the following integral representations [Zheng, Matsumoto, Takahashi, and Chen (2011)].

$$\begin{aligned} \frac{1}{2}p(x) + \int_{\Gamma-\Gamma_x} q^*(x,y)p(y)\,d\Gamma(y) &= \int_{\Gamma-\Gamma_x} p^*(x,y)q(y)\,d\Gamma(y) \\ &+ \int_{\Omega} p^*(x,y)s(y)\,d\Omega(y) + \frac{\mathrm{i}}{2k}\left(1 - \int_{\partial\Gamma_x} \mathrm{e}^{\mathrm{i}kR}\frac{\varepsilon_{ijm}s_i n_j R_{,m}}{2\pi R}\,dl\right)q(x) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \frac{1}{2}q(x) + \int_{\Gamma-\Gamma_x} \tilde{q}^*(x,y)p(y)\,d\Gamma(y) &= \int_{\Gamma-\Gamma_x} \tilde{p}^*(x,y)q(y)\,d\Gamma(y) \\ &+ \int_{\Omega} \tilde{p}^*(x,y)s(y)\,d\Omega(y) - \left(\frac{\mathrm{i}k}{2} - \int_{\partial\Gamma_x} \mathrm{e}^{\mathrm{i}kR}\frac{\varepsilon_{ijm}s_i n_j R_{,m}}{4\pi R^2}\,dl\right)p(x) \end{aligned} \tag{13}$$

where R denotes the distance between the collocation point x and a point x_0 on the edge $\partial\Gamma_x$ of the boundary Γ_x in the neighborhood of the collocation point as shown in Fig. 1, dl the differential arc length along $\partial\Gamma_x$, ε_{ijm} the permutation symbol, s_i the unit tangential vector along $\partial\Gamma_x$ at x_0, n_j the unit outward normal vector to the boundary Γ at x_0, and $R_{,m}$ is the derivative of R with respect to x_m component of the coordinate of x_0. Einstein's summation convention is applied to the repeated indices in Eqs.(12) and (13).

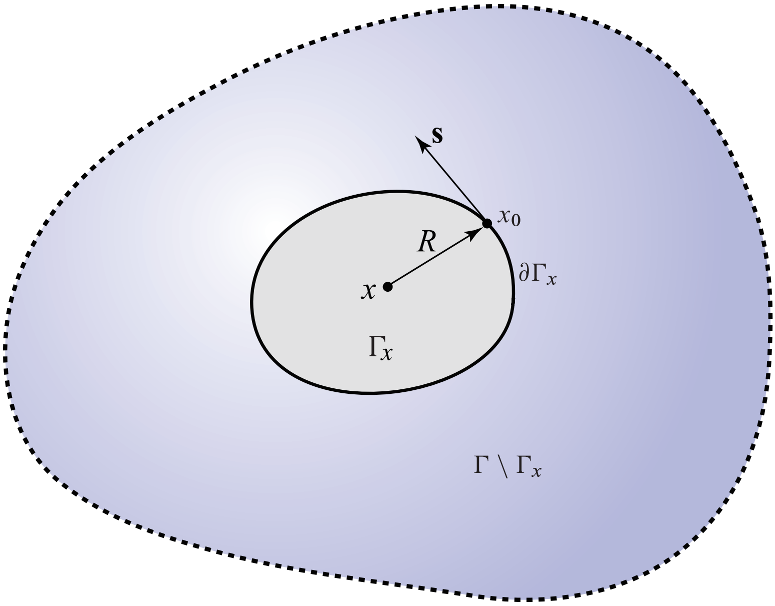

Figure 1: A neighborhood of the collocation point lying on a smooth part of the boundary of a three-dimensional space.

The linear combination of Eqs.(10) and (11), or (12) and (13), are discretized employing appropriate boundary elements, and then results in the following system of linear algebraic equation.

$$\mathbf{Hp} = \mathbf{Gq} + \mathbf{f} \tag{14}$$

After applying the boundary condition, by rearranging Eq.(14) so that the unknown nodal values come on the left-hand side, we obtain

$$\mathbf{Ax} = \mathbf{y} \tag{15}$$

By solving Eq.(15), we obtain all the boundary quantities and by using them in Eq.(9), we can calculate gradients of sound pressure at internal points.

3 Sensitivity analysis

3.1 Shape sensitivity

We assume the objective functions of shape and topology optimization problems can be written in the following form:

$$I = \int_\Gamma f(p,\bar{p},q,\bar{q})\,d\Gamma + \int_\Omega g(p,\bar{p})\,d\Omega \tag{16}$$

where f is a function defined with boundary sound pressure and its normal derivative, g is a function defined with internal sound pressure, and an over-scribed bar $(\bar{\ })$ denotes a complex conjugate.

Then, the sensitivity of I with respect to an arbitrary shape parameter becomes

$$\begin{aligned} I' = {} & \int_\Gamma \left(\frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial \bar{p}}\dot{\bar{p}} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \bar{q}}\dot{\bar{q}} \right) d\Gamma + \int_\Gamma f(\dot{d\Gamma}) \\ & + \int_\Omega \left(\frac{\partial g}{\partial p}\dot{p} + \frac{\partial g}{\partial \bar{p}}\dot{\bar{p}} \right) d\Omega + \int_\Omega g\,\dot{x}_{i,i}\,d\Omega \end{aligned} \tag{17}$$

where a dot $(\dot{\ })$ denotes a material derivative [Haug, Choi, and Komkov (1986); Arora (1993)].

Notice that we implicitly assume

$$\frac{\partial f}{\partial \bar{p}} = \overline{\left(\frac{\partial f}{\partial p}\right)}, \qquad \frac{\partial f}{\partial \bar{q}} = \overline{\left(\frac{\partial f}{\partial q}\right)}, \qquad \frac{\partial g}{\partial \bar{p}} = \overline{\left(\frac{\partial g}{\partial p}\right)} \tag{18}$$

Then, we have

$$\begin{aligned} I' &= 2\mathrm{Re}\left[\int_\Gamma \frac{\partial f}{\partial p}\dot{p}\,d\Gamma + \int_\Gamma \frac{\partial f}{\partial q}\dot{q}\,d\Gamma + \int_\Omega \frac{\partial g}{\partial p}\dot{p}\,d\Omega \right] + \int_\Gamma f(\dot{d\Gamma}) + \int_\Omega g\,\dot{x}_{i,i}\,d\Omega \\ &\equiv 2\mathrm{Re}[I_1'] + 2\mathrm{Re}\left[\int_\Omega \frac{\partial g}{\partial q_i}\dot{q}_i\,d\Omega \right] + \int_\Gamma f(\dot{d\Gamma}) + \int_\Omega g\,\dot{x}_{i,i}\,d\Omega \end{aligned} \tag{19}$$

where

$$I_1' \equiv \int_\Gamma \frac{\partial f}{\partial p}\dot{p}\,d\Gamma + \int_\Gamma \frac{\partial f}{\partial q}\dot{q}\,d\Gamma + \int_\Omega \frac{\partial g}{\partial p}\dot{p}\,d\Omega \tag{20}$$

We now augment the objective function, as shown below, so that the sound pressure satisfies the Helmholtz equation.

$$J = I + R + \bar{R} = I + 2\,\mathrm{Re}[R] \tag{21}$$

where

$$R \equiv \int_{\Omega} \lambda(x) \left[p_{,ii}(x) + k^2 p(x) + s(x)\right] d\Omega \tag{22}$$

and $\lambda(x)$ is a Lagrange multiplier.

We can use J as the objective function instead of I because p is the solution of Eq.(1) thus R identically equals zero.

Integrating R by parts once yields

$$R = \int_{\Gamma} \lambda q \, d\Gamma - \int_{\Omega} \lambda_{,i} p_{,i} \, d\Omega + \int_{\Omega} k^2 \lambda p \, d\Omega + \int_{\Omega} \lambda s \, d\Omega \tag{23}$$

The sensitivity of R with respect to an arbitrary shape change parameter is

$$\begin{aligned} R' = & \int_{\Gamma} \lambda \dot{q} \, d\Gamma + \int_{\Gamma} \lambda_{,i} q \dot{x}_i \, d\Gamma + \int_{\Gamma} \lambda q \, (\dot{d\Gamma}) \\ & - \left(\int_{\Omega} \lambda_{,i} p_{,i} \, d\Omega\right)' + \left(\int_{\Omega} k^2 \lambda p \, d\Omega\right)' + \left(\int_{\Omega} \lambda s \, d\Omega\right)' \end{aligned} \tag{24}$$

Recall that we have the following relationships concerning material derivative [Haug, Choi, and Komkov (1986); Arora (1993)]:

$$\dot{p} = p' + p_{,i} \dot{x}_i \tag{25}$$

$$(\dot{p_{,i}}) = (p_{,i})' + p_{,ij} \dot{x}_j = p'_{,i} + p_{,ij} \dot{x}_j \tag{26}$$

$$(\dot{p})_{,i} = (p' + p_{,j} \dot{x}_j)_{,i} = p'_{,i} + p_{,ij} \dot{x}_j + p_{,j} \dot{x}_{j,i} \tag{27}$$

$$(\dot{p_{,i}}) = (\dot{p})_{,i} - p_{,j} \dot{x}_{j,i} \tag{28}$$

where in this case a prime $(')$ denotes a differentiation at the original position, and for Lagrange multiplier (adjoint variable), λ, as follows:

$$\dot{\lambda} = \lambda' + \lambda_{,j} \dot{x}_j = \lambda_{,j} \dot{x}_j \tag{29}$$

$$(\dot{\lambda_{,i}}) = (\lambda_{,i})' + \lambda_{,ij} \dot{x}_j = \lambda_{,ij} \dot{x}_j \tag{30}$$

$$\dot{\lambda}_{,i} = \lambda_{,ij} \dot{x}_j + \lambda_{,j} \dot{x}_{j,i} \tag{31}$$

$$(\dot{\lambda_{,i}}) = \dot{\lambda}_{,i} - \lambda_{,j} \dot{x}_{j,i} \tag{32}$$

Also, the material derivatives of differential area $d\Gamma$ and differential volume $d\Omega$ are as follows:

$$(\dot{d\Gamma}) = (\dot{x}_{i,i} - \dot{x}_{i,j} n_i n_j) \, d\Gamma \tag{33}$$

$$(\dot{d\Omega}) = \dot{x}_{i,i} \, d\Omega \tag{34}$$

Then, after some manipulations, the last three sensitivity terms of Eq.(24) become

$$\begin{aligned}\left(\int_\Omega \lambda_{,i} p_{,i}\, d\Omega\right)' &= \int_\Omega (\dot{\lambda_{,i}}) p_{,i}\, d\Omega + \int_\Omega \lambda_{,i} (\dot{p_{,i}})\, d\Omega + \int_\Omega \lambda_{,i} p_{,i}\, (\dot{d\Omega}) \\ &= \int_\Gamma \lambda_{,i} p_{,i} \dot{x}_j n_j\, d\Gamma + \int_\Gamma \mu \dot{p}\, d\Gamma - \int_\Gamma \mu p_{,j} \dot{x}_j\, d\Gamma \\ &\quad - \int_\Omega \lambda_{,ii} \dot{p}\, d\Omega + \int_\Omega \lambda_{,ii} p_{,j} \dot{x}_j\, d\Omega \end{aligned} \tag{35}$$

where $\mu \equiv \partial\lambda/\partial n$, and

$$\begin{aligned}\left(\int_\Omega k^2 \lambda p\, d\Omega\right)' &= \int_\Omega k^2 \dot{\lambda} p\, d\Omega + \int_\Omega k^2 \lambda \dot{p}\, d\Omega + \int_\Omega k^2 \lambda p \dot{x}_{j,j}\, d\Omega \\ &= \int_\Omega k^2 \lambda \dot{p}\, d\Omega + \int_\Gamma k^2 \lambda p \dot{x}_j n_j\, d\Gamma - \int_\Omega k^2 \lambda p_{,j} \dot{x}_j\, d\Omega \end{aligned} \tag{36}$$

$$\left(\int_\Omega \lambda s\, d\Omega\right)' = \int_\Omega \lambda_{,i} s \dot{x}_i\, d\Omega + \int_\Omega \lambda \dot{s}\, d\Omega + \int_\Omega \lambda s \dot{x}_{j,j}\, d\Omega \tag{37}$$

Therefore, we have

$$\begin{aligned} R' = &\int_\Gamma \lambda_{,i} q \dot{x}_i\, d\Gamma + \int_\Gamma \lambda q\, (\dot{d\Gamma}) - \int_\Gamma \lambda_{,i} p_{,i} \dot{x}_j n_j\, d\Gamma + \int_\Gamma \mu p_{,j} \dot{x}_j\, d\Gamma \\ &- \int_\Omega (\lambda_{,ii} + k^2\lambda) p_{,j} \dot{x}_j\, d\Omega + \int_\Gamma k^2 \lambda p \dot{x}_j n_j\, d\Gamma + \int_\Omega \lambda_{,i} s \dot{x}_i\, d\Omega + \int_\Omega \lambda \dot{s}\, d\Omega \\ &+ \int_\Omega \lambda s \dot{x}_{j,j}\, d\Omega + \int_\Gamma \lambda \dot{q}\, d\Gamma - \int_\Gamma \mu \dot{p}\, d\Gamma + \int_\Omega (\lambda_{,ii} + k^2\lambda) \dot{p}\, d\Omega \end{aligned} \tag{38}$$

Then, we observe

$$\begin{aligned} I_1' + R' = &\int_{\Gamma_p \cup \Gamma_q} \left(\lambda + \frac{\partial f}{\partial q}\right) \dot{q}\, d\Gamma - \int_{\Gamma_p \cup \Gamma_q} \left(\mu - \frac{\partial f}{\partial p}\right) \dot{p}\, d\Gamma \\ &+ \int_\Omega \left(\lambda_{,ii} + k^2\lambda + \frac{\partial g}{\partial p}\right) \dot{p}\, d\Omega \\ &+ \int_\Gamma \lambda_{,i} q \dot{x}_i\, d\Gamma + \int_\Gamma \lambda q (\dot{d\Gamma}) - \int_\Gamma \lambda_{,i} p_{,i} \dot{x}_j n_j\, d\Gamma \\ &+ \int_\Gamma \mu p_{,j} \dot{x}_j\, d\Gamma - \int_\Omega (\lambda_{,ii} + k^2\lambda) p_{,j} \dot{x}_j\, d\Omega + \int_\Gamma k^2 \lambda p \dot{x}_j n_j\, d\Gamma \\ &+ \int_\Omega \lambda_{,i} s \dot{x}_i\, d\Omega + \int_\Omega \lambda \dot{s}\, d\Omega + \int_\Omega \lambda s \dot{x}_{j,j}\, d\Omega \end{aligned} \tag{39}$$

Here we assume λ satisfies the following differential equation and the boundary conditions:

$$\lambda_{,ii} + k^2\lambda + \frac{\partial g}{\partial p} = 0 \qquad \text{in } \Omega \tag{40}$$

$$\lambda = -\frac{\partial f}{\partial q} \qquad \text{on } \Gamma \tag{41}$$

$$\mu = \frac{\partial \lambda}{\partial n} = \frac{\partial f}{\partial p} \qquad \text{on } \Gamma \tag{42}$$

Then, Eq.(39) becomes

$$\begin{aligned} I_1' + R' = & \int_{\Gamma_q} \left(\lambda + \frac{\partial f}{\partial q}\right) \dot{q}\, d\Gamma - \int_{\Gamma_p} \left(\mu - \frac{\partial f}{\partial p}\right) \dot{p}\, d\Gamma \\ & + \int_\Gamma \lambda_{,i} q \dot{x}_i\, d\Gamma + \int_\Gamma \lambda q (\dot{d\Gamma}) - \int_\Gamma \lambda_{,i} p_{,i} \dot{x}_j n_j\, d\Gamma \\ & + \int_\Gamma \mu p_{,j} \dot{x}_j\, d\Gamma + \int_\Omega \frac{\partial g}{\partial p} p_{,j} \dot{x}_j\, d\Omega + \int_\Gamma k^2 \lambda p \dot{x}_j n_j\, d\Gamma \\ & + \int_\Omega \lambda_{,i} s \dot{x}_i\, d\Omega + \int_\Omega \lambda \dot{s}\, d\Omega + \int_\Omega \lambda s \dot{x}_{j,j}\, d\Omega \end{aligned} \tag{43}$$

Thus, the gradient of the augmented objective function becomes as follows:

$$\begin{aligned} J' = & I' + 2\,\mathrm{Re}[R'] = 2\mathrm{Re}[I_1' + R'] + 2\mathrm{Re}\left[\int_\Omega \frac{\partial g}{\partial p} \dot{p}\, d\Omega\right] + \int_\Omega g \dot{x}_{i,i}\, d\Omega \\ = & 2\,\mathrm{Re}\left[\int_{\Gamma_q} \left(\lambda + \frac{\partial f}{\partial q}\right) \dot{q}\, d\Gamma - \int_{\Gamma_p} \left(\mu - \frac{\partial f}{\partial p}\right) \dot{p}\, d\Gamma + \int_\Gamma \lambda_{,i} q \dot{x}_i\, d\Gamma + \int_\Gamma \lambda q (\dot{d\Gamma}) \right. \\ & - \int_\Gamma \lambda_{,i} p_{,i} \dot{x}_j n_j\, d\Gamma + \int_\Gamma \mu p_{,j} \dot{x}_j\, d\Gamma + \int_\Gamma k^2 \lambda p \dot{x}_j n_j\, d\Gamma \\ & \left. + \int_\Omega \frac{\partial g}{\partial p} p_{,j} \dot{x}_j\, d\Omega + \int_\Omega \lambda_{,i} s \dot{x}_i\, d\Omega + \int_\Omega \lambda \dot{s}\, d\Omega + \int_\Omega \lambda s \dot{x}_{j,j}\, d\Omega \right] \\ & + \int_\Gamma f\, (\dot{d\Gamma}) + \int_\Omega g \dot{x}_{i,i}\, d\Omega \end{aligned} \tag{44}$$

Assume that there are concentrated sound sources located in the field, and s can be written in the form given by Eq.(7). Then, the sensitivity coefficient of s becomes as

$$\dot{s} = \sum_\alpha s^\alpha \delta_{,i}(x - z^\alpha)\, \dot{x}_i \tag{45}$$

provided that the positions of the concentrated sound sources do not change through shape changes. Then, we find

$$\begin{aligned}\left(\int_\Omega \lambda s\, d\Omega\right)' &= \sum_\alpha s^\alpha \int_\Omega \left[\lambda_{,i}\,\delta(x-z^\alpha)\,\dot{x}_i + \lambda\,\delta(x-z^\alpha)_{,i}\,\dot{x}_i + \lambda\,\delta(x-z^\alpha)\,\dot{x}_{i,i}\right] d\Omega \\ &= \sum_\alpha s^\alpha \int_\Omega \left[\lambda\,\delta(x-z^\alpha)\,\dot{x}_i\right]_{,i} d\Omega \\ &= \sum_\alpha s^\alpha \int_\Gamma \lambda\,\delta(x-z^\alpha)\,\dot{x}_i\, n_i\, d\Gamma \\ &= 0 \end{aligned} \tag{46}$$

Next, we assume that $g(p,\bar{p})$ is also defined at discrete points in the domain as

$$g = \sum_\beta G(p,\bar{p})\delta(x-z^\beta) \tag{47}$$

Then, the related terms in Eq.(44) become as follows:

$$\begin{aligned}&2\,\mathrm{Re}\left[\int_\Omega \frac{\partial g}{\partial p} p_{,j}\,\dot{x}_j\, d\Omega\right] + \int_\Omega g\,\dot{x}_{i,i}\, d\Omega \\ &\quad = 2\,\mathrm{Re}\left[\sum_\beta \int_\Omega \frac{\partial G}{\partial p} p_{,j}\,\dot{x}_j\,\delta(x-z^\beta)\, d\Omega\right] + \sum_\beta \int_\Omega G\,\dot{x}_{i,i}\delta(x-z^\beta)\, d\Omega \\ &\quad = 2\,\mathrm{Re}\left[\sum_\beta \frac{\partial G}{\partial p}(z^\beta)\, p_{,j}(z^\beta)\,\dot{z}^\beta\right] + \sum_\beta G(z^\beta)\,\dot{z}^\beta_{i,i} = 0 \end{aligned} \tag{48}$$

because z^β are assumed to be the measuring points and may not move under shape change, that is, $\dot{z}^\beta_j = 0$, $\dot{z}^\beta_{i,i} = 0$.
In these cases, we finally obtain

$$\begin{aligned} J' = 2\,\mathrm{Re}\Bigg[&\int_{\Gamma_q}\left(\lambda + \frac{\partial f}{\partial q}\right)\dot{q}\, d\Gamma - \int_{\Gamma_p}\left(\mu - \frac{\partial f}{\partial p}\right)\dot{p}\, d\Gamma \\ &+ \int_\Gamma \lambda_{,i}\, q\,\dot{x}_i\, d\Gamma + \int_\Gamma \lambda q\,(\dot{d\Gamma}) - \int_\Gamma \lambda_{,i}\, p_{,i}\,\dot{x}_j n_j\, d\Gamma \\ &+ \int_\Gamma \mu\, p_{,j}\,\dot{x}_j\, d\Gamma + \int_\Gamma k^2 \lambda\, p\,\dot{x}_j n_j\, d\Gamma\Bigg] + \int_\Gamma f\,(\dot{d\Gamma}) \end{aligned} \tag{49}$$

where λ, adjoint variable, is the solution of the following boundary value problem:

$$\lambda_{,ii}(x) + k^2\lambda(x) + \sum_{\beta} \frac{\partial G}{\partial p}\delta(x - z^{\beta}) = 0 \qquad \text{in } \Omega \tag{50}$$

$$\lambda(x) = -\frac{\partial f}{\partial q}(x) \qquad \text{on } \Gamma_p \tag{51}$$

$$\mu(x) = \frac{\partial \lambda}{\partial n}(x) = \frac{\partial f}{\partial p}(x) \qquad \text{on } \Gamma_q \tag{52}$$

4 Topology optimization

4.1 Level-set method

We use a level-set method approach [Yamada, Izui, Nishiwaki, and Takezawa (2010)] for controlling the shape and topology of the domain. Level-set function is a scalar function of the point in the domain. In order to obtain an optimum topology, a fixed design domain, in which the optimum domain is included, is usually defined. The level-set function $\phi(\mathbf{x})$ is defined in the fixed-design space and takes the value as follows:

$$0 < \phi(\mathbf{x}) \leq 1, \quad \forall \mathbf{x} \in \Omega \setminus \partial\Omega \tag{53}$$

$$\phi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega \tag{54}$$

$$-1 \leq \phi(\mathbf{x}) < 0, \quad \forall \mathbf{x} \in D \setminus \Omega \tag{55}$$

By considering this level-set function as the design variable, we can control the shape and topology of the domain. In the level-set approach in [Yamada, Izui, Nishiwaki, and Takezawa (2010)], the objective function is again augmented by adding a regularization term, as follows:

$$F = J + \int_D \tau |\nabla\phi|^2 \, d\Omega \tag{56}$$

The variation of the level set function with respect to fictitious time t is assumed to be proportional to the gradient of the objective function, i.e.,

$$\frac{\partial \phi}{\partial t} = -K(\phi)\frac{dF}{d\phi} = -K(\phi)\left(\frac{dJ}{d\phi} - \tau\nabla^2\phi\right) \tag{57}$$

Once $dJ/d\phi$ is obtained by BEM, the distribution of ϕ can be obtained by solving the above Eq.(57) by using FEM for a fixed design domain for which the level-set function ϕ is defined [Yamada, Izui, Nishiwaki, and Takezawa (2010)]. The fixed design domain is usually of simple geometry like a rectangular solid domain, and can be discretized with simple structured mesh. Therefore, FEM analysis for the fixed design domain is very simple and can be done efficiently.

4.2 Topological sensitivity

When using the boundary element method for topology optimizations of acoustic fields, topological sensitivity of the objective function is required. The topological sensitivity can be used as the sensitivity of the augmented objective function with respect to the level set function [Yamada, Izui, Nishiwaki, and Takezawa (2010)].

Let us assume that an infinitesimal spherical obstacle Ω_ε with a radius ε is placed in the domain. Let the boundary of Ω_ε be denoted by Γ_ε. Then the augmented objective function Eq.(21) for $\Omega \setminus \Omega_\varepsilon$ becomes as

$$\begin{aligned} J+\delta J = &\int_{\Gamma\cup\Gamma_\varepsilon}\left[f(p,\bar{p},q,\bar{q})+\frac{\partial f}{\partial p}\delta p+\frac{\partial f}{\partial \bar{p}}\delta \bar{p}+\frac{\partial f}{\partial q}\delta q+\frac{\partial f}{\partial \bar{q}}\delta \bar{q}\right]d\Gamma \\ &+\int_{\Omega\setminus\Omega_\varepsilon}\left[g(p,\bar{p})+\frac{\partial g}{\partial p}\delta p+\frac{\partial g}{\partial \bar{p}}\delta \bar{p}\right]d\Omega \\ &+\left[\int_{\Gamma\cup\Gamma_\varepsilon}\lambda\,(q+\delta q)\,d\Gamma-\int_{\Omega\setminus\Omega_\varepsilon}\lambda_{,i}\,(p_{,i}+\delta p_{,i})\,d\Omega\right. \\ &\left.+\int_{\Omega\setminus\Omega_\varepsilon}k^2\lambda(p+\delta p)\,d\Omega+\int_{\Omega\setminus\Omega_\varepsilon}\lambda s\,d\Omega\right] \end{aligned} \tag{58}$$

while, in view of Eqs.(21) and (23), J is given as

$$\begin{aligned} J = &\int_\Gamma f(p,\bar{p},q,\bar{q})\,d\Gamma+\int_\Omega g(p,\bar{p})\,d\Omega \\ &+2\mathrm{Re}\left[\int_\Gamma \lambda q\,d\Gamma-\int_\Omega \lambda_{,i}p_{,i}\,d\Omega+\int_\Omega k^2\lambda p\,d\Omega+\int_\Omega \lambda s\,d\Omega\right] \end{aligned} \tag{59}$$

By subtracting Eq.(59) from Eq.(58) and rearranging it appropriately, we obtain the variation of the augmented objective function as follows:

$$\begin{aligned} \delta J = 2\mathrm{Re}&\left[\int_{\Gamma_\varepsilon}\lambda q\,d\Gamma+\int_{\Omega_\varepsilon}\lambda_{,i}p_{,i}\,d\Omega-\int_{\Omega_\varepsilon}k^2\lambda p\,d\Omega\right. \\ &+\int_{\Gamma_p\cup\Gamma_q\cup\Gamma_\varepsilon}\left(\lambda+\frac{\partial f}{\partial q}\right)\delta q\,d\Gamma-\int_{\Gamma_p\cup\Gamma_q\cup\Gamma_\varepsilon}\left(\mu-\frac{\partial g}{\partial p}\right)\delta p\,d\Gamma \\ &\left.+\int_{\Omega\setminus\Omega_\varepsilon}\left(\lambda_{,ii}+k^2\lambda+\frac{\partial g}{\partial p}\right)d\Omega\right] \end{aligned} \tag{60}$$

where it is assumed that f and g are not defined on Γ_ε and Ω_ε, respectively, and s is the summation of concentrated sound sources and does not exist within Ω_ε.

Now, we consider an adjoint field λ obtained as the solution of the following boundary value problem:

$$\lambda_{,ii} + k^2\lambda + \frac{\partial g}{\partial p} = 0 \quad \text{in } \Omega \tag{61}$$

$$\lambda = -\frac{\partial f}{\partial g} \quad \text{on } \Gamma_p \tag{62}$$

$$\mu = \frac{\partial \lambda}{\partial n} = \frac{\partial f}{\partial p} \quad \text{on } \Gamma_q \tag{63}$$

Also, δp and δq should be zero on Γ_p and Γ_q, respectively, because they are the quantities specified as the boundary condition on these parts of the boundary. Then, we have

$$\delta J = 2\text{Re}\left[\int_{\Gamma_\varepsilon} \lambda q \, d\Gamma + \int_{\Omega_\varepsilon} \lambda_{,i} p_{,i} \, d\Omega - \int_{\Omega} k^2 \lambda p \, d\Omega\right] \tag{64}$$

When Ω_ε is a infinitesimal sphere, we can evaluate the integrals in Eq.(64), as follows:

$$\int_{\Gamma_\varepsilon} \lambda q \, d\Gamma = \int_0^{4\pi} \lambda^0 p^0_{,i} n_i \varepsilon^2 \, d\sigma = \lambda^0 p^0_{,i} \varepsilon^2 \int_0^{4\pi} n_i \, d\sigma = 0 \tag{65}$$

where σ is the solid angle of the sphere surface, a superscript 0 denotes that the quantity is at the center of the infinitesimal sphere, and

$$\int_{\Omega_\varepsilon} \lambda_{,i} p_{,i} \, d\Omega = \frac{4}{3}\pi\varepsilon^3 \lambda^0_{,i} p^0_{,i} \tag{66}$$

$$\int_{\Omega_\varepsilon} k^2 \lambda p \, d\Omega = \frac{4}{3}\pi\varepsilon^3 k^2 \lambda^0 p^0 \tag{67}$$

Therefore, the topological sensitivity of the augmented objective function finally becomes as

$$J' = 2\text{Re}\left[\lambda^0_{,i} p^0_{,i} - k^2 \lambda^0 p^0\right] \tag{68}$$

In Fig. 2 is shown a flow diagram of the topology optimization of acoustic field using the present boundary element topology sensitivity analysis.

The field where an acoustic field having the optimum topology is designed is called a fixed design space and is discretized with cubic structured mesh. By using the initial shape of the boundary of the field, initial values of the level-set function at the grid points of the structured mesh is determined. Then, boundary element analyses for the acoustic field and for the adjoint field are performed, and the value of the objective function and the topological sensitivities at the grid points are calculated. Eq.(57) is then solved for the fixed design domain using FEM and the distribution of the level-set function is updated. This process is repeated until the convergence is achieved.

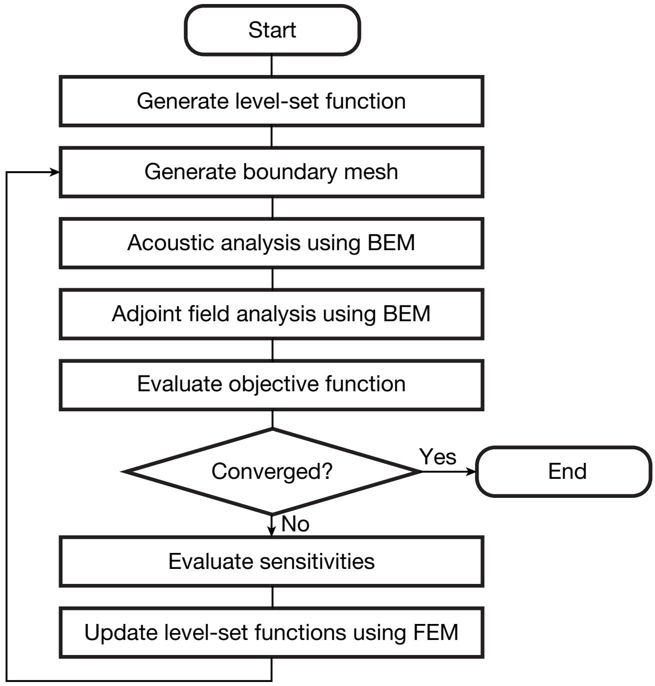

Figure 2: A flow diagram of topology optimization of acoustic field.

5 Numerical examples

5.1 2D example

We consider a rectangular region as shown in Fig. 3 as a test example model for 2-D sensitivity analysis. We define the following objective function to make the sound pressure at the center of the cavity close to a certain value p_0.

$$J = \int_{\Omega} \frac{1}{2} |p(x,y) - p_0|^2 \, \delta(x-2.5)\, \delta(y-0.5)\, d\Omega \tag{69}$$

The design variable is assumed to be the width L of the rectangular cavity. We divide the boundary of the rectangular cavity into 120 quadratic elements uniformly. The exact solution of the sound pressure is given as

$$p(x) = \tan(kL)\sin(kx) + \cos(kx) \tag{70}$$

where k is the wave number.

We assume that the target sound pressure is given as $p_0 = 0$. Then, the sensitivity

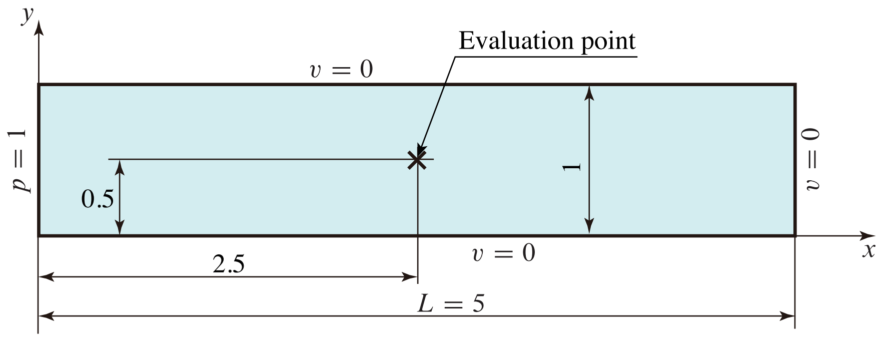

Figure 3: A rectangular cavity model.

of J with respect to L becomes

$$\frac{dJ}{dL} = \left[\tan(kL)\sin(kx) + \cos(kx)\right] \frac{k\sin(kx)}{\cos^2(kL)} \tag{71}$$

The governing equation and the boundary condition of the adjoint problem becomes, as follows:

$$\lambda_{,ii}(x,y) + k^2\lambda(x,y) + (p(x,y) - p_0)\delta(x-2.5)\delta(y-0.5) = 0, \quad (x,y) \in \Omega \tag{72}$$

$$\lambda = 0 \quad \text{on } \Gamma_p \tag{73}$$

$$\mu = \frac{\partial \lambda}{\partial n} = 0 \quad \text{on } \Gamma_q \tag{74}$$

We show in Fig. 4 the distribution of the adjoint solution obtained by BEM and in Tab. 1 the sensitivity values and their errors obtained for various discretization models. The sensitivity errors are found to decrease in accordance with the increase of the number of elements and become accurate.

Table 1: Obtained sensitivity values and their errors.

Number of elements	Sensitivity	Error [%]
48	-1.88909×10^{-1}	5.677
120	-1.79874×10^{-1}	0.623
240	-1.78997×10^{-1}	0.132
480	-1.78841×10^{-1}	0.045
960	-1.78814×10^{-1}	0.030

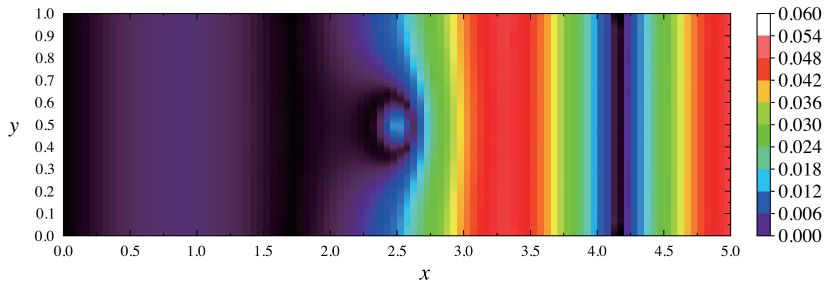

Figure 4: Distribution of the obtained adjoint variable λ.

5.2 *3D example for topology optimization*

Next we consider a sound scatterer to reduce the sound pressure at an observation point shown in Fig. 5 as an example of topology optimization. The fixed design space is the same as the initial sound scatterer. Hence, optimum topology of the scatterer is sought within this fixed design space. We also added a minimum volume constraint, 40% of the fixed design space, to the objective function. Fig. 6 shows the initial boundary mesh of the sound scatterer. Fast-multipole BEM for wide-band frequencies [Gumerov and Duraiswami (2009); Wolf and Lele (2010)] is used to calculate the original acoustic problem and the adjoint problem. We show in Fig. 7 the obtained geometry of the scatterer for 340Hz. We find that the present approach can be applied to such a practical problem with three-dimensional complicated geometry.

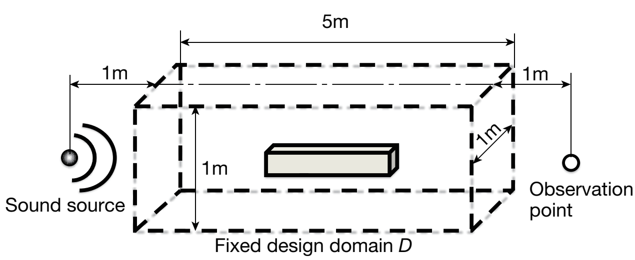

Figure 5: A sound scatterer model to reduce the sound pressure at the observation point.

6 Concluding remarks

An adjoint method approach based on BEM has been shown for shape and topology optimization of acoustic field. Because the BEM is based on boundary only dis-

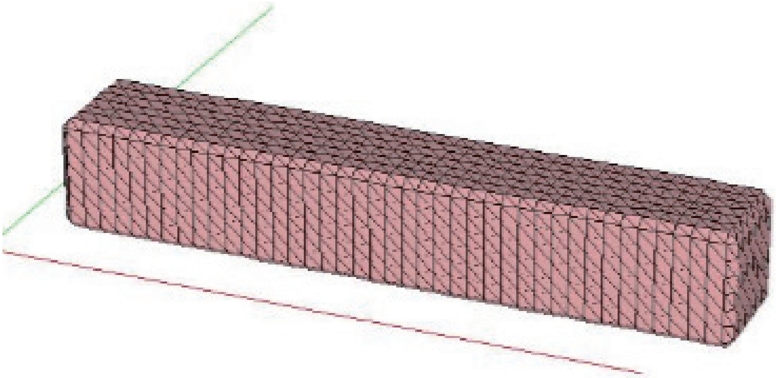

Figure 6: Boundary mesh of the initial shape of the sound scatterer.

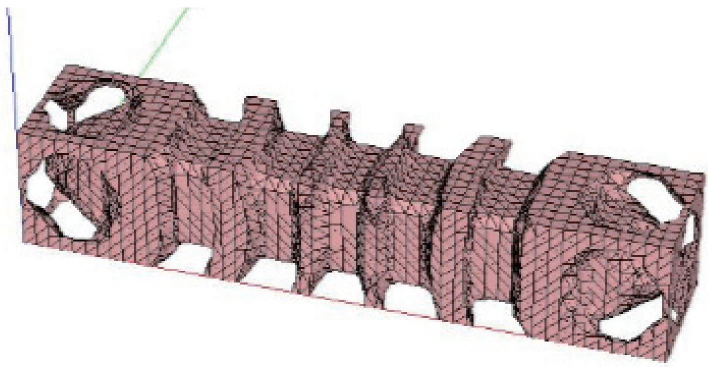

Figure 7: Obtained scatterer geometry.

cretization, the objective function is assumed to be a functional defined only with boundary sound pressure and particle velocity, and with quantities only at a finite number of internal points. An adjoint system is defined so that the unknown sensitivities of the sound pressure and particle velocity on the boundary and unknown quantities in the domain are eliminated from the gradient of the objective function. Then, the shape sensitivity and topological sensitivity expressions defined only with the original acoustic field and adjoint field are derived. On each calculation step of the gradient of the objective function, BEM calculations are repeated only for the original problem and the adjoint problem. Some numerical examples have been provided, and have demonstrated the effectiveness of the present approach.

References

Arora, J. S. (1993): An exposition of the material derivative approach for structural shape sensitivity analysis. *Comput. Methods. Appl. M.*, vol. 105, pp. 41–62.

Burton, A. J.; Miller, G. F. (1971): The application of integral equation methods to the numerical solution of some exterior boundary-value problems. *Proc. Roy. Soc. Lond. A.*, vol. 323, pp. 201–210.

Gumerov, N. A.; Duraiswami, R. (2009): A broadband fast multipole accelerated boundary element method for the three dimensional Helmholtz equation. *J. Acoust. Soc. Am.*, vol. 125, pp. 191–205.

Haug, E. J.; Choi, K. K.; Komkov, V. (1986): *Design Sensitivity Analysis of Structural Systems*. Academic Press.

Koo, B. W. (1997): Shape design sensitivity analysis of acoustic problems using a boundary element method. *Computers & Structures*, vol. 65, no. 5, pp. 713–719.

Matsumoto, T.; Tanaka, M.; Miyagawa, M.; Ishii, N. (1993): Optimum design of cooling lines in injection moulds by using boundary element design sensitivity analysis. *Finite Elem. Anal. Des.*, vol. 14, pp. 177–185.

Matsumoto, T.; Tanaka, M.; Yamada, Y. (1995): Design sensitivity analysis of steady-state acoustic problems using boundary integral equation formulation. *JSME Int. J.*, vol. 38C, pp. 9–16.

Rokhlin, V. (1985): Rapid solution of integral equations of classical potential theory. *J. Comput. Phys.*, vol. 60, pp. 187–207.

Wang, S. Y.; Lim, K. M.; Khoo, B. C.; Wang, M. Y. (2007): A geometric deformation constrained level set method for structural shape and topology optimization. *Computer Modeling in Engineering & Sciences*, vol. 18, pp. 155–181.

Wolf, W. R.; Lele, S. K. (2010): Wideband fast multipole boundary element method: Application to acoustic scattering from aerodynamic bodies. *J. Numer. Meth. Fluids*. doi: 10.1002/fld.2486.

Yamada, T.; Izui, K.; Nishiwaki, S.; Takezawa, A. (2010): A topology optimization method based on the level set method incorporating a fictitious interface energy. *Comput. Methods Appl. Mech. Engrg.*, vol. 199, pp. 2876–2891.

Zheng, C. J.; Matsumoto, T.; Takahashi, T.; Chen, H. B. (2011): Explicit evaluation of hypersingular boundary integral equations for acoustic sensitivity analysis based on direct differentiation method. *Eng. Anal. Bound. Elem.*, vol. 35, pp. 1225–1235.