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Some Fundamental Properties of Lattice Boltzmann Equation for Two Phase Flows

Qin Lou [1], Zhaoli Guo [1,2] and Chuguang Zheng[1]

Abstract: Due to the mesoscopic and kinetic nature, the lattice Boltzmann equation (LBE) method has become an efficient and powerful tool for modeling and simulating interfacial dynamics of multi-phase flows. In this work we discuss several fundamental properties of two-phase LBE models. Particularly, the effects of force discretization, spurious currents in the vicinity of interfaces, and checkerboard effects with the underlying lattices, are investigated.

Keywords: Two-phase flows, lattice Boltzmann equation.

1 Introduction

Multiphase flows are frequently encountered in the nature world and industrial applications. Owing to the complex interfacial dynamics involving large space and time scales, it is a challenging task to model and simulate such flows. Physically, the interfacial phenomena in multi-phase systems are the natural consequences of microscopic interactions among fluid molecules. Therefore, the multi-phase flows can be effectively captured once the microscopic inter-molecular interactions are correctly modeled.

Different numerical methods have been developed for simulating multi-phase flows from different viewpoints, such as phase field model [Jacqmin (1999)], volume of fluid method [Hirt and Nichols (1981)], level set method [Sussman, Smereka, and Osher (1994)], and the lattice Boltzmann equation (LBE) method [Shan and Chen (1993); Shan and Chen (1994); Gunstensen and Rothman (1991); Grunau, Chen, and Chen (1993); Swift, Osborn, and Yeomans (1995); Swift, Orlandini, Osborn, and Yeomans (1996); Inamuro, Konishi, and Ogino (2000); Inamuro, Ogata, Tajima, and Konishi (2004); He, Shan, and Doolen (1998); He, Chen, Zhang (1999)]. The former three methods are constructed based on continuous

[1] State Key Laboratory of Coal Combustion, Huazhong University of Science and Technology, Wuhan, China.
[2] Corresponding author: zlguo@hust.edu.cn

hydrodynamic equations, while the LBE method is based on mesoscopic kinetic theory, which has many advantages in modeling multi-phase flows.

Four type of multi-phase LBE models have been proposed in the past decades. The first type is the Color model proposed by Gunstensen *et al* [Gunstensen and Rothman (1991)] based on a lattice gas automation model [Rothman and Keller (1998)], in which two different fluids are denoted by red and blue particle distribution functions, and the inter-particle interactions are modeled by the local color gradient. Another type is pseudo-potential model proposed by Shan and Chen [Shan and Chen (1993); Shan and Chen (1994)], in which an artificial potential is included. The third type of LBE models for multi-phase flows was due to Swift *et al.* [Swift, Osborn, and Yeomans (1995); Swift, Orlandini, Osborn, and Yeomans (1996)], which is constructed based on the free-energy of a multi-phase system. The last type of multi-phase LBE models are base on certain kinetic equations [He, Shan, and Doolen (1998); He, Chen, Zhang (1999)], which has solid physical foundations.

Even though these models are developed in different ways, there are some common issues among them. In this work we will investigate several fundamental points in two-phase LBE models, aiming to clarify some confusion and misunderstandings about the LBE method.

2 LBE model for two-phase flows

Although the available two-phase LBE models were constructed from different viewpoints, it can be shown that they can all be regrouped into a standard evolution equation with a forcing term [He, Shan, and Doolen (1998)],

$$f_i(\mathbf{x}+\mathbf{c}_i\delta t, t+\delta t) - f_i(\mathbf{x},t) = -\frac{1}{\tau}[f_i(\mathbf{x},t) - f_i^{eq}(\mathbf{x},t)] + \delta t F_i, \quad i = 0 \sim b-1, \quad (1)$$

where $f_i(\mathbf{x},t)$ is the distribution function at position $\mathbf{x}$ and time t moving with discrete velocity $\mathbf{c}_i$, τ is the relaxation time, δt is the time step, b is the number of discrete velocities, and f_i^{eq} is the equilibrium distribution function given by

$$f_i^{eq} = w_i \rho \left[1 + \frac{\mathbf{c}_i \cdot \mathbf{u}}{c_s^2} + \frac{(\mathbf{c}_i \cdot \mathbf{u})^2}{2c_s^4} - \frac{\mathbf{u} \cdot \mathbf{u}}{2c_s^2}\right], \quad (2)$$

where w_i is the weighting factor, ρ and $\mathbf{u}$ are the local density and velocity, and c_s is the sound speed related to the lattice speed $c = \delta x/\delta t$, where δ_x is the lattice spacing; F_i is the forcing term representing the effect of the interaction force $\mathbf{F}$. Different formulations of F_i have been proposed, and here we consider the model

proposed by He *et al.* [He, Shan, and Doolen (1998)],

$$F_i = \left(1 - \frac{1}{2\tau}\right) \frac{(\mathbf{c}_i - \mathbf{u}) \cdot \mathbf{F}}{\rho c_s^2} f_i^{eq}. \tag{3}$$

For an isothermal system, the fluid density and velocity are defined as

$$\rho = \sum_i f_i, \quad \rho \mathbf{u} = \sum_i \mathbf{c}_i f_i + \frac{\delta t}{2} \mathbf{F}. \tag{4}$$

For a van der Waals fluid, the interaction force can be written as [He, Shan, and Doolen (1998)]

$$\mathbf{F} = \nabla \rho c_s^2 - \rho \nabla(\mu_0 - \kappa \nabla^2 \rho) = \nabla(\rho {c_s}^2 - p_0) + \rho \kappa \nabla \nabla^2 \rho, \tag{5}$$

where $\mu = \mu_0 - \kappa \nabla^2 \rho$ is the chemical potential with μ_0 being the chemical potential in the bulk region, κ is a parameter that controls the surface tension, and p_0 is the thermodynamic pressure related to the density through certain equation of state. In the literature the first and second expressions of the force in Eq. (5) are usually termed as "potential form" and "pressure form", respectively [Wagner (2003); Lee, Fischer (2006)]. From the thermodynamic relations $\nabla p_0 = \rho \nabla \mu$, it is apparent that the two formations are totally identical mathematically. However, their discrete versions may give quite different results due to numerical errors, and these numerical differences may have significant influences on the interfacial dynamics.

The interaction force is closely related with free-energy in two-phase systems. Without loss of generality, in this work we consider the following bulk free-energy [Jamet, Lebaigue, Coutris, and Delhaye (2001)],

$$E_0 \approx \beta(\rho - \rho_v^{sat})^2 (\rho - \rho_l^{sat})^2,$$

where the parameters β and κ are related to the thickness of an interface D and the surface-tension σ,

$$\kappa = \frac{\beta D^2 (\rho_l^{sat} - \rho_v^{sat})^2}{8}, \quad \sigma = \frac{(\rho_l^{sat} - \rho_v^{sat})^3}{6} \sqrt{2\kappa\beta},$$

where ρ_v^{sat} and ρ_l^{sat} are the vapor and liquid densities at saturation, respectively. From the free energy E_0 one can obtain the thermodynamic pressure p_0 and the chemical potential μ_0 [Lee, Fischer (2006)],

$$p_0(\rho) = \rho \frac{\partial E_0}{\partial \rho} - E_0, \quad \mu_0 = \frac{\partial E_0}{\partial \rho}.$$

3 Effects of force discretization

The gradients in the interaction force, either in potential formulation or pressure one, must be discretized in practical applications. Two finite-difference schemes are available in the literature. The first one is the second-order isotropic central scheme (ICS),

$$\nabla\psi \approx \nabla^c\psi(\mathbf{x}) = \sum_{i\neq 0} \frac{w_i \mathbf{c}_i \psi(\mathbf{x}+\mathbf{c}_i\delta t)}{c_s^2 \delta t}, \tag{6}$$

where ψ is an arbitrary function, and the discrete Laplacian operator is given by

$$\nabla^2\psi(\mathbf{x}) \approx \nabla_c^2\psi(\mathbf{x}) = \sum_{i\neq 0} \frac{2w_i[\psi(\mathbf{x}+\mathbf{c}_i\delta t)-\psi(\mathbf{x})]}{c_s^2 \delta t^2}. \tag{7}$$

The force $\mathbf{F}$ with the ICS is denoted by $\mathbf{F}^{ics}$, and the corresponding forcing term is denoted by F_i^{ics}. The fluid velocity is also calculated using this discrete force,

$$\rho\mathbf{u} = \sum_i \mathbf{c}_i f_i + \frac{\delta t}{2}\mathbf{F}^{ics}. \tag{8}$$

The second force-discretization method was proposed by Lee and Fisher, which is a mixed scheme (MS) [Lee, Fischer (2006)],

$$F_i^{ms} = \frac{1}{2}\left[\left(1-\frac{1}{\tau}\right)F_i^c + F_i^b\right], \tag{9}$$

where the subscripts c and b denote the second-order central-difference and biased-difference, respectively. F_i^c and F_i^b in potential form are defined by

$$F_i^c = \frac{1}{\rho c_s^2}\left[\nabla_i^c \rho c_s^2 - \rho\nabla_i^c\mu - u\cdot(\nabla^c\rho c_s^2 - \rho\nabla^c\mu)\right] f_i^{eq}, \tag{10a}$$

$$F_i^b = \frac{1}{\rho c_s^2}\left[\nabla_i^b \rho c_s^2 - \rho\nabla_i^b\mu - u\cdot(\nabla^b\rho c_s^2 - \rho\nabla^b\mu)\right] f_i^{eq}, \tag{10b}$$

where the directional derivatives ∇_i^c and ∇_i^b are defined by

$$\nabla_i^c\psi(\mathbf{x}) = \frac{\psi(\mathbf{x}+\mathbf{c}_i\delta t)-\psi(\mathbf{x}-\mathbf{c}_i\delta t)}{2\delta t}, \tag{11a}$$

$$\nabla_i^b\psi(\mathbf{x}) = \frac{-\psi(\mathbf{x}+2\mathbf{c}_i\delta t)+4\psi(\mathbf{x}+\mathbf{c}_i\delta t)-3\psi(\mathbf{x})}{2\delta t}. \tag{11b}$$

Expressions in pressure form can also be calculated similarly. It should be noted that the homogenous derivative $\nabla^c\psi$ in above equations is the same as the isotropic central scheme, and ∇^b is defined as

$$\begin{aligned}\nabla^b\psi(\mathbf{x}) &= \sum_{i\neq 0}\frac{w_i\mathbf{c}_i[-\psi(\mathbf{x}+2\mathbf{c}_i\delta t)+4\psi(\mathbf{x}+\mathbf{c}_i\delta t)-3\psi(\mathbf{x})]}{2c_s^2\delta t}\\ &= -\sum_{i\neq 0}\frac{w_i\mathbf{c}_i\psi(\mathbf{x}+2\mathbf{c}_i\delta t)}{2c_s^2\delta t}+2\sum_{i\neq 0}\frac{w_i\mathbf{c}_i\psi(\mathbf{x}+\mathbf{c}_i\delta t)}{c_s^2\delta t}. \end{aligned} \tag{12}$$

Equation (12) indicates that the biased scheme is actually a combination of two isotropic central schemes with the nearest and second nearest neighboring nodes as the stencils. It is also noted that the velocity in the Lee-Fischer model is still defined by Eq. (8) where only the isotropic central scheme is involved.

In the continuum limit, F_i^{ms} is identical to F_i^{ics}. However, they are different at discrete level due to numerical errors. It can be shown that the LBE with the mixed scheme does not conserve the total mass. For instance, for the D2Q9 LBE model it can be verified that

$$\begin{aligned}&\sum_i F_i^{ms} =\\ &-\frac{3c_s^2\delta t^3}{8}\left[(c_s^2+u_x^2)\partial_x^4\rho+(c_s^2+u_y^2)\partial_y^4\rho+4u_xu_y(\partial_x^3\partial_y\rho+\partial_x\partial_y^3\rho)\right.\\ &\left.+2(c_s^2+u_x^2+u_y^2)\partial_x^2\partial_y^2\rho\right]+\frac{3\rho\delta t^3}{8}\left[(c_s^2+u_x^2)\partial_x^4\mu+(c_s^2+u_y^2)\partial_y^4\mu\right.\\ &\left.+4u_xu_y(\partial_x^3\partial_y\mu+\partial_x\partial_y^3\mu)+2(c_s^2+u_x^2+u_y^2)\partial_x^2\partial_y^2\mu\right]+O(\delta t^4). \end{aligned} \tag{13}$$

Furthermore, it can be shown that $\sum_i \mathbf{c}_i F_i^{ms}$ contains some terms like $\mathbf{uu}\cdot\nabla\nabla^2\rho$, which means that F_i^{ms} is non-Galilean invariant. This phenomenon is shown in Fig. 1 where the potential form is used to represent the force. Initially, a circular droplet with density 1.0 and radius 20 is put at the center of a $L_x\times L_y = 128\times 127$ channel, whose walls move with a constant velocity along the x direction. The rest part of the channel is filled with the vapor with density 0.2. The system is set to be rest at first, and then the channel walls start to move with a constant velocity $\mathbf{u}_w = (u_x,\ u_y) = (0.01,\ 0.0)$. Periodic boundary conditions are employed at the inlet and outlet, and the half-way bounce back scheme [Ladd (1994)] is applied to the moving walls. It is observed that the drop remains a circular shape when the ICS is used, while large deformation is observed when the mixed scheme is employed. Similar phenomena are also observed when the force is expressed in pressure form.

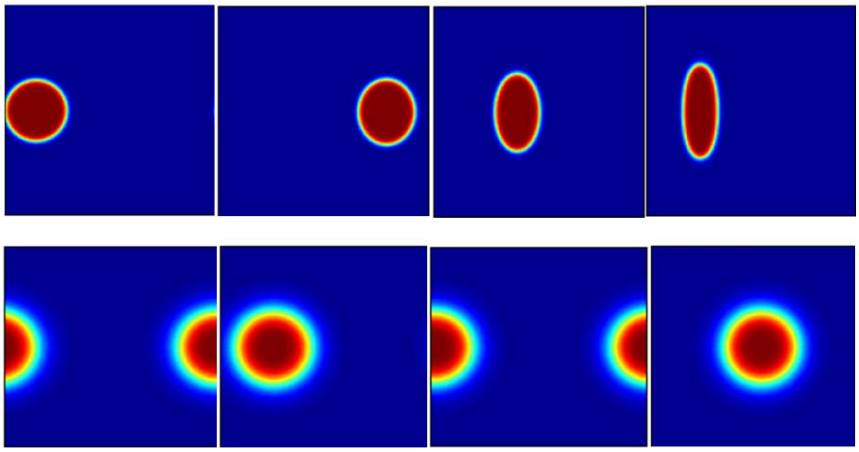

Figure 1: Evolution of a droplet in a moving channel (Force in potential form). Top: Mixed-scheme; Bottom: Isotropic-central-scheme.

3.1 Spurious currents

In LBE the interface between different phases usually takes several lattice grids, and therefore these two-phase LBE models can be viewed as certain diffuse-interface method. Consequently, LBE also shares the common shortcoming of diffuse-interface method that spurious currents will appear near an interface at equilibrium, which is described as "a small amplitude velocity field due to a slight unbalance between stresses in the interfacial region " [Lafaurie, Nardone, Scardovelli, Zaleski, and Zanetti (1994)]. These currents can lead to unphysical phenomena and numerical instability.

Many efforts have been devoted to reducing or eliminating spurious current and identifying its origin [Nourgaliev, Dinh, and Sehgal (2002); Lishchuk, Care, and Halliday (2003); Cristea and Sofonea (2003); Shan (2006); Seta and Okui (2007); Sbragaglia, Benzi, Biferale, Succi, Sugiyama, and Toschi (2007); Pooley and Furtado (2008); Tiribocchi, Stella, Gonnella, Lamura (2009); Wagner (2003); Lee, Fischer (2006)]. Nourgaliev *et al.* [Nourgaliev, Dinh, and Sehgal (2002)] related the origin of the spurious current to the local violation of the momentum conservation, and suggested to use of a "multifractional stepping" procedure for the advection operator together with an implicit trapezoidal discretization of the collision operator to remove it. Similarly, Cristea and Sofonea [Cristea and Sofonea (2003)] also attributed the spurious current to the discretization of the advection operator, and proposed a correction force term which is helpful to control the parasitic cur-

rent. By generalizing the usual lattice representations, Tiribocchi *et al.* [Tiribocchi, Stella, Gonnella, Lamura (2009)] used another finite-difference scheme to reduce the spurious currents. Shan [Shan (2006)] pointed out that spurious currents are caused by the insufficient isotropy of the discrete gradient operator in the interaction force, and argued that a discrete gradient operator with higher isotropy can effectively reduce the magnitude of the parasitic current. Later, Sbragaglia *et al.* [Sbragaglia, Benzi, Biferale, Succi, Sugiyama, and Toschi (2007)] extended the gradient operator proposed by Shan to further reduce spurious currents. Recently, Seta and Okui [Seta and Okui (2007)] suggested using an accurate forth-order form to calculate gradients in the pressure tensor so that the intensity of spurious current could decrease. Other than an accurate gradient operator, Pooley and Furtado [Pooley and Furtado (2008)] proposed to use a revised equilibrium distribution function to reduce spurious currents. Alternatively, Wagner [Wagner (2003)] attributed the origin of parasitic current to the different discretizations of the driving force for the order parameter and momentum equations, and argued that the use of potential form could remove the parasitic current. However, the numerical stability of this method was unsatisfied. Lee and Fisher [Lee, Fischer (2006)] found that the spurious currents can be effectively eliminated using the potential form of interaction force together with the mixed difference scheme.

Although all of the above methods can reduce the parasitic currents, these controversies also indicate that the origin of the spurious currents in LBE is not clear yet. In order to understand this unphysical phenomenon more clearly, we consider a two-dimensional flat interface parallel to the x direction where the density changes only in the y direction. For this problem, it is reasonable to assume that $\mathbf{u} = (u_x, u_y) = (0, v)$, $\mathbf{F} = (F_x, F_y) = (0, F)$, and $\partial_x \phi = 0$ for any fluid variable ϕ. We will apply the D2Q9 LBE model as an example to this problem.

We now try to find the analytical solution of the LBE for this flat interface problem without invoking the Chapman-Enskog expansion. In this case the LBE (1) can be written as

$$f_i(\mathbf{x} + \mathbf{c}_i \delta t) = f_i'(\mathbf{x}) \equiv f_i(\mathbf{x}) - \frac{1}{\tau}\left[f_i(\mathbf{x}) - f_i^{(eq)}(\mathbf{x})\right] + F_i(\mathbf{x}). \tag{14}$$

Particularly, we have $f_i = f_i'$ for $i = 0$, 1, and 3, from which we can obtain that

$$f_{013} \equiv f_0 + f_1 + f_3 = \frac{2}{3}\left(1 - \frac{v^2}{2c_s^2}\right)\left[\rho - \left(\tau - \frac{1}{2}\right)\frac{vF}{c_s^2}\right].$$

With this result and the definitions of ρ and v, i.e.,

$$f_{256} + f_{478} = \rho - f_{013}, \quad (f_{256} - f_{478}) = \rho v - \frac{1}{2}F,$$

where $f_{256} \equiv f_2 + f_5 + f_6$ and $f_{478} \equiv f_4 + f_7 + f_8$, we can obtain that

$$f_{256} = \frac{\rho}{6} + \frac{\rho v^2}{2} + \frac{\rho v}{2} - \frac{F}{4} + \left(\tau - \frac{1}{2}\right)\left[1 - \frac{3v^2}{2}\right] Fv, \tag{15a}$$

$$f_{478} = \frac{\rho}{6} + \frac{\rho v^2}{2} - \frac{\rho v}{2} + \frac{F}{4} + \left(\tau - \frac{1}{2}\right)\left[1 - \frac{3v^2}{2}\right] Fv. \tag{15b}$$

Multiplying 1 and c_{iy} on both sides of Eq. (14) and taking summation over i, respectively, we can obtain mass and momentum conservative equations in discrete form as,

$$f_{256}(j+1) + f_{478}(j-1) + f_{013}(j) = \rho_j, \tag{16a}$$

$$f_{256}(j+1) - f_{478}(j-1) = \rho v_j + \frac{\delta}{2} F_j, \tag{16b}$$

where j is the label of the grid number along the y direction. With the expressions of f_{013}, f_{256}, and f_{478}, we can obtain from Eq. (16) that

$$-\frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{3} + \frac{F_{j+1} - F_{j-1}}{2} = R_1(v_j), \tag{17a}$$

$$-\frac{\rho_{j+1} - \rho_{j-1}}{3} + \frac{F_{j+1} + 2F_j + F_{j-1}}{2} = R_2(v_j), \tag{17b}$$

where R_1 and R_2 are two collective terms of v,

$$R_1(v_j) = (\rho_{j+1} v_{j+1} - \rho_{j-1} v_{j-1}) + (4\tau - 2)(F_{j+1} v_{j+1} - 2F_j v_j + F_{j-1} v_{j-1}) + O(v^3), \tag{18a}$$

$$\begin{aligned} R_2(v_j) &= (\rho_{j+1} v_{j+1} - 2\rho_j v_j + \rho_{j-1} v_{j-1}) + (\rho_{j+1} v_{j+1}^2 - \rho_{j-1} v_{j-1}^2) \\ &\quad + (2\tau - 1)(F_{j+1} v_{j+1} - F_{j-1} v_{j-1}) + O(v^3). \end{aligned} \tag{18b}$$

It is obvious that $R_1 = R_2 = 0$ if $v = 0$, and therefore a "necessary condition" for vanishing spurious current is,

$$\frac{F_{j+1} - F_{j-1}}{2} = \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{3}, \tag{19a}$$

$$\frac{F_{j+1} + 2F_j + F_{j-1}}{4} = \frac{c_s^2[\rho_{j+1} - \rho_{j-1}]}{2}. \tag{19b}$$

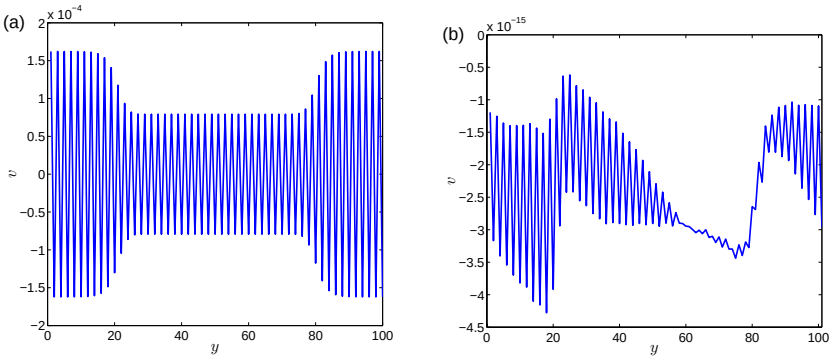

Figure 2: Velocity profiles of standard two-phase LBE model for the flat interface at steady state. (a)$N_x \times N_y$=4 $\times$ 100; (b)$N_x \times N_y$=5 $\times$ 101;

These two equations can be viewed as numerical schemes for the following equations,

$$\partial_y(F - c_s^2 \partial_y \rho) = 0, \quad F - c_s^2 \partial_y \rho = 0, \tag{20}$$

which are consistent with each other.

The terms on the left and right hand side of Eqs. (19a) and (19b) are linear combinations of the interaction force and fluid density, respectively. For non-ideal gases, it is well understood that the pressure given by the equation of state is usually nonlinear function of the density that varies nonlinearly in the interfacial region, and so is the interaction force F. Furthermore, because the gradients in the force given by Eq. (5) should be evaluated numerically point by point in the LBE (1), the terms on the left hand side of Eqs. (19a) and (19b) may involve some terms like ρ_{j+2} and/or ρ_{j-2}, which are absent on the right hand side. These implies that in general the necessary condition (19) cannot be fulfilled exactly no matter what formulations of the force (either pressure form or potential form) are adopted and what numerical schemes for the gradients are used, and the small force unbalance due to the numerical errors will produce spurious currents inevitably. Therefore, it can be concluded that *spurious current is an intrinsic nature of LBE due to the nonlinear interaction force.* Although the above arguments are made based on the LBE model derived from an extended Boltzmann equation, they also apply to other types of LBE models in that their evolution equations can also be reformulated as Eq. (1) with different force formulations.

3.2 Checkerboard effects

Checkerboard effects were first found in the lattice gas automata (LGA) method for single-phase flows [Zanetti (1989)]. Such effects are due to the finite symmetry of the discrete velocity set and the simple collision and streaming dynamics on the lattice. Checkerboard effects are also found in LBE models [Ladd (1994); Qian (1997); Kandhai, Koponen, Hoekstra, Kataja, Timonen, and Sloot (1999); d' Humiéres, Bouzidi, and Bouzidi (2001)], which are excited by solid boundaries in single-phase flows [Ladd (1994)].

For two-phase LBE models, the phase interface may act like a boundary and then excite the checkerboard invariance. It is known that checkerboard invariants depend on the parity of the number of grids with periodic boundary conditions (Kandhai, Koponen, Hoekstra, Kataja, Timonen, and Sloot , 1999). When the grid number in each direction is even (even lattice), a checkerboard invariant may be decoupled into two sub-populations; On the other hand, the sub-populations will mix on the boundaries if the grid numbers in all directions are odd (odd lattice).

An example is shown in Fig. 2, in which a liquid slab with density $\rho_l = 1.0$ and thickness 50 (in lattice unit) is put at the center of the lattice with $N_x \times N_y$ nodes, where x and y are set to be the directions parallel and normal to the interface, respectively. Other parts of the lattice are filled with the gas with $\rho_v = 0.2$. The checkerboard effect on the velocity field is clearly shown in Fig. 2(a) on the even lattice. On the other hand, the magnitude of the velocity is much weaker on the odd lattice, although some small oscillations also appears, which means that the checkerboard effects are effectively removed.

In order to eliminate the checkerboard effects in two-phase LBE, one need to mix the two sub-populations. Qian has proposed a fraction propagation method to remove staggered invariant for single-phase LBE models [Qian (1997)], which has been extended later [Guo, Zheng, Zhao (2001)]. We will try to employ the Lax-Wendroff (LW) scheme to control the checkerboard effects, which can be expressed as two sub-steps [Guo, Zheng, Zhao (2001)]:

$$\text{Collison:} \quad f_i'(\mathbf{x},t) = f_i(\mathbf{x},t) - \frac{1}{\tau'}\left[f_i(\mathbf{x},t) - f_i^{(eq)}(c';\mathbf{x},t)\right] + \delta_t' F_i(\mathbf{x},t), \tag{21}$$

$$\text{Streaming:} \quad f_i(\mathbf{x},t) = \alpha_0 f_i'(\mathbf{x},t) + \alpha_1 f_i'(\mathbf{x}+\mathbf{e}_i\delta_x,t) + \alpha_{-1} f_i'(\mathbf{x}-\mathbf{e}_i\delta_x,t), \tag{22}$$

where $\alpha_0 = 1-A^2$, $\alpha_1 = A(A+1)/2$, and $\alpha_{-1} = A(A-1)/2$, with $0 < A \leq 1$ being a parameter. Clearly as $A = 1$ the LW-LBE reduces to the standard LBE. It should be noted that in the LW-LBE scheme the lattice speed is defined as $c' = A\delta_x/\delta_t'$, so if we take δ_x and c' as the length and velocity units, respectively, the time step should be given by $\delta_t' = A\delta_x = A\delta_t$. On the other hand, it can be shown that LW-

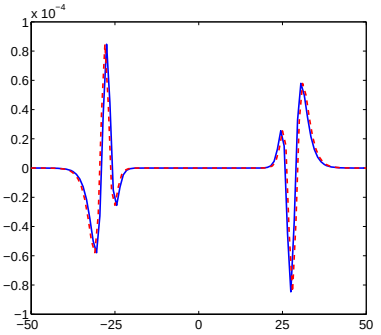

Figure 3: Velocity profiles of LW-LBE model for the flat interface at steady state ($A = 0.99$). Solid line: $N_x \times N_y$=4 × 100; Dashed line: $N_x \times N_y$=5 × 101

LBE is $\nu = c_s^2(\tau' - 0.5)\delta_t'$. Therefore in order to keep the same viscosity as the standard LBE, the relaxation time τ' should take $\tau' = 0.5 + (\tau - 0.5)/A$.

In Fig. 3 the velocity profiles predicted by the LW-LBE based on two lattices are shown. It is clear that the results on the odd and even lattices are nearly the same, meaning that checkerboard effects are effectively canceled on both grids, even with $A = 0.99$.

4 Conclusions

The LBE method has been recognized as an efficient tool for studying two-phase flows, and a variety of models have been developed based on different physical pictures. However, there are still some fundamental problems that should be clarify before applying such methods. In this work, we have investigated some of them.

First, we show that the discretization methods for the interaction force have significant influences on the phase behaviors of LBE, although in the continuous limit they are identical mathematically. Particularly, it is found that the mixed scheme, which is believed to be able to reduce the spurious currents in LBE, does not conserve mass and is non-Galilean invariant.

Second, a necessary condition for vanishing spurious currents in LBE is obtained, which cannot be fulfilled at discrete level due to discrete errors generally. This suggests that spurious current is an inherent nature of LBE.

Finally, we found that two-phase LBE models suffer from checkerboard effects, which are excited by the interfaces. A Lax-Wendroff scheme was shown to be able to suppress such effects effectively.

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