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Stress Function of Rock Surrounding the Circular Roadway with Uniform and Local Support by Natural BEM

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Abstract: Rock surrounding the circular roadway with uniform and local support is one of the most common phenomenons in roadway support engineering, which needs to be studied thoroughly at the theoretical level. The existing literatures on stress field function of rock surrounding the roadway is largely restricted to analytical solutions of stress for roadways with a uniform support or no support at all, the corresponding stress solution under conditions of local support has not been provided. Based on the mechanical models of uniform support and local support, the methods of the complex variable function and the complex Fourier series, using the boundary integral equations which is obtained by the natural boundary reduction, this paper obtains the rock surrounding's Airy stress functions of the circular roadway with the whole equilibrium uniform support and the non-equilibrium partial local support respectively, as well as the analytical and numerical solutions to the each stress field functions. We also analyze the rules of different distribution for the two stress fields varying with the lateral pressure coefficient and the support angle by comparison. The results of calculation show that, with the increasing of the lateral pressure coefficient and the support angle of the circular roadway with local support, the peak value of the compressive stress declines, thus the stability of the overlaying strata is improved.

Keywords: natural boundary element method (BEM); circular roadway with uniform and local support; stress function; rock surrounding; boundary integral equation; Airy stress function.

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1 Introduction

In mining engineering, the stress distribution inside rock surrounding is an important foundation from which to study the stability of roadways under a variety of supporting loads. Deep underground roadways of various shapes can be simplified as holes in the infinitely elastic body, usually by plane strain models to analyze the rock stress and deformation fields in the rock surrounding. The analysis of stress and deformation in rock surrounding is the key important basis theory to determine the stability of the roadway.

For the circular roadway, the stress analysis of rock surrounding is generally treated as the problems of plane outside the circle [Bowie (1956), Gao and Zhang (2000), Kubair and Bhanu-Chandar (2008),Wang and Gao (2008), Yang et al. (2008), Yang, Gao and Chen (2010)], but which usually only provide the analytical or numerical solution of stress for rock surrounding the roadway with primary hole (no support) and uniform support, and no stress solution (analytical and numerical) for the roadway is local support.

In theory, it is convenient to analyze the problems of rock surrounding the circular roadway under various boundary conditions (including the roadway with local support) using the natural boundary element method (BEM).

The natural BEM [Yu (1993)] is a branch of a number of BEMs, based on a complex variable method, a method using a Fourier series, or a Green functional method to induce a Dirichlet boundary value problem as a differential equation into Poisson integration equation of the studied area or to induce Neumann boundary value problem of differential equation into a strong singular boundary integral equation [Yu (1993)]. The natural BEM is widely used to solve problems of a circular interior and exterior domain and other plane and engineering problems [Hartmann (1989), Rencis and Jong (1989), Brebbia and Dominguez (1992), Jou and Liu (1999), Soares Jr. and Vinagre (2008), Zalewski and Mullen (2009a,b)]. Several researchers [Yu and Du (2003), Liu and Yu (2008)] have investigated coupling methods between natural BEM and finite element methods. Based on the natural BEM on the boundary value problem of a bi-harmonic equation of a circular exterior domain, a boundary integration equation of the Airy stress function in polar coordinates is obtained.

This paper uses the surface forces on the roadway boundary to calculate the stress function and its normal derivative, which are substituted into the integration equation, thus obtained the specific expression of a stress function under various boundary (supporting) conditions, thus permits the analysis of stress and related deformation inside the rock surrounding, and then the rules of different distribution for the two stress fields varying with the lateral pressure coefficient and the support angle are analyzed.

2 Boundary integration equation of stress function in rock surrounding the circular roadway with uniform support

2.1 Mechanical model with uniform support

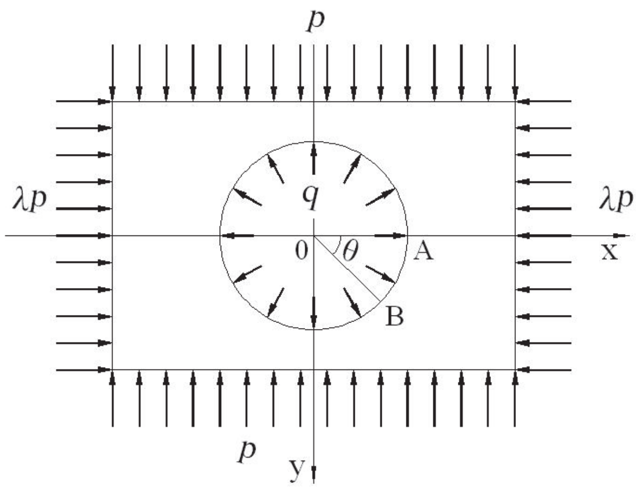

Figure 1: Mechanical model of circular roadway with uniform support

The stress function of rock surrounding the circular roadway with uniform support can be deduced using natural BEM [Yu (1993)]. In order to facilitate the analysis, we assume the radius of the roadway to be unit one. As shown in Fig. 1, the roadway boundary is supported by the balance of uniform load, we can obtain the boundary integration equation of stress function based on Gu Sa formulas and the complex Fourier series with natural boundary reduction method [Yu (1993), Xu (2005)] as Eq. (1).

$$\phi(r,\theta)_{uni} = \left(\frac{C}{2}\cos 2\theta - \frac{D}{2}\sin 2\theta\right)(r^{-2}+r^2-2) + (2E+d_0)\left(\frac{r^2-1}{2} - \ln r\right) + f(r,\theta), \quad (r>1) \quad (1)$$

where, C, D and E are determined by the principal stresses σ_1, σ_2 in infinity and the angle α between the principal stress σ_1 and the x-axis [Xu (2005)], namely,

$$C + iD = -\frac{1}{2}(\sigma_1 - \sigma_2)e^{-2i\alpha}, \quad (2)$$

$$E = \frac{1}{4}(\sigma_1 + \sigma_2), \tag{3}$$

$$f(r,\theta) = \int_0^{2\pi} \left\{ \frac{(r^2-1)^2\,[r\cos(\theta-\theta')-1]}{2\pi[1+r^2-2r\cos(\theta-\theta')]^2}\phi_0(\theta') - \frac{(r^2-1)^2\phi_n(\theta')}{4\pi[1+r^2-2r\cos(\theta-\theta')]} \right\} d\theta', \tag{4}$$

$$d_0 = \frac{1}{2\pi}\int_0^{2\pi} \phi_n(\theta)d\theta. \tag{5}$$

In Eqs. (4) and (5), $\phi_0(\theta)$ and $\phi_n(\theta)$are the stress function of the roadway boundary and the normal derivative of the stress function on the roadway boundary, respectively, and the numerical values can be obtained by the known surface force $\bar{X}$ and $\bar{Y}$on the roadway boundary.

The Eq. (1) is the boundary integration equation of stress function in rock surrounding the circular roadway with uniform support, and the numerical solutions can be obtained as the following calculation example.

2.2 Calculation of the boundary stress function

In Fig 1, let rock surrounding the circular roadway with uniform support is q at the hole boundary, with an infinite vertical ground stress p and infinite lateral stress λp, where λ is the lateral pressure coefficient and let $0 \le \lambda \le 1$.

By analyzing Fig. 1, we obtain: $\sigma_1 = -\lambda p > \sigma_2 = -p$, the positive angle is $\alpha = \pi$ between the principal stress σ_1 and the x-axis, substituting it into Eqs. (2) and (3), we obtain:

$$C = -\frac{1-\lambda}{2}p, \quad D = 0, \quad E = -\frac{1+\lambda}{4}p. \tag{6}$$

Now, we need to determine the boundary stress function $\phi_0(\theta)$ and its normal derivative $\phi_n(\theta)$ according to the known surface force $\bar{X}$ and $\bar{Y}$ on the roadway boundary. Firstly, a base point A is selected on the roadway boundary [Xu (2005)], namely

$$\phi_A = 0, \quad \left(\frac{\partial\phi}{\partial x}\right)_A = 0, \quad \left(\frac{\partial\phi}{\partial y}\right)_A = 0. \tag{7}$$

Then for a random point B on the boundary, we have

$$\phi_0(\theta) = \phi_B = -\int_A^B (x-x_B)\bar{Y}\mathrm{d}s - \int_A^B (y_B-y)\bar{X}\mathrm{d}s, \tag{8}$$

$$\phi_n(\theta) = (-\cos\theta)\left(\frac{\partial\phi}{\partial x}\right)_B + (-\sin\theta)\left(\frac{\partial\phi}{\partial y}\right)_B = -\cos\theta\int_A^B \bar{Y}\mathrm{d}s + \sin\theta\int_A^B \bar{X}\mathrm{d}s. \tag{9}$$

Because rock surrounding the circular roadway with uniform support is q at the hole boundary, so $\phi_0(\theta)$ and $\phi_n(\theta)$ can be rewritten as:

$$\phi_0(\theta) = \phi_B = -\int_0^\theta qR^2\sin(\theta-\alpha)\mathrm{d}\alpha = q(\cos\theta - 1), \tag{10}$$

$$\phi_n(\theta) = -\cos\theta\int_0^\theta qR\sin\alpha\mathrm{d}\alpha + \sin\theta\int_0^\theta qR\cos\alpha\mathrm{d}\alpha = q(1-\cos\theta). \tag{11}$$

Substituting Eqs. (10) and (11) into Eq. (4), and the integral becomes:

$$f(r,\theta) = q\left(r\cos\theta - \frac{r^2+1}{2}\right). \tag{12}$$

Based on Eq. (5), we have:

$$d_0 = \frac{1}{2\pi}\int_0^{2\pi}\phi_n(\theta)d\theta = q. \tag{13}$$

So, based on Eq. (1), we obtain the stress function:

$$\phi(r,\theta)_{uni} = -\frac{p-\lambda p}{4}\left(r^{-2}+r^2-2\right)\cos 2\theta + \frac{p+\lambda p}{2}\left(\ln r - \frac{r^2-1}{2}\right) + qr\cos\theta - q(1+\ln r). \tag{14}$$

The stress function $\phi_{uni}(r,\theta)$ at any point inside the surrounding rock can be obtained by the following equations:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \tag{15}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}, \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).$$

Thus the related stress analysis can be carried out [Xu (2005)]. Especially, in Eq. (14), if $q = 0$, namely, the analytical stress solution of the rock surrounding can be directly obtained for the excavated roadway without any boundary support. Fig. 2 shows the variation of the normal radial stress of roadway boundary σ_θ/p with angle θ at $p = 15q$, for different lateral pressure coefficient λ, respectively.

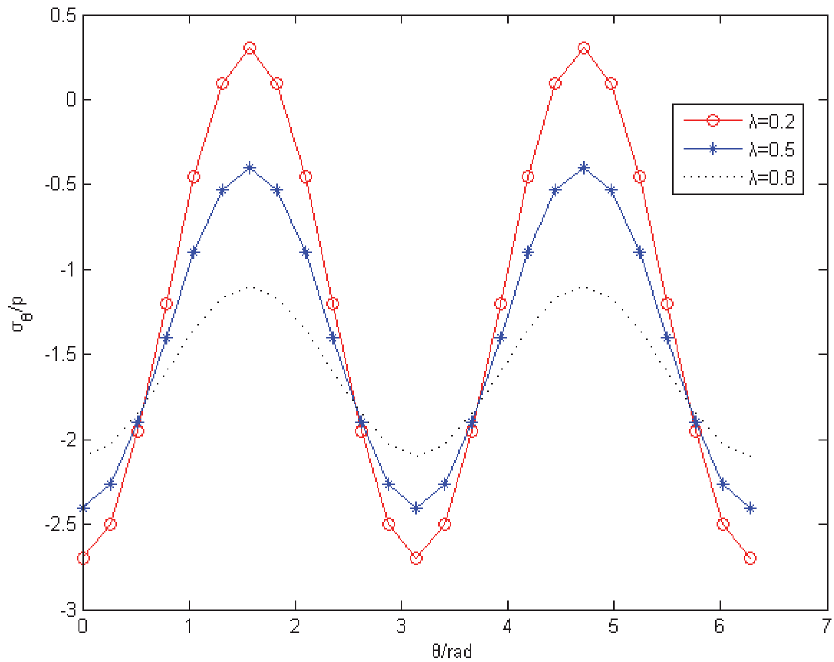

Figure 2: Variation of the normal radial stress of roadway boundary σ_θ/p with angle θ

3 Boundary integration equation of stress function in rock surrounding the circular roadway with local support

3.1 Mechanical model with local support

Fig. 3 is a schematic diagram of the roadway under the conditions with local boundary support, in order to facilitate the analysis, we assume the radius of the roadway to be unit one. In Fig. 3, X and Y are principal vectors of the surface force on the roadway boundary along the x-axis and y-axis directions, while $X=0, Y\neq 0$, with an infinite vertical ground stress p and infinite lateral stress λp. So we can obtain the boundary integration equation of stress function with local support by natural

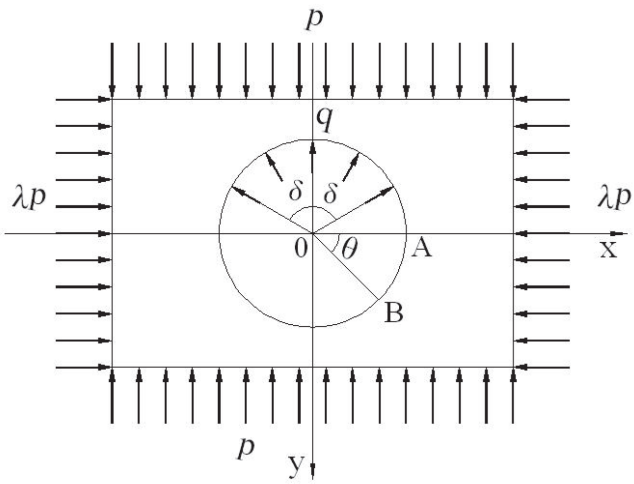

Figure 3: Mechanical model of circular roadway with local support

BEM as Eq. (16).

$$\phi(r,\theta)_{loc} = \frac{r\theta\cos\theta}{2\pi}Y + \frac{(2-2\mu)\ln r-(3-\mu)}{8\pi}rY\sin\theta + \frac{r^{-1}\sin\theta}{4\pi}Y\left(\frac{1-\mu}{2}+r^2\right)+\bar{f}(r,\theta) + \left(\frac{C}{2}\cos 2\theta - \frac{D}{2}\sin 2\theta\right)\left(r^{-2}+r^2-2\right)+(2E+d_0)\left(\frac{r^2-1}{2}-\ln r\right) + f(r,\theta), \quad (r \geq 1) \quad (16)$$

where, C, D, E and $f(r,\theta)$ can be obtained by Eqs. (2), (3) and (4). Thus the d_0 and $\bar{f}(r,\theta)$ are determined as Eqs. (17) and (18).

$$d_0 = \frac{1}{2\pi}\int_0^{2\pi}\left[\phi_n(\theta)+\frac{\theta\cos\theta}{2\pi}Y\right]d\theta. \quad (17)$$

$$\bar{f}(r,\theta) = -\int_0^{2\pi}\frac{(r^2-1)^3}{4\pi[1+r^2-2r\cos(\theta-\theta')]^2}\frac{\theta'\cos\theta'}{2\pi}Yd\theta'. \quad (18)$$

It can be seen from stress function of rock surrounding the circular roadway with local support, namely, Eq. (16) has four more polynomials than stress function of rock surrounding the circular roadway with uniform support, Eq. (1). Namely,

$$\phi(r,\theta)_{loc} = \phi(r,\theta)_{uni} + \frac{r\theta\cos\theta}{2\pi}Y + \frac{(2-2\mu)\ln r-(3-\mu)}{8\pi}rY\sin\theta + \frac{r^{-1}\sin\theta}{4\pi}Y\left(\frac{1-\mu}{2}+r^2\right) + \bar{f}(r,\theta).(r\geq 1) \quad (19)$$

Obviously, from Eq. (16), $\phi(r,\theta)_{loc} = \phi(r,\theta)_{uni}$ if the principal vector $Y = 0$, namely, Eq. (16) can be determined as the stress function of rock surrounding the circular roadway with uniform support Eq. (1).

3.2 Calculation of the boundary stress function

In Fig 3, let rock surrounding the circular roadway with local support on the roadway boundary within the range of angle 2δ and its supporting surface force is q, with an infinite vertical ground stress p and infinite lateral stress λp.

We can calculate the principal vector Y of the roadway boundary load as:

$$Y = -\int_{-\delta}^{\delta} q\cos\alpha d\alpha = -2q\sin\delta. \quad (20)$$

Based on the similar computational method as part 2.2, we obtain:

$$C = -\frac{1-\lambda}{2}p, \quad D = 0, \quad E = -\frac{1+\lambda}{4}p. \quad (21)$$

For a random point B on the boundary, we can obtain the stress function $\phi_0(\theta)$ and its normal derivative $\phi_n(\theta)$:

$$\phi_0(\theta) = \phi_B = \begin{cases} -q[1+\sin(\delta+\theta)], & \frac{3}{2}\pi-\delta\leq\theta\leq\frac{3}{2}\pi+\delta \\ -2q\sin\delta\cos\theta, & \frac{3}{2}\pi+\delta\leq\theta\leq 2\pi \end{cases} \quad (22)$$

$$\phi_n(\theta) = \left.\frac{\partial\phi}{\partial n}\right|_B = \begin{cases} q[1+\sin(\delta+\theta)], & \frac{3}{2}\pi-\delta\leq\theta\leq\frac{3}{2}\pi+\delta \\ 2q\sin\delta\cos\theta, & \frac{3}{2}\pi+\delta\leq\theta\leq 2\pi \end{cases} \quad (23)$$

Substituting Eqs. (20) and (23) into Eq. (17), and the integral becomes:

$$d_0 = \frac{1}{2\pi}\int_0^{2\pi}\left[\phi_n(\theta) + \frac{\theta\cos\theta}{2\pi}(-2q\sin\delta)\right]d\theta = q\delta/\pi. \quad (24)$$

Substituting Eqs. (20), (22) and (23) into Eqs. (4) and (18), we obtain:

$$f(r,\theta) = -\frac{\left(r^2-1\right)^3 q}{4\pi} \left[\int_{\frac{3\pi}{2}-\delta}^{\frac{3\pi}{2}+\delta} \frac{1+\sin(\delta+\theta')}{[1+r^2-2r\cos(\theta-\theta')]^2} d\theta' + \int_{\frac{3\pi}{2}+\delta}^{2\pi} \frac{2\sin\delta\cos\theta'}{[1+r^2-2r\cos(\theta-\theta')]^2} d\theta'\right], \tag{25}$$

$$\bar{f}(r,\theta) = -\frac{\left(r^2-1\right)^3 q}{4\pi} \left[\int_0^{2\pi} \frac{-2(1-\mu)\theta'\sin\delta\sin\theta'}{4\pi[1+r^2-2r\cos(\theta-\theta')]^2} d\theta'\right]. \tag{26}$$

Substituting the analytic integrals above into Eq. (16), using the computing software MATLAB to calculate the numerical values, the stress function $\phi_{loc}(r,\theta)$ and its numerical value at any point inside the surrounding rock can be obtained as shown in Fig. 4.

Let the Poisson's ratio as $\mu = 0.3, p = 15q, \delta = \pi/4$, Fig. 4 shows the variation of the normal radial stress of roadway boundary σ_θ/p with angle θ and different lateral pressure coefficient λ, respectively.

4 Results and discussion

Compare Fig. 2 and Fig. 4, we can obtain the following results:

(1) As shown in Fig. 2, when the circular roadway boundary of rock surrounding is supported by a uniform load, the normal radial stress of roadway boundary σ_θ is symmetry on the x-axis;

(2) As shown in Fig. 4, the symmetry becomes an end while the stress on the roadway boundary is local support. For the performance, when the lateral pressure coefficient $\lambda < 0.5$, there appeared peak value of the local tensile stress on the floor ($\theta = \pi/2$), however, there appeared peak value of the local compressive stress on the floor while the lateral pressure coefficient $\lambda > 0.5$;

(3) When λ, p and q have same values, the maximum of compressive stress with local support (Fig. 4) is always bigger than the stress with uniform support (Fig. 2).

Fig. 5 shows the variation of the normal radial stress of roadway boundary σ_r/p with angle θ at $\lambda = 0.8, r = 2$ and different supporting angle δ, respectively. It can be seen that: there appeared peck value of the compressive stress near the roof ($\theta = 3\pi/2 = 4.712$) and floor ($\theta = \pi/2 = 1.571$), while appeared minimum value on the bilateral roadway. With the increasing of the support angle δ, the peak value

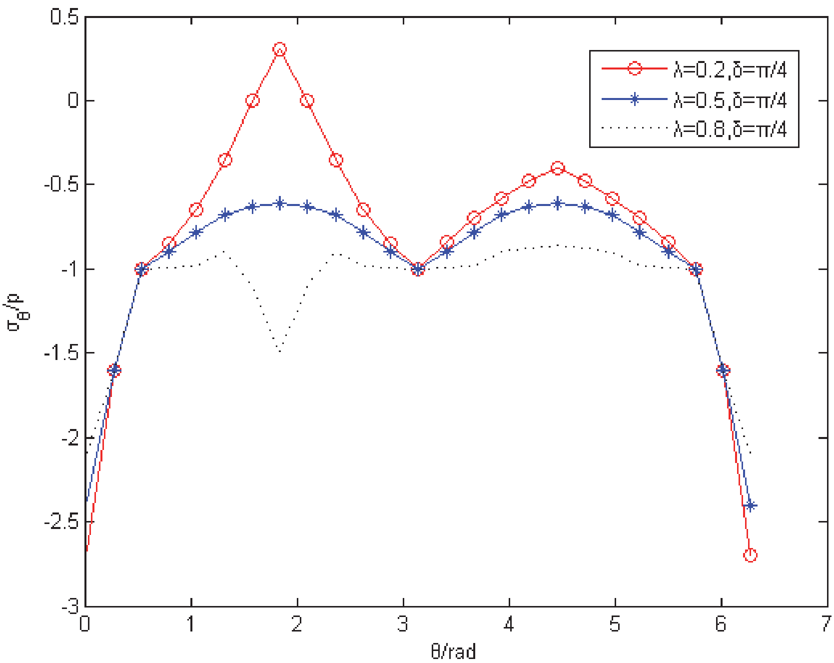

Figure 4: Variation of the normal radial stress of roadway boundary σ_θ/p with angle θ

of the compressive stress declines obviously on the roof ($\theta = 3\pi/2 = 4.712$), thus the stability of the overlaying strata is improved.

5 Conclusions and future research

The analytical and numerical solution of stress function of rock surrounding the circular roadway with uniform and local support has been studied by Natural BEM, based on the discussion above, we can obtain the following conclusions:

(1) Based on a complex variable function method, a complex Fourier series method and by a natural boundary reduction, we obtained the boundary integration equation of stress function in rock surrounding the circular roadway with uniform and local support. Using surface force on the roadway boundary to calculate the stress functions and their normal derivative under the two conditions, and substituting them into the integration equation, the specific expression, as well as the analyses solution and numerical solution of stress functions have been obtained.

(2) The rules of different distribution for the two stress fields varying with the lateral

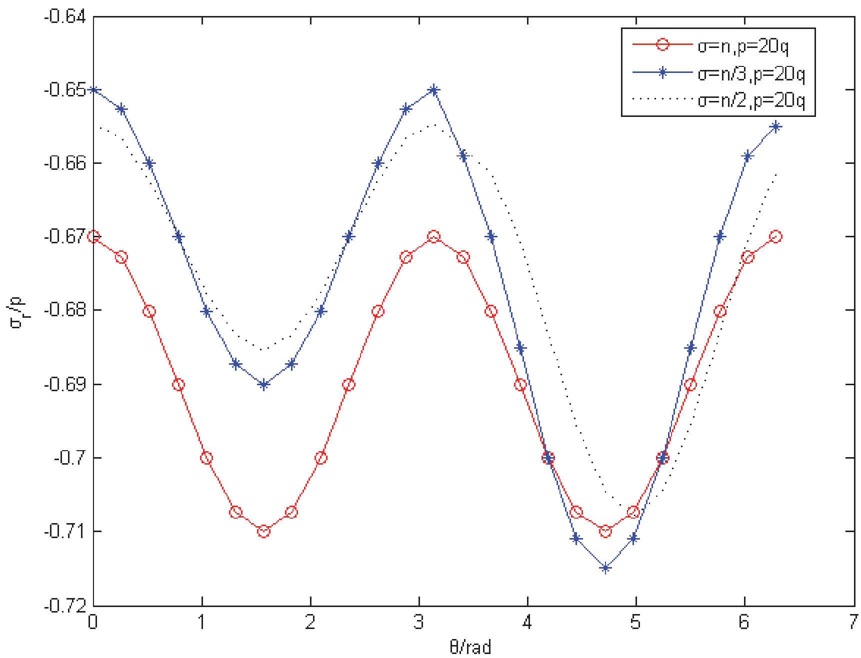

Figure 5: Variation of σ_r/p with angle θ at $\lambda = 0.8, r = 2$ and different supporting angle δ

pressure coefficient and the support angle are analyzed. Compared the variation of the normal radial stress of roadway boundary σ with different lateral pressure coefficient λ and different supporting angle δ, respectively, the results show that: with the increasing of the lateral pressure coefficient λ and the support angle δ of the circular roadway with local support, the peak value of the compressive stress declines, thus the stability of the overlaying strata is improved.

For future research, the stress function of the circular, oblong, etc. roadways with different supports will be analyzed, as well as the numerical simulation and the related engineering application.

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