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Reliability-Based Multiobjective Design Optimization under Interval Uncertainty

Fangyi Li[1,2], Zhen Luo[3] and Guangyong Sun[4]

Abstract: This paper studies the reliability-based multiobjective optimization by using a new interval strategy to model uncertain parameters. A new satisfaction degree of interval, which is significantly extended from [0, 1] to [–∞, +∞], is introduced into the non-probabilistic reliability-based optimization. Based on a predefined satisfaction degree level, the uncertain constraints can be effectively transformed into deterministic ones. The interval number programming method is applied to change each uncertain objective function to a deterministic two-objective optimization. So in this way the uncertain multiobjective optimization problem is transformed into a deterministic optimization problem and a reliability-based multiobjective optimization is then established. For sophisticated engineering problems, the objectives and constraints are modeled by using the response surface (RS) approximation method to improve the optimization efficiency. Thus the reliability-based multiobjective optimization is combined with the RS approximation models to form an approximation optimization problem. For the multiobjective optimization, the Pareto sets can be obtained with different satisfactory degree levels. Two numerical examples and one real-world crashworthiness design for vehicle frontal structure are presented to demonstrate the effectiveness of the proposed approach.

Keywords: Reliability-based optimization; Multiobjective optimization; Satisfaction degree of interval; Response surface method (RSM); Approximation model

[1] School of Automotive and mechanical Engineering, Changsha University of Science and Technology, Changsha, 410114, China. Corresponding Author: Telephone: +86-731-8525-8630; Fax: +86-731-8525-8630; Email: lfy703@sina.com (Fangyi Li)

[2] Key Laboratory of Manufacture and Test Techniques for Automobile Parts, Ministry of Education, Chongqing University of Technology, Chongqing, 400054,China.

[3] School of Electrical, Mechanical and Mechatronic Systems Faculty of Engineering and Information Technology, University of Technology, Sydney, NSW 2007, Australia

[4] State Key Laboratory of Advanced Design and Manufacture for Vehicle Body, Hunan University, Changsha, 410082, China

1 Introduction

Most real-world problems often involve multiple conflicting objectives, where improving one objective may sacrifice the performance of one or more of other objectives. For this reason, multiobjective optimization plays an important role in many real-world optimization applications. It can be found that most traditional multiobjective optimization methods [e.g. Marler and Arora (2004); Liu and Frangopol (2004); Luo, Yang and Chen (2006); Lin, Luo and Tong (2010); Liao, Li, Yang, Zhang and Li (2008)] have been deterministically-based, in which parameters involved are given some definite values. However, uncertainties are inevitably involved in loading conditions, material properties, geometric dimension, and manufacturing precision in many real-life engineering problems [Schuëller and Jensen (2008)]. An optimized deterministic design without considering uncertainties might be unreliable and in the worst case scenario may lead to catastrophic failure of the design. For this reason, to obtain a reliable design the effects of the various uncertainty factors should be taken into account and reliability-based design optimization (RBDO) will be of particular significance.

The most common approaches to study various uncertainties in reliability-based multiobjective optimization (RBMO) problems are probabilistic-based methods [Schuëller and Jensen (2008)]. For instance, Sinha (2007) presented a methodology for RMO, which was performed using the approximate moment and reliability index approaches. Nariman-zadeh, Jamali and Hajiloo (2007) proposed a RBO method for the Pareto optimum of proportional integral derivative (PID) controllers for systems with probabilistic uncertainty. A two-stage approach was also proposed by Li, Liao and Coitc (2009) for solving RMO problems. One common feature of the abovementioned reliability analysis and design adopts the probabilistic approach.

However, in practical multi-objective optimization, the probabilistic methods have shortcomings. Especially, the complete probabilistic information may not be precisely known. The evaluation of the probabilistic characteristics of system responses can present some significant mathematical and numerical difficulties because of extra computational cost. So it is difficult to specify precise probability distribution function for uncertain parameters in multi-objective problems, because in most cases only limited uncertain information is available. It is noted that even small derivations of probability distributions may result in unacceptable errors [Ben-Haim and Elishakoff (1990)] in probabilistic methods. In such cases, non-probabilistic methods [Moens and Vandepitte (2005); Möller and Beer (2008)], such as the convex model including interval [e.g. Jiang, Han and Liu (2007a); Li and Azarm (2008); Gao, Song and Tin-Loi (2009)] and ellipsoid [e.g. Pantelides and Ganzerli (1998); Luo, Kang and Luo (2009); Kang and Luo (2009)] methods have been developed as beneficial supplements to the conventional probabilistic methods. By the way, the fuzzy set theory is also applied to structural optimization problems considering system uncertainties [e.g. Luo, Chen and Yang (2006); Luo, Yang and Chen (2006)].

Ben-Haim (1994) and Elishakoff (1995) should be the first few researchers who initiated the concept of non-probabilistic reliability design. The anti-optimization process was developed by Lombardi, and etc. (1995) for composite structures, and then by Barbieri, Cinquini and Lombardi (1997) for structural shape optimization of trusses. Pantelides and Ganzerli (1998) applied the non-probabilistic ellipsoidal convex model for two different objectives involving weight and displacement minimizations. Later, Ganzerli and Pantelides (2000) proposed a superposition method to determine the structural responses subjected to bounded load uncertainties. Qiu and Elishakoff (1998) also presented the anti-optimization for the structures with large uncertain-but-nonrandom parameters by the interval analysis. Jiang, Chen and Xu (2007) developed a semi-analytical approach for calculating reliability index based on the interval models. Recently, Kang and his coworkers extended the convex ellipsoid model to linear and nonlinear topology optimization problems [Luo, Kang and Luo (2009); Kang and Luo (2009; 2010)]. In particular, the interval model is experiencing popularity due to its conceptual simplicity and many other merits [Elishakoff (1995); Qiu and Elishakoff (1998); Jiang, Han and Liu (2008a); Gao, Song and Tin-Loi (2009)]. Interval model bounds all possible values of an uncertain parameter in a convex set without the requirement of knowing precise probability distribution. The interval bounds for an uncertain parameter can be easily determined compared to the identity of a precise probability distribution. The interval model has been applied to many different optimization problems involving uncertain-but-bounded parameter variations [e.g. Qiu and Elishakoff (1998); Jiang, Han and Liu (2007a)].

It is noted that most of the convex model based non-probabilistic optimizations have been carried out by using the worst-case criterion, which is relatively conservative and can make the treatment of constraints over strict. For this reason, Jiang Han and Liu (2007a) proposed a satisfaction degree of interval to transform the uncertain constraints into deterministic ones. The restricting degree of the uncertain constraints can be relaxed to a certain extent according to specific problem. The range of the values is limited within the scope of [0, 1] rather than the entire real number field. A new satisfactory degree based on the interval order was presented [Jiang, and et al. (2010)], through which the value range is extended from [0, 1] to $[-\infty, +\infty]$. In this way, the range of the comparing values is extended to the whole real number space, which made it possible to compare any pairs of intervals on the real number field. In particular it can be used to express the degree of how much better an interval is over another. We will further extend this approach [Jiang, and et al. (2010)] to the reliability-based multiobjective optimization for both overlapped and completely separated intervals. More details will be given in the subsequent Section of this paper for the integrity of this paper.

The other key issue is that the abovementioned non-probabilistic reliability-based optimization problems are mainly involved in single objective designs. To date, there still lack of detailed study to investigate the reliability-based multiobjective optimization, which is in general more complicated than the single objective counterpart. To take the advantage of the simplest approaches most multiobjective optimization schemes are to aggregate these different objectives into an equivalent single function in terms of different formulations, e.g. weighted linear average. However, the complex nature of multiobjective optimization may not always guarantee the feasibility of formulating a single cost function [Marler and Arora (2004)]. For example, for some advanced multiobjective optimization problems, the weighted average scheme sometimes may not work properly to find all Pareto solution points for non-convex Pareto front, though this is generally not a major concern in many practical engineering designs. For this reason, it is important to develop more effective algorithms for reliability-based multiobjective optimization problems.

This paper aims to explore the reliability-based multiobjective optimization problem on the basis of a new satisfaction degree of interval. The remaining portion of the paper is organized as follow. In Section 2, a new satisfaction degree of interval is introduced. In Section 3, Limit state function based on satisfactory degree of interval is introduced. In Section 4, non-probabilistic reliability-based multiobjective optimization is formulated and implemented. The surrogate models are constructed for design objectives and constraints for engineering problems, thus reliability-based multiobjective approximation optimization is implemented in Section 5. Two numerical examples and one practical crashworthiness problem are used to illustrate the proposed method in Section 6. And finally some conclusions are obtained in Section 7.

2 Satisfaction degree of interval

The satisfaction degree of the interval represents the possibility whether one interval is greater or smaller than another, and in other words, it can be used to compare different intervals. In general, there will be six representative position relations between C^I and D^I (Fig. 1). Thus, three possible satisfaction degrees of interval were proposed [Facchinetti, Ricci and Muzzioli (1998); Liu and Da (1999); Xu and Da 2003)]:

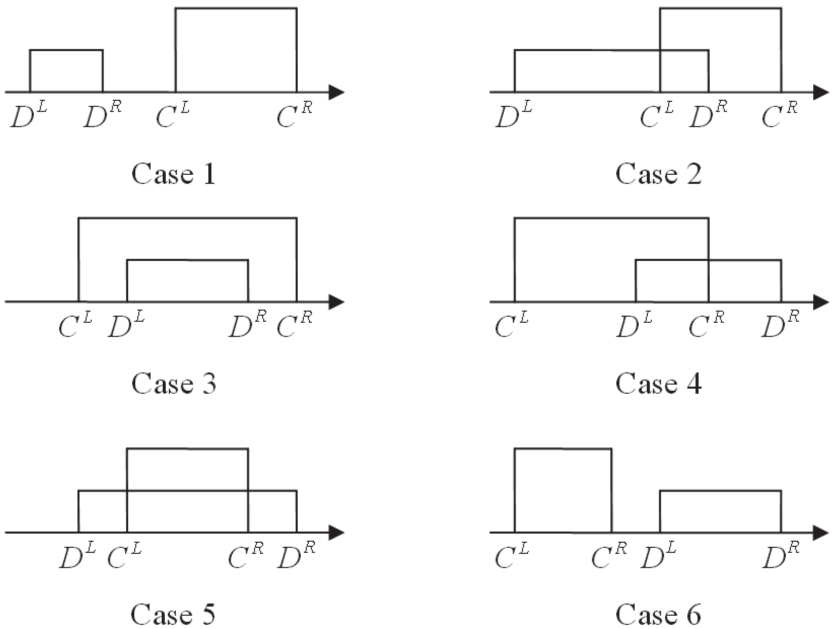

Figure 1: Six representative position relations of two intervals

Relation 1:

$$p(C^I \leq D^I) = \min\left\{\max\left\{\frac{D^R - C^L}{2C^w + 2D^w}, 0\right\}, 1\right\} \tag{1}$$

Relation 2:

$$p(C^I \leq D^I) = \frac{\max\left\{0, 2C^w + 2D^w - \max\left\{C^R - D^L, 0\right\}\right\}}{2C^w + 2D^w} \tag{2}$$

Relation 3:

$$p(C^I \leq D^I) = \frac{\min\left\{2C^w + 2D^w, \max\left\{D^R - C^L, 0\right\}\right\}}{2C^w + 2D^w} \tag{3}$$

Actually, the above three relations have been proven to be equivalent in representing the satisfaction degree of intervals [Xu and Da (2003)]. $p(C^I \leq D^I)$ has the following properties:

$0 \leq p(C^I \leq D^I) \leq 1;$

$$p(C^I \leq D^I) = 1 \Leftrightarrow C^R \leq D^L \Rightarrow C^I \leq D^I;$$

$$p(C^I \leq D^I) = 0 \Leftrightarrow C^L \geq D^R \Rightarrow C^I \geq D^I;$$

$$p(C^I \leq D^I) + p(C^I \leq D^I) = 1;$$

$$p(C^I \leq D^I) = 0.5 \Leftrightarrow C^L + C^R = D^L + D^R.$$

To a certain extent, the satisfaction degree of interval is of the probability implication for ranking the interval numbers.

In the abovementioned method, there are some limitations to describe the possibility degree of the interval. The possibility degree of interval can work well only for the cases when the intervals are partially or fully overlapped (Cases 2-5 in Fig. 1). If two intervals are separate (Case 1 and Case 6 in Fig. 1), the same value 0 or 1 will be given for the above satisfaction degree of interval regardless of their relative positions. That is, the conventional [0, 1] possible degree of intervals cannot represent the exact reliability information of two different intervals. Further, the two bound points are generally non-differentiable. However, in practical engineering problems, different relative positions of parameter intervals generally show different reliabilities of structure or system. There exist two inflection points of 0 and 1 for the abovementioned possibility degree of interval, and it will lead to non-differentiability for the concerned functions. To overcome these problems, a new satisfactory degree of interval can be formulated as follows [Jiang, and et al. (2010)]:

$$p(C^I \leq D^I) = \frac{D^R - C^L}{2C^w + 2D^w} \tag{4}$$

In Eq. (4), the value of $p(C^I \leq D^I)$ allows to relax the interval range from [0, 1] to $[-\infty, +\infty]$. In this way, the interval methods can be used for the reliability analysis of practical engineering problems. Based on this new satisfactory of interval, an effective mathematical tool can be used for calculating engineering reliability. It can be seen that the major limitation has been eliminated in this new satisfaction degree of interval.

When interval D^I is degenerated into a real number D, Eq. (4) can be rewritten:

$$p(C^I \leq D) = \frac{D - C^L}{2C^w} \tag{5}$$

As a result, $p(C^I \leq D^I)$ has the following properties:

$-\infty \leq p(C^I \leq D^I) \leq +\infty$;

$p(C^I \leq D^I) \geq 1 \Leftrightarrow C^R \leq D^L \Rightarrow C^I \leq D^I$, namely, on the real line, C^I is completely on the left of D^I;

$p(C^I \leq D^I) \leq 0 \Leftrightarrow C^L \geq D^R \Rightarrow C^I \geq D^I$, namely, on the real line, C^I is completely on the right of D^I;

If $p(C^I \leq D^I) = q$, then $p(C^I \geq D^I) = 1 - q$, where $q \in [-\infty, +\infty]$;

$p(C^I \leq D^I) = 0.5 \Leftrightarrow C^L + C^R = D^L + D^R$.

Note that the application of the satisfactory degree of interval to all position relations of C^I and D^I in Fig. 1 will play a crucial role in the following interval reliability analysis.

3 Limit state function based on satisfactory degree of interval

Define the state function that represents the working state as:

$$M = g(\mathbf{a}) = g(a_1, a_2, ...a_q) \tag{6}$$

When $\mathbf{a}$ is an uncertain vector to be modeled in terms of interval vector, the state function is represents as:

$$M^I = g(\mathbf{a}^I) \tag{7}$$

In the probability optimization, the constraints are generally made to satisfy a certain predetermined confidence level. Similarly, the state function in Eq. (7) satisfied with a certain satisfaction degree level:

$$p(M_j^I \leq B_j^I) \geq \eta_j, \quad M^I = [g_j^L(\mathbf{x}), g_j^R(\mathbf{x})], \quad B_j^I = [v_j^L, v_j^R], \quad j = 1, 2, ..., m, \tag{8}$$

where η_j is a predetermined satisfaction degree level of the jth constraint. B_j^I denotes the allowable interval number of the jth constraint. M_j^I is the interval of the jth constraint at $\mathbf{x}$ due to the uncertainty, and $g_i^L(\mathbf{x})$ and $g_i^R(\mathbf{x})$ are the lower and upper bounds of this interval, respectively as,

$$g_j^L(\mathbf{x}) = \min_{\mathbf{a}\in\Gamma} g_j(\mathbf{x},\mathbf{a}), g_j^R(\mathbf{x}) = \max_{\mathbf{a}\in\Gamma} g_j(\mathbf{x},\mathbf{a}), \quad j = 1,2,\cdots,m, \tag{9}$$

$$\Gamma = \{\mathbf{a}|\mathbf{a}^{\mathbf{L}} \le \mathbf{a} \le \mathbf{a}^{\mathbf{R}}\}$$

The satisfaction degree $p(M_j^I \le B_j^I)$ can be calculated as per Eq. (4). Through Eq. (9), the optimization method can be used to obtain the interval of the jth constraint. η_j can be adjusted to control the feasible domain of $\mathbf{x}$. When η_j is becoming larger, the inequality constraints Eq. (8) are restricted more strictly and the feasible domain of $\mathbf{x}$ become smaller. For $\eta_j > 1$, a leftmost interval of constraints is apart from the allowable interval. It reflects a better case of reliability than other cases. For $\eta_j = 1$, the interval of constraints just separates the allowable interval. It requires the constraints to be satisfied for all the possible combinations of the uncertain parameters, which is actually the worst-case criterion adopted in the literature [Lombardi, and et al. (1995); Barbieri, Cinquini and Lombardi (1997); Qiu and Elishakoff (1998)]. For $0 < \eta_j < 1$, the interval of constraints is partially overlapped the allowable interval. When η_j is 0, Eq. (8) is absolutely satisfied and it is actually an unconstraint treatment. $\eta_j < 0$ is added to define a negative satisfactory degree level. Obviously, a greater η_j indicates a greater extent of parameter variation. Different η_j values reflect different cases of constraints according to different reliability criteria, in which the worst case stands for a special case of the method.

4 Reliability-based multiobjective optimization

The reliability-based multiobjective optimization problem can be generally formulated as:

$$\min_{\mathbf{x}} \{f_1(\mathbf{x},\mathbf{a}), f_2(\mathbf{x},\mathbf{a}), \cdots, f_k(\mathbf{x},\mathbf{a})\}$$

$$\text{s.t} \quad p(M_j^I \le B_j^I) \ge \eta_j, j = 1,2,\cdots,m, \tag{10}$$

$$\mathbf{a} \in \mathbf{a}^{\mathbf{I}} = [\mathbf{a}^{\mathbf{L}}, \mathbf{a}^{\mathbf{R}}], a_i \in a_i^I = [a_i^L \le a_i \le a_i^R], i = 1,2,...,q,$$

$$\mathbf{x}_l \le \mathbf{x} \le \mathbf{x}_u$$

where $M^I = [g_j^L(\mathbf{x}), g_j^R(\mathbf{x})] = [\min_{\mathbf{a}\in\Gamma} g_j(\mathbf{x},\mathbf{a}), \max_{\mathbf{a}\in\Gamma} g_j(\mathbf{x},\mathbf{a})]$.and $B_j^I = [v_j^L, v_j^R]$

where $j = 1,2,\cdots,m$ and $i = 1,2,\cdots,k$. f_i and g_j are the objective and constraint functions, respectively. $\mathbf{x}$ is an n-dimensional design vector, and $\mathbf{x}_l$ and $\mathbf{x}_u$ denote the lower and upper bounds of $\mathbf{x}$, respectively. $\mathbf{a}$ denotes a q-dimensional uncertain vector and $m$is the total number of constraints. $p(M_j^I \le B_j^I)$ denotes the satisfactory degree associated with the performance constraints of $M_j^I \le B_j^I$, and η_j is the predefined satisfactory degree level of the jth state function.

In literature (e.g. Jiang, Han and Liu 2008b), an order relation "$\le_{mw}$" was adopted to treat the objective functions in terms of interval parameters. Thus it is expected to find an optimal vector for generating an objective interval, which has not only the smallest midpoint but also the smallest radius for minimization problems. Here the uncertain objective functions in Eq. (10) can be transformed into a deterministic optimization problem by also using the order relation $\le_{mw}$:

$$\min_{\mathbf{x}} [m(f_i(\mathbf{x},\mathbf{a})), w(f_i(\mathbf{x},\mathbf{a}))]$$

$$\begin{aligned} & m(f_i(\mathbf{x},\mathbf{a})) = \tfrac{1}{2}(f_i^L(\mathbf{x}) + f_i^R(\mathbf{x})) \\ & w(f_i(\mathbf{x},\mathbf{a})) = \tfrac{1}{2}(f_i^R(\mathbf{x}) - f_i^L(\mathbf{x})) \\ & i = 1,2,\cdots,k, \end{aligned} \tag{11}$$

where m and w represent the midpoint and radius of interval, respectively. At a specific $\mathbf{x}$, the bounds of the objective functions caused by uncertainty can be obtained as

$$\begin{aligned} & f_i^L(\mathbf{x}) = \min_{a\in\Gamma} f_i(\mathbf{x},\mathbf{a}), \\ & f^R(\mathbf{x}) = \min_{a\in\Gamma} f_i(\mathbf{x},\mathbf{a}), \\ & i = 1,2,\cdots,k \\ & \Gamma = \left\{ \mathbf{a} \,\middle|\, a_i^L \le a_i \le a_i^R,\ i = 1,2,...,q \right\} \end{aligned} \tag{12}$$

Thus the uncertain vector $\mathbf{a}$ is eliminated and the deterministic objective functions are formulated.

Based on the linearly weighted scheme, these two objective functions in Eq. (11) can be formulated as a assessment functionf_{di} as

(13)

where $0.0 \le \beta \le 1.0$ is a weighting factor and different values are used to measure the different weights of the objective functions. ξ denotes a factor, making $m(f(\mathbf{x},\mathbf{a})) + \xi$ and $w(f(\mathbf{x},\mathbf{a})) + \xi$ non-negative. φ and ψ are the normalization factors of these two objectives.

Through the above treatments, the reliability-based multiobjective optimization problem (10) can be transformed into the following deterministic multiobjective optimization problem:

$$\min_{\mathbf{x}} \{f_{d1}(\mathbf{x},\mathbf{a}), f_{d2}(\mathbf{x},\mathbf{a}), ...f_{dk}(\mathbf{x},\mathbf{a})\}$$

$$\text{s.t} \quad p(M^I \leq B^I) \geq \eta_j, j = 1,2,\cdots,m,$$

$$c_{\mathbf{a}} \in \mathbf{a}^I = a_i^I = [a_i^L \leq a_i \leq a_i^R], i = 1,2,...,q \tag{14}$$

$$\mathbf{x}_l \leq \mathbf{x} \leq \mathbf{x}_r,$$

where i, $i = 1,2,\cdots,k$,

5 Reliability-based multi-objective optimization based on a surrogate model

To formulate the assessment functions in optimization process in some real-word engineering problems efficiently, a surrogate model is adopted to replace sophisticated simulation model. As such, the optimization is performed at a lower computational cost using such approximation as:

$$\min_{\mathbf{x}} \left\{\tilde{f}_1(\mathbf{x},\mathbf{a}), \tilde{f}_2(\mathbf{x},\mathbf{a}), \cdots, \tilde{f}_k(\mathbf{x},\mathbf{a})\right\}$$

$$\text{s.t} \quad p(\tilde{M}_j^I \leq B_j^I) \geq \eta_j, j = 1,2,\cdots,m, \tag{15}$$

$$\mathbf{a} \in \mathbf{a^I} = [\mathbf{a^L}, \mathbf{a^R}],$$

$$a_i \in a_i^I = [a_i^L \leq a_i \leq a_i^R], i = 1,2,...,q$$

$$\mathbf{x}_l \leq \mathbf{x} \leq \mathbf{x}_u$$

where $\tilde{M}^I = [\tilde{g}_j^L(\mathbf{x}), \tilde{g}_j^R(\mathbf{x})] = [\min_{\mathbf{a}\in\Gamma} \tilde{g}_j(\mathbf{x},\mathbf{a}), \max_{\mathbf{a}\in\Gamma} \tilde{g}_j(\mathbf{x},\mathbf{a})]$ and $B_j^I = [v_j^L, v_j^R]$

where Γ and $\tilde{g}_j$ denote the surrogate models of the ith objective function and the jth constraint, respectively, and they are both some forms of explicit functions with respect to $\mathbf{x}$ and $\mathbf{a}$. Then through the interval programming method given in section 4, Eq. (15) can be formulated as a following deterministic optimization problem like Eq. (14):

$$\min_{\mathbf{x}} \left\{ \tilde{f}_{d1}(\mathbf{x},\mathbf{a}), \tilde{f}_{d2}(\mathbf{x},\mathbf{a}), \ldots \tilde{f}_{dk}(\mathbf{x},\mathbf{a}) \right\}$$

$$\text{s.t} \quad p(\tilde{M}_j^I \le B_j^I) \ge \eta_j, j = 1,2,\cdots,m,$$

$$\mathbf{a} \in \mathbf{a^I} = [\mathbf{a^L}, \mathbf{a^R}], a_i \in a_i^I = [a_i^L \le a_i \le a_i^R], i = 1,2,\ldots,q \tag{16}$$

$$\mathbf{x}_l \le \mathbf{x} \le \mathbf{x}_u$$

where $\tilde{f}_{di}$ presents the assessment function based on the surrogate models. The quadratic polynomial response surface methodology (RSM) technique [Draper and Smith (1998)] is employed herein to create the surrogate models for objective functions and constraints.

To generate the surrogate model, a number of sample points are needed to represent the functional space properly. It has been showed that the selection of sample points is very important yet challenging from the modeling accuracy and efficiency perspectives. In this paper, the Design of Experiments (DOE) is used to sample the points for constructing the surrogate models. More specifically, the Latin Hypercube Sampling (LHD) scheme [Morris and Mitchell (1995)] is adopted to generate the points over the design and uncertain spaces. LHD, to a considerable extent, can ensure a well-representative distribution of points over the design and uncertain spaces of variables via regular intervals to maximize the minimum distance between points.

To further clarify the numerical procedure, Fig. 2 shows an optimization flowchart of the present method. Obviously, this is a typical nesting optimization problem. In uncertain space and design domain, a set of sampling points are generated by LHD. After inputting the sample points into the actual simulation models, the samples can be obtained to construct the surrogate models of the objective functions and constraints, respectively. As a result, the optimization can be performed based on these surrogate models. To do so, the Non-dominated Sorting Genetic Algorithm II (NSGA-II) [Deb (2001); Deb, and et al. (2002)] and sequential quadratic programming (SQP) [Boggs and Tolle (1995)] are used as the outer layer and inner layer optimization solvers, respectively. The outer optimizer is used to optimize the design vector $\mathbf{x}$, and the inner optimizer is used to compute the bounds of the

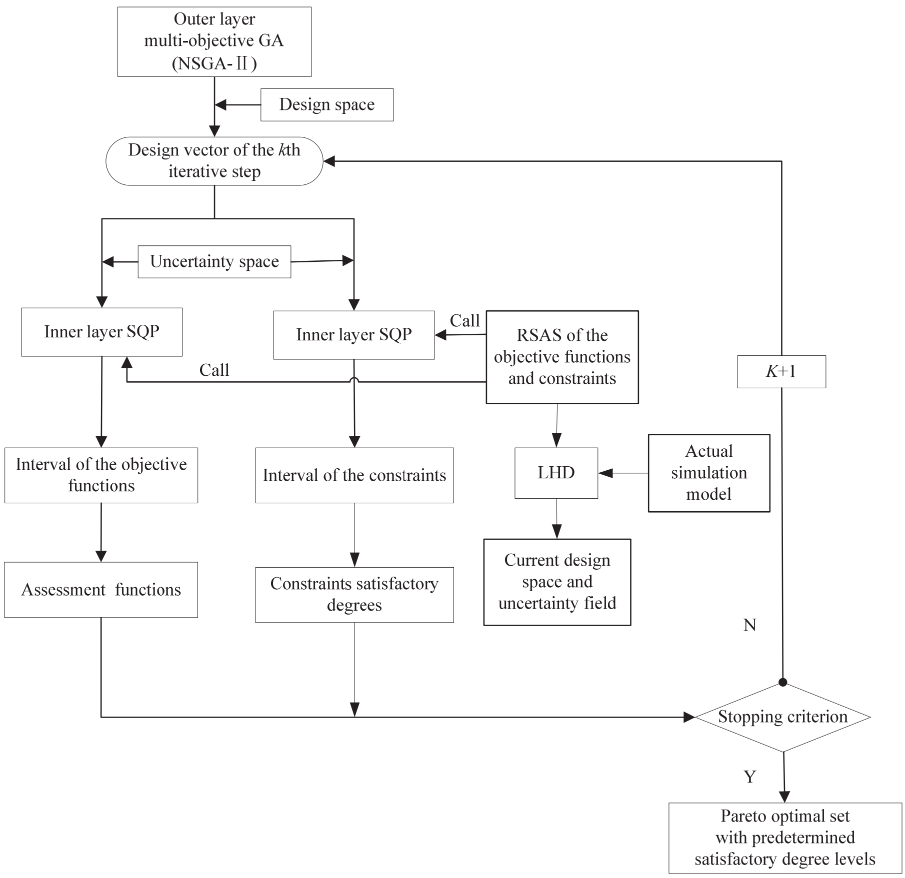

Figure 2: Flowchart of the present method nesting optimization based on the actual simulation models

objective functions and constraints induced by the uncertainty. Thus the Pareto set of Eq. (16) can be obtained with prescribed satisfactory degree levels.

6 Numerical examples and practical problem

6.1 Numerical example 1

The first numerical example serves as a benchmark numerical example to demonstrate the proposed method:

$$\begin{aligned} &\min_{\mathbf{x}} f_1(\mathbf{x},\mathbf{a}) = a_1(x_1+x_2-7.5)^2 + a_2^2(x_2-x_1+3)^2/4 \\ &f_2(\mathbf{x},\mathbf{a}) = a_1^2(x_1-1)^2/4 + a_2^3(x_2-4)^2/2 \end{aligned} \tag{17}$$

s.t.

$$
\begin{aligned}
&g_1(x,a) = a_1^2(x_1-2)^3/2 + a_2x_2 - 2.5 \leq [0,0.3] \\
&g_2(x,a) = a_1^3x_2 + a_2^2x_1 - 3.85 - 8a_2^2(x_2 - x_1 + 0.65)^2 \leq [0,0.3] \\
&0 \leq x_1 \leq 5, 0 \leq x_2 \leq 3 \\
&a_1 \in [0.9,1.1], a_2 \in [0.9,1.1]
\end{aligned}
$$

The parameters in NSGA-II are specified as Table 1. β, ξ, φ and ψ are set as 0.5, 0, 0 and 0, respectively.

Table 1: Details of NSGA-II specific parameters used

GA parameter name	Value
Population size	50
Number of generation	200
Probability of crossover	0.9
Distribution index for crossover	20
Distribution index for mutation	20

As shown in Fig. 3, the Pareto set under different satisfactory degree levels can be obtained. It can be seen that the Pareto set shifts and the range varies with different satisfactory degrees. The results of the minimum f_{d1} and f_{d2} are listed in Table 2. It can be found that the minimum values of f_{d1} and f_{d2} increase along with the increase of the satisfactory degree levels. This is because a higher satisfactory degree level results in a smaller feasible zone, thereby leading to a worse result of the objectives. From Table 2, it can be found that for η=0.6, the interval of the constraint 1 is partially overlapped with the corresponding allowable interval. For $\eta = 1.5$, the interval of the first constraint is completely separated with the corresponding allowable intervals. Generally speaking, the satisfactory degree of interval η only equals 1, and cannot work well for this case. In conclusion, when η<1, the interval of constraints has overlapped each other. But for $\eta \geq 1$, constraint interval has apart from the allowable intervals, which indicates a better reliability of the constraints. In other words, a greater satisfactory degree level indicates a better reliability. The iteration history is given in Fig. 4, from which we can see that the initial 30 to 50 iterations can only be used to find a portion of the solutions. Most solution points can be obtained after 100 iterations, and the whole Pareto solution points can be obtained around 200 iteration. So the Pareto solutions are sequentially converged after 200 iterations.

Table 2: Computation results under different satisfactory degree levels

η_1, η_2	only Min f_{d1} or Min f_{d2}	Optimal design vector	Interval of the objective 1	Interval of the objective 2	Interval of the constraint 1	Interval of the constraint2	Satisfactory degree of interval
0.6,0.6	3.933	(3.045, 1.983)	[6.267, 7.866]	[2.331, 3.976]	[-0.255, 0.370]	[-1.044, 0.821]	0.60,0.62
	0.333	(0.999, 3.000)	[16.088, 21.039]	[0.365, 0.666]	[-0.405, 0.395]	[-68.437, -44.557]	0.64,2.84
0.8,0.8	4.305	(2.9905, 1.8965)	[6.881, 8.609]	[2.415, 4.143]	[-0.399, 0.174]	[-1.323, 0.384]	0.80,0.80
	0.335	(0.883, 3.000)	[17.076, 22.310]	[0.367, 0.670]	[-0.643, 0.236]	[-74.702, 0.236]	0.80,2.86
1.0,1.0	4.674	(2.942, 1.813)	[7.491, 9.348]	[2.507, 4.323]	[-0.530, 0]	[-1.630, -0.094]	1.00,1.05
	0.343	(0.745, 3.000)	[18.280, 23.860]	[0.378, 0.685]	[-0.995, 0]	[-82.435, -53.928]	1.00,2.87
1.2,1.2	5.042	(2.907, 1.718)	[8.103, 10.083]	[2.635, 4.567]	[-0.652, -0.159]	[-2.130, -0.865]	1.20,1.55
	0.356	(0.610, 3.000)	[19.501, 25.432]	[0.395, 0.712]	[-1.425, -0.288]	[-90.378, -59.245]	1.20,2.88
1.5,1.5	5.583	(2.871, 1.570)	[9.006, 11.166]	[2.861, 4.989]	[-0.819, -0.373]	[-3.334, -2.181]	1.50,2.50
	0.387	(0.400, 3.000)	[21.483, 27.982]	[0.438, 0.775]	[-2.280, -0.860]	[-103.446, -67.993]	1.50,2.90

6.2 Numerical example 2

The second example further tests the method proposed with the following two objectives and constraints:

$$\begin{aligned}
&\min_{\mathbf{x}} f_1(\mathbf{x},\mathbf{a}) = 2 + a_1^2(x_1-2)^2 + (2a_2^2 - a_1)(x_2-2)^2 \\
&f_2(\mathbf{x},\mathbf{a}) = 9(2a_2^3 - a_1)x_1 - a_2^2(x_2-1)^2 \\
&s.t. g_1(\mathbf{x},\mathbf{a}) = a_2^3 x_1^2 + a_1^2 x_2^2 \leq [215, 235] \\
&g_2(\mathbf{x},\mathbf{a}) = a_1^2 x_1 - 3a_2^2 x_2 + 25 \leq [0, 0.3]
\end{aligned} \tag{18}$$

$a_1 \in [0.9, 1.1], \quad a_2 \in [0.9, 1.1]$

As shown in Fig. 5, it can be clearly seen that the 'span' of Pareto fronts with larger satisfactory degree level is much smaller than those with a smaller value in this nu-

Table 3: Computation results under different satisfactory degree levels

η_1, η_2	Min f_{d1} only / Min f_{d2} only	Optimal design vector	Interval of the objective 1	Interval of the objective 2	Interval of the constraint 1	Interval of the constraint2	Satisfactory degree of interval
1.8,0.8	41.799	(-2.718, 8.435)	[49.826, 83.598]	[-112.194, -50.847]	[63.021, 95.93]	[-8.909, 2.301]	3.25,0.80
	-43.990	(-2.401 11.000)	[68.491, 143.548]	[-163.438, -87.980]	[103.995, 156.7452]	[-18.197, -3.917]	1.80,1.27
1.8,1.0	51.907	(-1.656, 9.737)	[49.297,103.813]	[-119.960, -65.527]	[78.794, 118.367]	[-12.349, -0.002]	2.622,1.000
	-44.546	(-1.421, 11.300)	[61.126,142.922]	[-152.031,-89.092]	[104.887,157.172]	[-17.735,-3.608]	1.80,1.25
1.8,1.2	65.292	(-3.419, 10.303)	[73.381,130.584]	[-161.714, 77.740]	[94.511,144.012]	[-16.5382, -2.807]	2.021,1.2
	-42.719	(-3.255, 10.824)	[75.908,142.7202]	[-171.037,-85.438]	[106.622,155.8648]	[-18.230,-3.939]	1.81,1.27

merical example. The optimization results under the different satisfaction degree levels are listed in Tables 3. Obviously, the minimum value of f_{d1} increases with the increase of the satisfactory degree level value. This is understandable because the increase of the satisfactory degree level leads to a much safer and conservative design. The minimum f_{d2} does not seem to exactly follow up such a pattern. This may be the result when the constraint dominates the objective function [Sinha (2007)]. For the case of $\eta_1 = 1.8$ and $\eta_2 = 0.8$, the interval of the second constraint is partially overlapped with the corresponding allowable interval. However, for the cases with $\eta_1 = 1.8$, $\eta_2 = 1.0$ and $\eta_1 = 1.8, \eta_2 = 1.2$, the interval of constraints apart from the allowable interval, which indicates a better reliability.

6.3 Practical problem

The bumper is a key energy absorption part for the vehicles frontal crash. It is a very important design issue to optimize the bumper structure to improve the crashworthiness performance of vehicles. In this section, the optimization of the bumper impacting onto a rigid wall with the initial velocity of 13.6m/s is considered (Fig. 6). The time duration of the impacting is 18ms. The crashworthiness of bumper structure is commonly represented in terms of the deformation mode, the energy absorption and peak acceleration, and etc, which can determine the overall crash-

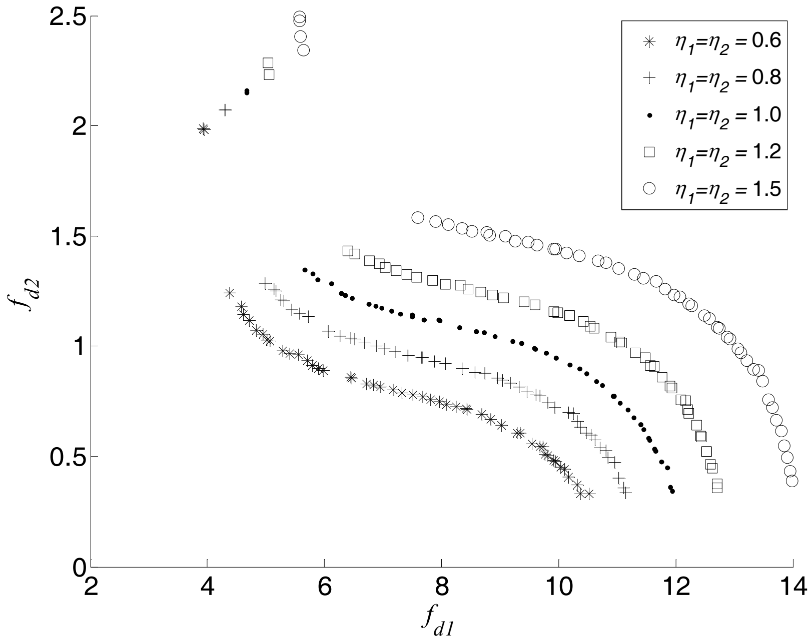

Figure 3: The Pareto optimal fronts with different satisfactory degree levels

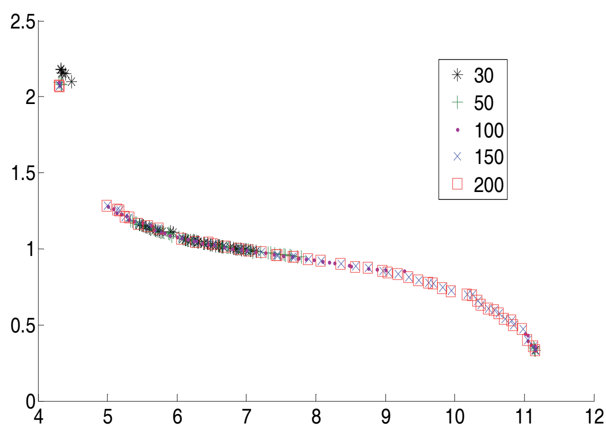

Figure 4: The plot of the iteration history

ing performance of the vehicle [Jiang, and et al. (2007b); Hou, and et al. (2008); Sun, and et al. (2010)]. In addition, the lightweight of vehicle should be taken

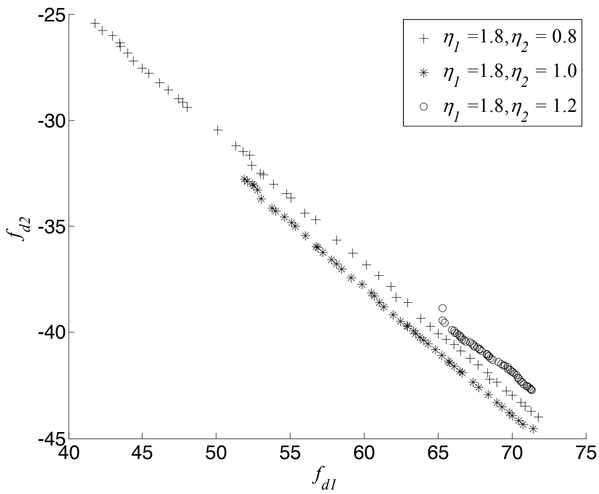

Figure 5: The Pareto optimal fronts with different satisfactory degree levels

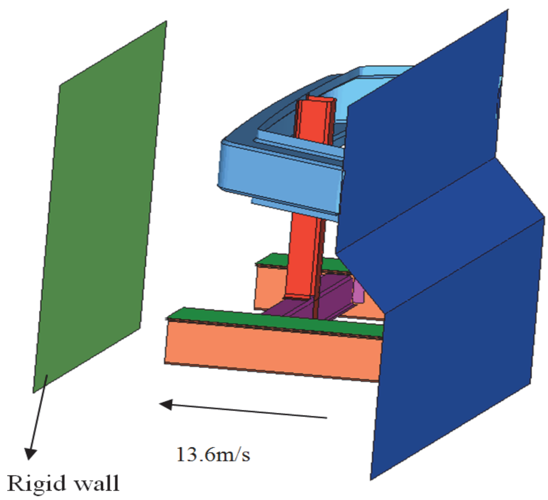

Figure 6: A bumper impacting the rigid wall

into account in the design process as well. Therefore, Maximum internal energy E_d and the weight of the bumper are used as the objective functions. The peak deceleration is treated as a constraint. As shown in Fig. 7, the sheet metal thickness of the structure is chosen as the design variables. The nominal values of Yield stress σ_s, Young's Modulus E, and Poisson's ratio ν are 167.1MPa, 207000MPa

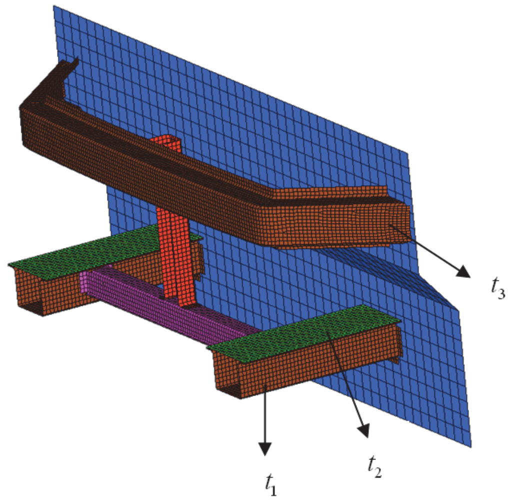

Figure 7: The finite model of the bumper

and 0.3, respectively. Due to the manufacturing and measurement errors, Yield stress σ_s, Young's Modulus E, and Poisson's ratio ν are considered as the uncertain parameters, and their uncertainty levels are 10% off from their normal values: $\sigma_s \in [150.39\text{MPa}, 183.81\text{MPa}]$, $E \in [186300\text{MPa}, 227700\text{MPa}]$, $\nu \in [0.27, 0.33]$. As a result, the reliability-based multiobjective optimization can be formulated as:

$$\min_{t_1,t_2,t_3} \quad f_1(t_1,t_2,t_3) = W$$

$$f_2(t_1,t_2,t_3,\sigma_s,E,\nu) = -E_d$$

s.t.

$$Acc(t_1,t_2,t_3,\sigma_s,E,\nu) \leq [500\text{m / s}^2, 550\text{m / s}^2] \tag{19}$$

$$\sigma_s \in [438.93\text{MPa}, 536.47\text{MPa}]$$

$$E \in [186300\text{MPa}, 227700\text{MPa}]$$

$\mu \in [0.27, 0.33]$

$1mm \leq t_1, t_2, t_3 \leq 3mm$

The finite element simulation for the crashworthiness process is carried out using the commercial software LS-DYNA. The structure comprises 13343 nodes and 13551 (mostly shell) elements. A concentrated mass 677.6kg is attached to the end of the bumper in order to supply enough energy for crashing. Fig. 7 shows the original FE model. A possible deformation of the bumper is shown in Fig. 8. A single simulation takes about 15 min with four processors computer in this problem.

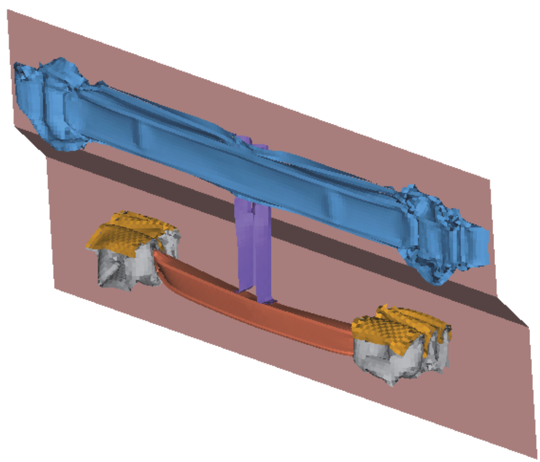

Figure 8: A typical deformation the finite element model

Initially, 60 sample points are selected through LHD to construct RS surrogate models for the objectives and constraints both in the design domain and uncertainty space. The surrogate model of weight W is a linear function of design variable $\mathbf{t}$, but the surrogate models of E_d and Acc are nonlinear functions with respect to $\mathbf{t}$, σ_s, E, and ν. According to the classical RSM theory, the larger the values of R^2 and R^2_{adj}, the better the model accuracy [Xiang, Wang, Fan and Fang (2006); Fang, Rais-Rohani, Liu and Horstemeyer (2005)]. The regression analysis results

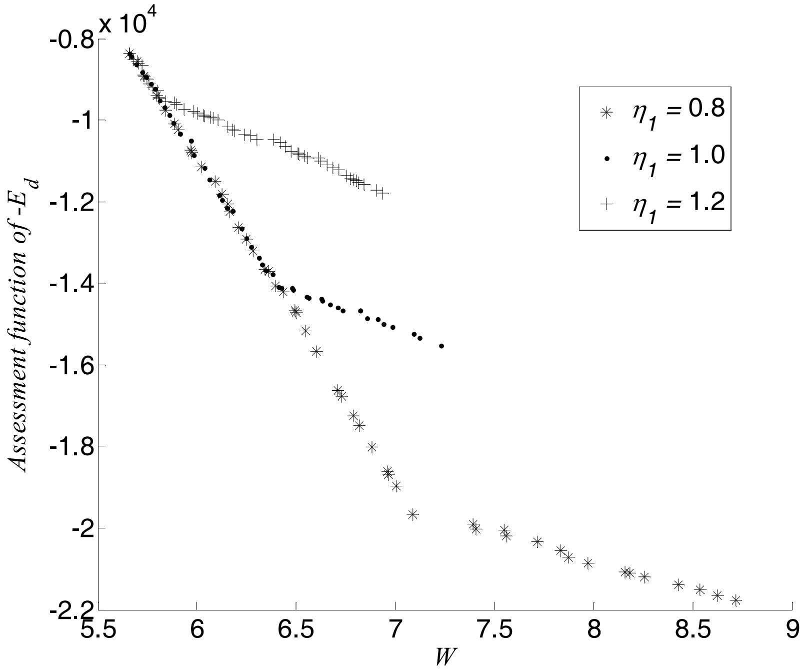

Figure 9: Pareto optimal front points with different satisfactory degree levels

Table 4: Results of regression analysis

	R^2	R^2_{adj}	Multiple R
W	1.0000	1.0000	1.0000
E_d	0.9993	0.9987	0.9997
Acc	0.9521	0.9118	0.9758

are given in Table 4. Obviously, the surrogate models for the weight W, maximum internal energy E_d and peak acceleration Acc exhibit a desirable accuracy.

Fig. 9 shows that the range and shape of the Pareto set varies for the different satisfactory degrees. It is clear that the minimum energy access function increases as the satisfactory degree increases. Similarly, this is because that a higher satisfactory degree can lead to a smaller feasible zone. It can be found the assessment function for $-E_d$ increases when increasing the satisfactory degree levels, while the $W$is not very sensitive to the different satisfactory degree levels. For $\eta = 0.8$, the obtained interval of Acc is $[368.2\text{m / s}^2, 545.3\text{m / s}^2]$, which is partially overlapped with the

Table 5: Summary of results for the thin-walled beam

η_1	Min W Min E_d	Design vector (mm)	Interval of E_d (J)	Interval of Acc (m/s^2)	Satisfactory degree of Acc
0.8	5.66	(1.00,1.00,1.00)	[-12103,-8375]	[208.1,431.4]	1.25
	-2.176×10^4	(1.72,1.00,1.86)	[-2.6670,-21764]	[368.2,545.3]	0.80
1.0	5.66	(1.00,1.00,1.00)	[-12103,-8375]	[208.1,431.4]	1.25
	-1.552×10^4	(1.39,1.00,1.43)	[-19872,-15519]	[299.1,500]	1.00
1.2	5.66	(1.00,1.00,1.00)	[-12103,-8375]	[208.1,431.4]	1.25
	-1.179×10^4	(1.00,1.00,1.57)	[-15462,-11795]	[265.5,452.7]	1.20

corresponding allowable interval $[500\text{m / s}^2, 550\text{m / s}^2]$. However, for $\eta = 1.2$, the obtained interval of Acc is $[265.5\text{m / s}^2, 452.7\text{m / s}^2]$, which is completely separated from allowable interval $[500\text{m / s}^2, 550\text{m / s}^2]$. So, this practical is also show the evidence that a higher satisfactory degree level might result in a better reliability in the multiobjective optimization designs.

7 Conclusion

This paper proposes a reliability-based multiobjective optimization in terms of a new satisfaction degree of interval. For engineering optimization applications, when the precise probabilistic information for the uncertain parameters is unavailable, the interval model is an alternative choice to describe the uncertainty conveniently and effectively. In reliability-based analysis, it is more appropriate to give a close interval constraint for reliability than a precise real value. Satisfaction degree of interval provides a general way to deal with the uncertain constraints. Most conventional satisfactory degree methods work well for the overlapped interval, with the interval values ranging from 0 to 1. This work introduces a new satisfactory degree of interval into the reliability-based multiobjective optimization. Thus, a new non-probabilistic reliability-based model is developed for multiobjective design optimization problem. Based on interval number programming, the reliability-based multiobjective problem is transformed into deterministic optimization. For practical engineering problems, the surrogate modeling method is incorporated into the reliability-based multiobjective optimization for objectives and constraints to improve the efficiency. Then, the Pareto set can be obtained with the different satisfactory levels. Two numerical examples and one typical engineering structural design are investigated to demonstrate the effectiveness of the current method. The significance of this study is the reliability-based multiobjective optimization works well not only for overlapped intervals but also the completely separated intervals. Furthermore, this study provides a possibility to apply the interval method to structural optimization problems where the gradient-based sensitivity information is required. Two problems are still remaining open for further investigation: (1) the reliability-based multiobjective formulation is a double-loop nested optimization, and how to improve its computational efficiency is an important topic for the future research, (2) more effective interval uncertain methods, rather than the relaxation of interval from [0, 1] to [-∞, +∞], to overcome the non-differentiability at the interval bounds are still in demand for advanced structural optimization of continuum structures.

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