 CMES, vol.73, no.3, pp.311-329, 2011

The Coupling FEM and Natural BEM for a Certain Nonlinear Interface Problem with Non-Matching Grids

Ju'e Yang[1] and Dehao Yu[2]

Abstract: In this paper, we introduce a domain decomposition method with non-matching grids for a certain nonlinear interface problem in unbounded domains. To solve this problem, we discuss a new coupling of finite element method(FE) and natural boundary element(NBE). We first derive the optimal energy error estimate of finite element approximation to the coupled FEM-NBEM problem. Then we use a dual basis multipier on the interface to provide the numerical analysis with non-matching grids.Finally, we give some numerical examples further to confirm our theoretical results.

Keywords: nonlinear interface problem, domain decomposition, natural boundary reduction, coupling, non-matching grids.

1. Introduction

The coupling method of finite element method (FEM) and boundary element method (BEM) [Hsiao (1988); Yu (1983, 1992); Han (1990)] was developed as a generalization of the standard finite element method to problems in unbounded domains with complicated geometry shapes. It keeps all advantages of the finite element in treating the complicated bounded domains as well as the boundary element in treating unbounded domains. Many authors have made contributions to the coupling method of this kind and there are many constructive research in this direction both in theory and practical computation(see Hu and Yu (2001b); Liu and Yu (2008) etc.). The standard coupling procedure can be described as following: the domain is decomposed into two subdomains, one bounded subdomain in which the standard finite element method will be used and the other unbounded subdomain where the boundary element method is applied. Finally, the unbounded domain problem is reduced to a bounded domain problem.

[1] School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China. Email:yangje@lzu.edu.cn.
[2] Institute of Computational Mathematics and Scientific/Engineering Computing Academy of Mathematical and System Sciences, The Chinese Academy of Sciences, Beijing 100080, China.

It is well known that domain decomposition methods (DDMs) are important numerical techniques for solving partial differential equations. When a domain is divided into some subdomains and artificial boundaries which called 'interface', the underlying partial differential equations can be solved independently in each subdomain. To get proper solution, the appropriate boundary condition must be given on the interface of sub-domains. When an unbounded domain is divided into some subdomains, there is at least one unbounded domain. In this case, the boundary reduction will be needed [see Yu (1995, 1996)]. There are many ways to accomplish boundary reduction for unbounded domain problems. the natural boundary reduction proposed by Feng and Yu [Feng and Yu (1983)] has some distinctive advantages over the usual boundary reduction methods: the preservation of the symmetry and coerciveness, simplification of the discrete problem and the preservation of the optimal estimates and the numerical stability.

The existing coupling methods of FEM and natural BEM or the domain decomposition methods based on the natural boundary element reduction requires that the approximate solutions exactly satisfy the matching conditions over the interface or on the artificial boundary. This leads to some restrictions for the finite element discretizations on subdomain. Especially in the case of singularities of the solution, which has strong singularity near the concave vertex [see Hu and Yu (2001b)]. Therefore, we cannot expect that the approximate solution has an $O(h)$ estimation for the discretization error. In order to obtain the approximation of the solution which possesses satisfactory accuracy, it is necessary to use high refinements of the finite element grids near the concave vertices. The DDMs with non-matching grids can couple different variational problems in different sub-domains, see, for example, Belgacem and Maday (1999); C. Bernardi and Patera (1994). One important character of this method is to introduce a Lagrange multiplier space on the interface such that the matching conditions across the interface is replaced by a weaker one, i.e. the pointwise matching is replaced by the integral matching. Most importantly, the relaxation of the matching conditions on the interface still yields optimal approximation. In recent years, domain decomposition with non-matching grids have attracted a lot of attention from computational mathematicians and engineers (see,Wohlmuth (2000); Hu (2005) and Ju'e Yang and Yu (2005)). In [Ju'e Yang and Yu (2005)], we first use the technique of nonmatching grid to deal with the Dirichlet exterior boundary problem on unbounded domain. To our knowledge there seems no study in the literature of the case of nonlinear problems in unbounded domain.

In the present paper, we try to extend the DDMs with non-matching grids to a certain nonlinear interface problems in unbounded domains by the coupling of FEM and natural BEM. In Yu and Huang (2008), the artificial boundary method has been used to this problem efficiently. For simplicity of exposition, we consider only the case with two dimensions in the paper. In our method, the multiplier space is spanned by the dual basis multipliers presented in Wohlmuth (2000). Such choice of the multiplier space can avoid the computation of the L^2 projector on the interface. We derive an optimal error estimate of the resulting nonconforming approximation. It will be shown that the iteration possesses a convergence rate independent of the mesh sizes.

Our paper is organized as following. In section 2, we introduce the interface problem in its strong and weak forms and derive a nonlinear system of coupled FEM-NBEM equations. In section 3 we make a discretization for the resulting coupled system based on the non-matching grids. In section 4, we give the error estimate for the approximates and obtain the optimal accuracy. Finally, the numerical experiments testify the theoretical results.

2. The Coupled FEM-NBEM Systems

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with Lipschitz-continuous boundary Γ and $\Omega_c := \mathbb{R}^2 \backslash \Omega$ be the exterior unbounded domain of Ω.

Assume that $p \in C^1(\mathbf{R}_+)$ satisfy the condition $p_1 < p(t) < p_2$ and $\alpha < p(t) + tp'(t) \leq \beta$ for some global constants $p_1,\ p_2,\ \alpha,\ \beta > 0$. Given the function $f : \Omega \to \mathbb{R}$ and $u_0,\ t_0 : \Gamma \to \mathbb{R}$, we consider the following nonlinear interface problem(see [Mund and Stephan (1999)]).

As the interior part, we consider the nonlinear partial differential equation

$$-div(p(|\nabla u|) \cdot \nabla u) = f \qquad \text{in } \Omega. \tag{2.1}$$

In the exterior part, we consider the Laplace equation

$$\Delta u = 0 \qquad \text{in } \Omega_c, \tag{2.2}$$

with the radiation condition

$$u(x) = a + o(1) \ \ (|x \to \infty|) \tag{2.3}$$

which a is a given real constant. We consider the transmission condition on Γ

$$u|_\Gamma - u_c|_\Gamma = u_0 \quad \text{ and } \quad p(|\nabla u|)\left.\frac{\partial u}{\partial n}\right|_\Gamma - \left.\frac{\partial u_c}{\partial n}\right|_\Gamma = t_0, \tag{2.4}$$

where n denote the unit normal on Γ defined almost everywhere pointing from Ω into Ω_c. Define the inner products in $L^2(\Omega_1)$ and $L^2(\Gamma)$

$$(u,v) := \int_\Omega u(x)v(x)dx \qquad \forall\ u(x),\ v(x) \in\ L^2(\Omega) \tag{2.5}$$

$$\langle u,v\rangle := \int_\Gamma u(x)v(x)ds \qquad \forall\, u(x),\ v(x) \in\ L^2(\Gamma) \tag{2.6}$$

Since the boundary Γ is not necessarily a circle, we draw a auxiliary circle Γ_1 with radius r, such that the circle disc Ω' with the boundary Γ_1 contains the domain $\bar{\Omega}$. Then The auxiliary boundary divides the exterior region of Ω_c into two nonoverlapping subdomains and we set $\Omega_1 = \Omega_c \bigcap \Omega'$, $\Omega_2 = \mathbb{R}^2 \backslash \bar{\Omega}'$, and define $u_1 := u_c|_{\Omega_1}$, $u_2 := u_c|_{\Omega_2}$(For the picture see Figure 1).

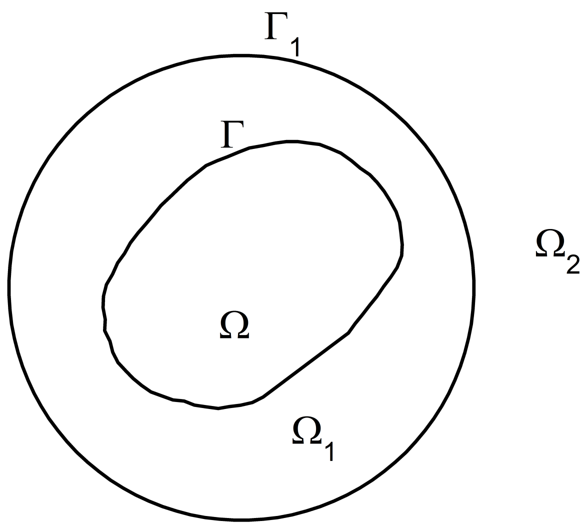

Figure 1: $\Omega' = \Omega \bigcup \Omega_1$, and Γ_1 is an auxiliary circle

Then we rewrite our nonlinear interface problem (2.1-2.4) as follows: Find the function u, u_1, u_2 such that

$$\begin{cases} -\mathrm{div}(p(|\nabla u|)\cdot\nabla u) \ = \ f, & \text{in } \Omega, \\ u - u_1 = u_0,\ \ p(|\nabla u|)\dfrac{\partial u}{\partial n} - \dfrac{\partial u_1}{\partial n} = t_0, & \text{on } \Gamma, \\ -\Delta u_1 \ = \ 0, & \text{in } \Omega_1, \\ u_1 \ = \ u_2,\ \ \dfrac{\partial u_1}{\partial n} \ = \ \dfrac{\partial u_2}{\partial n}, & \text{on } \Gamma_1, \\ -\Delta u_2 \ = \ 0, & \text{in } \Omega_2, \\ u_2(x) \ = a + \ O(1), & \text{for } |x| \to \infty \end{cases} \tag{2.7}$$

To obtain a variational formulation of Eq.2.7, we first consider the unbounded domain Ω_2. We need to define a symmetric and positive definite boundary operator.

Let $E(x,y)$ be the fundamental solution for the Laplacian, i.e.

$$E(x,y) = -\frac{1}{2\pi}\log|x-y|,\ x,y \in \mathbb{R}^2,$$

We have the following four classical boundary integral operators

$$Vt(x) = \int_{\Gamma_1} t(y)\cdot E(x,y)ds_y \quad (V : H^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)),$$

$$Ku(x) = \int_{\Gamma_1} u(y)\cdot \frac{\partial}{\partial n_y}E(x,y)ds_y \quad (K : H^{\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)),$$

$$K't(x) = \int_{\Gamma_1} t(y)\cdot \frac{\partial}{\partial n_x}E(x,y)ds_y \quad (K' : H^{-\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)),$$

$$Du(x) = -\frac{\partial}{\partial n_x}\int_{\Gamma_1} u(y)\cdot \frac{\partial}{\partial n_y}E(x,y)ds_y \quad (D : H^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)),$$

Define

$$\mathscr{K} u(x) = -\frac{\partial}{\partial n_x}\int_{\Gamma_1} \frac{\partial}{\partial n_y}G(x,y)u(y)ds_y, \tag{2.8}$$

which $G(x,y)$ is the Green's function for the Laplace equation on the domain Ω_2. The operator $\mathscr{K}$ is just the natural integral operator, i.e. the Dirichlet-Neumann operator (Steklov-Poincar*é* operator)(see [Yu (1993)](in Chinese) and [Yu (2002)](in English)).

Using the Green's formula we can obtain

$$\mathscr{K} = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) \quad (\mathscr{K} : H^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)). \tag{2.9}$$

It is well known that $V : H^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is symmetric and positive definite, and $D : H^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$ is symmetric and positive semidefinite. Thus, the operator $\mathscr{K} : H^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$ is also symmetric and positive definite with respect to the inner product $\langle\cdot,\cdot\rangle_{\Gamma_1}$.

Decomposing $\int_{\Omega_c}|\nabla u_c|^2dxdy$ into two parts

$$\int_{\Omega_c}|\nabla u_c|^2dxdy = \int_{\Omega_1}|\nabla u_1|^2dxdy + \int_{\Omega_2}|\nabla u_2|^2dxdy, \tag{2.10}$$

and applying the natural boundary element theory to Ω_2, we obtain the natural integral equation to the auxiliary boundary Γ_1

$$\frac{\partial u_2}{\partial n} = -\mathscr{K}(u_2|_{\Gamma_1}) \tag{2.11}$$

Next, we multiply the divergence partial differential equation in Eq.2.7 by any test function $v \in H^1(\Omega)$ and apply the Green formula to yield

$$\int_\Omega p(|\nabla u|)\nabla u \nabla v dx - \int_\Gamma p(|\nabla u|)\frac{\partial u}{\partial n} v ds = \int_\Omega f v dx, \ \ \forall v \in H^1(\Omega), \tag{2.12}$$

According to the interface condition (2.4), we have

$$\int_\Omega p(|\nabla u|)\nabla u \nabla v dx - \int_\Gamma \frac{\partial u_1}{\partial n} v ds = \int_\Omega f v dx + \int_\Gamma t_0 v ds, \ \ \forall v \in H^1(\Omega) \tag{2.13}$$

Let w_0 be the harmonic function in the domain Ω_1 which satisfy the following equation

$$\begin{cases} \Delta w_0 = 0 & \text{in } \Omega_1 \\ w_0 = u_0 & \text{on } \Gamma \\ w_0 = 0 & \text{on } \Gamma_1 \end{cases} \tag{2.14}$$

Set $w = u_1 + w_0$, and in the same way, on Ω_1, we have

$$\int_{\Omega_1} \nabla w \cdot \nabla v dx + \int_\Gamma \frac{\partial u_1}{\partial n} v ds - \int_{\Gamma_1} \frac{\partial u_2}{\partial n} v ds = \int_{\Omega_1} \nabla w_0 \cdot \nabla v dx, \ \ \forall v \in H^1(\Omega_1) \tag{2.15}$$

which, due to the natural integral equation (2.11), becomes

$$\int_{\Omega_1} \nabla w \cdot \nabla v dx + \int_\Gamma \frac{\partial u_1}{\partial n} v ds + \int_{\Gamma_1} v \mathscr{K} u_2 ds = \int_{\Omega_1} \nabla w_0 \cdot \nabla v dx, \ \ \forall v \in H^1(\Omega_1) \tag{2.16}$$

Define

$$H := \{(u,w) \in H^1(\Omega) \times H^1(\Omega_1) : u = w \ \text{ on } \ \Gamma\}. \tag{2.17}$$

Then we define the nonlinear functional

$$A((u,w),(v,\sigma)) := \int_\Omega p(|\nabla u|)\nabla u \cdot \nabla v dx dy, \quad (u,w),(v,\sigma) \in H, \tag{2.18}$$

and the bilinear functional

$$B((u,w),(v,\sigma)) := \int_{\Omega_1} \nabla w \cdot \nabla \sigma dx dy + \int_{\Gamma_1} \mathscr{K} w \cdot \sigma ds, \quad (u,w),(v,\sigma) \in H, \tag{2.19}$$

Adding (2.13) and (2.16), we obtain the coupled FEM-NBEM variational problem of Eq.2.7

$$\begin{cases} \text{Find } (u,w) \in H, \text{such that} \\ A((u,w),(v,\sigma)) + B((u,w),(v,\sigma)) = F(v,\sigma), \quad \forall (v,\sigma) \in H, \end{cases} \tag{2.20}$$

which F is the linear functional

$$F(v,\sigma) := \int_{\Omega} f v dxdy + \int_{\Omega_1} \nabla w_0 \cdot \nabla \sigma dxdy + \int_{\Gamma} t_0 \cdot v ds, \quad \forall (v,\sigma) \in H. \tag{2.21}$$

Define the dorm $||\cdot||_H$ by

$$||(v,\sigma)||_H := (|v|^2_{1,\Omega} + |\sigma|^2_{1,\Omega} + ||\sigma||^2_{\frac{1}{2},\Gamma_1})^{\frac{1}{2}}, \quad (v,\sigma) \in H. \tag{2.22}$$

Remark 2.1 *The natural integral operator $\mathscr{K} : H^{\frac{1}{2}}(\Gamma_1) \longrightarrow H^{-\frac{1}{2}}(\Gamma_1)$ is just the Dirichlet-Neumann operator(Steklov-Poincaré operator) for the exterior domain Ω_2. So, it is symmetric and semi-positive definite with respect to the inner product $< \cdot,\cdot >_{\Gamma_1}$, i.e. there are positive constant c and $\hat{c}$ such that(refer to [Yu (1993, 2002)])*

$$\langle \mathscr{K}\mu, \mu \rangle \geq c||\mu||^2_{\frac{1}{2},\Gamma_1}, \quad \forall \mu \in H^{\frac{1}{2}}(\Gamma_1)/P_0, \tag{2.23}$$

and

$$\langle \mathscr{K}\mu, \mu \rangle \leq \hat{c}||\mu||^2_{\frac{1}{2},\Gamma_1}, \quad \forall \mu \in H^{\frac{1}{2}}(\Gamma_1)/P_0, \tag{2.24}$$

where P_0 denotes the set of all constants.

We note that the bilinear form B is bounded and satisfies some positiveness condition. In fact, for any functions $(u,w),(v,\sigma) \in H$ there exist positive constant $\alpha > 0, \beta > 0$ such that

$$|B((u,w),(v,\sigma))| \leq \alpha(|u|_{1,\Omega}|v|_{1,\Omega} + ||w||_{\frac{1}{2},\Gamma_1}||\sigma||_{\frac{1}{2},\Gamma_1}) \leq \alpha||(u,w)||_H||(v,\sigma)||_H, \tag{2.25}$$

$$|B((v,\sigma),(v,\sigma))| \geq \beta(|v|^2_{1,\Omega} + ||\sigma||^2_{\frac{1}{2},\Gamma_1}) \geq \beta||(v,\sigma)||^2_H \tag{2.26}$$

Now, we introduce the nonlinear operators $T : H \to H*$, which is defined by

$$[T(u,w),(v,\sigma)] := A((u,w),(v,\sigma)) + B((u,w),(v,\sigma)) \tag{2.27}$$

for all $(v,\sigma) \in H$. Thus, the weak formulation (2.20) can be written in the form of an operator equation:

$$\begin{cases} \text{Find } (u,w) \in H, \text{such that} \\ T((u,w),(v,\sigma)) = [F,(v,\sigma)], \quad \forall (v,\sigma) \in H, \end{cases} \tag{2.28}$$

From the assumptions on the function $p(t)$, we infer that T is bounded and uniformly monotone with respect to the norm $||(\cdot,\cdot)||$. Therefore, the variational problem (2.28) has a unique solution $(u,w) \in H$.

3. Finite Element Discretization with Non-matching Grids

In this section, we make a finite element discretization for the subdomains and introduce the non-matching grids method and the construction of basis functions of the Larange multiplier space. The main motivation for us to do this is that we can couple different discretizations in different subdomains in this way. It seems very reasonable especially for the case of singularities of the solution.

Families of finite element triangulations $\mathscr{T}_{h_i}, i = 1,2$, are associated with Ω and Ω_1(e.g. some regular quasi-uniform triangles and curved triangles at the interfaces). We denotes by h_i the maximum diameter of the elements of $\mathscr{T}_{h_1}, i = 1,2$. But in most real calculation, the curved triangle nearby the interfaces are approximated by the straight triangles which has the same nodes with the curved triangles. This simplified method generates only small error. Here, we note that the meshes may not match at the interface between any two subdomains, which means that the finite element nodes on Γ respect to the triangulation $\mathscr{T}_{h_1}$ don't coincide with the boundary nodes on Γ respect to the triangulation $\mathscr{T}_{h_2}$. A similar case is to the artificial boundary Γ_1. Therefore, the continuity conditions on the interface between any two subdomains are broken, which is required for the usual coupling of FEM and NBEM. It is pointed out in [C. Bernardi and Patera (1994); Belgacem and Maday (1999); **?**] that some weaker continuity condition across the interface can guarantee the optimal error estimate provided that the solution u is smooth enough. Now, our interfaces are some circles and our problem consists of a nonlinear second order elliptic equation in divergence form in a bounded inner region, and the Laplace equation in the corresponding unbounded exterior region, in addition to appropriate boundary and transmission conditions.

Let $V_{h_i}(\Omega_i) \subset H^1(\Omega_i), i = 1,2$, be the piecewise linear finite element spaces on Ω_i with respect to $\mathscr{T}_{h_1}, i = 1,2$. Next, we discretize the auxiliary circle Γ_1. Given $n \in \mathbb{N}$, we let $0 = a_0 \leq a_1 \leq \cdots \leq a_n = 2\pi$ be a uniform partition of $[0,2\pi]$ with $h_3 = a_{j+1} - a_j = \frac{2\pi}{n}, j = 0,1,\cdots,n-1$, which generates a division $\mathscr{T}_{h_3}$ on the artificial boundary Γ_1. We denote this boundary element space by $V_{h_3}(\Gamma_1)$. The division in $\Omega_i, i = 1,2$, leads to a division on the interface Γ and Γ_1, so we set

$$V_{h_i}(\Gamma) := \{v|_\Gamma : v \in V_{h_i}(\Omega_i), i = 1,2\} \text{ and } V_{h_2}(\Gamma_1) := \{v|_{\Gamma_1} : v \in V_{h_2}(\Omega_2)\} \quad (3.29)$$

The parameter h is set equal to the 3-tuple(h_1,h_2,h_3) and the Lagrange multiplies space defined at the interfaces are denoted by $M_h(\Gamma)$ and $M_h(\Gamma_1)$ which will be discussed at detail later.

We set the product spaces

$$Q_h := V_{h_1}(\Omega_1) \times V_{h_2}(\Omega_2) \times V_{h_3}(\Gamma_1) \quad (3.30)$$

Define

$$V_h := \{v_h = (v_{h_1}, v_{h_2}, v_{h_3}) \in Q_h : \int_{\Gamma \bigcup \Gamma_1} [v_h] \cdot \mu ds = 0, \ \forall \mu \in M_h(\Gamma), M_h(\Gamma_1)\} \quad (3.31)$$

where $[\cdot]$ denotes the jump of the function v_h across the interfaces. We apply FEM and NBEM to compute approximations to the finite element solution u and the boundary element solution ϕ. For this purpose we denote $\mathscr{T}_h := \mathscr{T}_{h_1} + \mathscr{T}_{h_2}$ as the finite element space and τ_h as the boundary element space.

Then we obtain the following non-conforming variational problem associated with (2.20)

$$\begin{cases} \text{Find } u_h := (u_{h_1}, u_{h_2}, \phi_{h_3}) \in V_h, \text{such that} \\ (p(|\nabla u_{h_1}|)\nabla u_{h_1}, \nabla v_{h_1})_\Omega + (\nabla u_{h_2}, \nabla v_{h_2})_{\Omega_1} + \langle \mathscr{K} u_{h_3}, v_{h_3} \rangle_{\Gamma_1} \\ = (f, v_{h_1})_\Omega + (\nabla w_{0h_2}, \nabla v_{h_2})_{\Omega_1} + \langle t_0, v_{h_1} \rangle_\Gamma, \quad \forall v_h := (v_{h_1}, v_{h_2}, v_{h_3}) \in V_h, \end{cases} \quad (3.32)$$

Let $T_h : V_h \longrightarrow V_h^*$ be an operator that approximates T on V_h, and let $F_h \in V_h^*$ be an approximation of F on V_h. Then, the above variational problem is equivalent to the following discrete operator equation:

$$\begin{cases} \text{Find } u_h \in V_h, \text{such that} \\ T_h(u_h, v_h) = F_h(v_h), \quad \forall v_h \in V_h, \end{cases} \quad (3.33)$$

Since T is bounded and uniformly monotone on H, it can prove that T_h holds the same properties on V_h. Hence, the coupled discrete operator equation (3.33) has a unique solution $u_h \in V_h$.

As we have seen, the definition of Lagrange multiplier space is of great importance for the unique solvability. Here we'll use the dual basis (refer to Wohlmuth (2000)) to define a new type of multiplier space for unbounded domain problems.

Here and below we only discuss the interface Γ_1. a similar definition is to the interface Γ. To avoid confusion for the subscript, let us denote the interface by Γ which consists of the two interfaces Γ and Γ_1.

Let N be the number of nodes on Γ and $\{\theta_i\}_{i=0}^N, \theta_0 = \theta_N$ be the set of nodal points in Γ, and let γ be a segmental arc on Γ and $\{l_i^\gamma\}_{i=1}^2, l_i^\gamma \in P_1(\gamma)$ be a linear basis on the element γ. Using linear Lagrange interpolation it is easy to know

$$l_1^\gamma(\theta) = \frac{N}{2\pi}(\theta_i - \theta), \quad l_2^\gamma(\theta) = \frac{N}{2\pi}(\theta - \theta_{i-1}) \quad (3.34)$$

Define the test functions $\{\phi_i^\gamma\}_{i=1}^2$ satisfying

$$\left\langle l_i^\gamma(\theta), \phi_j^\gamma(\theta) \right\rangle_\gamma = \delta_{i,j} \left\langle l_i^\gamma(\theta), 1 \right\rangle_\gamma, \ \ 1 \le i, j \le 2, \quad (3.35)$$

where $\delta_{i,j}$ is the Kronecker delta symbol. Furthermore, we have

$$\left\langle l_i^\gamma(\theta), \left(\sum_{j=1}^{2} \phi_j^\gamma(\theta) - 1\right)\right\rangle_\gamma = 0, \quad 1 \le i \le 2. \tag{3.36}$$

Therefore, from (3.34) and (3.35) we deduce

$$\phi_1^\gamma(\theta) = 2l_1^\gamma(\theta) - l_2^\gamma(\theta) = \frac{N}{2\pi}(2\theta_i + \theta_{i-1} - 3\theta), \tag{3.37}$$

$$\phi_2^\gamma(\theta) = -l_1^\gamma(\theta) + 2l_2^\gamma(\theta) = \frac{N}{2\pi}(3\theta + \theta_i - 2\theta_{i-1}). \tag{3.38}$$

Let $\{\Phi_i\}_{i=1}^N$ and $\{\Psi_i(\theta)\}_{i=1}^N$ be the global nodal basis functions and the dual basis functions for Γ, respectively. As a consequence, we set $M_h(\Gamma) = span\{\Phi_i(\theta), 1 \le i \le N\}$. Under uniform subdivision the piecewise linear basis functions(see Figure 2) are

$$\Phi_i(\theta) = \begin{cases} \frac{N}{2\pi}(3\theta - \theta_i - 2\theta_{i-1}), & \theta_{i-1} \le \theta\theta_i, \\ \frac{N}{2\pi}(2\theta_{i+1} + \theta_i - 3\theta), & \theta_i \le \theta \le \theta_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.39}$$

where $i = 1, 2, \cdots, N$ and $\theta_i = \frac{2\pi i}{N}$.

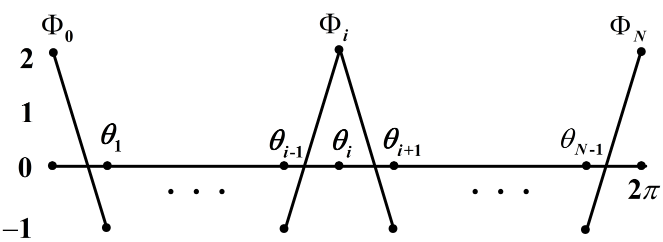

Figure 2: Dual basis functions of $M_h(\Gamma)$ with a circle interface Γ

Since $V_{h_1}(\Gamma) \subset H^{1/2}(\Gamma)$, the test functions space $M_h(\Gamma)$ may be embedded in the dual space of $H^{1/2}(\Gamma)$ with respect to the L^2-inner product. Then, we obtain $M_h(\Gamma) \subset H^{-1/2}(\Gamma)$. Therefore, we call $\{\Phi_i(\theta)\}_{i=1}^N$ as the dual basis on Γ. From Figure 2, for any fixed node θ_i on Γ, the dual basis function $\Phi_i(\theta)$ has its support on two mesh intervals and decreases linearly from 2 to -1 on the second interval. Similar to (3.35), $\Psi_i(\theta)$ and $\Phi_i(\theta)$ also satisfy the following global property

$$\langle \Phi_i(\theta), \Psi_i(\theta) \rangle_\Gamma = \delta_{i,j} \langle \Phi_i(\theta), 1 \rangle_\Gamma, \ \ 1 \le i, j \le N \tag{3.40}$$

Before we begin the analysis of error estimate, we'll introduce two important projection operators. Similar to (C. Bernardi and Patera (1994)), since each interface has two sides, we denote by Γ_{12} and Γ_{21}. We define the projection operator Π_h in such way: it maps the space $V_h(\Gamma_{12})$ into $V_h(\Gamma_{21})$ or maps $V_h(\Gamma_{21})$ into $V_h(\Gamma_{12})$. The choice of side is rather arbitrary. In our case, we choose the fine mesh side as the beginning such as Γ_{12}. That is to say, Given $v \in L^2(\Gamma)$, the values of $\Pi_h v \in V_h(\Gamma_{21})$ can be determined by

$$\int_\Gamma (v - \Pi_h v) \cdot \mu ds = 0, \ \ \forall \mu \in M_h(\Gamma). \tag{3.41}$$

Next, define by $P_h : L^2(\Gamma) \to M_h(\Gamma)$ the usual orthogonal projection operator. We recall its approximation properties in the following lemma. We can verify it in the standard manner and do not include the proof here.

Lemma 3.1 *For any real number s, $0 \le s \le 1$, there exists a constant c such that the following estimate holds for any function v in $H^s(\Gamma)$:*

$$||v - P_h v||_{0,\Gamma} \le ch^s ||v||_{s,\Gamma}, \tag{3.42}$$

$$||v - P_h v||_{(H^{\frac{1}{2}}(\Gamma))'} \le ch^{s+\frac{1}{2}} ||v||_{s,\Gamma}. \tag{3.43}$$

Here the dual norm is defined by

$$||f||_{X'} := \sup_{v \in X} \frac{\langle f, v \rangle}{||v||_X}. \tag{3.44}$$

where X' is the dual space of the Hilbert space X. The definition of operator P_h yieds to

$$\int_\Gamma (v - P_h v) \mu ds = 0, \quad \forall \mu \in M_h(\Gamma), \tag{3.45}$$

where $P_h v \in M_h(\Gamma)$.

The next lemma shows the stability property of the projection operator Π_h in $L^2(\Gamma)$ and $H^1(\Gamma)$.

Lemma 3.2 *There exists a constant $c,c'>0$ such that for $\forall v\in L^2()\Gamma$*

$$||\Pi_h v||_{0,\Gamma}\le c||v||_{0,\Gamma}, \tag{3.46}$$

Assumed that $v\in H^1(\Gamma)$, for uniform meshes we have

$$|\Pi_h v|_{1,\Gamma}\le c'|v|_{1,\Gamma}. \tag{3.47}$$

Proof. See (Ju'e Yang and Yu (2005)).

Then, by an interpolation argument, the following estimate holds for any function v in $H^{\frac{1}{2}}(\Gamma)$:

$$||\Pi_h v||_{\frac{1}{2},\Gamma}\le C||v||_{\frac{1}{2},\Gamma}. \tag{3.48}$$

which C is some positive constant.

4. Analysis of Error

Now, we give an approximate property of u_h.

Define the norm

$$||v_h||=\left(||v_{h_1}||^2_{1,\Omega}+||v_{h_2}||^2_{1,\Omega_1}+||v_{h_3}||^2_{\frac{1}{2},\Gamma_1}\right)^{\frac{1}{2}},\ \ \forall v_h\in V_h. \tag{4.49}$$

From the assumptions on the function $p(t)$, we infer that T_h is uniformly strongly monotone and we obtain (see Hu and Yu (2001a))

$$T_h(u_h,u_h-v_h)-T_h(v_h,u_h-v_h)\ge\alpha||u_h-v_h||^2,\ \ \forall u_h,v_h\in V_h, \tag{4.50}$$

and

$$T_h(u_h,v_h-u)-T_h(u,v_h-u)\le\beta||u-u_h||_{1,\Omega}|u-v_h|_{1,\Omega},\ \ \forall v_h\in V_h. \tag{4.51}$$

From the well-known second Strang's lemma, we have

$$||u-u_h||\le C\inf_{\forall v_h\in V_h}||u-v_h||+\tilde{C}\sup_{\forall v_h\in V_h}\frac{\int_{\Gamma\bigcup\Gamma_1}\frac{\partial u}{\partial n}[v_h]ds}{||v_h||} \tag{4.52}$$

where $[v_h]$ denotes the jump of this function through the interfaces Γ and Γ_1, and the constant $\tilde{C}$ is associated to coefficients $p(t)$. We note that the first term of the right hand of (4.52) is the best approximation error, while the second term is the consistency error. The best approximation error can be estimated by using interpolation inequalities for conforming finite elements and stability property of the projection Π_h; For estimation of the consistency error, we use the fact the jump of the solution is orthogonal to the multiplier space M_h. Thus we have the following theorem.

Theorem 4.1 *Assume that the solution u of Eq.(2.7) satisfies, for any real number s,* $\frac{1}{2} < \varepsilon_i \leq 1, i = 1,2, u|_{\Omega_1} \in H^{1+\varepsilon_1}(\Omega_1), u|_{\Omega_2} \in H^{1+\varepsilon_2}(\Omega_2)$ *and* $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$. *Then there exists a function* $v_h \in V_h$ *such that*

$$||u_h - v_h||_H \leq C(h_1^{\varepsilon_1}|u|_{1+\varepsilon,\Omega_1} + h_2^{\varepsilon_2}|u|_{1+\varepsilon_2,\Omega_2} + h_3||u||_{3/2,\Gamma_1}) \tag{4.53}$$

where $C > 0$ *is a constant independent of the mesh parameters* $h_i, i = 1,3$.

Proof: First, we estimate the error bound on the artificial boundary Γ. Let $\pi_{h_i}, i = 1,2$ are the Lagrange interpolation operators in $\Omega_i, i = 1,2$, respectively.
Then we define v_h by

$$v_{h_1} = \pi_{h_1}u_1, \qquad v_{h_2} = \pi_{h_2}u_2 + \Pi_h[\pi_{h_1}(u_1|_\Gamma) - \pi_{h_2}(u_2|_\Gamma)]$$

. Recalling that the projection operator Π_h have been defined in (3.45), we have

$$\begin{aligned} \langle v_{h_1} - v_{h_2}, \mu\rangle|_\Gamma &= \langle\{\pi_{h_1}(u_1|_\Gamma) - \pi_{h_2}(u_2|_\Gamma)\} - \Pi_h\{\pi_{h_1}(u_1|_\Gamma) - \pi_{h_2}(u_2|_\Gamma)\}, \mu\rangle_\Gamma \\ &= 0 \end{aligned} \tag{4.54}$$

Also, we can deal with the jump of solution in the interface Γ_1. Then the trace theorem and the stability properties of Π_h lead to

$$\begin{aligned} \inf_{\forall v_h \in V_h}||u - v_h|| &\leq \inf_{\forall v_h \in V_h}\left(||u_1 - v_{h-1}||_{1,\Omega_1} + ||u_2 - v_{h_2}||_{2,\Omega_2} + ||u_3 - v_{h_3}||_{\frac{1}{2},\Gamma_1}\right) \\ &\leq ||u - \pi_{h_1}u||_{1,\Omega_1} + ||u - \pi_{h_2}u||_{2,\Omega_2} + ||u - \pi_{h_3}u||_{\frac{1}{2},\Gamma_1} \\ &+ ||\Pi_h(\pi_{h_1}u_1 - \pi_{h_2}u_2)||_{\frac{1}{2},\Gamma} + ||\Pi_h(\pi_{h_2}u_2 - \pi_{h_3}u_3)||_{\frac{1}{2},\Gamma_1} \\ &\leq ch_1^{\varepsilon_1}||u_1||_{1+\varepsilon_1,\Omega_1} + ch_2^{\varepsilon_2}||u_2||_{1+\varepsilon_2,\Omega_2} + ch_3||u_3||_{\frac{3}{2},\Gamma_1} \end{aligned} \tag{4.55}$$

For the second part of (4.52), the consistency error, we first fix our attention to the interface Γ. Using the definition of the projection operators P_h and Π_h, we have

$$\begin{aligned} \left|\int_\Gamma \frac{\partial u}{\partial n}[v_h]ds\right| &= \left|\int_\Gamma \frac{\partial u}{\partial n}(v_{h_1 - \Pi_h v_{h_1}})ds\right| \\ &= \left|\int_\Gamma \left(\frac{\partial u}{\partial n} - P_h\frac{\partial u}{\partial n}\right)(v_{h_1} - \Pi_h v_{h_1})ds\right| \\ &\leq \left|\left|\frac{\partial u}{\partial n} - P_h\frac{\partial u}{\partial n}\right|\right|_{-\frac{1}{2},\Gamma}||v_{h_1} - \Pi_h v - h_1||_{\frac{1}{2},\Gamma} \\ &\leq \left|\left|\frac{\partial u}{\partial n} - P_h\frac{\partial u}{\partial n}\right|\right|_{-\frac{1}{2},\Gamma}\left(||v_{h_1}||_{\frac{1}{2},\Gamma} + ||v_{h_2}||_{\frac{1}{2},\Gamma}\right) \end{aligned} \tag{4.56}$$

From the Lemma 3.1 and the trace theorem for v_{h_i} we deduce that

$$\begin{aligned}\left|\int_\Gamma \frac{\partial u}{\partial n}[v_h]ds\right| &\leq Ch_1^{\varepsilon_1}\left|\left|\frac{\partial u}{\partial n}\right|\right|_{\frac{1}{2}+\varepsilon_1}(||v_{h_1}||_{1,\Omega_1}+||v_{h_2}||_{1,\Omega_2}) \\ &\leq Ch_1||u||_{1+\varepsilon_1,\Omega_1}(||v_{h_1}||_{1,\Omega_1}+||v_{h_2}||_{1,\Omega_2}).\end{aligned} \tag{4.57}$$

For the consistency error of the artificial interface Γ_1 we have the error bound [see Ju'e Yang and Yu (2005)]:

$$\left|\int_{\Gamma_1}\frac{\partial u}{\partial n}[v_h]ds\right| \leq Ch_3||u||_{\frac{3}{2},\Gamma_1}(||v_{h_2}||_{1,\Omega_2}+||v_{h_3}||_{\frac{1}{2},\Gamma_1}). \tag{4.58}$$

Combining (4.55), (4.57) and (4.58), we obtain the error estimate (4.53).

Remark 4.1 *In order to obtain the optimal error estimation in V_h, we should balance the finite element grids in $\Omega_i, i=1,2$ and boundary element grids in Γ_1 such that the fine mesh size $h_i, i=1,3$ satisfy $h_1^{\varepsilon_1}\approx h_2^{\varepsilon_2}\approx h_3$.*

5. Numerical examples

In this section, we give some numerical results to illustrate the theoretical results obtained in the paper. For numerical testing we consider the circular domain Ω with radius R and its exterior unbounded domain $\Omega_c=\mathbb{R}^2\backslash(\Omega\bigcup\partial\Omega)$. Γ is the boundary of Ω. First, we consider the linear case.

Example 1 Let $p(t)=1$ and the exact solution of (2.1–2.2) is

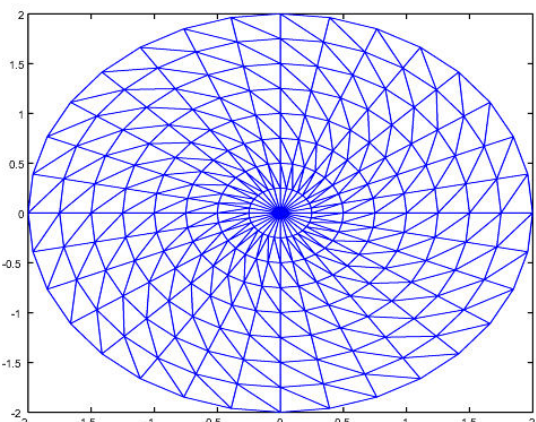

Figure 3: The finite element triangulation of Ω

$$u(r,\theta)=r^2\sin 2\theta \quad \text{and} \quad u_c(r,\theta)=\frac{3\cos\theta-15\sin\theta}{r} \tag{5.59}$$

where (r,θ) is a polar coordinate. Substituting u, u_c, p into the interface condition (2.4), we can compute u_0 and t_0

$$u_0 = r^2 \sin 2\theta|_{r=R} - \frac{3\cos\theta - 15\sin\theta}{r}|_{r=R}, \tag{5.60}$$

$$t_0 = 2r\sin 2\theta|_{r=R} + \frac{3\cos\theta - 15\sin\theta}{r^2}|_{r=R}. \tag{5.61}$$

Now, we make a triangulation in Ω whose number of elements is NEM, associated with N nodes on Γ(see Fig3.), then independently, we divide Γ into M equal segmental arcs. In this test we use piecewise linear finite element in Ω and take $\theta_n = \theta = 0.5$ and initial guess $\lambda^0 = 0.0$. The numerical solution u_h is compared with the true solution u with respect to L^2-error and H^1-error, respectively, in the tables (1-3).

Table 1: $R = 2.0$, $N = M$

N	M	NEM	$\| u-u_h \|_0$	$ratio$	$\| u-u_h \|_1$	$ratio$	$\| u-u_h \|_\infty$	$ratio$
16	16	80	2.8399×10^{-1}	—	2.1525		5.8073×10^{-2}	—
32	32	288	9.2896×10^{-2}	3.057	1.2166	1.769	1.5007×10^{-2}	3.870
64	64	1088	2.7165×10^{-2}	3.420	0.6562	1.854	4.8327×10^{-3}	3.105
128	128	4224	7.3936×10^{-3}	3.674	0.3423	1.917	1.3898×10^{-3}	3.477

Table 2: $R = 2.0$,$N = 2M$

N	M	NEM	$\| u-u_h \|_0$	$ratio$	$\| u-u_h \|_1$	$ratio$	$\| u-u_h \|_\infty$	$ratio$
16	8	80	3.2077×10^{-1}	—	2.2460		2.2291×10^{-1}	—
32	16	288	9.3626×10^{-2}	3.426	1.2212	1.839	3.0557×10^{-2}	7.295
64	32	1088	2.7230×10^{-2}	3.438	0.6564	1.860	6.6575×10^{-3}	4.898
128	64	4224	7.3968×10^{-3}	3.681	0.3423	1.918	1.5977×10^{-3}	4.167

We observe that the energy error is of order h and the error in the L^2-norm is of order h^2.

Example 2 The exact solution is given by

$$u(x,y) = xy(1.0-x)(1.0-y) \quad \text{and} \quad u_c(x,y) = \frac{x}{x^2+y^2}, \tag{5.62}$$

Table 3: $R = 2.0$, $N = 3M$

N	M	NEM	$\|\| u-u_h \|\|_0$	$ratio$	$\|\| u-u_h \|\|_1$	$ratio$	$\|\| u-u_h \|\|_\infty$	$ratio$
18	6	90	6.1633×10^{-1}	—	2.8566		6.0748×10^{-1}	—
36	12	324	1.1322×10^{-1}	5.443	1.3059	2.187	1.1865×10^{-1}	5.120
72	24	1224	2.8019×10^{-2}	4.041	0.6627	1.971	2.6890×10^{-2}	4.412
144	48	4752	7.2256×10^{-3}	3.878	0.3388	1.956	6.4238×10^{-3}	4.186

with

$$f(x,y) = 2(x+y-x^2-y^2). \tag{5.63}$$

Then we compute u_0 and t_0

$$u_0(\theta) = \left(\frac{1}{2}\sin 2\theta(1-r\sin\theta)(1-r\cos\theta) - \frac{\cos\theta}{r}\right)|_{r=R}, \tag{5.64}$$

$$t_0 = \left(r\sin 2\theta - \frac{3}{2}r^2\sin 2\theta(\cos\theta+\sin\theta) + r^3\sin^2 2\theta + \frac{\cos\theta}{r^2}\right)|_{r=R}. \tag{5.65}$$

In Table (4-5) order h for the energy norm and the order h^2 for the L^2-norm can be observed.

Table 4: $R = 2.0, N = 2M$

N	M	NEM	$\|\| u-u_h \|\|_0$	$ratio$	$\|\| u-u_h \|\|_1$	$ratio$	$\|\| u-u_h \|\|_\infty$	$ratio$
16	8	80	4.2707	—	7.3896		1.4566	—
32	16	288	1.3731	3.110	3.7343	1.979	0.4619	3.153
64	32	1088	0.4142	3.315	1.9386	1.926	0.1403	3.292
128	64	4224	0.1203	3.443	0.9990	1.941	0.0405	3.464

Table 5: $R = 2.0, N = 3M$

N	M	NEM	$\|\| u-u_h \|\|_0$	$ratio$	$\|\| u-u_h \|\|_1$	$ratio$	$\|\| u-u_h \|\|_\infty$	$ratio$
18	6	90	3.5583	—	6.7260		1.3582	—
36	12	324	1.1443	3.110	3.5378	1.901	0.4054	3.350
72	24	1224	0.3451	3.316	1.8689	1.893	0.1215	3.337
144	48	4752	0.0999	3.454	0.9695	1.928	0.0348	3.491

Example 3 For nonlinear cases, let $p(t) = 2 + \frac{1}{1+t}, t > 0$. Then we have

$$2 \le p(t) \le 3, \qquad 2 \le p(t) + tp'(t) = 2 + \frac{1}{(1+t)^2} \le 3. \tag{5.66}$$

Submitting $p(t)$ to Eq.(2.1), we obtain

$$-2\Delta u - div\left(\frac{\nabla u}{1+|\nabla u|} = f(x,y)\right). \tag{5.67}$$

If we set the solution

$$u_1(x,y) = \frac{1}{2}(x^2+y^2), \quad \text{in } \Omega, \tag{5.68}$$

$$u_c(x,y) = \frac{x}{x^2+y^2}, \quad \text{in } \Omega_c, \tag{5.69}$$

and

$$f(x,y) = -4 - \frac{1}{1+\sqrt{x^2+y^2}} - \frac{1}{(1+\sqrt{x^2+y^2})^2}, \tag{5.70}$$

then, we have

$$u_0(\theta)|_\Gamma = \frac{1}{2}R^2 - \frac{\cos\theta}{R}, \quad t_0(\theta)|_\Gamma = R + \frac{\cos\theta}{R^2}. \tag{5.71}$$

Table 6: $R = 1.0, N = 3M$

N	M	NEM	$\|\| u-u_h \|\|_0$	$ratio$	$\|\| u-u_h \|\|_1$	$ratio$
18	6	90	1.4761×10^{-1}	—	0.3645	
36	12	324	5.0936×10^{-2}	2.898	0.1507	2.419
72	24	1088	1.6586×10^{-2}	3.071	0.0713	2.114
144	48	4224	5.0892×10^{-3}	3.259	0.0353	2.020
288	96	18720	1.3157×10^{-3}	3.868	0.0177	1.994

The numerical solution u_h is compared with the true solution u with respect to L^2-error and the energy error, respectively, in Table (6). We can observe that the error of L^2-norm is of order h^2 and the energy error is of order h.

Acknowledgement: The authors would like to thank Prof. Qiya Hu for many constructive comments. The first author was supported by the Fundamental Research Funds for the Central Universities of China 860360.

References

Belgacem, F. B.; Maday, Y. (1999): The mortar finite element method with Lagrange multipliers. *Numer. Math.*, vol. 84, pp. 173–197.

C. Bernardi, Y. M.; Patera, A. (1994): A new nonconforming approach to domain decomposition:the mortar element method. *In:Nonlinear partial differential equations and their applications.College de France Seminar,Longman Sci. Tech., Harlow*, vol. Vol. XI (Paris, 1989-1991), pp. 13–51.

Feng, K.; Yu, D. (1983): Canonical integral equations of elliptic boundary value problems and their numerical solutions. *Proceeding of China-France symposium on the Finite Element Method (1982), Science Press, Beijing*, pp. 211–252.

Han, H. D. (1990): A new class of variational formulations for the coupling of finite element and boundary element methods. *J. Comp. Math.*, vol. 8, pp. 223–232.

Hsiao, G. (1988): The coupling of bem and fem - a brief review. *Boundary Element X 1*, vol. 0, pp. 431–446.

Hu, Q. (2005): Numerical integrations and unit resolution multipliers for domain decoposition methods with nonmatching grids. *Computing*, vol. 74, pp. 101–129.

Hu, Q.; Yu, D. (2001): A solution method for a certain nonlinear interface problem in unbounded domains. *Computing*, vol. 67, pp. 119–140.

Hu, Q.; Yu, D. (2001): Solving singularity problems in unbounded domains by couplings of natural bem and composite grid fem. *Appl. Numer. Math.*, vol. 37, pp. 127–143.

Ju'e Yang, Q. H.; Yu, D. (2005): Domain decomposition with non-matching grids for coupling of FEM and natural BEM. *J.Syst.Sci. Complex*, vol. 18, pp. 529–542.

Liu, D. J.; Yu, D. H. (2008): The coupling method of natural boundary element and mixed finite element for stationary n-s equation in unbounded domains. *CMES-Computer Modeling in Engneering & Sciences*, vol. 37:3, pp. 305–330.

Mund, P.; Stephan, E. P. (1999): An adaptive two-level method for the coupling of nonlinear fem-bem equations. *SIAM J. Numer. Anal.*, vol. 36, pp. 1001–1021.

Wohlmuth, B. I. (2000): A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, vol. 38, pp. 989–1012.

Yu, D. (1983): Coupling canonical boundary element method with FEM to solve harmonic problem over cracked domain. *J. Comp. Math.*, vol. 1:3, pp. 195–202.

Yu, D. (1992): The coupling of natural BEM and FEM for Stokes problem on unbounded domain. *Numer. Math. and Appl.*, vol. 14:4, pp. 111–120.

Yu, D. (1993): Mathematical Theory of Natural Boundary Element Method. *Science Press, Beijing*, vol. 0, pp. 0.

Yu, D. (1995): A domain decomposition method based on the natural boundary reduction over unbounded domain. *J. of Numer. Math. and Appl.*, vol. 17:1, pp. 95–105.

Yu, D. (1996): Discretization of non-overlapping domain decomposition method for unbounded domains and its covergence. *Math. Numer. Sinica*, vol. 18:3, pp. 328–336.

Yu, D. (2002): Natural Boundary Integral Method and Its Applications. *Science Press/Kluwer Academic Publishers*, vol. 0, pp. 0.

Yu, D. H.; Huang, H. Y. (2008): The artificial boundary method for a nonlinear interface problem on unbounded domain. *CMES-Computer Modeling in Engineering & Sciences*, vol. 35:3, pp. 227–252.