A High-Fidelity Cable-Analogy Continuum Triangular Element for the Large Strain, Large Deformation, Analysis of Membrane Structures

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The analysis of a continuum membrane by means of a discrete net-Abstract: work of cables or bars is an efficient and readily tractable approach to the solution of a complex mechanics problem. However, is so doing, compromises are made in the quality of the approximation of the strain field. It is shown in this paper that the original form of the cable-analogy continuum triangle formulation is degraded by an inherent assumption of small strains in the underlying equations, in which the term "small" is shown to be "negligibly small". A revised version of this formulation is proposed in which a modification to the basic formulation may be made to reduce the error. However, this approach is shown to have convergence challenges, but provides a "patch" to "fix" existing codes presently used in fabric architecture engineering practice. Based on Green's strain, a new three node triangular element is presented in this paper and provides a rigorous solution to the problem of assumed small strains. The formulation has been developed within the cable-analogy framework, and makes use of the dynamic relaxation solver. It is shown to be accurate, efficient, and capable of dealing with issues such as wrinkling.

Keywords: Membrane structure analysis; small strains; large deformations; constant strain triangle; architectural engineering.

Notation

Α	Element surface area
a_x, a_y, a_z	The sub-terms in the function of the transformation between the global nodal coordinate system
α^{c}	An iteration coefficient for deriving B matrix of meso CST element.

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$lpha_w$	The angle of wrinkling direction and local X axis
b_x, b_y, b_z	The sub-terms in the function of the transformation between the global nodal coordinate system
B, B_L, B_L	<i>NL</i> Strain-displacement matrix, linear and non-linear components
c_x, c_y, c_z	The sub-terms in the function of the transformation between the global nodal coordinate system
δ_i	Displacement vector of presedo cable
$\delta_1, \delta_2, \delta_3$	Extension of triangular element sides
Δ	A function of all the higher order terms in the small strain CST element.
δ_{pq}	The nodal displacement of p^{th} node in the q^{th} direction.
E_X, E_Y	Young's Modulus along orthogonal axes
$[E_{mod}]$	A modified elastic stiffness matrix for wrinkling
F	Applied load vectors
<i>F</i> _{pre}	Pre-defined designed prestress
G_{XY}	The shear modulus across X and Y
ŶXY	The local shear stress across local axis X and Y Displacement along X axis
J	Jacobian
$[K_E^{tr}]$	The elastic stiffness matrix of the triangular element
$[K^{pc}_{\delta}]$	The geometric stiffness matrix
K_{pq}	The p^{th} nodal stiffness selected from the terms of the element stiffness matrix of node q
l	A cable element of current length
M_{pq}	The fictitious nodal mass of p^{th} node in the q^{th} direction
[N]	Element shape functions

v_{YX}	Possion's ratio in Y direction etc.		
ω	Modification coefficient on the stiffness matrix due to wrinkling		
Р	The penalization parameter applied the modification of element stiffness in case of wrinkling		
P_N	Axial force in linear element		
P_A, P_B	Force vectors at the geometry configuration A and B		
P_{pq}	The external load vector of p^{th} node in the q^{th} direction.		
[R]	The transformation matrix applied in the wrinkling procedure		
R^m	Membrane element reaction force vectors		
R^c	Cable element reaction force vectors		
R_{pq}	The out-of-balance nodal force (or residual) of p^{th} node in the q^{th} direction		
σ	A surface element of isotropic stress		
σ_I	Maximum element principal stress		
σ_{II}	Minimum element principal stress		
σ_X	Element Stress along X axis		
σ_Y	Element Stress along Y axis		
σ_i	Element strains in presedo cable		
σ_n	The normal stress		
σ_I, σ_{11}	The maximum element principal stress		
σ_{II}, σ_{22}	The minimum element principal stress		
σ_0	Stress vectors from the pretension		
σ_E	Stress vectors from the elastic deformation		
σ^p_{min}	Minimum principal membrane stress		
σ_{per}^p	A predefined lower limit of membrane stresses		

Т	Cable element tension
[T]	Transformation Matrix
T_1, T_2, T_3	The element side force
T_{c1}, T_{c2}, T_{c3}	The pseudo cable element forces
$\{T_c\}$	The linear element force vector
T^c	Transformation vector for cable elements
$ au_{XY}$	Element Shear Stress between X and Y axis
$ heta_p$	The angle between wrinkle direction and local X axis
U_d	The nodal displacements in local coordinate system
<i>u</i> _d	The nodal displacements in global coordinate system
u_A, u_B	Displacements at the geometry configuration A and B
x_i, y_i, z_i	The global nodal coordinates of <i>i</i> th node.
x_p, y_p, z_p	The global coordinates of a single point p along the local Y axis

1 Introduction - membrane structure analysis and design: principles and practice

The analysis and design of membrane structures is clearly not in its infancy. The initial use of physical models Otto (1971) (Figure 1) has long been replaced by numerical simulation. Arguably these methods are generally based on the principles of the finite element method in which, in the present context, an arbitrarily shaped continua for which an exact closed-form solution is not normally available, is replaced by combining smaller "elements" whose behaviour is prescribed within a possible set. An approximate solution is obtained to the original problem by solving a set of equations developed from these combined "elements".

The analysis of membrane-type structures was not a priority during the initial development of the finite element method. Challenges in the field of numerical simulation were more associated with plates and shells, material non-linearity, and dynamics. Geometric non-linearity was also not a major focus of research in a development context as it was less significant than plasticity for many engineering applications. This is with the exception of aeronautical engineering, for example, but where geometric non-linearity is associated with plates and shells as opposed



Figure 1: Exploring potential membrane geometries with different boundaries using soap film

to purely membrane-type structural responses.

Concurrent with the development of simulation methodologies, analysis methods and associated computer technologies aimed to achieve analyses of sufficient detail and complexity but within acceptable timescales. Day Day (1965) proposed the use of the dynamic relaxation method as a means of solving non-linear static problems in structural mechanics. The primary advantage of the approach arose from the need to store only the diagonal terms of the structure stiffness matrix and the decoupling, or linearisation, of the equilibrium equations, negating the need for matrix inversion. From a computational perspective, the numerical model was much more compact and tractable than an equivalent matrix-based formulation. The issue of selecting appropriate damping coefficients to damp the (non-physical) oscillations of the simulated structure was circumvented by the introduction of the concept of kinetic damping, resulting in a highly efficient & generally robust solution algorithm for non-linear problems.

In the present context, dynamic relaxation was initially used for the analysis of cable and cable net structures, with geometric non-linearity introduced through the ratio of the axial load in the cable and the cable length. This approach proved to be computationally highly efficient and accurate. Barnes Barnes (1980) using the same basic principles to analyse membrane structures as cable nets. The key to the approach was to define a cable-analogy in which the cable stiffnesses and axial loads represented an acceptable approximation to a continuum. A suitable choice was the constant strain triangle, in which the element sides exhibit constant forces and stiffnesses along their lengths. As such, the continuum constant strain triangle, may be replaced by a set of three cables (or bars in the case of a compressive side force). This is different to assuming cables orientated in the direction of the fabric warp and fill, such as in the case of the force density method, for example, where the shear of the fabric, and perhaps Poisson's effects cannot be represented. Whilst it is generally accepted that the constant strain triangle may not be the best element for stress analysis, using the cable-analogy it does provide the basis for the development of a very efficient computational methodology for the geometrically non-linear analysis of membrane structures. Consistent with the principle of all finite element-based approaches, accuracy increases as the mesh is refined.

Barnes Barnes (1976) showed that it is possible to represent the continuum constant strain triangle using a triplet of cables by the repeated application of the expression describing strains at an inclined plane. In so doing, the continuum strain field may then be represented by the extensions of the element sides which can then be used as the degrees-of-freedom in a finite element strain-displacement type matrix to define the elastic stiffness matrix. The element geometric stiffness matrix is derived from the element side forces and lengths, with the former being functions of the continuum prestress and strains.

Therefore, the basis of converting a continuum membrane analysis into an equivalent discrete cable net analysis is the adoption of a constant strain triangle, where the continuum strains may be represented by the element side extensions. As such, at the element level, and as is shown in this paper, the quality of the numerical model relies initially on the the expression describing strains at an inclined plane.

The principle of the transformation between a continuum to a discrete network has been shown to be computationally effective and efficient in the analysis of membrane structures. Subsequent developments extending the basic capability to include boundary cables in the analysis automatically and the formulation of beam elements to enable the coupled analysis of the supporting structure with the membrane, led to its early adoption in engineering practice and its continued extensive use Barnes (1999). However, analysis and solution anomalies have been observed during this period, some of which are described in this paper. It is demonstrated in the initial section of this paper that these solution anomalies are related to the expression describing strains at an inclined plane and forming the basis of the element formulation. This expression assumes small strains. It is shown that for even very small deformations, strain errors are introduced into the element stiffness matrices that then result in significantly inaccurately predicted stresses. In practice, the solution anomalies have been interpreted as part of the simulation and design process, leading to the final design. As will all design, this involves varying degrees of engineering judgment. The process is also supported by the adoption a safety factors, which naturally may be increased as the level of uncertainty is increased. In addition to uncertainties in loading and material properties, this uncertainty will also include modelling uncertainty, ranging from the nature of the constitutive model used to describe the fabric in the present context, through to the quality of the mesh discretisation and assumptions made about support structures and details. Clearly, in reducing any uncertainty a more efficient design would normally be expected to result. However, the quality of a simulation tool will be expected to have far greater significance in the near future.

CEN250 working group 5 has been convened to develop a Eurocode for the analysis and design of membrane structures. Eurocode 0 regulates the basic reliability requirement for different types of structures, defined according to the importance of the structure. Whilst a membrane structure is generally considered to be a new structure type, it must also be compliant with safety requirements in the building and construction system. Of course, Eurocode 0 only provides a statement of a general reliability requirement. The exact safety index specific for a given membrane structure may be calculated on a case-by-case basis, and will generally vary from one to another. It is the limiting value, and its calculation, that is of importance. As such, it is essential that the analysis tool is accurate.

In this paper we concentrate on analysis formulations in the context of the Eurocode framework requirements for accuracy of the simulation. We present the principles of the 3-node constant strain triangular element based on a geometrically nonlinear cable analogy approach with a linear strain function and an assumption of small strains. The anomalies exhibited by the element formulation in simulating the behavior of a continuum in the presence of shear strains in particular is demonstrated and discussed. A modified version of this basic element is proposed based on including higher-order strain terms. The revised element is shown to perform better than the original version, but as not all higher-order terms can be included, large strains fail to be represented accurately. Convergence is not always assured. This element is denoted here as a meso-strain formulation.

It should be noted that strictly, neither of the aforementioned formulations pass the patch test. To achieve this most fundamental of requirements, the cable-analogy principle is retained, but the relationship between the element strains and side lengths is completely revised. In defining the strain-displacement relations, the original expression describing strains at an inclined plane has been replaced by a classical finite element approach based on Green's strain for large deformations. Therefore, from a practical perspective, the implementation of the new element formulation can be achieved by modifying existing analysis codes. Dynamic relaxation is used to solve the resulting state equations.

In addition to summarising the principles of the existing constant strain triangule (CST) element, the two new CST formulations (termed meso and large strain) are presented in this paper. The original small strain formulation has been enhanced by the inclusion of some higher-order terms to create the meso-strain version of the CST. A large deformation CST element formulation is detailed that abandons the original strain-displacement approach and is based on a typical finite element philosophy, with all higher-order terms included in a non-linear continuum framework. The capabilities of the elements are demonstrated through a number of benchmark problems, including wrinkling.

2 CST small strains formulation

The original form of the constant strain triangle element was, arguably, motivated by a requirement for computational efficiency and also by a desire to analyse a membrane as a cable net, for which the computational mechanics was more straightforward. Consequently, the CST became a good candidate element because of the characteristic of having constant values of strain along each of its three sides, meaning that the membrane could be analysed as a geometrically non-linear cable net or truss. We summarise the development of the element in its original form to demonstrate the principle of the formulation.

Referring to Figure 2, (A'C'-AC)/AC describes the strain normal to the plane FB arising from the normal stress σ_n . Considering the triangles ACD and A'C'D', and defining *du* as the increase in length from AD to A'D', and *dv* the increase in length from CD to C'D', then,

$$A'D' = AD + dU = AD\left(1 + \frac{dU}{AD}\right) = AD(1 + \varepsilon_X)$$

$$C'D' = CD + dV = CD\left(1 + \frac{dV}{CD}\right) = CD(1 + \varepsilon_Y)$$
(1)

in which, $\varepsilon_X = \frac{\partial U}{\partial X}$ and $\varepsilon_Y = \frac{\partial V}{dY}$. Similarly,

$$A'C' = AC(1 + \varepsilon_n) \tag{2}$$



Figure 2: Deformed plane element.

In the triangle A'C'D' we have:

$$A'C'^{2} = A'D'^{2} + C'D'^{2} - 2A'D' \cdot C'D'\cos(\frac{\pi}{2} + \gamma_{XY}),$$
(3)

or,

$$AC^{2}(1+\varepsilon_{n})^{2} = AD^{2}(1+\varepsilon_{X})^{2} + CD^{2}(1+\varepsilon_{Y})^{2} + 2AD(1+\varepsilon_{X})CD(1+\varepsilon_{Y})\sin(\gamma_{XY})$$
(4)

If it is assumed that strains are very small, $sin(\gamma_{XY}) \approx \gamma_{XY}$ and second order powers may be neglected, we obtain:

$$AC^{2}(1+2\varepsilon_{n}) = AD^{2}(1+2\varepsilon_{X}) + CD^{2}(1+2\varepsilon_{Y}) + 2AD \cdot CD\gamma_{XY}$$
(5)

which, with $AC^2 = AD^2 + CD^2$, reduces to,

$$AC^{2}(2\varepsilon_{n}) = AD^{2}(2\varepsilon_{X}) + CD^{2}(2\varepsilon_{Y}) + 2AD \cdot CD\gamma_{XY}$$
(6)

Dividing through by $2AC^2$ and introducing $\cos^2\theta = \frac{AD^2}{AC^2}$ and $\sin^2\theta = \frac{CD^2}{AC^2}$,

$$\varepsilon_n = \varepsilon_X \cos^2 \theta + \varepsilon_Y \cos^2 \theta + \gamma_{XY} \cos \theta \sin \theta \tag{7}$$

If the direct strain in the element side *i* (Figure 3) is denoted as ε_i , and local orthogonal strains defined as $\{\varepsilon\}^T = \{\varepsilon_X \quad \varepsilon_Y \quad \gamma_{XY}\}$, then,

$$\varepsilon_i = \varepsilon_X \cos^2 \theta_i + \varepsilon_Y \sin^2 \theta_i + \gamma_{XY} \sin \theta_i \cos \theta_i \tag{8}$$

where, θ_i is the anti-clockwise angle between the element side *i* and the local *X* axis, and $i = 1 \rightarrow 3$, and ε_X and ε_Y are the direct strains in the local *X* and *Y* directions, respectively, with the local shear stress γ_{XY} . The extensions of the side lengths can thus be expressed as,

$$\{\delta^{tr}\} = \begin{cases} \delta_1\\ \delta_2\\ \delta_3 \end{cases} = \begin{cases} L_1 \varepsilon_1\\ L_2 \varepsilon_2\\ L_3 \varepsilon_3 \end{cases}$$
(9)

Writing Eqn. 8 for each side of the triangular element leads to:

$$\varepsilon_{1} = \varepsilon_{X} \cos^{2} \theta_{1} + \varepsilon_{Y} \sin^{2} \theta_{1} + \gamma_{XY} \sin \theta_{1} \cos \theta_{1} = \frac{\delta_{1}}{L_{1}}$$

$$= \varepsilon_{X} a_{1} + \varepsilon_{Y} b_{1} + \gamma_{XY} c_{1}$$

$$\varepsilon_{2} = \varepsilon_{X} a_{2} + \varepsilon_{Y} b_{2} + \gamma_{XY} c_{2} = \frac{\delta_{2}}{L_{2}}$$

$$\varepsilon_{3} = \varepsilon_{X} a_{3} + \varepsilon_{Y} b_{3} + \gamma_{XY} c_{3} = \frac{\delta_{3}}{L_{3}}$$
(10)

or,

$$\begin{cases} \boldsymbol{\varepsilon}_{1} \\ \boldsymbol{\varepsilon}_{2} \\ \boldsymbol{\varepsilon}_{3} \end{cases} = \begin{cases} \frac{\boldsymbol{\delta}_{1}}{L_{1}} \\ \frac{\boldsymbol{\delta}_{2}}{L_{2}} \\ \frac{\boldsymbol{\delta}_{3}}{L_{3}} \end{cases} = \begin{bmatrix} a_{1} & b_{1} & c_{1} \\ a_{2} & b_{2} & c_{2} \\ a_{3} & b_{3} & c_{3} \end{bmatrix} \begin{cases} \boldsymbol{\varepsilon}_{X} \\ \boldsymbol{\varepsilon}_{Y} \\ \boldsymbol{\gamma}_{XY} \end{cases}$$
(11)

Solving for the continuum strains, then,

$$\varepsilon = \begin{cases} \varepsilon_{X} \\ \varepsilon_{Y} \\ \gamma_{XY} \end{cases}$$

$$= \frac{1}{\det [A]} \begin{bmatrix} (b_{2}c_{3} - b_{3}c_{2})L_{1}^{-1} & (b_{3}c_{1} - b_{1}c_{3})L_{2}^{-1} & (b_{1}c_{2} - b_{2}c_{1})L_{3}^{-1} \\ (a_{3}c_{2} - a_{2}c_{3})L_{1}^{-1} & (a_{1}c_{3} - a_{3}c_{1})L_{2}^{-1} & (a_{2}c_{1} - a_{1}c_{2})L_{3}^{-1} \\ (a_{2}b_{3} - a_{3}b_{2})L_{1}^{-1} & (a_{3}b_{1} - a_{1}b_{3})L_{2}^{-1} & (a_{1}b_{2} - a_{2}b_{1})L_{3}^{-1} \end{bmatrix} \begin{cases} \delta_{1} \\ \delta_{2} \\ \delta_{3} \end{cases}$$

$$(12)$$

or,

$$\{\varepsilon\} = [B^{tr}]\{\delta\}^{tr}$$
where det[A] = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. (13)

 $[B^{tr}]$ can be recognised as the classically expressed strain-displacement relationship which may be used to define the element elastic stiffness matrix $[K_E^{tr}]$, the geometric stiffness matrix $[K_{\delta}^{pc}]$, and the element force vector $\{T_c\}$ as follows. The element local stresses are defined as:

$$\{\sigma\} = \begin{cases} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{cases} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \cdot \begin{cases} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{cases} = [E][B^{tr}] \begin{cases} \delta_1 \\ \delta_2 \\ \delta_3 \end{cases}$$
(14)

where, for an isotropic material,

$$d_{11} = d_{22} = \frac{E}{(1 - v^2)}, d_{12} = d_{21} = v \cdot d_{11}, d_{33} = \frac{E}{2(1 + v)}$$

in which, *E* is Young's Modulus and *v* Possion's ratio, and for an orthotropic material,

$$d_{11} = \frac{E_X \cdot E_Y}{E_Y - E_X \cdot v_{YX}^2}, \ d_{12} = d_{21} = \frac{E_X \cdot E_Y \cdot v_{XY}}{E_Y - v_{XY}^2 \cdot E_X}, \ d_{22} = \frac{E_Y^2}{E_Y - E_X \cdot v_{YX}^2}, \ d_{33} = G_{XY}$$

with E_X, E_Y are Young's Modulus along orthogonal axes X and Y. G_{XY} is the shear modulus across X and Y, and v_{YX} is the Possion's ratio in Y direction etc.

For a 3 node CST element, the elastic stiffness matrix is easily shown to be,

$$[K^{tr}] = [B^{tr}]^T [E] [B^{tr}] \times V \tag{15}$$

where V is the volume of the element. $[K_E^{tr}]$ is a 3×3 matrix acting on the D.O.F δ_1, δ_2 and δ_3 (e.g the side extensions of the element). Knowing $[B^{tr}]$ (e.g. Eqn. 13) the element elastic stiffness matrix can be readily obtained. Acting on the 3 D.O.F δ_1, δ_2 and δ_3 , means that the triangular element has effectively been replaced by a set of three cables or bars with the diagonal terms of $[K_E^{tr}]$ effectively terms of the type EA/L, with A the cross-section area of the fictitious pseudo cable/bar line elements.

It is convenient to develop this analogy further in the derivation of the geometric stiffness matrix. The geometric stiffness is derived from a combination of the pseudo-cable natural (axial) force and a change in orientation in the form of rigidbody rotation. Without the need to provide a full derivation in this paper, it suffices to state the geometric stiffness of the bar/cable element shown in Fig.4 as,

$$[K_{\sigma}^{pc}] = \frac{P_N}{L} \begin{bmatrix} [I_3] - [C][C]^T & -[I_3] + [C][C]^T \\ -[I_3] + [C][C]^T & [I_3] - [C][C]^T \end{bmatrix}$$
(16)

where P_N is the axial force in the bar/cable of length L, $[I_3]$ is a 3×3 identity matrix, and

$$[C] = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \frac{1}{L} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$
(17)

Clearly, the relationship between the natural force P_N of the three pseudo cables/bars describing the triangular element and the continuum stresses σ_x , σ_y and τ_{xy} is required.

Combing the elastic stiffness matrix $[K_E^{tr}]$ (Eqn. 15) with the element side extensions δ_1, δ_2 and δ_3 leads to the element side force T_1, T_2 and T_3 (Fig. 5). as in,

$$\{T\} = \begin{cases} T_{c1} \\ T_{c2} \\ T_{c3} \end{cases} = [B^{tr}][E][B^{tr}]V \begin{cases} \delta_1 \\ \delta_2 \\ \delta_3 \end{cases}$$
(18)

Noting Eqn. 14 and pre-multiplying it by $[B^{tr}]^T$ and V, then,

$$V \times [B^{tr}]^T \{\delta\} = V \times [B^{tr}]^T [E] [B^{tr}] \begin{cases} \delta_1 \\ \delta_2 \\ \delta_3 \end{cases}$$
(19)

The right-hand-side of Eqn. 19 is then identical to the right-hand-side of the Eqn. 18, such that we have the definition:

$$\{T\} = \begin{cases} T_{c1} \\ T_{c2} \\ T_{c3} \end{cases} = V \times [B^{tr}]^T \begin{cases} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{cases}$$
(20)

Eqn. 20 provides the link between the pseudo cable element forces T_{c1} , T_{c2} and T_{c3} (viz. P_N in Eqn. 16) and the triangular element continuum stresses σ_X , σ_Y and τ_{XY} .

Both the elastic stiffness and geometric stiffness matrices are functions of the straindisplacement matrix $[B^{tr}]$. The limitations of the original CST formulation $[B^{tr}]$ (Eqn. 13 with Eqn. 12) are demonstrated in section 7. To improve the CST-pseudo cable element formulation it remains to establish "better" forms of $[B^{tr}]$. Two alternatives are described in sections 3 and 4, and assessed in section 7.

3 Enhanced CST meso strains formulation

As identified in the section 2, in deriving the formulation $\varepsilon_i = \varepsilon_X \cdot \cos^2 \theta_i + \varepsilon_Y \cdot \sin^2 \theta_i + \gamma_{XY} \sin \theta_i \cdot \cos \theta_i$, high order terms are neglected. A revised formulation is presented in this section that endeavours to include these missing terms with the aim of enhancing the resulting CST. Basing the formulation on the pseudo-cable approach, only the derivation of $[B^{tr}]$ is required to use in Eqns. 15 and 20.

The length of side [1] in the deformed triangular element in figure 6 (depicted with a dotted line) is,

$$OA^{2}[1+\varepsilon_{1}]^{2} = OB^{2}[1+\varepsilon_{X}]^{2} + AB^{2}[1+\varepsilon_{Y}]^{2} - 2OB \cdot AB[1+\varepsilon_{X}][1+\varepsilon_{Y}] \cdot \cos(90^{\circ}+\gamma_{XY})$$
(21)

while the original length is $OA^2 = OB^2 + AB^2$. With the definitions: $\cos\theta = \frac{OB}{OA}$, $\sin\theta = \frac{AB}{OA}$, and using the necessary simplifying assumption as in section 2: $(\sin(\gamma_{XY}) \approx \gamma_{XY})$, then:

$$2\varepsilon_{1} + \varepsilon_{1}^{2} = \cos^{2}\theta_{1}(2\varepsilon_{X} + \varepsilon_{X}^{2}) + \sin^{2}\theta_{1}(2\varepsilon_{Y} + \varepsilon_{Y}^{2}) + 2\cos\theta_{1} \cdot \sin\theta_{1} \cdot (1 + \varepsilon_{X} + \varepsilon_{Y} + \varepsilon_{X} \cdot \varepsilon_{Y})\gamma_{XY}$$
(22)

If we assume that the formulation follows the same form as Eqn. 8, then:

$$\varepsilon_1 = \varepsilon_X \cos^2 \theta_1 + \varepsilon_Y \cos^2 \theta_1 + \gamma_{XY} \sin \theta_1 \cos \theta_1 + \Delta$$
(23)

where Δ represents all the higher order terms. The role of Δ can be examined as follows. Equation 23 can be used in the first term on the left side of Eqn. 22. In substituting Eqn. 23 into the second term of left side of Eqn. 22, it is mathematically necessary to omit the higher order terms. Consequently,

$$\varepsilon_1^2 = (\varepsilon_X \cdot \cos^2 \theta_1 + \varepsilon_Y \cdot \sin^2 \theta_1 + \gamma_{XY} \cdot \sin \theta_1 \cdot \cos \theta_1 + \Delta)^2 \approx (\varepsilon_X \cdot \cos^2 \theta_1 + \varepsilon_Y \cdot \sin^2 \theta_1 + \gamma_{XY} \cdot \sin \theta_1 \cdot \cos \theta_1)^2 \quad (24)$$

or,

$$\varepsilon_{1}^{2} \approx \cos^{4}\theta_{1} \cdot \varepsilon_{X}^{2} + \sin^{4}\theta_{1} \cdot \varepsilon_{Y}^{2} + \sin^{2}\theta_{1} \cdot \cos^{2}\theta_{1} \cdot \gamma_{XY}^{2} + 2\varepsilon_{X} \cdot \varepsilon_{Y} \cdot \sin^{2}\theta_{1} \cdot \cos^{2}\theta_{1} + 2\varepsilon_{Y} \cdot \gamma_{XY} \cdot \sin^{3}\theta_{1} \cdot \cos\theta_{1} + 2\gamma_{XY} \cdot \varepsilon_{X} \cdot \cos^{3}\theta_{1} \cdot \sin\theta_{1} \quad (25)$$

We substitute Eqn. 25 into the second term on the left side of Eqn. 22. Therefore, using equation 23 and 25 as described above, then:

$$\begin{split} & 2\Delta \approx \cos^2\theta_1 \cdot \varepsilon_X^2 + \sin^2\theta_1 \cdot \varepsilon_Y^2 + 2(\varepsilon_X + \varepsilon_Y + \varepsilon_X \cdot \varepsilon_Y) \cdot \gamma_{XY} \cdot \sin\theta_1 \cdot \cos\theta_1 - \varepsilon_1^2 \\ & \approx (\cos^2\theta_1 - \cos^4\theta_1) \cdot \varepsilon_X^2 + (\sin^2\theta_1 - \sin^4\theta_1) \cdot \varepsilon_Y^2 + 2(1 - \sin^2\theta_1) \cdot \sin\theta_1 \cdot \cos\theta_1 \cdot \gamma_{XY} \cdot \varepsilon_X \\ & + 2(1 - \cos^2\theta_1) \cdot \sin\theta_1 \cdot \cos\theta_1 \cdot \gamma_{XY} \cdot \varepsilon_Y + 2(\gamma_{XY} - \sin\theta_1 \cdot \cos\theta_1) \cdot \varepsilon_X \cdot \varepsilon_Y \cdot \sin\theta_1 \cdot \cos\theta_1 \\ & -\gamma_{XY}^2 \cdot \cos^2\theta_1 \cdot \sin^2\theta_1 \\ & \approx \sin^2\theta_1 \cdot \cos^2\theta_1 \cdot \varepsilon_X^2 + \sin^2\theta_1 \cdot \cos^2\theta_1 \cdot \varepsilon_Y^2 + 2\cos^3\theta_1 \cdot \sin\theta_1 \cdot \gamma_{XY} \cdot \varepsilon_X + 2\sin^3\theta_1 \cdot \cos\theta_1 \cdot \varepsilon_Y \\ & + 2(\gamma_{XY} - \sin\theta_1 \cdot \cos\theta_1) \cdot \varepsilon_X \cdot \varepsilon_Y \cdot \sin\theta_1 \cdot \cos\theta_1 - \gamma_{XY}^2 \cdot \cos^2\theta_1 \cdot \sin^2\theta \\ & \approx \frac{1}{4}(\varepsilon_X^2 + \varepsilon_Y^2)\sin^22\theta_1 + \gamma_{XY} \cdot (\varepsilon_Y \cdot \cos^2\theta_1 + \varepsilon_X \cdot \sin^2\theta_1)\sin2\theta_1 \\ & + \varepsilon_X \cdot \varepsilon_Y (\gamma_{XY} - \frac{1}{2}\sin2\theta_1)\sin2\theta_1 - \frac{1}{4}\gamma_{XY}^2 \cdot \sin^22\theta_1 \end{split}$$

$$\approx \left[\frac{1}{4}(\varepsilon_X - \varepsilon_Y)^2 - \frac{1}{4}\gamma_{XY}^2\right] \cdot \sin^2 2\theta_1 + \gamma_{XY} \cdot (\varepsilon_Y \cdot \cos^2 \theta_1 + \varepsilon_X \cdot \sin 2\theta_1 + \varepsilon_X \cdot \varepsilon_Y) \sin 2\theta_1$$

such that,

$$\Delta \approx \left[\frac{1}{8}(\varepsilon_X - \varepsilon_Y)^2 - \frac{1}{8}\gamma_{XY}^2\right] \cdot \sin^2 2\theta_1 + \frac{1}{2}\gamma_{XY} \cdot (\varepsilon_Y \cdot \cos^2 \theta_1 + \varepsilon_X \cdot \sin^2 \theta_1 + \varepsilon_X \cdot \varepsilon_Y) \cdot \sin 2\theta_1$$
$$= \Delta(o^2)$$

and, therefore,

$$\varepsilon_{i} \approx \varepsilon_{X} \cos^{2} \theta_{i} + \varepsilon_{Y} \cos^{2} \theta_{i} + \varepsilon_{XY} \sin \theta_{i} \cos \theta_{i} + \left[\frac{1}{8}(\varepsilon_{X} - \varepsilon_{Y})^{2} - \frac{1}{8}\gamma_{XY}^{2}\right] \cdot \sin^{2} 2\theta_{i} + \frac{1}{2}\gamma_{XY} \cdot (\varepsilon_{Y} \cdot \cos^{2} \theta_{i} + \varepsilon_{X} \cdot \sin^{2} \theta_{i} + \varepsilon_{X} \cdot \varepsilon_{Y}) \cdot \sin 2\theta_{i}$$
(26)

Eqn. 8 describes a linear relationship between cartesian strains ε_X , ε_Y , and γ_{XY} and a strain orientated in a direction θ from the local X axis. Eqn. 26 attempts to capture second order components of strains, whilst the omission of o^3 strains and above is a necessary simplification. Subtracting Eqn. 8 from Eqn. 26 and representing the result graphically for a range of strains (see fig. 7), the simplification of the o^2 strain terms can be visualized. Quantitatively, the error is significant. For example, with direct strains ε_X and ε_Y up to 25% and a shear strain of 0.25 or 14°, the maximum difference between Eqn. 8 and 26 is of the order of 500%. Whilst the nonlinear formulation Eqn. 26 clearly improves the prediction of the values of strains at arbitrary values of θ , its use in generating the [B^{tr}] matrix is not appropriate, because of its complicated nonlinear form. Instead it is necessary to return to the fundamental definitions.

Assuming $\varepsilon_X = \varepsilon_3$ (figure 6), then with Eqn. 21, we have:

$$OA^{2}(1 + \varepsilon_{1})^{2} = OB^{2}(1 + \varepsilon_{3})^{2} + AB^{2}(1 + \varepsilon_{Y})^{2} - 2OB \cdot AB(1 + \varepsilon_{3}) \cdot (1 + \varepsilon_{Y})\cos(\gamma_{XY} + 90^{\circ})$$
(27)
If $OA^{2} = OB^{2} + AB^{2}$, and $OC = OA \cdot \cos\theta_{1}, AC = OA \cdot \sin\theta_{1}$, then,
 $(1 + \varepsilon_{1})^{2} =$

$$\cos^2\theta_1 \cdot (1+\varepsilon_3)^2 + \sin^2\theta_1 \cdot (1+\varepsilon_y)^2 + 2(1+\varepsilon_3)(1+\varepsilon_y)\sin\gamma_{xy}\sin\theta_1\cos\theta_1 \quad (28)$$

Similarly,

$$(1+\varepsilon_2)^2 = \cos^2\theta_2 \cdot (1+\varepsilon_3)^2 + \sin^2\theta_2 \cdot (1+\varepsilon_y)^2 + 2(1+\varepsilon_3)(1+\varepsilon_y)\sin\gamma_{xy}\sin\theta_2\cos\theta_2 \quad (29)$$

To obtain a strain-displacement relationship that is linear in ε_X , ε_Y , γ_{XY} , then it is necessary to assume that $\sin \gamma_{XY} \approx \gamma_{XY}$ and

$$(1 + \varepsilon_Y)^2 = 1 + 2\varepsilon_Y + \alpha^c \cdot \varepsilon_Y$$

= 1 + (2 + \alpha^c)\varepsilon_Y (30)

Using the notation in Eqn. 30, then, Eqns. 28 and 29 become,

$$2\varepsilon_{1} + \varepsilon_{1}^{2} = \cos^{2}\theta_{1} \cdot (2\varepsilon_{3} + \varepsilon_{3}^{2}) + \sin^{2}\theta_{1} \cdot (2 + \alpha^{c}) \cdot \varepsilon_{Y} + 2\gamma_{XY}(1 + \varepsilon_{3})(1 + \varepsilon_{Y})\sin\theta_{1}\cos\theta_{1} \quad (31)$$

$$2\varepsilon_{2} + \varepsilon_{2}^{2} = \cos^{2}\theta_{2} \cdot (2\varepsilon_{3} + \varepsilon_{3}^{2}) + \sin^{2}\theta_{2} \cdot (2 + \alpha^{c}) \cdot \varepsilon_{Y} + 2\gamma_{xy}(1 + \varepsilon_{3})(1 + \varepsilon_{Y})\sin\theta_{2}\cos\theta_{2} \quad (32)$$

Solving Eqns. 31 and 32 simultaneously for ε_Y , then,

$$\varepsilon_Y = \frac{(2\varepsilon_1 + \varepsilon_1^2)\sin\theta_2\cos\theta_2 - (2\varepsilon_2 + \varepsilon_2^2)\sin\theta_1\cos\theta_1}{((2\varepsilon_1 + \varepsilon_3^2) \cdot (\cos^2\theta_1\sin\theta_2\cos\theta_2 - \cos^2\theta_2\sin\theta_1\cos\theta_1)}$$

Defining

_

 $a_2 = \sin\theta_2 \cos\theta_2, b_2 = -\sin\theta_1 \cos\theta_1, c_2 = \cos^2\theta_2 \sin\theta_1 \cos\theta_1 - \cos^2\theta_1 \sin\theta_2 \cos\theta_2$ $A^c = \sin^2\theta_1 \sin\theta_2 \cos\theta_2 - \sin^2\theta_2 \sin\theta_1 \cos\theta_1, \text{ then,}$

$$\varepsilon_Y = \frac{a_2(2\varepsilon_1 + \varepsilon_1^2) + b_2(2\varepsilon_2 + \varepsilon_2^2) + c_2(2\varepsilon_3 + \varepsilon_3^2)}{A^c(2 + \alpha^c)}$$

Similarly, solving Eqns (31) and (32) for γ_{XY} , then

$$\gamma_{XY} = \frac{a_3(2\varepsilon_1 + \varepsilon_1^2) + b_3(2\varepsilon_2 + \varepsilon_2^2) + c_3(2\varepsilon_3 + \varepsilon_3^2)}{2(1 + \varepsilon_3)[a_2(2\varepsilon_1 + \varepsilon_1^2) + b_2(2\varepsilon_2 + \varepsilon_2^2) + c_2(2\varepsilon_3 + \varepsilon_3^2) + (2 + \alpha^c)A^c]}$$

with, $a_3 = \sin_2^{\theta}$, $b_3 = -\sin^2 \theta_1$, $c_3 = -[\cos^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_2 \sin^2 \theta_1]$. Finally,

$$\begin{cases} \boldsymbol{\varepsilon}_{x} \\ \boldsymbol{\varepsilon}_{y} \\ \boldsymbol{\gamma}_{xy} \end{cases} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{a_{2}(2+\varepsilon_{1})}{A^{c}(2+\alpha^{c})} & \frac{b_{2}(2+\varepsilon_{2})}{A^{c}(2+\alpha^{c})} & \frac{c_{3}(2+\varepsilon_{3})}{A^{c}(2+\alpha^{c})} \\ \frac{a_{3}(2+\varepsilon_{1})}{2(1+\varepsilon_{3})B} & \frac{b_{3}(2+\varepsilon_{2})}{2(1+\varepsilon_{3})B} & \frac{c_{3}(2+\varepsilon_{3})}{2(1+\varepsilon_{3})B} \end{bmatrix} \begin{cases} \boldsymbol{\varepsilon}_{1} \\ \boldsymbol{\varepsilon}_{2} \\ \boldsymbol{\varepsilon}_{3} \end{cases}$$
(33)

or $\{\varepsilon\} = [B^{tr}]\{\delta\}^{tr}$, where,

$$[B^{tr}] = \begin{bmatrix} 0 & 0 & L_3^{-1} \\ \frac{a_2(2+\varepsilon_1)\cdot L_1^{-1}}{A^c(2+\alpha^c)} & \frac{b_2(2+\varepsilon_2)\cdot L_2^{-1}}{A^c(2+\alpha^c)} & \frac{c_3(2+\varepsilon_3)\cdot L_3^{-1}}{A^c(2+\alpha^c)} \\ \frac{a_3(2+\varepsilon_1)\cdot L_1^{-1}}{2(1+\varepsilon_3)B^c} & \frac{b_3(2+\varepsilon_2)\cdot L_2^{-1}}{2(1+\varepsilon_3)B^c} & \frac{c_3(2+\varepsilon_3)\cdot L_3^{-1}}{2(1+\varepsilon_3)B^c} \end{bmatrix}$$
(34)

in which, collecting all definitions together,

 $a_2 = \sin\theta_2 \cos\theta_2, \ b_2 = -\sin\theta_1 \cos\theta_1, \ c_2 = \cos^2\theta_2 \sin\theta_1 \cos\theta_1 - \cos^2\theta_1 \sin\theta_2 \cos\theta_2$ $a_3 = \sin^2\theta_2, \ b_3 = -\sin^2\theta_1, \ c_3 = -[\cos^2\theta_1 \sin^2\theta_2 - \cos^2\theta_2 - \cos^2\theta_2 \sin^2\theta_1], \ A = \sin^2\theta_1 \sin\theta_2 \cos\theta_2 - \sin^2\theta_2 \sin\theta_1 \cos\theta_1, \text{ and,}$

$$B^c = \frac{a_2(2\varepsilon_1 + \varepsilon_1^2) + b_2(2\varepsilon_2 + \varepsilon_2^2) + c_3(\varepsilon_3 + \varepsilon_3^2) + (2 + \alpha_i^c)A^c}{A^c(2 + \alpha_i^c)}.$$

 α^{c} is an iteration coefficient, which has an initial value of zero and is subsequently updated according to,

$$\alpha_i^c = \frac{a_2(2\varepsilon_1^{i-1} + \varepsilon_1^{i-1^2}) + b_2(2\varepsilon_2^{i-1} + \varepsilon_2^{i-1^2}) + c_3(2\varepsilon_3^{i-1} + \varepsilon_3^{i-1^2})}{A(2 + \alpha_{i-1}^c)}$$

Eqn. 31 is in a suitable form to be used in Eqns. 15 and 16 to define the element stiffness matrices and associated Eqn. 18. It should be noted that this formulation relies upon the assumption in Eqn. 1 that $\varepsilon_X = \frac{\partial U}{\partial X}$ and $\varepsilon_Y = \frac{\partial V}{\partial Y}$. No assumption is made about the form of γ_{XY} , but it is necessarily assumed that $\sin(\gamma_{XY}) \cong \gamma_{XY}$.

4 CST with large strain formulation

The preceding CST formulations are characterized by the element strains ε_X and ε_Y defined as a linear deformation gradient (e.g. $\frac{\partial U}{\partial X}$) as in Eqn. 1, and the shear strain γ_{XY} approximated by $\sin \gamma_{XY} = \gamma_{XY}$. These assumptions are invalid for large deformations and large strains where the following Green's definitions may be more appropriate:

$$\varepsilon_{X} = \frac{dU}{dX} + \frac{1}{2} \left(\left(\frac{dU}{dX} \right)^{2} + \left(\frac{dV}{dX} \right)^{2} + \left(\frac{dW}{dX} \right)^{2} \right)$$

$$\varepsilon_{Y} = \frac{dV}{dY} + \frac{1}{2} \left(\left(\frac{dU}{dY} \right)^{2} + \left(\frac{dV}{dY} \right)^{2} + \left(\frac{dW}{dY} \right)^{2} \right)$$

$$\gamma_{XY} = \frac{dU}{dY} + \frac{dV}{dX} + \frac{1}{2} \left(\frac{dU}{dX} \frac{dU}{dY} + \frac{dV}{dX} \frac{dV}{dY} + \frac{dW}{dX} \frac{dW}{dY} \right)$$
(35)

in which, ε_X , ε_Y and γ_{XY} are element strains, and U, V, W are displacements of the element in the local coordinate system XY. Deriving the element equations using a "standard" finite element philosophy, displacements are interpolated from nodal values using shape functions [N] as in;

$$U(X,Y) = \sum_{i=1}^{3} N_i U_i, \quad V(X,Y) = \sum_{i=1}^{3} N_i V_i, \quad W(X,Y) = \sum_{i=1}^{3} N_i W_i$$
(36)

where $N_1 = \xi_1$, $N_2 = \xi_2$, $N_3 = \xi_3$, and ξ_1 , ξ_2 , ξ_3 are area co-ordinates as defined in figure 8:

Given that,

$$\frac{d\xi_1}{dX} = \frac{Y_{23}}{2A} \quad \frac{d\xi_2}{dX} = \frac{Y_{31}}{2A} \quad \frac{d\xi_3}{dX} = \frac{Y_{12}}{2A}$$
$$\frac{d\xi_1}{dY} = \frac{X_{32}}{2A} \quad \frac{d\xi_2}{dY} = \frac{X_{13}}{2A} \quad \frac{d\xi_3}{dY} = \frac{X_{21}}{2A}$$

where A is the area of the element triangle, and for example $Y_{23} = Y_2 - Y_3$, then displacement derivatives are:

$$\frac{dU}{dX} = \frac{Y_{32}}{2A}U_1 + \frac{Y_{13}}{2A}U_2 + \frac{Y_{21}}{2A}U_3 \quad \frac{dU}{dY} = \frac{X_{23}}{2A}U_1 + \frac{X_{31}}{2A}U_2 + \frac{X_{12}}{2A}U_3
\frac{dV}{dX} = \frac{Y_{32}}{2A}V_1 + \frac{Y_{13}}{2A}V_2 + \frac{Y_{21}}{2A}V_3 \quad \frac{dV}{dY} = \frac{X_{23}}{2A}V_1 + \frac{X_{31}}{2A}V_2 + \frac{X_{12}}{2A}V_3
\frac{dW}{dX} = \frac{Y_{32}}{2A}W_1 + \frac{Y_{13}}{2A}W_2 + \frac{Y_{21}}{2A}W_3 \quad \frac{dW}{dY} = \frac{X_{23}}{2A}W_1 + \frac{X_{31}}{2A}W_2 + \frac{X_{12}}{2A}W_3$$
(37)

If node 1 is set at the origin of the local *XY* coordinate system in which the element is co-planar, and the local X axis is aligned with the base of the triangular element (see fig. 9), then $U_1 = V_1 = W_1 = 0$ and $U_3 = \delta_3, W_2 = 0, V_3 = W_3 = 0$, such that Eqns. 37 simplify to:

$$\frac{dU}{dX} = \frac{Y_{13}}{2A}U_2 + \frac{Y_{21}}{2A}\delta_3 \qquad \frac{dU}{dY} = \frac{X_{31}}{2A}U_2 + \frac{X_{12}}{2A}\delta_3$$
(38)

$$\frac{dV}{dX} = \frac{Y_{13}}{2A}V_2 \qquad \frac{dV}{dY} = \frac{X_{31}}{2A}V_2$$
(39)

To ensure consistency with Eqns. 15 and 18 (and to achieve the associated computational efficiency), nodal displacements U_i and V_i are rewritten as functions of the element side extensions δ_1, δ_2 , and δ_3 as in:

$$U_{2} = L_{1}' \cos \theta_{1}' - L_{1} \cos \theta_{1}$$

$$= L_{1}' \frac{L_{1}'^{2} + L_{3}'^{2} - L_{2}'^{2}}{2L_{1}' \cdot L_{3}'} - L_{1} \frac{L_{1}^{2} + L_{3}^{2} - L_{2}^{2}}{2L_{1} \cdot L_{3}}$$

$$= \frac{(L_{1} + \delta_{1})^{2} + (L_{3} + \delta_{3})^{2} - (L_{2} + \delta_{2})^{2}}{2(L_{3} + \delta_{3})} - \frac{L_{1}^{2} + L_{3}^{2} - L_{2}^{2}}{2L_{3}}$$

$$= \frac{\delta_{1} + 2L_{1}}{2(L_{3} + \delta_{3})} \delta_{1} - \frac{\delta_{2} + 2L_{2}}{2(L_{3} + \delta_{3})} \delta_{2} + \frac{L_{3}^{2} - L_{1}^{2} + L_{2}^{2} + \delta_{3} \cdot L_{3}}{2(L_{3} + \delta_{3})L_{3}} \delta_{3}$$

$$= a_{1} \cdot \delta_{1} + a_{2} \cdot \delta_{2} + a_{3} \cdot \delta_{3}$$
(40)

$$\begin{split} & V_{2} = h' - h \\ &= \frac{2Area_{123'}}{L_{3}'} - \frac{2Area_{123}}{L_{3}} \\ &= \frac{\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{3}^{2} - L_{1}^{2}}{2})^{2}}}{L_{3}} - \frac{\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{3}^{2} - L_{1}^{2}}{2})^{2}}}{L_{3}} \\ &= \frac{L_{3}\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{1}^{2} - L_{1}^{2}}{2})^{2}} - L_{3}^{2}\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{2}^{2} - L_{1}^{2}}{2})^{2}}}{L_{3}L_{3}' \\ &= \frac{L_{3}^{2}[L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{1}^{2} - L_{1}^{2}}{2})^{2} - L_{3}^{2}\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{2}^{2} - L_{1}^{2}}{2})^{2}}}{L_{3}L_{3}' (L_{3} - \sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{1}^{2} - L_{1}^{2}}{2})^{2} + L_{3}^{2}\sqrt{L_{2}^{2}L_{3}^{2} - (\frac{L_{2}^{2} + L_{2}^{2} - L_{1}^{2}}{2})^{2}} \\ &= \frac{(-[L_{3} \frac{L_{1}^{2} + L_{1}^{2} - L_{1}^{2} - L_{3}^{2} + \frac{L_{1}^{2} - L_{1}^{2}}{2}) \cdot [L_{3}^{\frac{L_{1}^{2} + L_{2}^{2} - L_{1}^{2}}{2} + L_{3}^{2} + L_{3}^{2} - L_{1}^{2}}])}{L_{3}L_{3}(L_{3} \cdot 2A' + L_{3}' \cdot 2A)} \\ &= \frac{(-[L_{3} \frac{L_{1}^{2} + L_{1}^{2} - L_{1}^{2} - L_{3}^{2} + \frac{L_{1}^{2} - L_{1}^{2}}{2}}] \cdot [L_{3}^{\frac{L_{1}^{2} + L_{2}^{2} - L_{1}^{2}}{2}}] \cdot [L_{3}^{\frac{L_{1}^{2} + L_{2}^{2} - L_{1}^{2}}}{2}] \cdot BB} \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{2}' + L_{2})\delta_{2} - [L_{3} \frac{L_{2}^{2} + L_{1}^{2} - L_{1}^{2}}{2} - L_{3}^{2} + L_{3}^{2} - L_{1}^{2}}] \cdot BB} \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{2}' + L_{2})\delta_{2} - [L_{3} \frac{L_{1}^{2} + L_{1}^{2} - L_{1}^{2}}{2} - L_{3}^{2} - L_{3}^{2} + L_{3}^{2} - L_{1}^{2}}] \cdot BB} \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{2}' + L_{2})\delta_{2} - (L_{3} - L_{3}' - L_{3})(L_{3}' + L_{3}) - (L_{1}' - L_{1})(L_{1}' + L_{1}))] \cdot BB} \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{2}' + L_{2})\delta_{2} + (L_{2}' - L_{3}^{2} - L_{3})\delta_{3}} \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{2}' + L_{2})\delta_{2} + (L_{3}' - L_{3}' - L_{3})\delta_{3}}{L_{3}L_{3}(L_{3} \cdot 2A' + L_{3}' \cdot 2A)} \\ \\ &= \frac{L_{3}^{2}L_{3}^{2}(L_{1}' + L_{1})BB\delta_{1} + (L_{3}^{2}L_{3}^{2} - L_{3}^{2} - B)(L_{2} + L_{2})\delta_{2} + (L_{2}^{2} - L_{1}^{2} - L_{3}^{2})B}) \\ \\ &= \frac{L_{3}^{2}L_{4}^{2}(L_{1}' + L_{1})BB\delta_{1} + (L_{3}^{2}L_{3}^{2} - L_{3}^$$

Substituting eqns. 38-41 into 35, we get:

$$\{\varepsilon\} = \begin{cases} \varepsilon_{x} \\ \varepsilon_{y} \\ \gamma_{xy} \end{cases} = [B] \cdot \{\delta\} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \cdot \begin{cases} \delta_{1} \\ \delta_{2} \\ \delta_{3} \end{cases}$$
(42)

in which,

$$\begin{split} B_{11} &= B_{12} = 0 \qquad B_{13} = \frac{1}{L_3} + \frac{\delta_3}{2L_3^2} \\ B_{21} &= E_1 \cdot b_1 + E_2 \cdot a_1 \quad B_{22} = E_1 \cdot b_2 + E_2 \cdot a_2 \quad B_{23} = E_1 \cdot b_3 + E_2 \cdot a_3 + E_3 \\ B_{31} &= F_1 \cdot a_1 \qquad B_{32} = F_1 \cdot a_2 \qquad B_{33} = F_1 \cdot a_3 + F_2 \\ \text{The coefficients are defined as:} \\ L_1' &= L_1 + \delta_1 \qquad L_2' = L_2 + \delta_2 \qquad L_3' = L_3 + \delta_3 \\ AA &= L_3' \cdot L_3(L_3\sqrt{L_3'^2 \cdot L_2'^2 - 0.25(L_2'^2 + L_3'^2 - L_1'^2)^2} \\ &+ L_3' \cdot \sqrt{L_2^2 \cdot L_3^2 - 0.25(L_2^2 + L_3^2 - L_1^2)^2} \\ BB &= 0.5L_3(L_2'^2 + L_3'^2 - L_1'^2) + 0.5L_3'(L_2^2 + L_3^2 - L_1^2) \\ a_1 &= \frac{\delta_1 + 2L_1}{2L_3} \qquad b_1 = \frac{BB(2L_1L_3 + L_3\delta_1)}{2AA} \\ a_2 &= -\frac{\delta_2 + 2L_2}{2L_3'} \qquad b_2 = \frac{L_3'^2 \cdot L_3'(L_2' + L_2)}{AA} - \frac{BB(2L_2L_3 + \delta_2 \cdot L_3)}{2AA} \\ d_3 &= \frac{L_3^2 - L_1^2 + L_2^2 + \delta_3 \cdot L_3}{2L_3' \cdot L_3} \qquad b_3 = \frac{BB(L_2^2 - L_1^2 - L_3^2 - L_3 \cdot \delta_3)}{2AA} \\ U_2 &= a_1\delta_1 + a_2\delta_2 + a_3\delta_3 \qquad V_2 = b_1\delta_1 + b_2\delta_2 + b_3\delta_3 \qquad U_3 = \delta_3 \\ E_1 &= \frac{X_{31}}{2A} + \frac{X_{31}^2}{8A^2}V_2 \qquad E_2 = \frac{X_{31}X_{12}}{2A} \qquad U_3 + \frac{X_{31}}{8A^2}U_2 \qquad E_3 = \frac{X_{12}^2}{8A^2}\delta_3 \\ F_1 &= \frac{X_{31}}{2A} \qquad F_2 = \frac{X_{12}}{2A} + \frac{Y_{21}}{2A}(\frac{X_{31}}{2A} + U_2 + \frac{X_{12}}{2A}\delta_3) \qquad F = L_3'^2 \cdot L_3'^2 \cdot L_3' \cdot L_1' \end{aligned}$$

with A and A' the areas of the undeformed and deformed triangles respectively. Eqn. 42 may be substituted into Eqn. 15 and 18 to define the element characteristic matrices.

5 Generic Stiffness Matrix Definitions

Only the diagonal terms of the stiffness matrices $[K_E^{tr}]$ and $[K_{\sigma}^{pc}]$ are required by the dynamic relaxation solution procedure. $[K_E^{tr}]$ is the 3×3 elastic stiffness matrix defined in Eqn. 15 acting on the side extensions δ_1 , δ_2 and δ_3 , and written symbolically as:

$$\begin{bmatrix} K_E^{tr} \end{bmatrix} = \begin{bmatrix} K_{11}^{tr} & K_{12}^{tr} & K_{13}^{tr} \\ K_{21}^{tr} & K_{22}^{tr} & K_{23}^{tr} \\ K_{31}^{tr} & K_{32}^{tr} & K_{33}^{tr} \end{bmatrix}$$
(43)

The diagonal terms of the global elastic stiffness matrix $[K_E^t]$ acting on nodal displacements $\{u_1 \ v_1 \ w_1 \ u_2 \ v_2 \ w_2 \ u_3 \ v_3 \ w_3\}$ are:

$$[K_E^{t1}] = K_{11}^{tr} \cdot \begin{bmatrix} c_{x1}^2 \\ c_{y1}^2 \\ c_{z1}^2 \end{bmatrix} + K_{33}^{tr} \cdot \begin{bmatrix} c_{x3}^2 \\ c_{y3}^2 \\ c_{z3}^2 \end{bmatrix}$$
(44)

$$[K_E^{t2}] = K_{11}^{tr} \cdot \begin{bmatrix} c_{x1}^2 \\ c_{y1}^2 \\ c_{z1}^2 \end{bmatrix} + K_{22}^{tr} \cdot \begin{bmatrix} c_{x2}^2 \\ c_{y2}^2 \\ c_{z2}^2 \end{bmatrix}$$
(45)

$$[K_E^{t3}] = K_{22}^{tr} \cdot \begin{bmatrix} c_{x2}^2 \\ c_{y2}^2 \\ c_{z2}^2 \end{bmatrix} + K_{33}^{tr} \cdot \begin{bmatrix} c_{x3}^2 \\ c_{y3}^2 \\ c_{z3}^2 \end{bmatrix}$$
(46)

From the Eqn. 16, the diagonal terms of $[K_{\sigma}^{pc}]$ are:

$$[K^{pc}_{\sigma 11}] = \frac{T_{ci}}{l_i} - \frac{T_{ci}}{l_i} c_{xi}^2$$
(47)

$$[K^{pc}_{\sigma 22}] = \frac{T_{ci}}{l_i} - \frac{T_{ci}}{l_i} c_{yi}^2$$
(48)

$$[K^{pc}_{\sigma 33}] = \frac{T_{ci}}{l_i} - \frac{T_{ci}}{l_i} c_{zi}^2$$
(49)

where c_{xi} , c_{yi} and c_{zi} are the direction cosines of the pseudo cable *i* in the global x, y, z co-ordinate system, and $i = 1 \rightarrow 3$. The values of T_{c1} , T_{c2} , T_{c3} are calculated from Eqn. 18.

In a similar format to the elastic stiffness terms, the diagonal terms of the global geometric stiffness matrix $[K_{pc}^t]$ may be written as:

$$[K_{pc}^{t1}] = [I_3](\frac{T_{c1}}{L_1} + \frac{T_{c3}}{L_3}) - \frac{T_{c1}}{L_1} \cdot \begin{bmatrix} c_{x_1}^2\\ c_{y_1}^2\\ c_{z_1}^2 \end{bmatrix} - \frac{T_{c3}}{L_3} \cdot \begin{bmatrix} c_{x_3}^2\\ c_{y_3}^2\\ c_{z_3}^2 \end{bmatrix}$$
(50)

$$[K_{pc}^{t2}] = [I_3](\frac{T_{c1}}{L_1} + \frac{T_{c2}}{L_2}) - \frac{T_{c1}}{L_1} \cdot \begin{bmatrix} c_{\chi_1}^2\\ c_{\chi_1}^2\\ c_{\chi_2}^2\\ c_{\chi_2}^2 \end{bmatrix} - \frac{T_{c2}}{L_2} \cdot \begin{bmatrix} c_{\chi_2}^2\\ c_{\chi_2}^2\\ c_{\chi_2}^2\\ c_{\chi_2}^2 \end{bmatrix}$$
(51)

$$[K_{pc}^{t3}] = [I_3](\frac{T_{c2}}{L_2} + \frac{T_{c3}}{L_3}) - \frac{T_{c2}}{L_2} \cdot \begin{bmatrix} c_{\chi^2}^2\\ c_{\chi^2}^2\\ c_{\chi^2}^2 \end{bmatrix} - \frac{T_{c3}}{L_3} \cdot \begin{bmatrix} c_{\chi^3}^2\\ c_{\chi^3}^2\\ c_{\chi^3}^2 \end{bmatrix}$$
(52)

The total global stiffness matrix $[K_T^t]$ can be expressed as the summation of $[K_E^t]$ and $[K_{pc}^t]$:

$$[K_T^t] = [K_E^t] + [K_{pc}^t]$$
(53)

where the nine diagonal terms are given by summing the corresponding components of eqns. 44-46 and 50-52.

Similarly the nodal forces of the CST element in the coordinate system xyz can be calculated from:

$$f_{1} = \begin{cases} f_{1}^{x} \\ f_{1}^{y} \\ f_{1}^{z} \end{cases} = -T_{c1} \cdot \begin{bmatrix} c_{x1} \\ c_{y1} \\ c_{z1} \end{bmatrix} - T_{c3} \cdot \begin{bmatrix} c_{x3} \\ c_{y3} \\ c_{z3} \end{bmatrix}$$
(54)

$$f_2 = \begin{cases} f_2^x \\ f_2^y \\ f_2^z \end{cases} = T_{c1} \cdot \begin{bmatrix} c_{x1} \\ c_{y1} \\ c_{z1} \end{bmatrix} + T_{c2} \cdot \begin{bmatrix} c_{x2} \\ c_{y2} \\ c_{z2} \end{bmatrix}$$
(55)

$$f_{3} = \begin{cases} f_{3}^{x} \\ f_{3}^{y} \\ f_{3}^{z} \end{cases} = -T_{c2} \cdot \begin{bmatrix} c_{x2} \\ c_{y2} \\ c_{z2} \end{bmatrix} + T_{c3} \cdot \begin{bmatrix} c_{x3} \\ c_{y3} \\ c_{z3} \end{bmatrix}$$
(56)

with Eqn. 18 defining the values of T_{c1} , T_{c2} , and T_{c3} .

6 Wrinkling

With relatively negligible flexural stiffness, vanishing of tensile stresses in an arbitrary position or direction of the membrane surface will immediately lead to buckling in the form of wrinkles in the membrane material. In this case, the membrane will completely or partially lose stiffness and load resistance in the wrinkled area. From the perspective of either aesthetics or structural safety, wrinkling can be regarded as a type of structural (serviceability) failure, and should be inadmissible during membrane structural design. Structural analysis taking into account wrinkling is sophisticated because the detailed wrinkling pattern not only depends on the stress state but the imperfection of membrane material introduced during the fabrication process. Therefore, in this section, the main aim of the finite element formulation taking into account wrinkling concentrates on the prediction of wrinkling under loading as opposed to simulating the physical forms of the wrinkles.

Wrinkling criteria developed by OttoOtto (1962), MillerMiller, Hedgepeth, Weingarten, Das, Kahyai (1985) and Roddeman Roddeman (1987) (summarised in Table 1) are based on principal stress (denoted σ_p), strain (ε_p) or combined principal strain and stress (σ_p , ε_p). According to these criteria, the membrane state can be described as taut(no wrinkle), wrinkled(uniaxial wrinkling) or slack (biaxial wrinkling).

	Wrinkling criteria		Wrinkling	Membrane		
σ_p	$arepsilon_p$	σ_p, ε_p	state	state		
$\sigma_{II} > 0$	$\varepsilon_I \geq 0$ and $\varepsilon_{II} \geq \upsilon \varepsilon_I$	$\sigma_{II} > 0$	None	Taut		
$\sigma_I > 0$ and $\sigma_{II} < 0$	$\varepsilon_I \geq 0$ and $\varepsilon_{II} \leq -\upsilon \varepsilon_I$	$\varepsilon_I \ge 0$ and $\sigma_{II} \le 0$	Uniaxial	Wrinkled		
$\sigma_I \leq 0$	$\epsilon_I \leq 0$	$arepsilon_I \leq 0$	Biaxial	Slack		
Table 1: Wrinkling criteria based on principal stresses						

 Table 1: Wrinkling criteria based on principal stresses

For the local stresses $\sigma_X, \sigma_Y, \tau_{XY}$, the major principal stress σ_I and minor principal stress σ_{II} are,

$$\sigma_{I,II} = \frac{\sigma_X + \sigma_Y}{2} \pm \sqrt{\left(\frac{\sigma_X - \sigma_Y}{2}\right)^2 + \gamma_{XY}^2}$$
(57)

and corresponding principal strains are:

$$\varepsilon_{I,II} = \frac{\varepsilon_X + \varepsilon_Y}{2} \pm \sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2}$$
(58)

In zones where wrinkling occurs, the direction of the major principal strain (ε_I) is colinear with the wrinkle direction. If θ_p is the angle between wrinkle direction and local X axis, then:

$$\theta_p = \frac{1}{2} tan^{-1} \left(\frac{2\gamma_{XY}}{\varepsilon_X - \varepsilon_Y} \right) \tag{59}$$

When calculating the element stiffness matrix, the existence of wrinkling should be taken into account. In this case it is not sufficient to use

$$\{\sigma\} = \left\{\sigma_X \quad \sigma_Y \quad \tau_{XY}\right\}^T = [E] \left\{ \begin{array}{c} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{array} \right\} = [E][B]\{U\}$$
(60)

in which, $\{U\}$ is the vector of nodal displacements. If compressive stresses are not permitted and therefore the stiffness normal to the wrinkle direction is zero, solution convergence is not always smooth and sometimes may not be achieved Rossi (2005); Liu (2001); Rossi (2003). Rossi Rossi (2005) proposed an algorithm for the stabilization of the material manipulation. Using his method, if the membrane is in a "wrinkled state", a modified elastic stiffness matrix $[E_{mod}]$ is defined as:

$$\begin{bmatrix} E_{mod} \end{bmatrix} = \begin{bmatrix} E_{rot,11} & P \cdot E_{rot,12} & E_{rot,13} \\ P \cdot E_{rot,21} & P \cdot E_{rot,22} & P \cdot E_{rot,23} \\ E_{rot,31} & P \cdot E_{rot,32} & E_{rot,33} \end{bmatrix} = \begin{bmatrix} E_{rot} \end{bmatrix} \times \begin{bmatrix} P \end{bmatrix}$$
(61)

in which, P is the penalization parameter, and

$$[E_{rot}] = [R]^T [E][R]$$
(62)

[R] is a transformation matrix. Denoting the orientation of the principal stress to the local X axis is α_w , then

$$c = \cos(\alpha_w); s = \sin(\alpha_w); [R] = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}$$
(63)

If the penalization is constant, the performance of the wrinkling procedure may be compromised. An alternative definition of P to improve the stability is to make P a function of the maximum (σ_{max}) and effective compressive stresses (σ_2)Rossi (2005):

$$P_{\sigma} = \frac{\sigma_{max}}{\sigma_2} \to \begin{cases} P_{\sigma} > P & \to P = P_{\sigma} \\ P_{\sigma} > 1 \text{ or } P_{\sigma} < 0 & \to P = 1.0 \end{cases}$$
(64)

If the modification makes the state change (from wrinkled or slack to taut), then *P* is increased by a factor ω , where $\omega = 10$ is recommended (Rossi (2005)). The elastic stiffness matrix then becomes:

$$K_{E,mod} = \int_{v} B^{T}[E_{mod}] B dv$$
(65)

In addition to the obvious change in the element stiffness matrix, the equivalent nodal load vector is also changed to be of the form:

$$f_e = \int_V B_0(E_{mod} \cdot \varepsilon + \sigma_0) dV \tag{66}$$

In this method, a small compressive stress is allowed in the analysis to enhance the stability of solution procedure and the accuracy. It is necessary to note that there is no guarantee that the "fictitious" compressive stresses are removed during each iteration and at the final configuration, but wrinkling can be predicted with an acceptable accuracy and economical computational cost using this approach Rossi (2005).

7 Numerical examples

Three types of numerical example are presented in this section that aim to not only demonstrate the capabilities of the element formulations proposed in this paper, but also to facilitate discussion of topics relevant to the analysis of membrane structures.

7.1 In-plane shear test

The basis of the small and meso-strain CST formulations is that $\sin \gamma_{XY} \approx \gamma_{XY}$. In addition, approximations are made that truncate the definition of the strain descriptions in these two elements. No such approximations are made in the large strain CST element. The capability of the element formulations to simulate combined direct and shear stresses are examined by the simulation of an in-plane shear test depicted in fig. 10.

The elastic modulus of the orthotropic patch is $E_x = E_y = 600kN/m$, G = 30kN/m, with Possion's ratio v = 0.3 and a thickness t of unity. The fictitious patch is discretized with a mesh of 192 CST elements. When subjected to an edge traction (10), the patch deforms in combined direct and shear strain mode (11). Wrinkling is assumed not to occur in this example. Whilst the applied load of 75 kN/m is clearly high, generating maximum shear strains of approximately 60° , this numerical example is used to explore potential severe scenarios in the vicinity of clamp plates and edge cables, for example, where shear strains can be very significant.

Fig. 12-14 illustrate the stress σ_x obtained from the small, meso, and large strain CST formulations, respectively. Within the context of a constant strain field within each element, the mesh-scale field continuity of σ_x predicted by the large strain CST formulation (fig. 14) is greater than that from the small amd meso strain CST (fig. 12 and 13). With increasing distance from the upper boundary, the trends of stress σ_x are similar, but near the bottom line of the patch, the small and meso strain CST models produce much lower values of σ_x , with a target value expected to be close to the applied load $F_x = 75kN/m$ with pure tension in x direction at the lower boundary. Similar characteristics are displayed by σ_y .

The shear stresses illustrated in fig. 15-17 are absolute values, such that the maximum shear stresses from each of the formulations can be compared more directly. The discontinuity at the top right corner of the mesh produces peak shear stresses. The two small strain CST forumlations significantly overestimate the stresses in this area compared with the predictions obtained from the large strain CST (and an independent solution obtained from a discretisation of 6-node plane stress triangular elements featuring geometric non-linearity but not stress stiffening - fig. 18).

It may also be noted that the small-strain and meso-strain CST formulations produce a chequerboard style stress distribution. This phenomenon is erroneous and is symptomatic of a type of solution instability, in this case introduced by truncation and small shear assumptions within the strain-displacement components of the cable-analogy formulation. Similar, undesirable solution characteristics are not reproduced by the CST triangle based on the large strain formulation (c.f. 16 with 17). It is clear that adding higher-order terms to the basic CST forumlation $\varepsilon_i = \varepsilon_X \cdot \cos^2 \theta_i + \varepsilon_Y \cdot \sin^2 \theta_i + \gamma_{XY} \sin \theta_i \cdot \cos \theta_i$ whilst maintaining the necessary approaximation $\sin(\gamma_{XY}) \approx \gamma_{XY}$ to generate the meso-strain formulation does not lead to an improvement in the predicted stress field under condistions of severe shear strain.

Furthermore, relating the element strains and side extensions in the meso-strain formulation not straight-forward, requiring the introduction of the tertiary coefficient α . The accuracy of the strain field represented by the side extensions is a function of α . However, the solution process to determine α is not necessarily smooth, and in many cases, the value of α that correctly defines the strain field is difficult to find. As such, the application of the meso-strain formulation may be limited to analyses where α is known *a priori*, either because the strain state is well defined, or from experience. Moreover, the analysis of membrane structures is generally a complicated nonlinear solution process, normally requiring a stable strain-deformation relation at each iteration. The application of the meso-strain

formulation may result in slow rates of convergence or numerical divergence.

7.2 Analysis of the Newcastle University biaxial test cruciform

The design and manufacture of a fabric structure normally relies on a biaxial test to determine the compensation factors to be applied to the cutting patterns, the aim of which is to achieve a prescribed stress state at the point at which the fabric is fully installed on the supporting structure. For large projects, or when the membrane material is of a new type or composition, for example, more extensive biaxial tests are commissioned to establish the stiffness characteristics of the fabric. Owing to the complex nature of the behaviour of architectural fabrics, the design of the biaxial test protocol and the test specimen and methodology are not simple. In the following example, we use the analysis of a current biaxial test cruciform to demonstrate the capabilities of the existing and proposed CST element formulations, and the types of complex stress fields that are obtained for a simple, fictitious, orthotropic "fabric".

In the simulation of the membrane biaxial test illustrated in fig. 19, the material properties are: $E_x = 600kN/m$, $E_y = 600kN/m$, G = 30kN/m, and Poisson's ratio v = 0.3. The loads and mesh are shown in figures 19 and 20. The biaxial specimen is characterised by eleven slits, 150mm long, in each of the four strips around the central square (see fig. 19).



Figure 3: Constant strain riangular element defined as 3 cables (bars).



Figure 4: Pseudo cable/bar element.



Figure 5: The pseudo cable forces





Figure 7: Error in the linear cable analogy formulation under large strains



Figure 8: Area coordinate system



Figure 9: Local coordinate system for large strain CST formulation



Figure 10: Shear patch test and the element mesh



Figure 11: Deformed mesh — large CST with $F_x=75$ kN/m



Figure 12: Stress σ_x — small strain CST



Figure 13: Stress σ_x — meso-strain CST



Figure 14: Stress σ_x — large strain CST



Figure 15: Shear τ_{xy} — small strain CST



Figure 16: Shear τ_{xy} — meso-strain CST

y z x



Figure 17: Shear τ_{xy} — large strain CST



Figure 18: Shear stress τ_{xy} — plane stress, linear strain



Figure 19: Numerical representation of membrane biaxial test



Figure 20: CST discretisation of the biaxial cruciform

The distribution of the maximum principal stress σ_x of the biaxial cruciform membrane under the load $F_x = 30kN/m$, $F_y = 30kN/m$, as predicted by the original (small strain) CST formulation is depicted in fig. 21. The same simulation solution with the displayed stress range limited to the range 28-30 kN/m for clarity is presented in fig. 22. It is clear in fig. 21 and 22 that the solution for σ_x to this symmetrical problem are not symmetrical, with the maximum shear stress $\leq 5kN/m$. With the load increased to $F_x = F_y = 60kN/m$ and hence the strains similarly increased, the asymmetry of the geometrically nonlinear result is reinforced (c.f. fig. with 22 with 23).

Figures 24 and 25 present the maximum principal stress results obtained from the CST formulation assuming large deformations and strains. From the large strain CST model, the maximum principal stress in the biax cruciform is similar to that predicted by the small strain formulation. However, in contrast to the small strain element, the large strain formulation is shown to be fully symmetrical, not only about x,y axis, but also about the diagonal line A-B (fig. 25). The solution also remains fully symmetrical for higher values of strain (e.g. when $F_x = 60kN/m$ 26). The symmetry is well captured by the eye, as colour patterns are easily seen in the symmetrical solution. It should also be noted that the small and large strain solutions are based on the same mesh of elements.

The analysis of the cruciform biaxial test specimen clearly serves as a useful informal benchmark, making use of the essential symmetry. It also provides potential insight into the stress state within the biaxial test specimen. For example, using the large strain formulation it is possible to estimate the stress state associated with the location of strain measurement for the purposes of establishing material stiffnesses. In the present example it is clear that less than 100% of the applied load appears as a stress at the centre of the specimen, leading to an over estimate of the predicted fabric stiffness. Furthermore, the region of relatively uniform stress can be used to guide the placement and extent of strain measurement, with clear impacts on the accuracy of the experimental test methodology.

7.3 Wrinkling analysis

The prediction of wrinkling (loss of tautness) within a fabric membrane is an important necessary capability for the practical application of a simulation code in this field. The potential capability of the proposed large strain CST formulation is demonstrated using the combined stress-strain criterion and computation algorithm summarised in section 6 using the reference shear test Ishii (1989) defined in fig. 27.



Figure 21: Principal stresses, σ_x , for $F_x = F_y = 30$ kN/m - small strain



Figure 22: Principal stresses σ_x in the range 28-30 kN/m for $F_x = F_y = 30$ kN/m - small strain



Figure 23: Principal stresses σ_x in the range 55-60 kN/m for $F_x = F_y = 60$ kN/m - small strain



Figure 24: Principal stresses σ_x for $F_x = F_y = 30$ kN/m - large strain



Figure 25: Principal stresses σ_x in the range 28-30 kN/m for $F_x = F_y = 30$ kN/m - large strain



Figure 26: Principal stresses σ_x in the range 55-60 kN/m for $F_x = F_y = 60$ kN/m - large strain



Figure 27: Geometry, boundary and loading conditions for the shear test calculation

In the virtual shear test the membrane is pre-stressed in the x-direction by a displacement (prestrain) of $u_x = 1mm$. This displacement is held fixed for the subsequent shear loading which induces a maximum displacement in the y direction of $u_y = 10mm$. An isotropic ETFE-foil membrane material is assumed (ETFE-foil thickness $t = 200\mu$, Young's modulus is $E = 600N/mm^2$ and the Poisson's ratio is $\mu = 0.45$). The corresponding numerical results from reference Ishii (1989) are shown in fig. 28, in which, u_x and u_y denote the orthogonal displacements in the membrane plane respectively. Solid hexahedral elements were used to model the foil, with the aim of capturing the full 3-D physical behaviour as the wrinkles developed, including out-of-plane deformations. These are represented by the displacement measure u_z , reflecting the depth of the wrinkles in the membrane.

As illustrated in fig. 28, the wrinkles start to arise at the two ends, then propagate to the middle of the patch. It is also observed that wrinkling develops before the lateral displacement u_y reaches 1.65 mm, with the wrinkle depths u_z less than 0.0035mm, subsequently from $u_y = 1.65mm$ the wrinkles start to develop dramatically, and the maximum wrinkle depth increases up to 0.62mm with a small increment ($\Delta u_y = 2.0 - 1.65 = 0.35mm$) in the lateral displacement. After $u_y = 2.0mm$, the wrinkles develop proportionally, and propagate to the majority of the foil when $u_y = 2.2mm$. This numerical test is repeated using a large strain CST element discretisation comprising 64 elements (fig. 29). The element mesh is coarse compared with the meshes used in the reference solutionIshii (1989) (at least 10×20) which aimed at determining the physical details of the wrinkles (e.g depths) using hexahedral elements. In the context of the introduction of this paper, it is notable that in the reliability analysis of fabric structures using membrane finite elements, the existence of wrinkling is regarded as one of the structural failure modes. Therefore, a primary



Figure 28: Displacement u_z [mm] normal to the membrane plane for the development of wrinkles for selected shear displacements u_y in reference test

goal for the analysis system must be to predict the existence of wrinkling, and not the physical details of the wrinkles. At the same time, it must be capable of achieving this without recourse to a mesh density inconsistent with other requirements of the simulation. As such, it is desireable, from an engineering design perspective, not to be required to use an overly dense mesh to assess the the capabilities of the large strain CST element in initially predicting the onset of wrinkling and



simulating the development of wrinkled zones as the analysis proceeds.

The positions and directions of the predicted fabric wrinkles are represented by the short lines in fig. 30. The principal stresses are constant across the CST element, meaning that the wrinkling point and direction are single valued for each element.

Given the invariance of stress within each of the CST elements, the dependency of the prediction of the existence of wrinkling solution has been assessed. The test foil has been remeshed using the same number of D.O.F used in the mesh shown in fig. 29 (denote mesh I) to create the mesh depicted in fig. 31 (mesh II). The corresponding results are presented in fig. 32. Comparing the wrinkle patterns predicted using CST mesh I and II, the wrinkling distributions and directions are shown to correspond very closely.

Comparisons of figs. 30 and 32 with fig. 28, suggest that the onset and existence of the wrinkling in the patch can be predicted by the CST models accurately, even when they are not easy to observe in a three-dimensional (hexahedral) simulation (fig. 28) with wrinkle depths in the range $u_z = 0.002 - 0.0035mm$. The general propagation trend of the wrinkles can also be closely predicted.

8 Conclusions

The existing (original) CST formulation, currently used extensively in the engineering analysis of membrane structures, and based on the cable-analogy has been examined. The formulation is shown to assume small strains (e.g $\varepsilon_X = \frac{\partial U}{\partial X}$, $\varepsilon_Y = \frac{\partial V}{\partial Y}$ and $sin(\gamma_{XY}) = \gamma_{XY}$). Furthermore, to ensure a linear strain-displacement (D.O.F)



Figure 30: Detected wrinkle locations and directions for selected shear displacements u_y - mesh I.

relationship, a second simplifying assumption is made to limit the inclusion of only first-order strain components(Eqn. 8). The adoption of the cable-analogy leads to a compact formulation in which the element continuum strains are written as a function of the element side extensions. As such, the continuum is effectively replaced by a triangulated truss. However, when applying standard quality checks often used in finite element technologies, the formulation is shown to be deficient. For example, it does not pass a basic uniaxial patch test. Under moderate strains (5%-10% strains in the biax cruciform example), the deficiency of the existing CST formulation is implied by the development of non-symmetric stresses in the analysis of a symmetric problem.



Figure 31: Wrinkling numerical test - large strain CST mesh II.

A revised formulation – meso-strain CST has been derived by adding high order terms to the small strain CST from Eqn 4 - 7. Whilst maintaining the small strain assumption ($\varepsilon_X = \frac{\partial U}{\partial X}$ etc.), the modifications to the element aim to explore the significance of the second assumption referred to above. It should be noted that the small strain assumption appears only once (at the beginning) in the formulation. The adoption of the element side lengths as D.O.F help to introduce higher-order effects that are otherwise missing. The meso-strain CST formulation shows some improvements over the original form of the element. However, it fails to pass the same patch test and produces significant errors at large strains, again manifested as unexpected asymmetry.

From the perspective of computational compactness and ease of implementation of a new element into an existing engineering code, it is expedient to maintain the element side-lengths as D.O.F. Based on Green's strains and classical finite element philosophies, a large strain CST formulation has been successfully derived that satisfies tests associated with finite element technology standards. The proposed new element has been shown to successfully pass the patch test and achieve other desired benchmark results. It also serves to illustrate the significance and deficiency of assuming small strains in the existing CST element. The element has also been shown to successfully predict the onset and development of wrinkling, which is an important practical engineering requirement. The formulation satisfies rules of finite element technologies. Whilst maintaining the efficient "computational architecture" in the form of side-length D.O.F. coupled with the



Figure 32: Detected wrinkle locations and directions for selected shear displacements u_y - mesh II.

dynamic relaxation algorithm, the formulation can directly replace deficient small strain constant strain triangle cable-analogy elements currently used in engineering practice.

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