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Divergent Integrals in Elastostatics: Regularization in 3-D Case

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Abstract: In this article the divergent integrals, which arise when the boundary integral equation (BIE) methods are used for solution of the 3-D elastostatic problems is considered. The same approach for weakly singular, singular and hypersingular integral regularization is developed. The approach is based on theory of distribution and Green's theorems. This approach is applied for regularization of the divergent integrals over convex polygonal boundary elements (BE) in the case of piecewise constant approximation and over rectangular and triangular BE for piecewise linear approximation. The divergent integrals are transformed into the regular contour integrals that can be easily calculated analytically. Proposed methodology easy can be extended to other problems: elastodynamics, analytical calculation of the regular integrals, when collocation point situated outside the BE. Calculations of the divergent and regular integrals for square and triangle of the unit side are presented

Keywords: weakly singular, singular, hypersingular integrals, regularization, boundary integral equations.

1 Introduction

Boundary integral equation (BIE) is a very powerful method for solution of the mathematical problems in science and engineering, in particular for stress analysis in the theory of elasticity (Balas, Sladek J, Sladek V 1989; Hsiao, Wendland 2008; Guz, Zozulya, 1993). Since analytical solutions in 3-D theory of elasticity have been limited to the case of relatively simple geometry with a simple load, numerical methods such as a boundary element method (BEM) have been developed. The BIE and BEM methods are now established in many engineering disciplines as an alternative numerical technique to domain approaches, for example the finite element method. The attraction of BEM can be largely attributed to the reduction in

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the dimensionality of the problem. Another advantage of the BEM is a high accuracy of results especially for stress concentration problems. Namely, the solution at an internal point of analyzed domain is exactly expressed through the boundary values and no discretization of domain is required. This is the main reason why the BEM is the most accurate computational method for solution of the stress analysis problems. A familiar complication of BIE and BEM methods is, however, that they must in general be formulated in terms of divergent integral operators (Balas, Sladek J, Sladek V 1989; Chen, Hong 1999; Guz, Zozulya, 2001, Sladek V, Sladek J. 1998; Tanaka, Sladek V, Sladek J. 1994, etc.).

It is known that the overall accuracy of the BEM is largely dependent on the precision with which various integrals are evaluated. No doubt, the evaluation of divergent integrals requires much more sensitive treatment than that of regular integrals. Numerical methods developed for regular integrals calculation can not be used for their calculation. In mathematics divergent integrals have established theoretical basis. For example, the weakly singular integrals are considered as improper integrals, the singular integrals are considered in the sense of Cauchy principal value (*PV*), the hypersingular integrals had been considered by Hadamard as finite part integrals (*FP*). Usually different divergent integrals need different methods for their calculation. Analysis of the most known methods used for treatment of the different divergent integrals has been done in books Courant, Hilbert 1968; Gel'fand, Shilov 1964; Gunter 1967; Hadamard 1923; Michlin 1965; Sladek V, Sladek J. 1998; review articles Chen, Hong 1999; Guz, Zozulya, 2001, Tanaka, Sladek V, Sladek J. 1994 and resent papers Fata 2009; Han, Atluri 2007; Karlis, Tsinopoulos, Polyzos, Beskos, 2007, 2008; Marin 2008; Salvadori 2001; Sanz, Solis, Dominguez 2007; Zozulya 2006a,b 2008, 2010a-c. It has to be mentioned that direct method developed by Guiggiani and coauthors 1990, 1992 is widely applied for the divergent integral regularization in 2-D and 3-D cases. We will not discuss here advantages and disadvantages of these methods; it has already been done in the above mentioned review. We will consider here in more details method of the divergent integrals regularization, which is based on the theory of distributions and idea of finite part integrals according to Hadamard 1923.

We apply the approach based on the theory of distributions and finite part integrals for the problems of fracture mechanics firstly in Zozulya 1991. Then it was further developed for regularization of the hypersingular integrals in static and dynamic problems of fracture mechanics in Zozulya, Lukin 1998 and Zozulya, Men'shikov 2000 respectively. More applications of the devloped regularization method can be found in review articles Guz, Zozulya 2001, 2002; Gonzalez-Chi 2000 and resent papers Guz and Zozulya 2007; Guz, at all 2007. Further development of this approach and application of the Green's theorems in the sense of theory of distribution has bean done in Zozulya 2006a,b. Further development of this approach and application of the Green's theorems in the sense of theory of distribution has bean done in Zozulya 2008, 2010a-c. The equations presented in Zozulya 2006b, 2008, Zozulya, Gonzalez-Chi 1999 permit transform divergent weakly singular, singular and hypersingular integrals into the regular ones. The developed approach can be applied for regularization of a wide class of divergent integral. The developed approach can be applied not only for hypersingular integrals regularization but also for a wide class of divergent integral regularizations and any polynomial approximation. For example in gradient elasticity BIE contain more divergent integrals (see for details Karlis, Tsinopoulos, Polyzos, Beskos, 2007, 2008; Polyzos, Tsepoura, Beskos 2005; Polyzos, Tsepoura, Tsinopoulos, Beskos 2003; Tsepoura, Tsinopoulos, Polyzos, Beskos 2003). Methods developed in this and other our publications can been applied for such divergent integrals.

In the present paper, the above mentioned approach for the divergent integral regularization is further developed and applied for the case of 3-D elastostatic problems. For example in Zozulya 2008, 2010a we consider divergent integrals that appear in fracture mechanics, which are hypersingular. In the present paper we consider divergent integrals that appear in 3-D elastostatic which are weakly singular, singular and hypersingular. We obtain simple formulae which permit in the similar way consider the 2-D weakly singular, singular and hypersingular integrals which appear in 3-D elastostatics over arbitrary convex polygon for piecewise constant approximation and over rectangular and triangular BE for piecewise linear approximation. The regularized equations for the 2-D weakly singular, singular and hypersingular integrals calculation have been presented here for the case of 3-D elastostatics. In resented equations all calculations can be done analytically, no numerical integration is needed. It is important to mention that proposed methodology can be easy applied to dynamic problems, analytical calculation of the regular integrals when collocation point situated outside BE and for regularization of the divergent integrals in the case of quadratic and higher BE.

2 Main equations of elatostatics

Let consider a homogeneous, lineally elastic body, which in three-dimensional Euclidean space $\Re^3$ occupies volume V with smooth boundary ∂V. The region V is an open bounded subset of the three-dimensional Euclidean space $\Re^3$with a $C^{1,1}$ Lyapunov's class regular boundary ∂V. The boundary contain two parts ∂V_u and ∂V_p such that $\partial V_u \cap \partial V_p = \emptyset$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u are prescribed displacement $u_i(\mathbf{x})$ of the body points and on the part ∂V_p are prescribed traction $p_i(\mathbf{x})$respectively. The body may be affected by volume forces $b_i(\mathbf{x})$. We assume that displacements of the body points and their gradients are small, so its stress-strain state is described by small strain deformation tensor $\varepsilon_{ij}(\mathbf{x})$. The strain tensor and displacement vector are connected by Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{2.1}$$

where $\partial/\partial x_i$ is a derivative with respect to space coordinates x_i. The components of the strain tensor must also satisfy the Saint-Venant's relations

$$\partial^2_{kl}\varepsilon_{ij} - \partial^2_{il}\varepsilon_{kj} = \partial^2_{kj}\varepsilon_{il} - \partial^2_{ij}\varepsilon_{kl}$$

From the balance of impulse and moment of impulse lows follow that the stress tensor is symmetric one and satisfy the equations of equilibrium

$$\partial_j\sigma_{ij} + b_i = 0_i, \qquad \forall\mathbf{x} \in V. \tag{2.2}$$

Here and throughout the article the summation convention applies to repeated indices.

The tensor of deformation $\varepsilon_{ij}(\mathbf{x})$ and stress $\sigma_{ij}(\mathbf{x})$ are related by Hook's law

$$\sigma_{ij} = c_{ijkl}\varepsilon_{ij} \tag{2.3}$$

Here c_{ijkl} are elastic modules. In the case of homogeneous anisotropic medium they are symmetric

$$c_{ijkl} = c_{jikl} = c_{klij}$$

and satisfy condition of ellipticity

$c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha_1\varepsilon_{ij}\varepsilon_{ij} \quad , \quad \forall\varepsilon_{ij}$ and $\forall\alpha_1 > 0$

In the case of homogeneous isotropic medium the elastic modules have the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.4}$$

where λ and μ are Lame constants, $\mu > 0$ and $\lambda > -\mu$, δ_{ij} is a Kronecker's symbol. Throughout this paper we use the Einstein summation convention.

Substituting stress tensor in (2.2) and using Hook's law (2.3) and Cauchy relations (2.1) we obtain the differential equations of equilibrium in the form of displacements which may be presented in the form

$$A_{ij}u_j + b_i = 0, \qquad \forall\mathbf{x} \in V \tag{2.5}$$

The differential operatorA_{ij} for homogeneous anisotropic medium has the form

$$A_{ij} = c_{ikjl}\partial_k\partial_l \tag{2.6}$$

and for homogeneous isotropic medium has the form

$$A_{ij} = \mu\delta_{ij}\partial_k\partial_k + (\lambda+\mu)\partial_i\partial_j. \tag{2.7}$$

If the problem is defined in an infinite region, then solution of the equations (2.5) must satisfy additional conditions at the infinity in the form

$$u_j(\mathbf{x}) = O(r^{-1}), \quad \sigma_{ij}(\mathbf{x}) = O(r^{-2}) \text{ for } r \to \infty \tag{2.8}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance in the three-dimensional Euclidian space.
If the body occupied a finite region V with the boundary ∂V, it is necessary to establish boundary conditions. We consider the mixed boundary conditions in the form

$$\begin{aligned} u_i(\mathbf{x}) = {} & \varphi_i(\mathbf{x}) \,, \forall \mathbf{x} \in \partial V_u \,, \\ p_i(\mathbf{x}) = {} & \sigma_{ij}(\mathbf{x})n_j(\mathbf{x}) = P_{ij}[u_j(\mathbf{x})] = \psi_i(\mathbf{x}) \,, \forall \mathbf{x} \in \partial V_p \end{aligned} \tag{2.9}$$

The differential operator $P_{ij} : u_j \to p_i$ is called stress operator. It transforms the displacements into the tractions. For homogeneous anisotropic and isotropic medium they have the forms

$$P_{ij} = c_{ikjl}n_k\partial_l \qquad P_{ij} = \lambda n_i\partial_j + \mu\,(\delta_{ij}\partial_n + n_j\partial_i) \tag{2.10}$$

respectively. Here n_i are components of the outward normal vector, $\partial_n = n_i\partial_i$ is a derivative in direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V_p.
For additional information in linear elasticity refer to Gurtin 1972.

3 Integral representations for displacements and traction

In order to establish integral representations for the displacements and tractions let us consider bilinear form which depends on two fields of the strain tensor, that correspond to two fields of the displacements $\mathbf{u}$ and $\mathbf{u}^*$

$$a(\mathbf{u},\mathbf{u}^*) = c_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u}^*) \tag{3.1}$$

Obviously that
$a(\mathbf{u},\mathbf{u}^*) = a(\mathbf{u}^*,\mathbf{u})$ and $a(\mathbf{u},\mathbf{u}) = \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u}) \geq \alpha_1\varepsilon_{ij}(\mathbf{u})$

Integrating the equality (3.1) over the volume V and applying the Gauss-Ostrogradskii formula we will obtain

$$\int_V a(\mathbf{u},\mathbf{u}^*)dV = \int_V \sigma_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u}^*)dV = \int_{\partial V} \sigma_{ij}n_j u_i^* dS - \int_V u_i^* \partial_j \sigma_{ij} dV \tag{3.2}$$

Taking into account that $A_{ij}u_j = \partial_j\sigma_{ij}$ and $p_i = \sigma_{ij}n_j = P_{ij}[u_j]$ we will find the first Betty's theorem in the form

$$\int_V u_i^* A_{ij}u_j dV = \int_V a(\mathbf{u},\mathbf{u}^*)dV - \int_{\partial V} u_i^* P_{ij}[u_j]dS \tag{3.3}$$

We will replace u_i and u_i^* in the equation (3.3) and subtract resulting equation from the equation (3.3). Because of the form (3.1) is symmetrical one we will obtain the second Betty's theorem in the form

$$\int_V (u_i^* A_{ij}u_j - u_i A_{ij}u_j^*)dV = \int_{\partial V} (u_i P_{ij}[u_j^*] - u_i^* P_{ij}[u_j])dS \tag{3.4}$$

Taking into account definition of differential operator A_{ij} given in (2.5) and definition of differential operator P_{ij} given in (2.10) we obtain relation

$$\int_V (b_i u_i^* - b_i^* u_i)dV = \int_{\partial V} (p_i^* u_i - p_i u_i^*)dS \tag{3.5}$$

which is called the Betti's reciprocal theorem.

This theorem is usually used for obtain integral representations for the displacements and traction vectors. To do that we consider solution of the elliptic partial differential equation (2.5) in an infinite space for the body force $b_i^*(\mathbf{x}) \rightarrow \delta_{ij}\delta(\mathbf{x}-\mathbf{y})$

$$A_{ij}U_{kj}(\mathbf{x}-\mathbf{y}) + \delta_{ki}\delta(\mathbf{x}-\mathbf{y}) = 0, \quad \forall \mathbf{x},\mathbf{y} \in \Re^3 \tag{3.6}$$

Now considering that

$u_i^*(\mathbf{x}) \rightarrow U_{ij}(\mathbf{x}-\mathbf{y})$ and $p_i^*(\mathbf{x}) = P_{ij}[u_j^*(\mathbf{x})] \rightarrow W_{ij}(\mathbf{x},\mathbf{y})$

from (3.5) we obtain the integral representation for the displacements vector

$$u_i(\mathbf{y}) = \int_{\partial V} (p_i(\mathbf{x})U_{ij}(\mathbf{x}-\mathbf{y}) - u_j(\mathbf{x})W_{ij}(\mathbf{x},\mathbf{y}))dS + \int_V b_i(\mathbf{x})U_{ij}(\mathbf{x}-\mathbf{y})dV \tag{3.7}$$

which is called Somigliana's identity. The kernels $U_{ij}(\mathbf{x}-\mathbf{y})$ and $W_{ij}(\mathbf{x},\mathbf{y})$ are called fundamental solutions for elastostatics.

Applying to (3.7) the differential operator P_{ij} we will find integral representation for the traction in the form

$$p_i(\mathbf{y}) = \int\limits_{\partial V} (p_i(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y}) - u_j(\mathbf{x})F_{ij}(\mathbf{x},\mathbf{y}))dS + \int\limits_{V} b_i(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y})dV \tag{3.8}$$

The kernels $K_{ij}(\mathbf{x},\mathbf{y})$ and $F_{ij}(\mathbf{x},\mathbf{y})$ may be obtained applying the differential operator P_{ij} to the kernels $U_{ij}(\mathbf{x}-\mathbf{y})$ and $W_{ij}(\mathbf{x},\mathbf{y})$ respectively.

The integral representations (3.7) and (3.8) are usually used for direct formulation of the boundary integral equations in elastostatics. Refer to Balas, Sladek J, Sladek V 1989 for fore information regarding application of the BIE in the theory of elasticity.

4 Fundamental solutions

In order to find the fundamental solutions $U_{ij}(\mathbf{x}-\mathbf{y})$ for the differential operator A_{ij} we consider the differential equations of elastostatics in the form displacements (3.6). Solutions of these equations are called the fundamental solutions.

In 3-D elastostatics they have the form

$$U_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{16\pi\mu(1-\upsilon)r}((3-4\upsilon)\delta_{ij} + \partial_i r\partial_j r) \tag{4.1}$$

Here $r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$ is a distance between points $\mathbf{x}$ and $\mathbf{y}$ in 3-D Euclidean space $\Re^3$ and $\partial_i r = \frac{\partial r}{\partial x_i} = -\frac{\partial r}{\partial y_i} = \frac{x_i - y_i}{r}$ is a derivative in respect to x_i.

The kernels $W_{ij}(x,y)$ from (3.7) may be obtained by applying to $U_{ij}(x-y)$differential operator

$$P_{ik}[\bullet,(\mathbf{x})] = \lambda\ n_i(\mathbf{x})\partial_k[\bullet] + \mu\ \left[\delta_{ik}n_j(x)\partial_j[\bullet] + n_k(\mathbf{x})\partial_i[\bullet]\right] \tag{4.2}$$

as it is shown here

$$W_{ij}(x,y) = \lambda\ n_i(x)\partial_k U_{kj}(x,y) + \mu\ n_k(x)\left[\partial_k U_{ij}(x,y) + \partial_i U_{kj}(x,y)\right]. \tag{4.3}$$

Then after some transformations and simplifications the expression for the kernels $W_{ij}(x,y)$ will have the following form

$$W_{ij}(\mathbf{x},\mathbf{y}) = \frac{-1}{4\pi(1-\upsilon)r^2}(n_k(\mathbf{x})\partial_k r((1-2\upsilon)\delta_{ij} + 3\partial_i r\partial_j r) + (1-2\upsilon)(n_i(\mathbf{x})\partial_j r - n_j(\mathbf{x})\partial_i r)) \tag{4.4}$$

The kernels $K_{ij}(x,y)$ and $F_{ij}(x,y)$ from (3.8) may be obtained by applying differential operator

$$P_{ik}[\bullet,(y)] = \lambda\ n_i(y)\partial_k + \mu\ [\delta_{ik}n_j(y)\partial_j + n_k(y)\partial_i] \tag{4.5}$$

to $U_{ij}(x-y)$ and $W_{ij}(x,y)$ respectively

$$\begin{aligned} K_{ij}(x,y) &= \lambda\ n_i(y)\partial_k U_{jk}(x,y) + \mu\ n_k(y)\left[\partial_k U_{ji}(x,y) + \partial_i U_{jk}(x,y)\right]\ , \\ F_{ij}(x,y) &= \lambda\ n_i(y)\partial_k W_{jk}(x,y) + \mu\ n_k(y)\left[\partial_k W_{ji}(x,y) + \partial_i W_{jk}(x,y)\right]. \end{aligned} \tag{4.6}$$

Then after some transformations and simplifications the expression for the kernels $K_{ij}(x,y)$ will have the form

$$K_{ij}(\mathbf{x},\mathbf{y}) = \\ \frac{1}{4\pi(1-\upsilon)r^2}\left(n_k(\mathbf{y})\partial_k r\left((1-2\upsilon)\delta_{ij} + 3\partial_i r\partial_j r\right) + (1-2\upsilon)\left(n_i(\mathbf{y})\partial_j r - n_j(\mathbf{y})\partial_i r\right)\right) \tag{4.7}$$

and for the kernels $F_{ij}(x,y)$ will have the form

$$F_{ij}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi(1-\upsilon)r^3}\left(3n_k(\mathbf{x})\partial_k r\left((1-2\upsilon)n_i(\mathbf{y})\partial_j r + \upsilon\left(\delta_{ij}n_k(\mathbf{y})\partial_k r + n_j(\mathbf{y})\partial_i r\right) - \right.\right.$$

$$-5n_k(\mathbf{y})\partial_k r\partial_i r\partial_j r\right) + 3\nu\left(n_i(\mathbf{x})n_k(\mathbf{y})\partial_k r\partial_j r + n_k(\mathbf{x})n_k(\mathbf{y})\partial_i r\partial_j r\right) + \tag{4.8}$$

$$+(1-2\upsilon)\left(3n_j(\mathbf{x})n_k(\mathbf{y})\partial_k r\partial_i r + n_k(\mathbf{x})n_k(\mathbf{y})\delta_{ik} + n_i(\mathbf{x})n_j(\mathbf{y})\right) - (1-4\upsilon)n_j(\mathbf{x})n_i(\mathbf{y})\right)$$

The kernels (4.1) – (4.5) contain different kind singularities, therefore corresponding integrals are divergent. Here we will investigate there singularities and develop methods of divergent integrals calculation.

5 Singularities, boundary properties and boundary integral equations

Simple observation shows that kernels in the integral representations (3.7) and (3.8) tend to infinity when$r \to 0$. More detailed analysis of the equations (4.1), (4.4), (4.7), (4.8) give us the following results, when $\mathbf{x} \to \mathbf{y}$

$$U_{ij}(\mathbf{x}-\mathbf{y}) \to r^{-1}, W_{ij}(\mathbf{x},\mathbf{y}) \to r^{-2}, K_{ij}(\mathbf{x},\mathbf{y}) \to r^{-2}, F_{ij}(\mathbf{x},\mathbf{y}) \to r^{-3} \tag{5.1}$$

In order to investigate these functions and integrals with divergent kernels, following Michlin 1965 definition and classification of the integrals with various singularities will be presented here.

Definition 5.1. Let we consider two points with coordinated $\mathbf{x}, \mathbf{y} \in \Re^n$(where n =3 or n=2) and region V with smooth boundary ∂V of the class$C^{0,1}$. The boundary integrals of the types

$$\int_{\partial V} \frac{G(\mathbf{x}, \mathbf{y})}{r^\alpha} \varphi(\mathbf{x}) dS \,, \quad \alpha > 0 \tag{5.2}$$

where $G(\mathbf{x}, \mathbf{y})$ is a finite function in $\Re^n \times \partial V$ and $\varphi(\mathbf{x})$ is a finite function in ∂V, are *weakly singular* for $\alpha < n-1$, *strongly singular* for $\alpha = n-1$ and *hypersingular* for $\alpha > n-1$.

Definition 5.2. Let we consider two points with coordinated $\mathbf{x}, \mathbf{y} \in \Re^n$(where n =3 or n=2) and region V with smooth boundary ∂V of the class$C^{0,1}$. The boundary integrals of the types

$$\int_{\partial V} G(\mathbf{x}, \mathbf{y}) \ln(r) \varphi(\mathbf{y}) d\mathbf{y}, \tag{5.3}$$

where $G(\mathbf{x}, \mathbf{y})$ is a finite function in $\Re^n \times \partial V$ and $\varphi(\mathbf{x})$ is a finite function in ∂V, are weakly singular.

The integrals with singularities can not be considered in usual (Riemann or Lebegue) sense. In order to such integrals have sense it is necessary special consideration of them. We will apply these definitions of the integrals from (3.7) and (3.8) and give definition of weakly singular integrals as improper, strongly singular in sense of Cauchy principal values and hypersingular in sense of Hadamard's finite part (see Hadamard 1923, Michlin 1965, Lifanov, Poltavskii, Vainikko 2004).

Definition 5. 3. Integrals in (3.7) with kernels $U_{ij}(\mathbf{x} - \mathbf{y})$ are weakly singular and must be considered as improper

$$W.S. \int_{\partial V} p_i(\mathbf{x}) U_{ij}(\mathbf{x} - \mathbf{y}) dS = \lim_{\varepsilon \to 0} \int_{\partial V \setminus \partial V_\varepsilon} p_i(\mathbf{x}) U_{ij}(\mathbf{x} - \mathbf{y}) dS \tag{5.4}$$

Here ∂V_ε is a part of the boundary, projection of which on tangential plane is contained in the circle $C_\varepsilon(\mathbf{x})$ of the radio ε with center in the point $\mathbf{x}$.

Definition 5.4. Integrals in (3.7) and (3.8) with kernels $W_{ij}(\mathbf{x}, \mathbf{y})$ and $K_{ij}(\mathbf{x}, \mathbf{y})$ are singular and must be considered in sense of the Cauchy principal values as

$$P.V. \int_{\partial V} u_i(\mathbf{x}) W_{ij}(\mathbf{x}, \mathbf{y}) dS = \lim_{\varepsilon \to 0} \int_{\partial V \setminus \partial V(r<\varepsilon)} u_i(\mathbf{x}) W_{ij}(\mathbf{x}, \mathbf{y}) dS$$

$$P.V.\int\limits_{\partial V} p_i(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y})dS = \lim_{\varepsilon\to 0}\int\limits_{\partial V\backslash\partial V(r<\varepsilon)} p_i(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y})dS \tag{5.5}$$

Here $\partial V(r<\varepsilon)$ is a part of the boundary, projection of which on tangential plane is the circle $C_\varepsilon(\mathbf{x})$ of the radio ε with center in the point $\mathbf{x}$.

Definition 5.5. Integrals in (3.8) with kernels $F_{ij}(\mathbf{x},\mathbf{y})$are hypersingular and must be considered in sense of the Hadamard's finite part as

$$F.P.\int\limits_{\partial V} u_i(\mathbf{x})F_{ji}(\mathbf{x}-\mathbf{y})dS = \lim_{\varepsilon\to 0}\left(\int\limits_{\partial V\backslash\partial V(r<\varepsilon)} u_i(\mathbf{x})W_{ji}(\mathbf{x}-\mathbf{y})dS + 2u_j(\mathbf{x})\frac{f_j(\mathbf{x})}{\partial V(r<\varepsilon)}\right) \tag{5.6}$$

Here functions $f_j(\mathbf{x})$are chosen from the condition of the limit existence.

Singular character of the channels in (3.7) and (3.8) determine boundary properties of the corresponding potentials. Analysis of these formulae show that the boundary potentials with the kernels $U_{ij}(\mathbf{x}-\mathbf{y})$, are weakly singular and therefore they are continuous everywhere in the $\Re^n$ and, therefore, may be continuously extended on the boundary ∂V. The potentials with the kernels $W_{ij}(\mathbf{x},\mathbf{y})$ and $K_{ij}(\mathbf{x},\mathbf{y})$contain singular kernels and they jump when crossing the boundary ∂V. The potential with the kernels $F_{ij}(\mathbf{x},\mathbf{y})$ contain hypersingular kernels. They continuously cross the boundary ∂V.

Boundary properties of these potentials are well known Balas, Sladek J, Sladek V 1989; Hsiao, Wendland 2008. For smooth part of the boundary they may be expressed by the equations

$$\left(\int\limits_{\partial V} p_i(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS\right)^{\pm} = \left(\int\limits_{\partial V} p_i(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS\right)^{0} \left(\int\limits_{\partial V} u_i(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y})dS\right)^{\pm}$$

$$= \mp\frac{1}{2}u_j(\mathbf{y}) + \left(\int\limits_{\partial V} u_i(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y})dS\right)^{0}$$

$$\left(\int\limits_{\partial V} p_i(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS\right)^{\pm} = \pm\frac{1}{2}p_j(\mathbf{y}) + \left(\int\limits_{\partial V} p_i(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS\right)^{0} \tag{5.7}$$

$$\left(\int\limits_{\partial V} u_i(\mathbf{x})F_{ji}(\mathbf{x}-\mathbf{y})dS\right)^{\pm} = \left(\int\limits_{\partial V} u_i(\mathbf{x})F_{ji}(\mathbf{x}-\mathbf{y})dS\right)^{0}$$

The symbols " $\pm$" and " $\mp$ " denote that two equalities, one with the top sign and the other with the bottom sign, are considered. The up index "0" points out, that the direct value of the corresponding potentials on the surface ∂V should be taken.

Now using integral representations for displacements and traction and boundary properties of the potentials we can get boundary integral equations for elastostatics. Tending $\mathbf{y}$ in (3.7) and (3.8) to the boundary ∂V and taking into consideration boundary properties of the potentials (5.8) we obtain representation of the displacements and traction vectors on the boundary surface ∂V. On the smooth parts of the boundary they have the following form

$$\pm\frac{1}{2}u_i(\mathbf{y}) = \int\limits_{\partial V}(p_j(\mathbf{x})U_{ij}(\mathbf{x}-\mathbf{y}) - u_j(\mathbf{x})W_{ij}(\mathbf{x},\mathbf{y}))dS + \int\limits_{V} p_j(\mathbf{x})U_{ij}(\mathbf{x}-\mathbf{y})dV \quad (5.8)$$

$$\mp\frac{1}{2}p_i(\mathbf{y}) = \int\limits_{\partial V}(p_j(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y}) - u_j(\mathbf{x})F_{ij}(\mathbf{x},\mathbf{y}))dS + \int\limits_{V} p_j(\mathbf{x})K_{ij}(\mathbf{x},\mathbf{y})dV \quad (5.9)$$

The plus and minus signs in these equations are used for the interior and exterior problems, respectively. Together with boundary conditions they are used for compositing the BIE for the problems of elastoctatics.

In our previous publications Zozulya 1991, 2006a,b, Zozulya, Lukin 1998, Zozulya Gonzalez-Chi 1999 Zozulya Menshykov 2000 the approach for regularization of the divergent integrals has been developed. The approach is based on theory of generalized functions Gel'fand, Shilov 1964 and consists in application formula of part integration in 2-D case and second Green theorem in 3-D case. This approach can be applied for static and dynamic problems. Particularly in Zozulya 1991 it was shown that regularization of the divergent integrals in elastodynamics may be transformed to the ones in elastostatics.

Following approach developed in Zozulya 1991 we can transform the regularization over any curvilinear boundary element ∂V_ε to the regularization over its flat projectionΠ_ε. To do that let us introduce a Cartesian coordinate system such that its origin is at the point $\mathbf{y}$ and the y_3-axis coincides with the external normal to ∂V_ε at this point, and the other two axes lie in the tangential plane Π_ε. Orthogonal projection of the point $\mathbf{x} \in \partial V_n$ is defined as

$$\pi_i(\mathbf{x}) = x_i - n_i(\mathbf{y})n_j(\mathbf{y})(x_j - y_j) \quad (5.10)$$

Then integrals in (5.8) and (5.9) can be presented in the form

$$\int\limits_{\partial V} p_j(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS = \int\limits_{\partial V/\partial V_\varepsilon} p_j(\mathbf{x})U_{ji}(\mathbf{x}-\mathbf{y})dS + \int\limits_{\partial V_\varepsilon} U_{ji}(\mathbf{x}-\mathbf{y})(p_j(\mathbf{x}) - p_j(\mathbf{y}))dS -$$

$$-p_j(\mathbf{y})\int\limits_{\partial V_\varepsilon} U_{ji}(\mathbf{x}-\mathbf{y})-U_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS+p_j(\mathbf{y})\int\limits_{\Pi_\varepsilon} U_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

$$\int\limits_{\partial V} u_j(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y})dS=\int\limits_{\partial V/\partial V_\varepsilon} u_j(\mathbf{x})W_{ji}(\mathbf{x},\mathbf{y})dS+\int\limits_{\partial V_\varepsilon} W_{ji}(\mathbf{x},\mathbf{y})(u_j(\mathbf{x})-u_j(\mathbf{y}))dS-$$

$$-u_j(\mathbf{y})\int\limits_{\partial V_\varepsilon} W_{ji}(\mathbf{x},\mathbf{y})-W_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS+u_j(\mathbf{y})\int\limits_{\Pi_\varepsilon} W_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

$$\int\limits_{\partial V} p_j(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS=\int\limits_{\partial V/\partial V_\varepsilon} p_j(\mathbf{x})K_{ji}(\mathbf{x},\mathbf{y})dS+\int\limits_{\partial V_\varepsilon} K_{ji}(\mathbf{x},\mathbf{y})(p_j(\mathbf{x})-p_j(\mathbf{y}))dS+ \tag{5.11}$$

$$+p_j(\mathbf{y})\int\limits_{\partial V_\varepsilon} K_{ji}(\mathbf{y},\mathbf{x})-K_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS+p_j(\mathbf{y})\int\limits_{\Pi_\varepsilon} K_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

$$\int\limits_{\partial V} u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS=\int\limits_{\partial V/\partial V_\varepsilon} u_j(\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})dS$$

$$+\int\limits_{\partial V_\varepsilon} F_{ji}(\mathbf{x},\mathbf{y})(u_j(\mathbf{x})-u_j(\mathbf{y})-\partial_{\tau_i}u_j(\mathbf{y})(\mathbf{y}-\mathbf{x}))dS+$$

$$+u_j(\mathbf{y})\int\limits_{\partial V_\varepsilon} F_{ji}(\mathbf{x},\mathbf{y})-F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

$$+\partial_{\tau_i}u_j(\mathbf{y})\int\limits_{\partial V_\varepsilon} ((\mathbf{y}-\mathbf{x})F_{ji}(\mathbf{x},\mathbf{y})-(\pi(\mathbf{x})-\mathbf{y})F_{ji}(\pi(\mathbf{x}),\mathbf{y}))\,dS+$$

$$+\partial_{\tau_i}u_j(\mathbf{y})\int\limits_{\Pi_\varepsilon} (\pi(\mathbf{x})-\mathbf{y})F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS+u_j(\mathbf{y})\int\limits_{\Pi_\varepsilon} F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

where $\partial_{\tau_i}=\partial_i-n_i\partial_n$ is the derivative in tangential direction.

Because of ∂V_ε is sufficiently smooth, then the integrals on the right-hand side of these equalities are regular. The integrals over flat plane Π_ε in (5.11) contain kernels those are fundamental solutions of the static theory of elasticity

$$\int\limits_{\Pi_\varepsilon} U_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS,\quad \int\limits_{\Pi_\varepsilon} W_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS,\quad \int\limits_{\Pi_\varepsilon} K_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS \tag{5.12}$$

$$\int\limits_{\Pi_\varepsilon} (\pi(\mathbf{x})-\mathbf{y})F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS, \int\limits_{\Pi_\varepsilon} F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS$$

They are divergent and in introduced above specific system of coordinates their singularities have the form

$$\frac{1}{r^m}, \frac{(x_i-y_i)(x_j-y_j)}{r^m}, \quad m=1,2,3,4,5. \tag{5.13}$$

For such singular function following Zozulya 2006a we have regular representation of the form

$$F.P.\int\limits_V \frac{\varphi(\mathbf{x})}{r^m}dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int\limits_{\partial V}[\Delta^{k-i-1}\varphi(\mathbf{x})\partial_n\frac{P_i}{r^{m-2i}} - \frac{P_i}{r^{m-2i}}\partial_n\Delta^{k-i-1}\varphi(\mathbf{x})]dS + (-1)^k\int\limits_V \frac{1}{r^{m-2k}}\Delta^{k+1}\varphi(\mathbf{x})]dV, \tag{5.14}$$

where $P_k=(-1)^k\prod_{i=0}^{k-1}\frac{1}{(m+2i)^2}$ for $k,m>1$.

6 Projection method and the BEM equations

The main idea of the BEM consists in approximation of the BIE and further solution of that approximated finite dimensional BE system of equations. The mathematical essence of this approach is so-called projection method. Let us outline some results from mathematical theory of the projection methods related to the approximation of the BIE. For more information one can refer to Lebedev, Vorovich, Gladwell 1996, Zeidler 1997.

We consider two Banach spaces $\mathbf{X}$ and $\mathbf{Y}$ and functional equation in those spaces

$$\mathbf{A}\cdot\mathbf{u}=\mathbf{f},\ \mathbf{u}\in D(\mathbf{A})\subset\mathbf{X}\ \mathbf{f}\in R(\mathbf{A})\subset\mathbf{Y}. \tag{6.1}$$

Here $\mathbf{A}:\ \mathbf{X}\to\mathbf{Y}$ is the linear operator mapping from Banach space $\mathbf{X}$ in Banach space$\mathbf{Y}$, $D(\mathbf{A})$ is a domain and $R(\mathbf{A})$ is a range of the operator $\mathbf{A}$.The equation (6.1) is named the exact equation, and its solution is the exact solution. We denote $L(\mathbf{X},\mathbf{Y})$ Banach space of the linear operators mapping from $\mathbf{X}$ in $\mathbf{Y}$.

Let in $\mathbf{X}$ and $\mathbf{Y}$ act sequences of projection operators $\mathbf{P}_h$ and $\mathbf{P}'_h$ such that

$$\begin{gathered}\mathbf{P}_h^2=\mathbf{P}_h,\ \mathbf{P}_h\mathbf{X}=\mathbf{X}_h,\ \mathbf{X}_h\subset\mathbf{X},\\ \left(\mathbf{P}'_h\right)^2=\mathbf{P}'_h,\ \mathbf{P}'_h\mathbf{Y}=\mathbf{Y}_h,\ \mathbf{Y}_h\subset\mathbf{Y},\end{gathered} \tag{6.2}$$

where $\mathbf{X}_h$ and $\mathbf{Y}_h$ are finite dimension subspaces of the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, $h \in R^1$ is a parameter of discretization.

Now we consider operator$\mathbf{A}_h \in L(\mathbf{X}_h, \mathbf{Y}_h)$ mapping in finite dimensional subspaces $\mathbf{X}_h$ and $\mathbf{Y}_h$ and an approximate equation

$$\mathbf{A}_h \cdot \mathbf{u}_h = \mathbf{f}_h,\ \mathbf{A}_h = \mathbf{P}'_h \cdot \mathbf{A} \cdot \mathbf{P}_h,\ \mathbf{u}_h = \mathbf{P}_h \cdot \mathbf{u},\ \mathbf{f}_h = \mathbf{P}'_h \cdot \mathbf{f}. \tag{6.3}$$

Solution $\mathbf{u}_h$ of the equation (6.3) is the approximate solutions of the equation (6.1). The general scheme of the approached equations construction (6.3) is illustrated by the following diagram

$$\begin{aligned} &\mathbf{X} \supset D(\mathbf{A}) \xrightarrow{\mathbf{A}} R(\mathbf{A}) \subset \mathbf{Y} \\ &P_h \downarrow \quad P'_h \downarrow \\ &\mathbf{X}_h \supset D(\mathbf{A}_h) \xrightarrow{\mathbf{A}_h} R(\mathbf{A}_h) \subset \mathbf{Y}_h \end{aligned} \tag{6.4}$$

Now let us consider operator $\mathbf{A}_h \in L(\mathbf{X}_h, \mathbf{Y}_h)$ mapping in finite dimensional subspaces $\mathbf{X}_h$ and $\mathbf{Y}_h$ and an approximate equation

Existence of the exact solution, convergence of the approximate solution to the exact one and stability of the approximations are the main problems which arrived in application of the projection methods. In order to solve these problems we have to formulate them mathematically.

We assume, that projection operators $\mathbf{P}_h$ and $\mathbf{P}'_h$ converge to identity operators in $\mathbf{X}$ and $\mathbf{Y}$ respectively. It means that

$$\begin{aligned} &\lim_{h\to 0} \|\mathbf{P}_h \cdot \mathbf{u} - \mathbf{u}\|_{\mathbf{X}} = 0\ \forall \mathbf{u} \in \mathbf{X}, \\ &\lim_{h\to 0} \left\|\mathbf{P}'_h \cdot \mathbf{f} - \mathbf{f}\right\|_{\mathbf{Y}} = 0\ \forall \mathbf{f} \in \mathbf{Y}. \end{aligned} \tag{6.5}$$

Definition 3.1. Let conditions (6.5) are satisfied and stating from some $h = h_0 > 0$ for any $\mathbf{f} \in \mathbf{Y}$ the equation (6.3) has unique solution $\mathbf{u}_h$. In this case if

$$\lim_{h\to 0} \|\mathbf{A}_h \cdot \mathbf{u}_h - \mathbf{A} \cdot \mathbf{u}\|_{\mathbf{Y}} = 0, \tag{6.6}$$

than the solution of the approximate problem (6.3) converges to the exact solution (6.1). It means that the projective method presented on diagram (6.4) is applicable to the initial problem (6.1).

Definition 3.2. Let for some sequence of the operators $\{\mathbf{A}_h\}$ mapping from$\mathbf{X}_h$ into$\mathbf{Y}_h$ and there is a constant $\gamma > 0$ such, that stating from some $h = h_0 > 0$

$$\|\mathbf{A}_h \cdot \mathbf{u}_h - \mathbf{A} \cdot \mathbf{u}\|_{\mathbf{Y}} \geq \gamma \|\mathbf{u}_h\|_{\mathbf{X}}\ \forall \mathbf{u}_h \in \mathbf{X}_h. \tag{6.7}$$

than for sequence of the operators $\{\mathbf{A}_h\}$ the condition of stability of the approximate solution is satisfied.

Conditions (6.5) - (6.7) are very important for formulation conditions of existence, convergence and stability of the approximate solution. These conditions contain the following theorem Zeidler 1997.

Theorem 6.1. Let the following conditions are satisfied:

1) the projection operators $\mathbf{P}_h$ and $\mathbf{P}'_h$ converge to identity operators in the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$ respectively, as it stated in (6.5);

2) the sequence of approximate operators $\{\mathbf{A}_h\}$ converge to $\boldsymbol{A}$ on each exact solution;

3) the condition of stability (6.7) is satisfied for the sequence of operators $\{\mathbf{A}_h\}$.

Then the following consequences take place:

1) the exact solution exists and it is unique;

2) for all enough small h exists a unique solution $\mathbf{u}_h \in \mathbf{X}_h$ of the approximate equation (6.3);

3) the sequence of the approximate equation $\{\mathbf{u}_h\}$ converges to the exact one and take place the estimation

$$\|\mathbf{u}_h - \mathbf{P}_h \cdot \mathbf{u}\|_{\mathbf{X}_h} \leq \gamma^{-1} \left\|\mathbf{P}'_h \mathbf{A} \cdot \mathbf{u} - \mathbf{A}_h \mathbf{P}_h \mathbf{u}\right\|_{\mathbf{Y}_h}. \tag{6.8}$$

Thus, using a projective method instead of the exact solution of the equation (6.1) in functional space $\mathbf{X}$, we can find the solution of the approximate equation (6.3) in finite dimensional space $\mathbf{X}_h$. The functional spaces $\{\mathbf{X}, \mathbf{X}_h\}$ and $\{\mathbf{Y}, \mathbf{Y}_h\}$ are related by means of the projection operators $\mathbf{P}_h \in L(\mathbf{X}, \mathbf{X}_h)$ and $\mathbf{P}'_h \in L(\mathbf{Y}, \mathbf{Y}_h)$ respectively. It is also important to construct inverse operator $\mathbf{P}_h^{-1} \in L(\mathbf{X}_h, \mathbf{X})$ which maps the finite dimensional space $\mathbf{X}_h$ into initial functional space $\mathbf{X}$. Such operator refers to as the operator of interpolation. Because of $\mathbf{X}_h \subset \mathbf{X}$ the interpolation operator is not unique, moreover for any two functional spaces $\mathbf{X}_h$ and $\mathbf{X}$ it is possible to construct infinite set of interpolation operators.

Let us apply projection method to the BIE of elastostatics and construct corresponding finite dimensional BE equations. It is known Hsiao, Wendland 2008 that integral operators in (5.9) and (5.10) maps between two functional spaces $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$that are trace of displacements and traction on the boundary of the region in the following way

$$\mathbf{U}_{ij} \cdot \mathbf{p}_j = \int\limits_{\partial V} (U_{ij}(\mathbf{x},\mathbf{y})\ p_j(\mathbf{x}))dS\colon \mathbf{Y}(\partial V) \to \mathbf{X}(\partial V)$$

$$\mathbf{W}_{ij}\cdot\mathbf{u}_j = \int\limits_{\partial V} (W_{ij}(\mathbf{x},\mathbf{y})\,u_j(\mathbf{x}))dS : \mathbf{X}(\partial V)\to\mathbf{Y}(\partial V)$$

$$\mathbf{K}_{ij}\cdot\mathbf{p}_j = \int\limits_{\partial V} (K_{ij}(\mathbf{x},\mathbf{y})\,p_j(\mathbf{x}))dS\!: \mathbf{Y}(\partial V)\to\mathbf{X}(\partial V)$$

$$\mathbf{F}_{ij}\cdot\mathbf{u}_j = \int\limits_{\partial V} (F_{ij}(\mathbf{x},\mathbf{y})\,u_j(\mathbf{x}))dS : \mathbf{X}(\partial V)\to\mathbf{Y}(\partial V) \tag{6.9}$$

We have to construct finite dimensional functional spaces that correspond to $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$ and the corresponding projection operators. To construction finite dimensional functional spaces we shall apply approximation by finite functions and splitting ∂V into finite elements

$$\partial V = \overset{N}{\underset{n=1}{\cup}}\partial V_n,\ \partial V_n\cap\partial V_k = \emptyset,\ \text{if}\ n\neq k. \tag{6.10}$$

Because of ∂V is the boundary of the region these elements are called boundary elements. On each boundary element we shall choose Q nodes of interpolation. Local projection operators act from functional $\mathbf{X}(\partial V_n)$ and $\mathbf{Y}(\partial V_n)$ to the finite directional ones $\mathbf{X}_q(\partial V_n)$ and $\mathbf{Y}_q\partial V_n)$

$$\begin{aligned}&\mathbf{P}_q^u : \mathbf{X}(\partial V_n)\to\mathbf{X}_q(\partial V_n)\ \forall\mathbf{x}\in\partial V_n,\\&\mathbf{P}_q^p : \mathbf{Y}(\partial V_n)\to\mathbf{Y}_q(\partial V_n)\ \forall\mathbf{x}\in\partial V_n.\end{aligned} \tag{6.11}$$

Global projection operators are defined as the sum of the local projection operators

$$\mathbf{P}_{nq}^u = \sum_{n=1}^{N}\mathbf{P}_q^u,\ \mathbf{P}_{nq}^p = \sum_{n=1}^{N}\mathbf{P}_q^p. \tag{6.12}$$

They map $\mathbf{X}(\partial V)$ and $\mathbf{Y}(\partial V)$ to finite dimensional interpolations spaces

$$\begin{aligned}&\mathbf{P}_{nq}^u : \mathbf{X}(\partial V)\to\mathbf{X}_q(\bigcup_{n=1}^{N}\partial V_n)\ \forall\mathbf{x}\in\partial V,\\&\mathbf{P}_{nq}^p : \mathbf{Y}(\partial V)\to\mathbf{Y}_q(\bigcup_{n=1}^{N}\partial V_n)\ \forall\mathbf{x}\in\partial V\end{aligned} \tag{6.13}$$

The local projection operators $\mathbf{P}_n^u$ also $\mathbf{P}_n^p$ establish correspondence between vectors of displacements and traction and their value on the nodes of interpolation of the boundary elements ∂V_n

$$\begin{aligned}&\mathbf{P}_q^u\cdot\mathbf{u}_i(\mathbf{x}) = \left\{\mathbf{u}_i^n(\mathbf{x}_q),\ q=1,...,Q\right\}\ \forall\mathbf{x}\in\partial V_n,\\&\mathbf{P}_q^p\cdot\mathbf{p}_i(\mathbf{x}) = \left\{\mathbf{p}_i^n(\mathbf{x}_q),\ q=1,...,Q\right\}\ \forall\mathbf{x}\in\partial V_n.\end{aligned} \tag{6.14}$$

Similarly for the global operators we have

$$\mathbf{P}^u_{nq}\cdot\mathbf{u}_i(\mathbf{x})=\left\{\mathbf{u}^n_i(\mathbf{x}_q),\ q=1,...,Q;\ n=1,...,N\right\}\ \forall\mathbf{x}\in\partial V,$$
$$\mathbf{P}^p_{nq}\cdot\mathbf{p}_i(\mathbf{x})=\left\{\mathbf{p}^n_i(\mathbf{x}_q),\ q=1,...,Q;\ n=1,...,N\right\}\ \forall\mathbf{x}\in\partial V. \tag{6.15}$$

Let us construct local interpolation operators $\left(\mathbf{P}^u_q\right)^{-1}$ and $\left(\mathbf{P}^p_q\right)^{-1}$. For this purpose we will introduce systems of shape functions $\varphi_{nq}(\mathbf{x})$ in the finite dimensional functional spaces $\mathbf{X}_q(\partial V_n)$ and $\mathbf{Y}_q(\partial V_n)$. Then the vectors of displacements and traction on the boundary element ∂V_n will be represented approximately in the form

$$u_i(\mathbf{x})\approx\sum_{q=1}^{Q}u^n_i(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}),\ \mathbf{x}\in\partial V_n$$
$$p_i(\mathbf{x})\approx\sum_{q=1}^{Q}p^n_i(\mathbf{x}_q)\,\varphi_{nq}(\mathbf{x})\ \mathbf{x}\in\partial V_n \tag{6.16}$$

and on the whole crack surface ∂V in the form

$$u_i(\mathbf{x})\approx\sum_{n=1}^{N}\sum_{q=1}^{Q}u^n_i(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}),\ \mathbf{x}\in\bigcup_{n=1}^{N}\partial V_n,$$
$$p_i(\mathbf{x})\approx\sum_{n=1}^{N}\sum_{q=1}^{Q}p^n_i(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}),\ \mathbf{x}\in\bigcup_{n=1}^{N}\partial V_n \tag{6.17}$$

Finite-dimensional analogies for the integral operators (6.9) are operators which map the finite dimensional functional spaces $\mathbf{X}_q(\bigcup_{n=1}^{N}\partial V_n)$ and $\mathbf{Y}_q(\bigcup_{n=1}^{N}\partial V_n)$, from one to another

$$\mathbf{U}^{nq}_{ij}=\mathbf{P}^p_{nq}\cdot\mathbf{U}_{ij}\cdot\mathbf{P}^u_{nq}{:}\mathbf{Y}_q(\partial V_n)\to\mathbf{X}_q(\partial V_n),$$
$$\mathbf{W}^{nq}_{ij}=\mathbf{P}^p_{nq}\cdot\mathbf{W}_{ij}\cdot\mathbf{P}^u_{nq}{:}\mathbf{X}_q(\partial V_n)\to\mathbf{Y}_q(\partial V_n) \tag{6.18}$$
$$\mathbf{K}^{nq}_{ij}=\mathbf{P}^p_{nq}\cdot\mathbf{K}_{ij}\cdot\mathbf{P}^u_{nq}{:}\mathbf{Y}_q(\partial V_n)\to\mathbf{X}_q(\partial V_n),$$
$$\mathbf{F}^{nq}_{ij}=\mathbf{P}^p_{nq}\cdot\mathbf{F}_{ij}\cdot\mathbf{P}^u_{nq}{:}\mathbf{X}_q(\partial V_n)\to\mathbf{Y}_q(\partial V_n)$$

Note that in contrast to differential operators, the integral operators are global and they are defined in the entire space, i.e. at every boundary element.

Substitution of the expressions (6.17) in (2.5) gives us the finite-dimensional representations for the vectors of displacements and traction on the boundary in the form

$$\frac{1}{2}u_i(\mathbf{y}_r)=\sum_{n=1}^{N}\sum_{q=1}^{Q}\Big[U^q_{ji}(\mathbf{y}_r,\mathbf{x}_m)\,p^n_j(\mathbf{x}_m)-W^q_{ji}(\mathbf{x}_r,\mathbf{x}_m)\,u^n_j(\mathbf{x}_m)\ +U_i(\mathbf{f},\mathbf{y},V_n)$$

$$\frac{1}{2}p_i(\mathbf{y}_r) = \sum_{n=1}^{N}\sum_{q=1}^{Q}\left[K_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m)\,p_j^{n}(\mathbf{y}_m) - F_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m)\,u_j^{n}(\mathbf{x}_m) + K_i(\mathbf{f},\mathbf{y},V_n)\right. \tag{6.19}$$

where

$$U_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} U_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_{nq}(\mathbf{x})\,dS, \quad W_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} W_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_{nq}(\mathbf{x})\,dS,$$
$$K_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} K_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_{nq}(\mathbf{x})\,dS, \quad F_{ji}^{q}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} F_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_{nq}(\mathbf{x})\,dS. \tag{6.20}$$

The volume potentials $U_i(\mathbf{f},\mathbf{y},V_n)$ also $K_i(\mathbf{f},\mathbf{y},V_n)$ depend on discretization of the V domain. More detailed information on transition from the BIE to the BEM equations can be found in Balas, Sladek J, Sladek V 1989; Guz, Zozulya 1993.

7 Boundary elements and approximation

The BEM can be treated as the approximate method for the BIE solution, which includes approximation of the functions that belong to some functional space by discrete finite model. This model comprises finite number of values of the considered functions which are used for approximation of these functions by the shape functions determined on small sub domains called boundary elements. In this sense the BEM is closely related to a finite elements method where the functions also belong to corresponding functional spaces and are approximated by finite model. Below we shall speak about finite element approximations and finite elements (FE), keeping in mind that boundary elements are their specific case.

It is important to mention that local approximation of the considered function on one FE can be done independently from other FEs. It means, that it is possible to approximate function on a FE by means of its values on the nodes independently of the place will occupied considered the FE in the finite element model and how behave the function on other FEs. Hence, it is possible to create the catalogue of various FE or BE with arbitrary node values interpolation function. Then from this catalogue can be chosen FEs which are necessary for approximation the function and domain of its definition. The same FE can be used for discrete models of various functions or physical fields by determination of the necessary position of nodes in the model and further definition of the node values of the function or physical field. Thus, finite models of an area and its boundary is not depend on functions and physical fields for which that area can be a domain of definition.

Let us consider how to construct a FE model of an area $V \subset R^n$ and a BE model of its boundary $\partial V \subset R^{n-1}$(§ = 2, 3). We fix in the area V finite number of points

$\mathbf{x}^q$ $(q = 1, \ldots, Q)$, these points refer to as global nodes points $V(q) = \{\mathbf{x}^q \in V : q = 1, \ldots, Q\}$. We shall divide the area V into finite number of sub areas V_n $(n = 1, \ldots, N)$ which are FEs. They have to satisfy the following conditions

$$V_n \cup V_m = \emptyset,\ m \neq n,\ m,\ n = 1,\ 2,\ \ldots,\ N,\ V = \bigcup_{n=1}^{N} V_n. \tag{7.1}$$

On each FE we introduce a local coordinate system $\boldsymbol{\xi}$. The nodal points $\mathbf{x}^q \in V_n$ in the local system of coordinates we designate by $\boldsymbol{\xi}^q$. They are coordinates of nodal points in the local coordinate system. Local and global coordinate are related in the following way

$$\mathbf{x}^q = \sum_{n=1}^{N} \Lambda_n \boldsymbol{\xi}_n^q. \tag{7.2}$$

Functions Λ_n depend on position of the nodal points in the FE and BE. They join individual FE together in a FE model. Borders of the FEs and position of the nodal points should be such that, after joining together, separate elements form discrete model of the area V.

Having constructed FE model of the area$V$, we shall consider approximation of the function $f(\mathbf{x})$ that belong to some functional space. The FE model of the area V is the domain of function which should be approximated. We denote function $f(\mathbf{x})$ on the FE V_n by $f^n(\mathbf{x})$. Then

$$f(\mathbf{x}) = \sum_{n=1}^{N} f^n(\mathbf{x}). \tag{7.3}$$

On each FE the local functions $f^n(\mathbf{x})$ may be represented in the form

$$f^n(\mathbf{x}) \approx \sum_{q=1}^{Q} f^n(\mathbf{x}^q)\varphi_{nq}(\boldsymbol{\xi}), \tag{7.4}$$

where $\varphi_{nq}(\boldsymbol{\xi})$ are interpolation polynomials or shape functions of the FE with number n. In nodal point with coordinates $\mathbf{x}^q$ they are equal to 1 and in other nodal points are equal to zero. Taking into account (7.3) and (7.4) global approximation of the function $f(\mathbf{x})$ looks like

$$f(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} f^n(\mathbf{x}_q)\varphi_{nq}(\boldsymbol{\xi}). \tag{7.5}$$

If the nodal point q belongs to several FEs it is considered in these sums only once.

The FE and BE elements can be of various form and sizes, their surfaces can be curvilinear. The curvilinear FE are very important in BEM because of boundary surface is usually curvilinear. But it is more convenient to use standard FE, which surfaces coincide with coordinate planes of local coordinates system. Mathematically it means, that it is necessary to establish relation between local coordinates ξ_i in which element has a simple appearance, and global x_i where the FE represents more complex figure. Local coordinates ξ_i should be functions of global $(\xi_i(x_1, x_2, x_3))$ ones, and on the contrary global coordinates should be functions of $(x_i(\xi_1, \xi_2, \xi_3))$ ones. In order to these maps be one-to-one, it is necessary and sufficient that Jacobians of the transformations be nonzero

$$J = \det\left|\frac{\partial x_i}{\partial \xi_j}\right| \neq 0,\ J^{-1} = \det\left|\frac{\partial \xi_i}{\partial x_j}\right| \neq 0$$

The differential elements along coordinate axes are related by

$$\begin{aligned} dx_i &= \left(\partial x_i/\partial \xi_j\right) d\xi_j,\ d\mathbf{x} = J(\xi)d\boldsymbol{\xi}, \\ d\xi_i &= \left(\partial \xi_i/\partial x_j\right) dx_j,\ d\boldsymbol{\xi} = J^{-1}(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{7.6}$$

The volume element in the $\Re^3$ is transformed under the formula

$$dV = dx_1 dx_2 dx_3 = J(\boldsymbol{\xi}) d\xi_1 d\xi_2 d\xi_3,$$

and the area element in the $\Re^2$ is transformed under the formula

$$dA = dx_1 dx_2 = \det\left|\frac{\partial x_\alpha}{\partial \xi_\beta}\right| d\xi_1 d\xi_2,\ \alpha,\ \beta = 1,\ 2 \tag{7.7}$$

The differential of the surface located in the $\Re^3$ is defined by expression

$$dS = \left(n_1^2 + n_2^2 + n_3^2\right)^{1/2} d\xi_1 d\xi_2, \tag{7.8}$$

where

$$\begin{aligned} n_1 &= \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_2}\frac{\partial x_3}{\partial \xi_1}, \\ n_2 &= \frac{\partial x_3}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2}, \\ n_3 &= \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}. \end{aligned}$$

The element of length of a contour in the $\Re^2$ is defined by expression

$$dl = \left[(dx_1/d\xi_1)^2 + (dx_2/d\xi_1)^2 \right]^{1/2} d\xi_1. \tag{7.9}$$

It is important to point attention that it is quite enough to consider standard FE which can be transformed to the necessary form by suitable transformation of coordinates. The FE approximation has to be linear independent and compact in the corresponding functional space.

Applying representations (5.11) to discrete equations (6.20) we transform the regularization over any curvilinear boundary element ∂V_n to the regularization over its flat projection Π_n.

$$\int\limits_{\partial V_n} U_{ji}(\mathbf{x},\mathbf{y}_r)\,\psi_{nq}(\mathbf{x})J(\mathbf{x})dS =$$
$$\int\limits_{\partial V_n} U_{ji}(\mathbf{x},\mathbf{y}_r)\,(\psi_{nq}(\mathbf{x})J(\mathbf{x}) - \psi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r))\,dS + \psi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r)\int\limits_{\Pi_n} U_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS,$$

$$\int\limits_{\partial V_n} W_{ji}(\mathbf{x},\mathbf{y}_r)\,\varphi_{nq}(\mathbf{x})J(\mathbf{x})dS =$$
$$\int\limits_{\partial V_n} W_{ji}(\mathbf{x},\mathbf{y}_r)\,(\varphi_{nq}(\mathbf{x})J(\mathbf{x}) - \varphi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r))\,dS + \varphi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r)\int\limits_{\Pi_n} W_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS,$$

$$\int\limits_{\partial V_n} K_{ji}(\mathbf{x},\mathbf{y}_r)\,\psi_{nq}(\mathbf{x})J(\mathbf{x})dS =$$
$$\int\limits_{\partial V_n} K_{ji}(\mathbf{x},\mathbf{y}_r)\,(\psi_{nq}(\mathbf{x})J(\mathbf{x}) - \psi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r))\,dS + \psi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r)\int\limits_{\Pi_n} K_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS,$$

$$\int\limits_{\partial V_n} F_{ji}(\mathbf{x},\mathbf{y}_r)\,\varphi_{nq}(\mathbf{x})J(\mathbf{x})dS =$$
$$\int\limits_{\partial V_n} F_{ji}(\mathbf{x},\mathbf{y}_r)\,(\varphi_{nq}(\mathbf{x})J(\mathbf{x}) - (\varphi_{nq}(\mathbf{y}_r) - \partial_\tau\psi_{nq}(\mathbf{y}_r)\,(\mathbf{y}_r - \mathbf{x}))J(\mathbf{y}_r))\,dS +$$
$$+\,\partial_\tau\psi_{nq}(\mathbf{y}_r)\,J(\mathbf{y}_r)\int\limits_{\Pi_n}(\mathbf{y}_r - \mathbf{x})F_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS + \varphi_{nq}(\mathbf{y}_r)J(\mathbf{y}_r)\int\limits_{\Pi_n} W_{ji}(\pi(\mathbf{x}),\mathbf{y})\,dS \tag{7.10}$$

We consider here examples of piecewise constant and piecewise linear FE approximation, which are frequently used in the BEM.

8 Piecewise constant approximation

The piecewise constant approximation is the simplest one. Interpolation functions in this case do not depend on the FE form and dimension of the domain. They have the form

$$\varphi_q(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in S_n, \\ 0 & \forall \mathbf{x} \notin S_n. \end{cases} \tag{8.1}$$

In order to simplify situation we transform global system of coordinates such that the origin is located at the nodal point $\{x_1 = 0, x_2 = 0\}$, the coordinate axes x_1 and x_2 are located in the plane of the element, while the axis x_3 is perpendicular to that plane. In this case $x_3 = 0$ and $n_1 = 0 n_2 = 0 n_3 = 1$ and fundamental solutions have the following simple form

$$\begin{aligned}
U_{11}(\mathbf{x}-\mathbf{y}) &= \frac{1}{16\pi\mu(1-\upsilon)}\left(\frac{(3-4\upsilon)}{r} + \frac{x_1^2}{r^3}\right) \quad , \\
U_{22}(\mathbf{x}-\mathbf{y}) &= \frac{1}{16\pi\mu(1-\upsilon)}\left(\frac{(3-4\upsilon)}{r} + \frac{x_2^2}{r^3}\right), \\
U_{12}(\mathbf{x}-\mathbf{y}) &= \frac{1}{16\pi\mu(1-\upsilon)}\frac{x_1 x_2}{r^3}, \\
U_{33}(\mathbf{x}-\mathbf{y}) &= \frac{(3-4\upsilon)}{16\pi\mu(1-\upsilon)}\frac{1}{r} \\
W_{13}(\mathbf{x},\mathbf{y}) &= -K_{13}(\mathbf{x},\mathbf{y}) = -\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\frac{x_1}{r^3}, \\
W_{23}(\mathbf{x},\mathbf{y}) &= -K_{23}(\mathbf{x},\mathbf{y}) = -\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\frac{x_2}{r^3} \\
F_{11}(\mathbf{x},\mathbf{y}) &= \frac{\mu}{4\pi(1-\upsilon)}\left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon\frac{x_1^2}{r^5}\right], \\
F_{22}(\mathbf{x},\mathbf{y}) &= \frac{\mu}{4\pi(1-\upsilon)}\left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon\frac{x_2^2}{r^5}\right], \\
F_{12}(\mathbf{x},\mathbf{y}) &= \frac{\mu\upsilon}{4\pi(1-\upsilon)}\frac{x_1 x_2}{r^5}, \quad F_{33}(\mathbf{x},\mathbf{y}) = \frac{\mu}{4\pi(1-\upsilon)}\frac{1}{r^3}.
\end{aligned} \tag{8.2}$$

In order to calculate the divergent integrals in (6.20) with kernels of the type (8.2) the approach developed in Zozulya 2006a,b; Zozulya Gonzalez-Chi 1999; Zozulya, Lukin 1998; Zozulya, Men'shikov 2000 will be used. The approach is based of theory of generalized functions Gel'fand, Shilov 1964 and application of the Green theorems Courant, Hilbert 1968 and transformation of the divergent integrals into regular ones (see Zozulya 2006a,b for details). Regular representations for integrals with these kernels can be fond in above mentioned our publications. They have the form:

Weakly singular

$$J_1^{0,0} = W.S.\int_{S_n} \frac{dS}{r} = \int_{\partial S_n} \frac{r_n}{r} dl \quad ,$$

$$J_3^{2,0} = W.S.\int_{S_n} \frac{x_1^2}{r^3} dS = \frac{1}{3}\int_{\partial S_n} \left(\frac{x_1^2 r_n}{r^3} + \frac{2r_n}{r} - \frac{2x_1 n_1}{r} \right) dl,$$

$$J_3^{0,2} = W.S.\int_{S_n} \frac{x_2^2}{r^3} dS = \frac{1}{3}\int_{\partial S_n} \left(\frac{x_2^2 r_n}{r^3} + \frac{2r_n}{r} - \frac{2x_2 n_2}{r} \right) dl \tag{8.3}$$

$$J_3^{1,1} = W.S.\int_{S_n} \frac{x_1 x_2}{r^3} dS = \frac{1}{3}\int_{\partial S_n} \left(\frac{x_1 x_2 r_n}{r^3} - \frac{r_+}{r} \right) dl$$

Singular

$$J_3^{1,0} = P.V.\int_{S_n} \frac{x_1}{r^3} dS = \int_{\partial S_n} \left(\frac{x_1 r_n}{r^3} - \frac{n_1}{r} \right) dl, \tag{8.4}$$

$$J_3^{0,1} = P.V.\int_{S_n} \frac{x_2}{r^3} dS = \int_{\partial S_n} \left(\frac{x_2 r_n}{r^3} - \frac{n_2}{r} \right) dl$$

Hypersingular

$$J_3^{0,0} = F.P.\int_{S_n} \frac{dS}{r^3} = -\int_{\partial S_n} \frac{r_n}{r^3} dl,$$

$$J_5^{2,0} = F.P.\int_{S_n} \frac{x_1^2}{r^5} dS = \int_{\partial S_n} \left(\frac{x_1^2 r_n}{r^5} - \frac{2r_n}{3r^3} - \frac{2x_1 n_1}{3r^3} \right) dl \quad ,$$

$$J_5^{0,2} = F.P.\int\limits_{S_n} \frac{x_2^2}{r^5} dS = \int\limits_{\partial S_n} \left(\frac{x_2^2 r_n}{r^5} - \frac{2r_n}{3r^3} - \frac{2x_2 n_2}{3r^3} \right) dl \quad , \tag{8.5}$$

$$J_5^{1,1} = F.P.\int\limits_{S_n} \frac{x_1 x_2}{r^5} dS = \int\limits_{\partial S_n} \left(\frac{x_1 x_2 r_n}{r^5} - \frac{r_+}{3r^3} \right) dl,$$

where $r_n = x_\alpha n_\alpha$, $r_+ = x_1 n_2 + x_2 n_1$.

$$U_{ji}^n (\mathbf{y}_r, \mathbf{x}_q) = \int\limits_{S_n} U_{ji}(\mathbf{y}_r, \mathbf{x}) dS = \sum_{k=1}^{K} \int\limits_{l_k} U_{ji} (\mathbf{y}_r, \mathbf{x}) dl \quad ,$$

$$W_{ji}^n (\mathbf{y}_r, \mathbf{x}_q) = \int\limits_{S_n} W_{ji}(\mathbf{y}_r, \mathbf{x}) dS = \sum_{k=1}^{K} \int\limits_{l_k} W_{ji} (\mathbf{y}_r, \mathbf{x}) dl, \tag{8.6}$$

$$K_{ji}^n (\mathbf{y}_r, \mathbf{x}_q) = \int\limits_{S_n} K_{ji}(\mathbf{y}_r, \mathbf{x}) dS = \sum_{k=1}^{K} \int\limits_{l_k} K_{ji} (\mathbf{y}_r, \mathbf{x}) dl \quad ,$$

$$F_{ji}^n (\mathbf{y}_r, \mathbf{x}_q) = \int\limits_{S_n} F_{ji}(\mathbf{y}_r, \mathbf{x}) dS = \sum_{k=1}^{K} \int\limits_{l_k} F_{ji} (\mathbf{y}_r, \mathbf{x}) dl$$

Here indexes r and q indicate number of nodes.

Thus the divergent integrals in (6.20) with kernels (8.2) have been transformed into regular integrals (8.3)-(8.5) and may be easily calculated. For example regularization of the hypersingular integral $J_3^{0,0}$ for a circular area with the point ^{q} located in the center of circle leads to the following result

$$J_3^{0,0} = -\int\limits_{\partial S_n} \frac{r_n}{r^3} dl = -\frac{1}{r}\int\limits_0^{2\pi} d\varphi = -\frac{2\pi}{r} \tag{8.7}$$

Here polar coordinates are used, were r and φ are the circle radius and polar angle respectively.

In the application of the divergent integrals in the BEM, it is necessary to calculate the above integrals over any triangular, rectangular or polygonal elements. For that purpose these integrals must be transformed into a more convenient for calculation form.

Let us consider a polygonS_n with K vertexes as it is shown in Fig. 1. All the calculations will be done using the local rectangular coordinate system. Its origin is located in the point $x_1 = 0, x_2 = 0$, and the axes x_1 and x_2 are located in the plane of the polygon and the axis x_3 is perpendicular to this plane.

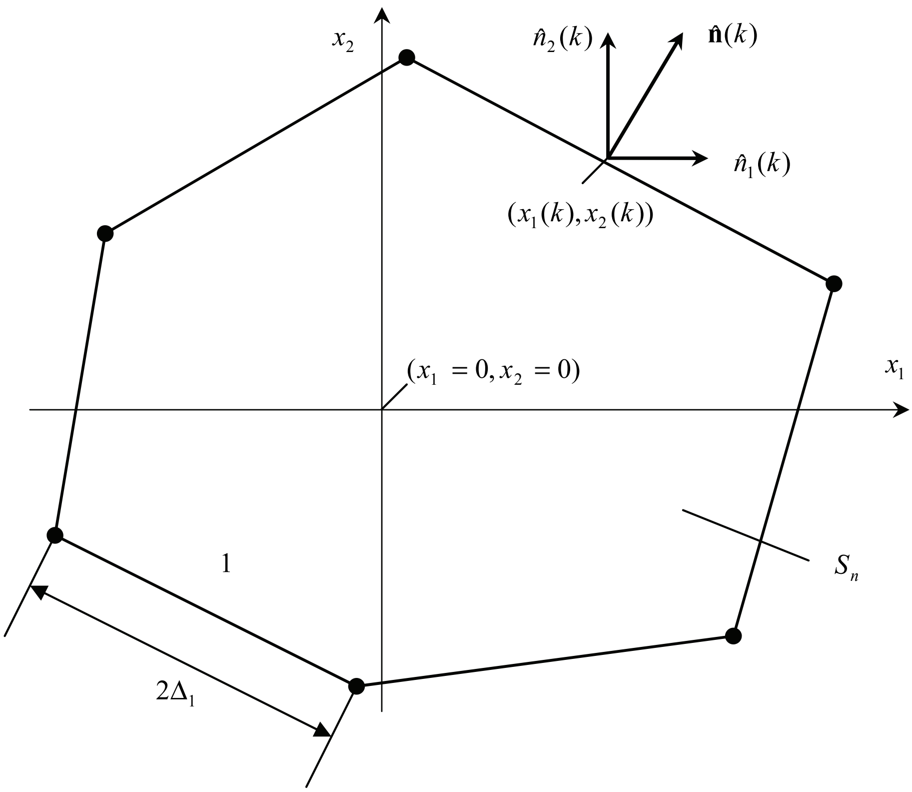

Figure 1:

Global coordinates of the vertexes of polygon are (x_1^k, x_2^k). The coordinates of an arbitrary point on the contour S_n may be represented in the form

$$x_1(\xi) = x_1(k) - \xi\Delta_k\hat{n}_2(k) \text{ and } x_2(\xi) = x_2(k) + \xi\Delta_k\hat{n}_1(k) \tag{8.8}$$

where $x_1(k)$ and $x_2(k)$ are the coordinates of the k-th side of the contour, $\hat{\mathbf{n}}(\hat{n}_1, \hat{n}_2)$ is a unit normal to the contour ∂S_n vector, $\xi \in [-1, 1]$ is a parameter of integration along the k-th side, and $2\Delta_k$ is the length of a k-th side.

Coordinates $x_1(k)$ and $x_2(k)$, unit vector $\mathbf{\hat{n}}(\hat{n}_1,\hat{n}_2)$ and length the contour ∂S_n can be calculate though the nodal points in the form

$$x_i(k)=\frac{x_i^{k+1}+x_i^k}{2}-y_1^q,\ \hat{n}_1(k)=\frac{x_2^{k+1}-x_2^k}{2\Delta_k}-y_2^q, \tag{8.9}$$
$$\hat{n}_2(k)=-\frac{x_1^{k+1}-x_1^k}{2\Delta_k},\ 2\Delta_k=\sqrt{(x_1^{k+1}-x_1^k)^2+(x_2^{k+1}-x_2^k)^2},$$

where y_1^q,y_2^q are coordinates of a collocation point.

These are some useful notations that will be used bellow

$$r(\xi)=\sqrt{\Delta_k^2\xi^2+2\xi\Delta_k r_-(k)+r^2(k)},\ r(k)=\sqrt{x_1^2(k)+x_2^2(k)},\ r_n(k)=x_\alpha(k)\hat{n}_\alpha(k),$$
$$r_+(k)=x_1(k)\hat{n}_2(k)+x_2(k)\hat{n}_1(k),\ r_n(\xi)=\ r_n(k)\ ,\ r_+(\xi)=r_+(k)+\xi\Delta_k(\hat{n}_1^2-\hat{n}_2^2). \tag{8.10}$$

Substituting (8.8)-(8.10) into (8.3)-(8.5) we obtain formulas for calculation of the corresponding integrals over each side of polygon in the form:

Weakly singular integrals

$$J_1^{0,0}(k)=\int\limits_{-1}^{1}\frac{r_n(k)}{r(\xi)}\Delta_k d\xi,$$

$$J_3^{2,0}(k)=\frac{1}{3}\int\limits_{-1}^{1}\left(\frac{r_n(k)}{r^3(\xi)}\left(x_1^2(k)-2\Delta_k\hat{n}_2(k)x_1(k)\xi+\Delta_k^2\hat{n}_2^2(k)\xi^2\right)\right.$$
$$\left.+\frac{2}{r(\xi)}\left(r_n(k)-x_1(k)\hat{n}_1(k)+\Delta_k\hat{n}_1(k)\hat{n}_2(k)\xi\right)\right)\Delta_k d\xi$$
$$J_3^{0,2}(k)=\frac{1}{3}\int\limits_{-1}^{1}\left(\frac{r_n(k)}{r^3(\xi)}\left(x_2^2(k)+2\Delta_k\hat{n}_1(k)x_2(k)\xi+\Delta_k^2\hat{n}_1^2(k)\xi^2\right)\right.$$
$$\left.+\frac{2}{r(\xi)}\left(r_n(k)-x_2(k)\hat{n}_2(k)-\Delta_k\hat{n}_1(k)\hat{n}_2(k)\xi\right)\right)\Delta_k d\xi$$
$$J_3^{1,1}(k)$$
$$=\frac{1}{3}\int\limits_{-1}^{1}\frac{r_n(k)}{r^3(\xi)}\left(x_1(k)x_2(k)+(\hat{n}_1(k)x_1(k)-\hat{n}_2(k)x_2(k))\Delta_k\xi-\ \hat{n}_1(k)\hat{n}_2(k)\Delta_k^2\xi^2\right)\Delta_k d\xi-$$

(8.11)

$$-\frac{1}{3}\int_{-1}^{1}\frac{r_+(k)+\Delta_k(\hat{n}_1^2(k)-\hat{n}_1^2(k))\xi}{r(\xi)}\Delta_k d\xi$$

Singular integrals

$$J_3^{1,0}(k)=\frac{1}{2}\int_{-1}^{1}\left(\frac{r_n(k)}{r^3(\xi)}\left(x_1(k)-\Delta_k\hat{n}_2(k)\xi\right)-\frac{\hat{n}_1(k)}{r(\xi)}\right)\Delta_k d\xi$$

$$J_3^{0,1}(k)=\frac{1}{2}\int_{-1}^{1}\left(\frac{r_n(k)}{r^3(\xi)}\left(x_2(k)+\Delta_k\hat{n}_1(k)\xi\right)-\frac{\hat{n}_2(k)}{r(\xi)}\right)\Delta_k d\xi \qquad (8.12)$$

Hypersingular integrals

$$J_3^{0,0}(k)=-\Delta_k\int_{-1}^{1}\frac{r_n(k)}{r^3(\xi)}d\xi$$

$$J_5^{2,0}(k)=\int_{-1}^{1}\frac{r_n(k)}{r^5(\xi)}\left(x_1^2(k)-2\Delta_k\hat{n}_2(k)x_1(k)\xi+\right.$$

$$\left.\Delta_k^2\hat{n}_2^2(k)^2\xi^2-\frac{2}{3r^3(\xi)}\left(r_n(k)+x_1(k)\hat{n}_1(k)-\Delta_k\hat{n}_1^2(k)\xi\right)\right)\Delta_k d\xi$$

$$J_5^{0,2}(k)=\int_{-1}^{1}\frac{r_n(k)}{r^5(\xi)}\left(x_2^2(k)+2\Delta_k\hat{n}_1(k)x_2(k)\xi+\right.$$

$$\left.\Delta_k^2\hat{n}_1^2(k)^2\xi^2-\frac{2}{3r^3(\xi)}\left(r_n(k)+x_2(k)\hat{n}_2(k)+\Delta_k\hat{n}_2^2(k)\xi\right)\right)\Delta_k d\xi$$

$$J_5^{1,1}(k)=\int_{-1}^{1}\frac{r_n(k)}{r^5(\xi)}\left(x_1(k)x_2(k)+(\hat{n}_1(k)x_1(k)-\hat{n}_2(k)x_2(k))\Delta_k\xi-\right. \qquad (8.13)$$

$$\hat{n}_1(k)\hat{n}_2(k)\Delta_k^2\xi^2\big)\,\Delta_k d\xi -$$

$$-\frac{1}{3}\int\limits_{-1}^{1}\frac{r_+(k)+\Delta_k(\hat{n}_1^2(k)-\hat{n}_1^2(k))\xi}{r^3(\xi)}\Delta_k d\xi$$

These formulae may be represented in more convenient for calculation form:

Weakly singular

$$J_1^{0,0}=\sum_{k=1}^{K} r_n(k)I_{1,0},$$

$$J_3^{2,0}=\frac{1}{3}\sum_{k=1}^{K}\big((x_1^2(k)I_{3,0}-2\hat{n}_2(k)x_1(k)I_{3,1}+\hat{n}_2^2(k)I_{3,2})r_n(k)$$

$$+2((r_n(k)-x_1(k)\hat{n}_1(k))I_{1,0}+\hat{n}_1(k)\hat{n}_2(k)I_{1,1}))$$

$$J_3^{0,2}=\frac{1}{3}\sum_{k=1}^{K}\big((x_2^2(k)I_{3,0}+2\hat{n}_1(k)x_2(k)I_{3,1}+\hat{n}_1^2(k)I_{3,2})r_n(k)$$

$$+2((r_n(k)+x_2(k)\hat{n}_2(k))I_{1,0}-\hat{n}_1(k)\hat{n}_2(k)I_{1,1}))$$

$$J_3^{1,1}=\frac{1}{3}\sum_{k=1}^{Q} r_n(k)\,(x_1(k)x_2(k)I_{3,0}+(\hat{n}_1(k)x_1(k)-\hat{n}_2(k)x_2(k))I_{3,1}-\hat{n}_1(k)\hat{n}_2(k)I_{3,2})- \tag{8.14}$$

$$-\big(r_+(k)I_{1,0}+(\hat{n}_1^2(k)-\hat{n}_1^2(k))I_{1,1}\big)$$

Singular

$$J_3^{1,0}=\frac{1}{2}\sum_{k=1}^{K}(r_n(k)\,(x_1(k)I_{3,0}-\hat{n}_2(k)I_{3,1})\,I_{3,1}-\hat{n}_1(k)I_{1,0})$$

$$J_3^{0,1}=\frac{1}{2}\sum_{k=1}^{K}(r_n(k)\,(x_2(k)I_{3,0}+\hat{n}_1(k)I_{3,1})\,I_{3,1}-\hat{n}_2(k)I_{1,0}) \tag{8.15}$$

Hypersingular

$$J_3^{0,0} = -\sum_{k=1}^{K} r_n(k) I_{3,0}$$

$$J_5^{2,0} = \sum_{k=1}^{K} \big((x_1^2(k) I_{5,0} - 2\hat{n}_2(k) x_1(k) I_{5,1} + \hat{n}_2^2(k) I_{5,2}) r_n(k)$$

$$-\frac{2}{3}((r_n(k) + x_2(k)\hat{n}_2(k)) I_{3,0} + \hat{n}_1(k)\hat{n}_2(k) I_{3,1})\Big)$$

$$J_5^{0,2} = \sum_{k=1}^{K} \big((x_2^2(k) I_{5,0} + 2\hat{n}_1(k) x_2(k) I_{5,1} + \hat{n}_1^2(k) I_{5,2}) r_n(k)$$

$$-\frac{2}{3}((r_n(k) + x_1(k)\hat{n}_1(k)) I_{3,0} - \hat{n}_1(k)\hat{n}_2(k) I_{3,1})\Big)$$

$$J_5^{1,1} = \sum_{k=1}^{Q} r_n(k) \left(x_1(k) x_2(k) I_{5,0} + (\hat{n}_1(k) x_1(k) - \hat{n}_2(k) x_2(k)) I_{5,1} - \hat{n}_1(k)\hat{n}_2(k) I_{5,2}\right) - \tag{8.16}$$

$$-\frac{1}{3}\left(r_+(k) I_{3,0} + (\hat{n}_1^2(k) - \hat{n}_1^2(k)) I_{3,1}\right)$$

Here we use the following notation for the integrals presented in (8.11)-(8.13)

$$I_{p,l} = (\Delta_k)^{l+1} \int_{-1}^{1} \frac{\xi^l}{r^p(\xi)} d\xi \tag{8.17}$$

The integrals in (8.17) may be calculated analytically. Formulae for their calculation are presented bellow

$$I_{1,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)} d\xi = \ln|r_-(k) + \Delta_k \xi + r(\xi)||_{-1}^{1},$$

$$I_{1,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(t)} d\xi = r(\xi)|_{-1}^{1} - r_=(k) I_{1,0},$$

$$I_{1,2} = \Delta_k \int_{-1}^{1} \frac{\xi^2}{r(\xi)} d\xi$$

$$= \frac{1}{2\Delta_k} \left(r(\xi)(\Delta_k\xi - r_-(k)) - (r^2(k) - 3r_-^2(k)) \ln|r_-(k) + \Delta_k\xi + r(\xi)| \right)\Big|_{-1}^{1}$$

$$I_{3,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)^3} d\xi = \left. \frac{\Delta_k\xi + r_-(k)}{(r^2(k) - r_-^2(k))r(\xi)} \right|_{-1}^{1},$$

$$I_{3,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(\xi)^3} d\xi = -\left. \frac{r_-(k)\Delta_k\xi + r^2(k)}{(r^2(k) - r_-^2(k))r(\xi)} \right|_{-1}^{1},$$

$$I_{3,2} = (\Delta_k)^3 \int_{-1}^{1} \frac{\xi^2}{r(\xi)^3} d\xi = I_{1,0} - r_-(k)I_{3,1} - \left. \frac{\Delta_k\xi}{r(\xi)} \right|_{-1}^{1},$$

$$I_{3,3} = (\Delta_k)^4 \int_{-1}^{1} \frac{\xi^3}{r(\xi)^3} d\xi$$

$$= -2\Delta_k\xi r_-(k)I_{3,1} - 3r_-(k)I_{3,2} + \left. \left(\frac{2r^2(k)r(\xi)}{r^2(k) - r_-^2(k)} - \frac{\Delta_k^2\xi^2}{r(\xi)} \right) \right|_{-1}^{1}$$

$$I_{5,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)^5} d\xi = \left. \frac{(\Delta_k\xi + r_-(k))(3r^2(k) + 2\Delta_k^2\xi^2 + 4\Delta_k\xi\, r_-(k) - r_-^2(k))}{3(r^2(k) - r_-^2(k)^2 r(\xi)^3} \right|_{-1}^{1},$$

$$I_{5,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(\xi)^5} d\xi$$

$$= -\left. \frac{r^4(k) + (3r^2(k)\Delta_k\xi + 2\Delta_k^3\xi^3)r_-(k) + (r^2(k) + 6\Delta_k^2\xi^2)r_-^2(k) + 3\Delta_k\xi\, r_-^3(k)}{3(r^2(k) - r_-^2(k)^2 r(\xi)^3} \right|_{-1}^{1},$$

$$I_{5,2} = (\Delta_k)^3 \int_{-1}^{1} \frac{\xi^2}{r(\xi)^5} d\xi \tag{8.18}$$

$$= \left. \frac{\begin{array}{c} r^2(k)\Delta_k^3\xi^3 + (2r^4(k) + 3r^2(k)\Delta_k^2\xi^2)r_-(k) \\ +(6r^2(k)\Delta_k\xi + \Delta_k^3\xi^3)r_-^2(k) + 3\Delta_k^2\xi^2 r_-^3(k) \end{array}}{3(r^2(k) - r_-^2(k)^2 r(\xi)^3} \right|_{-1}^{1}$$

$$I_{5,3} = (\Delta_k)^4 \int_{-1}^{1} \frac{\xi^3}{r(\xi)^5} d\xi = \frac{2r^2(k)r(\xi)^2 + r^2(k)\Delta_k^2\xi^2 + 3\Delta_k^2\xi^2 r_+^2(k)}{3\Delta_k r(\xi)^2} I_{3,1}|_{-1}^{1}$$

It is important to mention that above formulas can be applied for calculation normal nonsingular integrals. Obtained formulas valid for any collocation point situated inside or outside of the BE. Only for points situated at the vertexes of the boundary element special consideration is needed. It will be done in next section.

In the Tables 1 and 2 are presented results of the divergent and regular integrals calculations for the square and triangle of a unit side respectively.

Using these representations the integrals in (6.20) may be represented in a convenient form for the calculation.

In order to check validation of the above regularized equations we compare results for hypersingular integrals with the ones reported by Ioakimidis, 1982 and for weakly singular and regular integrals with results obtained using regular 2-D numerical calculation. Our calculations show that results of calculation obtained using presented here regularized equations agree with ones obtained by other methods. Also it is important to mention that there are two possibilities for calculation integrals in regular representations (8.11)-(8.16): the first one is to calculate corresponding integrals using formulas (8.17) and numerical integration and the second one is to calculate corresponding integrals using analytical formulas (8.18). Our calculations show that with analytical formulas (8.17) results are more accurate and time of calculation is 5-7 times faster in comparison with numerical formulas (8.17) and 8-12 times faster then obtained with 2-D numerical integration.

Now divergent integrals (8.6) with divergent kernels (8.2) may be represented though regular contour integrals in the form

$$U_{11}^n(\mathbf{y}_r, \mathbf{x}_q) = \frac{1}{16\pi\mu(1-\upsilon)} \left((3-4\upsilon)J_1^{0,0} + J_3^{2,0} \right),$$

$$U_{22}^n(\mathbf{y}_r, \mathbf{x}_q) = \frac{1}{16\pi\mu(1-\upsilon)} \left((3-4\upsilon)J_1^{0,0} + J_3^{0,2} \right),$$

Table 1: Divergent integrals calculated for unit square at collocation points: 1 -$y_1^0 = 0.0 y_2^0 = 0.0$, 2-$y_1^0 = 0.0 y_2^0 = 5.0$

Points	$J_1^{0,0}$	$J_3^{2,0}$	$J_3^{0,2}$	$J_3^{1,1}$	$J_3^{1,0}$	$J_3^{0,1}$	$J_3^{0,0}$	$J_5^{2,0}$	$J_5^{0,2}$	$J_5^{1,1}$
1	3.525	1.762	1.762	0.0	0.0	0.0	-11.31	-5.656	-5.656	0.0
2.	0.200	0.001	0.199	0.0	0.0	-0.040	0.008	0.000	0.008	0.0

Table 2: Divergent integrals calculated for unit triangle at collocation points: 1 -$y_1^0 = 0.0 y_2^0 = 0.0$, 2-$y_1^0 = 0.0 y_2^0 = 5.0$

Points	$J_1^{0,0}$	$J_3^{2,0}$	$J_3^{0,2}$	$J_3^{1,1}$	$J_3^{1,0}$	$J_3^{0,1}$	$J_3^{0,0}$	$J_5^{2,0}$	$J_5^{0,2}$	$J_5^{1,1}$
1	2.281	1.141	1.141	0.0	0.0	0.0	-18.0	-9.0	-9.0	0.0
2	0.086	0.0001	0.086	0.0	0.0	0.034	0.003	0.0	0.003	0.0

$$U_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}J_3^{1,1},\quad U_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{(3-4\upsilon)}{16\pi\mu(1-\upsilon)}J_1^{0,0}, \tag{8.19}$$

$$W_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}J_3^{1,0},$$

$$W_{23}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{23}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}J_3^{0,1},$$

$$F_{11}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)J_3^{0,0}+3\upsilon J_5^{2,0}\right],$$

$$F_{22}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)J_3^{0,0}+3\upsilon J_5^{0,2}\right],$$

$$F_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{\mu}{4\pi(1-\upsilon)}J_3^{0,0},$$

$$F_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}J_5^{1,1}.$$

All calculations here can be done analytically, no numerical integration is needed.

Singularities of the fundamental solutions in elastostatics and elastodynamics are the same, therefore obtained formulas can be easy applied for regularization of the divergent integrals in elastodynamics. For example following Zozulya, Men'shikov 2000 hypersingular fundamental solution in frequency domain can be presented in the form

$$F_{11}(\mathbf{y},\mathbf{x},\omega)=\frac{\mu}{4\pi(1-\upsilon)}\left((1-2\upsilon)\frac{1}{r^3}+3\upsilon\frac{(y_1-x_1)^2}{r^5}\right)+\frac{\mu\omega^2}{8\pi}\left(\frac{1}{c_2^2}+\frac{c_2^2}{c_1^4}\right)\frac{1}{r}$$
$$-\frac{\mu\omega^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_1-x_1)^2}{r^3}-$$
$$-\frac{\mu}{2\pi}\frac{1}{r^3}\sum_{n=3}^{\infty}(-i\omega r)^n\frac{(n-1)}{n!(n+1)}\left[\frac{n}{c_2^n}+2\frac{c_2^2}{c_1^{2+n}}\right]$$
$$-\frac{\mu}{4\pi}\frac{(y_1-x_1)^2}{r^5}\sum_{n=4}^{\infty}(-i\omega r)^n\frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n}+4\frac{c_2^2}{c_1^{2+n}}\right],$$

$$F_{12}(\mathbf{y},\mathbf{x},\omega)=F_{21}(\bar{x},\bar{y},\omega_m)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}\frac{(y_1-x_1)(y_2-x_2)}{r^5}$$
$$-\frac{\mu\omega^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_1-x_1)(y_2-x_2)}{r^3}-$$
$$-\frac{\mu}{4\pi}\frac{(y_1-x_1)(y_2-x_2)}{r^5}\sum_{n=4}^{\infty}(-i\omega r)^n\frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n}+4\frac{c_2^2}{c_1^{2+n}}\right],$$

$$F_{22}(\mathbf{y},\mathbf{x},\omega)=\frac{\mu}{4\pi(1-\upsilon)}\left((1-2\upsilon)\frac{1}{r^3}+3\upsilon\frac{(y_2-x_2)^2}{r^5}\right)+\frac{\mu\omega^2}{8\pi}\left(\frac{1}{c_2^2}+\frac{c_2^2}{c_1^4}\right)\frac{1}{r}$$
$$-\frac{\mu\omega^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_2-x_2)^2}{r^3}-\frac{\mu}{2\pi}\frac{1}{r^3}\sum_{n=3}^{\infty}(-i\omega r)^n\frac{(n-1)}{n!(n+2)}\left[\frac{n}{c_2^n}+2\frac{c_2^2}{c_1^{2+n}}\right]$$
$$-\frac{\mu}{4\pi}\frac{(y_2-x_2)^2}{r^5}\sum_{n=4}^{\infty}(-i\omega r)^n\frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n}+4\frac{c_2^2}{c_1^{2+n}}\right],\qquad(8.20)$$

$$F_{33}(\mathbf{y},\mathbf{x},\omega)=\frac{\mu}{4\pi\,(1-\nu)}\frac{1}{r^3}+\frac{\omega^2}{8\pi\mu}\left(\frac{\mu^2}{c_2^2}+\left(2\lambda^2+4\lambda\mu+3\mu^2\right)\frac{c_2^2}{c_1^4}\right)\frac{1}{r}-$$

$$-\sum_{n=3}^{\infty}\frac{(-i\omega r)^n}{r^3 4\pi\mu}\frac{(n-1)}{n!(n+2)}\left(\frac{4\mu^2(n-1)}{c_2^n}+\left[\lambda^2 n(n+2)+4\lambda\mu\,(n+2)+12\mu^2\right]\frac{c_2^2}{c_1^{2+n}}\right),$$

$$F_{13}(\mathbf{y},\mathbf{x},\omega)=F_{31}(\mathbf{y},\mathbf{x},\omega)=F_{23}(\mathbf{y},\mathbf{x},\omega)=F_{32}(\mathbf{y},\mathbf{x},\omega)=0\,.$$

First two members on the right had side are divergent, they can be easy calculated using above formulas.

9 Piecewise linear approximation

In order to calculate the divergent integrals in (6.20) with kernels of the type (8.2) for piecewise linear approximation the above mentioned approach developed in Zozulya 2006a,b,2008; Zozulya, Gonzalez-Chi 1999, 2000; Zozulya, Lukin 1998; Zozulya, Men'shikov 2000 will be used. In this case integrals in (6.20) have be calculated for each shape functions over appropriate BE. Because of after regularization calculation of the integrals over BE is replaced by calculation over each side of the BE finally we have to calculate the sum of integrals of the following type

$$U_{ji}^n(\mathbf{y}_r,\mathbf{x}_q)=\int_{S_n}U_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS=\sum_{k=1}^{K}\int_{l_k}U_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dl$$

$$W_{ji}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\int\limits_{S_n}W_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS=\sum_{k=1}^{K}\int\limits_{l_k}W_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dl$$

$$K_{ji}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\int\limits_{S_n}K_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS=\sum_{k=1}^{K}\int\limits_{l_k}K_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dl \qquad (9.1)$$

$$F_{ji}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\int\limits_{S_n}F_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS=\sum_{k=1}^{K}\int\limits_{l_k}F_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dl$$

Regular representations for the integrals with the kernels (8.2) can be fond in above mentioned our publications. They have the form:

Weakly singular

$$J_{q,1}^{0,0}=W.S.\int\limits_{S_n}\frac{\varphi_q(\xi)}{r}dS=\int\limits_{\partial S_n}[\varphi_q(\xi)\frac{r_n}{r}-r\partial_n\varphi_q(\boldsymbol{\xi})]dl$$

$$J_{q,3}^{2,0}=W.S.\int\limits_{S_n}\varphi_q(\xi)\frac{x_1^2}{r^3}dS$$

$$=\frac{1}{3}\int\limits_{\partial S_n}\left(\varphi_q(\boldsymbol{\xi})\left(\frac{x_1^2 r_n}{r^3}+\frac{2r_n}{r}-\frac{2x_1n_1}{r}\right)+\left(\frac{x_1^2}{r}-2r\right)\partial_n\varphi_q(\boldsymbol{\xi})\right)dl,$$

$$J_{q,3}^{0,2}=W.S.\int\limits_{S_n}\varphi_q(\xi)\frac{x_2^2}{r^3}dS$$

$$=\frac{1}{3}\int\limits_{\partial S_n}\left(\varphi_q(\boldsymbol{\xi})\left(\frac{x_2^2 r_n}{r^3}+\frac{2r_n}{r}-\frac{2x_2n_2}{r}\right)+\left(\frac{x_2^2}{r}-2r\right)\partial_n\varphi_q(\boldsymbol{\xi})\right)dl \qquad (9.2)$$

$$J_{q,3}^{1,1}=W.S.\int\limits_{S_n}\varphi_q(\boldsymbol{\xi})\frac{x_1x_2}{r^3}dS=\frac{1}{3}\int\limits_{\partial S_n}\left(\varphi_q(\boldsymbol{\xi})\left(\frac{x_1x_2r_n}{r^3}-\frac{r_+}{r}\right)+\frac{x_1x_2}{r}\partial_n\varphi_q(\boldsymbol{\xi})\right)dl$$

Singular

$$J_{q,3}^{1,0}=P.V.\int\limits_{S_n}\frac{\varphi_q(\boldsymbol{\xi})x_1}{r^3}dS=\int\limits_{\partial S_n}\left(\left(\frac{x_1r_n}{r^3}-\frac{n_1}{r}\right)+\frac{x_1}{r}\partial_n\varphi_q(\boldsymbol{\xi})\right)dl,$$

$$J_{q,3}^{0,1} = P.V.\int\limits_{S_n} \frac{\varphi_q(\boldsymbol{\xi})x_2}{r^3} dS = \int\limits_{\partial S_n} \left(\left(\frac{x_2 r_n}{r^3} - \frac{n_2}{r} \right) + \frac{x_2}{r} \partial_n \varphi_q(\boldsymbol{\xi}) \right) dl \quad . \tag{9.3}$$

Hypersingular

$$J_{q,3}^{0,0} = F.P.\int\limits_{S_n} \frac{\varphi_q(\xi)}{r^3} dS = -\int\limits_{\partial S_n} \left(\varphi_q(\xi) \frac{r_n}{r^3} + \frac{1}{r} \partial_n \varphi_q(\boldsymbol{\xi}) \right) dl,$$

$$J_{q,5}^{2,0} = F.P.\int\limits_{S_n} \varphi_q(\xi) \frac{x_1^2}{r^5} dS$$

$$= \int\limits_{\partial S_n} \left(\varphi_q(\boldsymbol{\xi}) \left(\frac{x_1^2 r_n}{r^5} - \frac{2r_n}{3r^3} - \frac{2x_1 n_1}{3r^3} \right) + \left(\frac{x_1^2}{3r^3} - \frac{2}{3r} \right) \partial_n \varphi_q(\boldsymbol{\xi}) \right) dl \quad ,$$

$$J_{q,5}^{0,2} = F.P.\int\limits_{S_n} \varphi_q(\xi) \frac{x_2^2}{r^5} dS$$

$$= \int\limits_{\partial S_n} \left(\varphi_q(\boldsymbol{\xi}) \left(\frac{x_2^2 r_n}{r^5} - \frac{2r_n}{3r^3} - \frac{2x_2 n_2}{3r^3} \right) + \left(\frac{x_2^2}{3r^3} - \frac{2}{3r} \right) \partial_n \varphi_q(\boldsymbol{\xi}) \right) dl,$$

$$J_{q,5}^{1,1} = F.P.\int\limits_{S_n} \varphi_q(\boldsymbol{\xi}) \frac{x_1 x_2}{r^5} dS = \int\limits_{\partial S_n} \left(\varphi_q(\xi) \left(\frac{x_1 x_2 r_n}{r^5} - \frac{r_+}{3r^3} \right) + \frac{x_1 x_2}{3r^3} \partial_n \varphi_q(\boldsymbol{\xi}) \right) dl. \tag{9.4}$$

Analysis of these expressions show that we have to calculate the sum of integrals of the following type

$$J_{q,p}^{l,m} = \int\limits_{\partial S_n} \varphi_q(\xi) \frac{x_1^l(\boldsymbol{\xi}) x_2^m(\boldsymbol{\xi})}{r^p(\xi)} dl(\boldsymbol{\xi}) = \sum_{k=1}^{K} \int\limits_{l_k} \varphi_q(\boldsymbol{\xi}) \frac{x_1^l(\boldsymbol{\xi}) x_2^m(\xi)}{r^p(\boldsymbol{\xi})} dl(\boldsymbol{\xi}). \tag{9.5}$$

This formula will be used for calculation over the rectangular and triangular BE of the corresponding divergent integrals that arrives when the problems of elastostatics are solved by the BIE method.

9.1 Rectangular boundary elements

Let us consider rectangular BE that is shown in Fig. 2. In order to simplify situation we transform global system of coordinates in such way that the origins of global and local systems of coordinates coincide. The coordinate axes x_1 and x_2 are located in the plane of the BE and coincide with the local ones ξ_1 and ξ_2, while the axis x_3 is perpendicular to that plane. In this case $x_3 = 0$ and $n_1 = 0$, $n_2 = 0$, $n_3 = 1$.

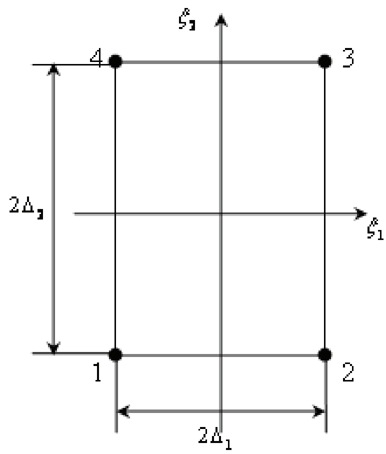

Figure 2:

Quadrilateral BE is defined by its angular nodes and its shape functions are

$$\varphi_1 = 1/4\,(1-\xi_1)\,(1-\xi_2)\,, \quad \varphi_2 = 1/4\,(1+\xi_1)\,(1-\xi_2)\,, \quad \xi_1 \in [0,1], \quad \xi_2 \in [0,1]$$

$$\varphi_3 = 1/4\,(1+\xi_1)\,(1+\xi_2)\,, \quad \varphi_4 = 1/4\,(1-\xi_1)\,(1+\xi_2) \quad . \tag{9.6}$$

Then global coordinates can be expressed as functions of local ones in the form

$$x_i(\xi_1,\xi_2) = \sum_{q=1}^{4} x_i^q \varphi_q(\xi_1,\xi_2) - y_i^q \quad . \tag{9.7}$$

Derivatives of the shape functions are

$$\frac{\partial \varphi_1(\boldsymbol{\xi})}{\partial \xi_1} = -1/4\,(1-\xi_2)\,, \quad \frac{\partial \varphi_1(\boldsymbol{\xi})}{\partial \xi_2} = -1/4\,(1-\xi_1)\,,$$

$$\frac{\partial\varphi_2(\boldsymbol{\xi})}{\partial\xi_1} = 1/4\,(1-\xi_2)\,, \quad \frac{\partial\varphi_2(\boldsymbol{\xi})}{\partial\xi_2} = -1/4\,(1+\xi_1)\,,$$
$$\frac{\partial\varphi_3(\boldsymbol{\xi})}{\partial\xi_1} = 1/4\,(1+\xi_2)\,, \quad \frac{\partial\varphi_3(\boldsymbol{\xi})}{\partial\xi_2} = 1/4\,(1+\xi_1)\,, \tag{9.8}$$
$$\frac{\partial\varphi_4(\boldsymbol{\xi})}{\partial\xi_1} = -1/4\,(1+\xi_2)\,, \quad \frac{\partial\varphi_4(\boldsymbol{\xi})}{\partial\xi_2} = 1/4\,(1-\xi_1)\,,$$
$$\partial_n\varphi_1(\boldsymbol{\xi}) = -1/4\,(1-\xi_2)\,\hat{n}_1 - 1/4\,(1-\xi_1)\,\hat{n}_2,$$
$$\partial_n\varphi_2(\boldsymbol{\xi}) = 1/4\,(1-\xi_2)\,\hat{n}_1 - 1/4\,(1+\xi_1)\,\hat{n}_2,$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = 1/4\,(1+\xi_2)\,\hat{n}_1 + 1/4\,(1+\xi_1)\,\hat{n}_2,$$
$$\partial_n\varphi_4(\boldsymbol{\xi}) = -1/4\,(1+\xi_2)\,\hat{n}_1 + 1/4\,(1-\xi_1)\,\hat{n}_2$$

Coordinates of the nodal points are
$(x_1^1 = -\Delta_1, x_2^1 = -\Delta_2)$, $(x_1^2 = \Delta_1, x_2^2 = -\Delta_2)$, $(x_1^3 = \Delta_1, x_2^3 = \Delta_2)$, $(x_1^4 = -\Delta_1, x_2^4 = \Delta_2)$.
We introduce here some more useful notations that will be used bellow

$$r(\boldsymbol{\xi}, \mathbf{y}^q) = \sqrt{x_1^2 + x_2^2} = \sqrt{(\Delta_1(1+\xi_1) - y_1^q)^2 + (\Delta_2(1+\xi_2) - y_2^q)^2} \quad ,$$
$$r_n = x_\alpha \hat{n}_\alpha, \quad r_+ = x_1\hat{n}_2 + x_2\hat{n}_1, \quad dl = \sqrt{\Delta_1^2 d\xi_1^2 + \Delta_2^2 d\xi_2^2}, \tag{9.9}$$
$$\hat{n}_1(k) = \frac{x_2^{k+1} - x_2^k}{\Delta_k}, \quad \hat{n}_2(k) = \frac{x_1^{k+1} - x_1^k}{\Delta_k}$$

Taking into account (9.5) calculation of divergent integrals will be done side by side using the formula

$$J_{q,p}^{l,m}(k) = \int\limits_{l_k} \varphi_q(\xi) \frac{x_1^l(\boldsymbol{\xi}) x_2^m(\boldsymbol{\xi})}{r^p(\xi)} dl(\boldsymbol{\xi}) \tag{9.10}$$

Details of the calculations are presented in the Appendix A. Final results of the calculations side by side are presented bellow.

Side 1-2. In this case the sums of the integrals (9.10) are

Weakly singular

$$J_{1,1}^{0,0}(1) = -\frac{\Delta_1}{3}, \quad J_{2,1}^{0,0}(1) = -\frac{2\Delta_1}{3}, \quad J_{3,1}^{0,0}(1) = \frac{2\Delta_1}{3}, \quad J_{4,1}^{0,0}(1) = \frac{\Delta_1}{3},$$

$$J_{1,3}^{2,0}(1) = -\frac{\Delta_1}{9}, \quad J_{2,3}^{2,0}(1) = -\frac{2\Delta_1}{9}, J_{3,3}^{2,0}(1) = \frac{2\Delta_1}{9}, \quad J_{4,3}^{2,0}(1) = \frac{\Delta_1}{9}, \tag{9.11}$$

$$J_{1,3}^{0,2}(1) = -\frac{2\Delta_1}{9}, \quad J_{2,3}^{0,2}(1) = -\frac{4\Delta_1}{9}, J_{3,3}^{0,2}(1) = \frac{4\Delta_1}{9}, \quad J_{4,3}^{0,2}(1) = \frac{2\Delta_1}{9},$$

$$J_{1,3}^{1,1}(1) = \frac{\Delta_1}{3}, \quad J_{2,3}^{1,1}(1) = \frac{\Delta_1}{3}, \quad J_{3,3}^{1,1}(1) = 0, J_{4,3}^{1,1}(1) = 0.$$

Singular

$$J_{1,3}^{1,0}(1) = \frac{1}{2}, \quad J_{2,3}^{1,0}(1) = \frac{1}{2}, J_{3,3}^{1,0}(1) = -\frac{1}{2}, \quad J_{4,3}^{1,0}(1) = -\frac{1}{2},$$

$$J_{1,3}^{0,1}(1) = 1, \quad J_{2,3}^{0,1}(1) = 1, \quad J_{3,3}^{0,1}(1) = 0, \quad J_{4,3}^{0,1}(1) = 0 \tag{9.12}$$

Hypersingular

$$J_{1,3}^{0,0}(1) = -\frac{1}{2\Delta_1}, J_{2,3}^{0,0}(1) = -\frac{1}{2\Delta_1}, \quad J_{3,3}^{0,0}(1) = \frac{1}{2\Delta_1}, \quad J_{4,3}^{0,0}(1) = \frac{1}{2\Delta_1},$$

$$J_{1,5}^{2,0}(1) = -\frac{1}{6\Delta_1}, \quad J_{2,5}^{2,0}(1) = -\frac{1}{6\Delta_1} J_{3,5}^{2,0}(1) = \frac{1}{6\Delta_1}, \quad J_{4,5}^{2,0}(1) = \frac{1}{6\Delta_1}, \tag{9.13}$$

$$J_{1,5}^{0,2}(1) = -\frac{1}{3\Delta_1}, J_{2,5}^{0,2}(1) = -\frac{1}{3\Delta_1}, J_{3,5}^{0,2}(1) = \frac{1}{3\Delta_1} J_{4,5}^{0,2}(1) = \frac{1}{3\Delta_1},$$

$$J_{1,5}^{1,1}(1) = -\frac{1}{12\Delta_1}, \quad J_{2,5}^{1,1}(1) = 0, \quad J_{3,5}^{1,1}(1) = 0, \quad J_{4,5}^{1,1}(1) = 0.$$

Side 2-3. In this case the sums of the integrals (9.10) are

Weakly singular

$$J_{1,1}^{0,0}(2) = \frac{1}{4}\left(I_{-1,0} - I_{-1,1}\right), J_{2,1}^{0,0}(2) = J_{3,1}^{0,0}(2) = \Delta_1\Delta_2\left(I_{1,0} - I_{1,1}\right) - \frac{1}{4}\left(I_{-1,0} - I_{-1,1}\right),$$

$$J_{3,1}^{0,0}(2) = \Delta_1\Delta_2\left(I_{1,0} + I_{1,1}\right) - \frac{1}{4}\left(I_{-1,0} + I_{-1,1}\right), J_{4,1}^{0,0}(2) = \frac{1}{4}\left(I_{-1,0} + I_{-1,1}\right),$$

$$J_{1,3}^{2,0}(2) = \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right) - \frac{1}{3}\Delta_1^2\left(I_{1,0} - I_{1,1}\right),$$

$$J_{2,3}^{2,0}(2) = \frac{4}{3}\Delta_1^3\Delta_2\left(I_{3,0} - I_{3,1}\right) - \frac{\Delta_1^2}{3}\left(I_{1,0} - I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right)$$

$$J_{3,3}^{2,0}(2) = \frac{4}{3}\Delta_1^3\Delta_2\left(I_{3,0} + I_{3,1}\right) - \frac{\Delta_1^2}{3}\left(I_{1,0} + I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right),$$

$$J_{4,3}^{2,0}(2) = \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right) - \frac{1}{3}\Delta_1^2\left(I_{1,0} + I_{1,1}\right),$$

$$J_{1,3}^{0,2}(2) = \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right) + \frac{\Delta_2^2}{12}\left(I_{1,0} + I_{1,1} - I_{1,2} - I_{1,3}\right), \tag{9.14}$$

$$J_{2,3}^{0,2}(2) = \frac{1}{3}\Delta_1\Delta_2^3\left(I_{3,0} + I_{3,1} - I_{3,2} - I_{3,3}\right) + \frac{1}{12}\Delta_2^2\left(I_{1,0} + I_{1,1} - I_{1,2} - I_{1,3}\right)$$

$$+\frac{2\Delta_1\Delta_2}{3}\left(I_{1,0} - I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right),$$

$$J_{3,4}^{0,2}(2) = \frac{1}{3}\Delta_1\Delta_2^3\left(I_{3,0} + 3I_{3,1} + 3I_{3,2} + I_{3,3}\right) + \frac{1}{12}\Delta_2^2\left(I_{1,0} + 3I_{1,1} + 3I_{1,2} + I_{1,3}\right)$$

$$+\frac{2\Delta_1\Delta_2}{3}\left(I_{1,0} + I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right)$$

$$J_{4,3}^{0,2}(2) = \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right) + \frac{\Delta_2^2}{12}\left(I_{1,0} + 3I_{1,1} + 3I_{1,2} + I_{1,3}\right),$$

$$J_{1,3}^{1,1}(2) = -\frac{1}{6}\Delta_1\Delta_2\left(I_{1,0} - I_{1,2}\right),$$

$$J_{2,3}^{1,1}(2) = \frac{\Delta_1^2\Delta_2^2}{3}\left(I_{3,0} - I_{3,2}\right) + \frac{\Delta_1\Delta_2 + \Delta_2^2}{6}\left(I_{1,0} - I_{1,2}\right),$$

$$J_{3,3}^{1,1}(2) = \frac{\Delta_1^2\Delta_2^2}{3}\left(I_{3,0} + 2I_{3,1} + I_{3,2}\right) + \frac{\Delta_1\Delta_2 + \Delta_2^2}{6}\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{4,3}^{1,1}(2) = \frac{1}{2}\Delta_1\Delta_2\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right).$$

Singular

$$J_{1,3}^{1,0}(2) = \frac{1}{2}\Delta_1\left(I_{1,0} - I_{1,1}\right), \quad J_{2,3}^{1,0}(2) = 2\Delta_1^2\Delta_2\left(I_{3,0} - I_{3,1}\right) - \frac{\Delta_1 - \Delta_2}{2}\left(I_{1,0} - I_{1,1}\right),$$

$$J_{3,3}^{1,0}(2) = 2\Delta_1^2\Delta_2\left(I_{3,0} + I_{3,1}\right) - \frac{\Delta_1 - \Delta_2}{2}\left(I_{1,0} + I_{1,1}\right), \tag{9.15}$$

$$J_{4,3}^{1,0}(2) = \frac{1}{2}\Delta_1\left(I_{1,0} + I_{1,1}\right),$$

$$J_{1,3}^{0,1}(2) = -\frac{1}{4}\Delta_2\left(I_{1,0} - I_{1,2}\right),$$

$$J_{2,3}^{0,1}(2) = \Delta_1\Delta_2^2\left(I_{3,0} - I_{3,2}\right) + \frac{\Delta_2}{4}\left(I_{1,0} - I_{1,2}\right),$$

$$J_{3,3}^{0,1}(2) = \Delta_1\Delta_2^2\left(I_{3,0} + 2I_{3,1} + I_{3,2}\right) + \frac{\Delta_2}{4}\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{4,3}^{0,1}(2) = -\frac{\Delta_2}{4}\left(I_{3,0} + 2I_{3,1} + I_{3,2}\right).$$

Hypersingular

$$J_{1,3}^{0,0}(3) = \frac{1}{4}\left(I_{1,0} - I_{1,1}\right), J_{2,3}^{0,0}(3) = -\Delta_1\Delta_2\left(I_{3,0} - I_{3,1}\right) - \frac{1}{4}\left(I_{1,0} - I_{1,1}\right),$$

$$J_{3,3}^{0,0}(3) = -\Delta_1\Delta_2\left(I_{3,0} + I_{3,1}\right) - \frac{1}{4}\left(I_{1,0} + I_{1,1}\right), J_{4,3}^{0,0}(3) = \frac{1}{4}\left(I_{1,0} + I_{1,1}\right), \tag{9.16}$$

$$J_{1,5}^{2,0}(3) = -\frac{1}{6}\left(I_{1,0} - I_{1,1}\right) + \frac{\Delta_1^2}{12}\left(I_{3,0} + I_{3,1} - I_{3,2} - I_{3,3}\right),$$

$$J_{2,5}^{2,0}(3) = -\frac{1}{6}\left(I_{1,0}+I_{1,1}\right)+\frac{\Delta_1^2}{12}\left(I_{3,0}+3I_{3,1}+3I_{3,2}+I_{3,3}\right)$$

$$J_{3,5}^{2,0}(3) = \frac{\Delta_1^3\Delta_2}{3}\left(I_{5,0}+3I_{5,1}+3I_{5,2}+I_{5,3}\right)+\frac{\Delta_1^2}{12}\left(I_{3,0}+3I_{3,1}+3I_{3,2}+I_{3,3}\right)$$

$$+\frac{3\Delta_1\Delta_2}{4}\left(I_{3,0}-I_{3,1}\right)-\frac{1}{6}\left(I_{1,0}-I_{1,1}\right)$$

$$J_{4,5}^{2,0}(2) = \frac{\Delta_1^3\Delta_2}{3}\left(I_{5,0}+I_{5,1}-I_{5,2}-I_{5,3}\right)+\frac{\Delta_2^2}{12}\left(I_{3,0}+I_{3,1}-I_{3,2}-I_{3,3}\right)$$

$$+\frac{3\Delta_1\Delta_2}{4}\left(I_{3,0}-I_{3,1}\right)-\frac{1}{6}\left(I_{1,0}-I_{1,1}\right),$$

$$J_{1,5}^{0,2}(3) = \frac{1}{6}\left(I_{1,0}-I_{1,1}\right)-\frac{\Delta_2^2}{3}\left(I_{3,0}-I_{3,1}\right),$$

$$J_{4,5}^{0,2}(3) = \frac{1}{6}\left(I_{1,0}+I_{1,1}\right)-\frac{\Delta_2^2}{3}\left(I_{3,0}+I_{3,1}\right),$$

$$J_{3,5}^{0,2}(3) = 4\Delta_1\Delta_2^3\left(I_{5,0}+I_{5,1}\right)-\frac{\Delta_2^2+4\Delta_1\Delta_2}{3}\left(I_{3,0}+I_{3,1}\right)-\frac{1}{4}\left(I_{1,0}+I_{1,1}\right),$$

$$J_{4,5}^{0,2}(3) = 4\Delta_1\Delta_2^2\left(I_{5,0}-I_{5,1}\right)-\frac{\Delta_2^2+4\Delta_1\Delta_2}{3}\left(I_{3,0}-I_{3,1}\right)-\frac{1}{4}\left(I_{1,0}-I_{1,1}\right),$$

$$J_{1,5}^{1,1}(3) = -\frac{1}{6}\Delta_1\Delta_2\left(I_{3,0}-I_{3,2}\right),$$

$$J_{2,5}^{1,1}(3) = -\frac{1}{2}\Delta_1\Delta_2\left(I_{5,0}+2I_{5,1}+I_{5,2}\right),$$

$$J_{3,5}^{1,1}(2) = 2\Delta_1^2\Delta_2^2\left(I_{5,0}+2I_{5,1}+I_{5,2}\right)-\frac{\Delta_1\Delta_2+\Delta_2^2}{6}\left(I_{3,0}-I_{3,2}\right),$$

$$J_{2,5}^{1,1}(2) = 2\Delta_1^2\Delta_2^2\left(I_{5,0}-I_{5,1}\right)-\frac{\Delta_1\Delta_2+\Delta_2^2}{6}\left(I_{3,0}-I_{3,2}\right).$$

Side 3-4. In this case the sums of the integrals (9.10) are

Weakly singular

$$J_{1,1}^{0,0}(3) = \frac{1}{4}\left(I_{-1,0} - I_{-1,1}\right), J_{2,1}^{0,0}(3) = \frac{1}{4}\left(I_{-1,0} + I_{-1,1}\right),$$

$$J_{3,1}^{0,0}(3) = -\Delta_1\Delta_2\left(I_{1,0} + I_{1,1}\right) - \frac{1}{4}\left(I_{-1,0} + I_{-1,1}\right)$$

$$J_{4,1}^{0,0}(3) = \Delta_1\Delta_2\left(I_{1,0} - I_{1,1}\right) + \frac{1}{4}\left(I_{-1,0} - I_{-1,1}\right),$$

$$J_{1,3}^{2,0}(3) = \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right) + \frac{1}{12}\Delta_1^2\left(I_{1,0} + I_{1,1} - I_{1,2} - I_{1,3}\right),$$

$$J_{2,3}^{2,0}(3) = \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right) + \frac{1}{12}\Delta_1^2\left(I_{1,0} + 3I_{1,1} + 3I_{1,2} + I_{1,3}\right),$$

$$J_{3,3}^{0,2}(3) = \frac{1}{3}\Delta_1^3\Delta_2\left(I_{3,0} + 3I_{3,1} + 3I_{3,2} + I_{3,3}\right) + \frac{1}{12}\Delta_1^2\left(I_{1,0} + 3I_{1,1} + 3I_{1,2} + I_{1,3}\right)$$
$$+\frac{2\Delta_1\Delta_2}{3}\left(I_{1,0} + I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right)$$

$$J_{2,4}^{0,2}(3) = \frac{1}{3}\Delta_1^3\Delta_1^3\left(I_{3,0} + I_{3,1} - I_{3,2} - I_{3,3}\right) + \frac{1}{12}\Delta_1^2\left(I_{1,0} + I_{1,1} - I_{1,2} - I_{1,3}\right)$$
$$+\frac{2\Delta_1\Delta_2}{3}\left(I_{1,0} - I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right),$$

$$J_{1,3}^{0,2}(3) = \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right) - \frac{\Delta_2^2}{3}\left(I_{1,0} - I_{1,1}\right),$$

$$J_{3,3}^{0,2}(3) = \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right) - \frac{\Delta_2^2}{3}\left(I_{1,0} + I_{1,1}\right)$$

$$J_{3,4}^{0,2}(3) = \frac{4}{3}\Delta_1\Delta_2^3\left(I_{3,0} + I_{3,1}\right) - \frac{\Delta_2^2}{3}\left(I_{1,0} + I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} + I_{-1,1}\right),$$

$$J_{4,3}^{0,2}(3) = \frac{4}{3}\Delta_1\Delta_2^3\left(I_{3,0} - I_{3,1}\right) - \frac{\Delta_2^2}{3}\left(I_{1,0} - I_{1,1}\right) - \frac{1}{6}\left(I_{-1,0} - I_{-1,1}\right), \quad (9.17)$$

$$J_{1,3}^{1,1}(3) = -\frac{1}{6}\Delta_1\Delta_2\left(I_{1,0} - I_{1,2}\right),$$

$$J_{2,3}^{1,1}(3) = -\frac{1}{6}\Delta_1\Delta_2\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{3,3}^{1,1}(3) = \frac{2\Delta_1^2\Delta_2^2}{3}\left(I_{3,0} + 2I_{3,1} + I_{3,2}\right) + \frac{\Delta_1\Delta_2 - \Delta_1^2}{6}\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{4,3}^{1,1}(3) = \frac{2\Delta_1^2\Delta_2^2}{3}\left(I_{3,0} - I_{3,2}\right) - \frac{\Delta_1\Delta_2 - \Delta_1^2}{6}\left(I_{1,0} - I_{1,2}\right).$$

Singular

$$J_{1,3}^{1,0}(3) = -\frac{1}{4}\Delta_2\left(I_{1,0} - I_{1,2}\right),$$

$$J_{2,3}^{1,0}(3) = \frac{\Delta_2}{4}\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{3,3}^{1,0}(3) = \Delta_1^2\Delta_2\left(I_{3,0} + 2I_{3,1} + I_{3,2}\right) + \frac{\Delta_1}{4}\left(I_{1,0} + 2I_{1,1} + I_{1,2}\right),$$

$$J_{4,3}^{1,0}(3) = \Delta_1^2\Delta_2\left(I_{3,0} - I_{3,2}\right) + \frac{\Delta_1}{4}\left(I_{1,0} - I_{1,2}\right).$$

$$J_{1,3}^{0,1}(3) = -\frac{1}{2}\Delta_2\left(I_{1,0} - I_{1,1}\right), \quad J_{2,3}^{0,1}(3) = -\frac{1}{2}\Delta_2\left(I_{1,0} + I_{1,1}\right), \tag{9.18}$$

$$J_{3,3}^{0,1}(3) = 2\Delta_1\Delta_2^2\left(I_{3,0} + I_{3,1}\right) - \frac{\Delta_1 - \Delta_2}{2}\left(I_{1,0} + I_{1,1}\right),$$

$$J_{4,3}^{0,1}(3) = 2\Delta_1\Delta_2^2\left(I_{3,0} - I_{3,1}\right) - \frac{\Delta_1 - \Delta_2}{2}\left(I_{1,0} - I_{1,1}\right),$$

Hypersingular

$$J_{1,3}^{0,0}(3) = \frac{1}{4}\left(I_{1,0} - I_{1,1}\right), \; J_{2,3}^{0,0}(3) = \frac{1}{4}\left(I_{1,0} + I_{1,1}\right),$$

$$J_{3,3}^{0,0}(3) = -\Delta_1\Delta_2\left(I_{3,0}+I_{3,1}\right) - \frac{1}{4}\left(I_{1,0}+I_{1,1}\right),$$

$$J_{4,3}^{0,0}(3) = -\Delta_1\Delta_2\left(I_{3,0}-I_{3,1}\right) - \frac{1}{4}\left(I_{1,0}-I_{1,1}\right), \tag{9.19}$$

$$J_{1,5}^{2,0}(2) = \frac{1}{4}\left(I_{1,0}-I_{1,1}\right) + \Delta_1^2\left(I_{3,0}-I_{3,1}\right),$$

$$J_{2,5}^{2,0}(2) = 4\Delta_1^3\Delta_2\left(I_{5,0}-I_{5,1}\right) - \frac{\Delta_1^2+4\Delta_1\Delta_2}{3}\left(I_{3,0}-I_{3,1}\right) - \frac{1}{4}\left(I_{1,0}-I_{1,1}\right)$$

$$J_{3,5}^{2,0}(2) = 4\Delta_1^3\Delta_2\left(I_{5,0}+I_{5,1}\right) - \frac{\Delta_1^2+4\Delta_1\Delta_2}{3}\left(I_{3,0}+I_{3,1}\right) - \frac{1}{4}\left(I_{1,0}+I_{1,1}\right),$$

$$J_{4,5}^{2,0}(2) = \frac{1}{4}\left(I_{1,0}+I_{1,1}\right) + \Delta_1^2\left(I_{3,0}+I_{3,1}\right),$$

$$J_{1,5}^{0,2}(2) = \frac{1}{2}\left(I_{1,0}-I_{1,1}\right) - \Delta_2^2\left(I_{3,0}-I_{3,1}\right),$$

$$J_{2,5}^{0,2}(2) = 4\Delta_1^3\Delta_2\left(I_{5,0}-I_{5,1}\right) - \frac{3\Delta_2^2+\Delta_1\Delta_2}{3}\left(I_{3,0}-I_{3,1}\right) - \frac{1}{2}\left(I_{1,0}-I_{1,1}\right),$$

$$J_{3,5}^{0,2}(2) = 4\Delta_1^3\Delta_2\left(I_{5,0}+I_{5,1}\right) - \frac{3\Delta_1^2+\Delta_1\Delta_2}{3}\left(I_{3,0}+I_{3,1}\right) - \frac{1}{2}\left(I_{1,0}+I_{1,1}\right),$$

$$J_{4,5}^{0,2}(2) = \frac{1}{2}\left(I_{1,0}+I_{1,1}\right) - \Delta_1^2\left(I_{3,0}+I_{3,1}\right),$$

$$J_{1,5}^{1,1}(2) = \frac{1}{2}\Delta_1\Delta_2\left(I_{3,0}-I_{3,1}\right),$$

$$J_{2,5}^{1,1}(2) = 2\Delta_1^2\Delta_2^2\left(I_{5,0}-I_{5,1}\right) - \frac{3\Delta_1\Delta_2+\Delta_1^2}{6}\left(I_{3,0}-I_{3,1}\right),$$

$$J_{3,5}^{1,1}(2) = 2\Delta_1^2\Delta_2^2\left(I_{5,0}+I_{5,1}\right) - \frac{3\Delta_1\Delta_2+\Delta_1^2}{6}\left(I_{3,0}+I_{3,1}\right),$$

$$J_{4,5}^{1,1}(2) = \frac{1}{2}\Delta_1\Delta_2\left(I_{3,0}+I_{3,1}\right).$$

Side 4-1. In this case the sums of the integrals (9.10) are

Weakly singular

$$J_{1,1}^{0,0}(4) = -\frac{\Delta_2}{3}, \quad J_{2,1}^{0,0}(4) = \frac{\Delta_2}{3}, \quad J_{3,1}^{0,0}(4) = \frac{2\Delta_2}{3}, \quad J_{4,1}^{0,0}(4) = -\frac{2\Delta_2}{3},$$

$$J_{1,3}^{2,0}(4) = -\frac{2\Delta_2}{9}, \quad J_{2,3}^{2,0}(4) = \frac{2\Delta_2}{9}, J_{3,3}^{2,0}(4) = \frac{4\Delta_2}{9}, \quad J_{4,3}^{2,0}(4) = -\frac{4\Delta_2}{9}, \tag{9.20}$$

$$J_{1,3}^{0,2}(4) = -\frac{\Delta_2}{9}, \quad J_{2,3}^{0,2}(4) = \frac{\Delta_2}{9}, J_{3,3}^{0,2}(4) = \frac{2\Delta_2}{9}, \quad J_{4,3}^{0,2}(4) = -\frac{2\Delta_2}{9},$$

$$J_{1,3}^{1,1}(4) = \frac{\Delta_2}{3}, \quad J_{2,3}^{1,1}(4) = 0, \quad J_{3,3}^{1,1}(4) = 0, J_{4,3}^{1,1}(4) = \frac{\Delta_2}{3}.$$

Singular

$$J_{1,3}^{1,0}(4) = 1, \quad J_{2,3}^{1,0}(4) = 0, \quad J_{3,3}^{1,0}(4) = 0, \quad J_{4,3}^{1,0}(4) = 1.$$

$$J_{1,3}^{0,1}(4) = \frac{1}{2}, \quad J_{2,3}^{0,1}(4) = -\frac{1}{2}, \quad J_{3,3}^{0,1}(4) = -\frac{1}{2}, \quad J_{4,3}^{0,1}(4) = \frac{1}{2}. \tag{9.21}$$

Hypersingular

$$J_{1,3}^{0,0}(4) = -\frac{1}{2\Delta_2}, J_{2,3}^{0,0}(4) = \frac{1}{2\Delta_2}, J_{3,3}^{0,0}(4) = \frac{1}{2\Delta_2}, J_{4,3}^{0,0}(4) = -\frac{1}{2\Delta_2},$$

$$J_{1,5}^{2,0}(4) = -\frac{1}{3\Delta_2}, \quad J_{2,5}^{2,0}(4) = \frac{1}{3\Delta_2} J_{3,5}^{2,0}(4) = \frac{1}{3\Delta_2}, \quad J_{4,5}^{2,0}(4) = -\frac{1}{3\Delta_2}, \tag{9.22}$$

$$J_{1,5}^{0,2}(4) = -\frac{1}{6\Delta_2}, \quad J_{2,5}^{0,2}(4) = \frac{1}{6\Delta_2}, J_{3,5}^{0,2}(4) = \frac{1}{6\Delta_2}, \quad J_{4,5}^{0,2}(4) = -\frac{1}{6\Delta_2},$$

$$J_{5}^{1,1}(4) = -\frac{1}{12\Delta_2}, \quad J_{2,5}^{1,1}(4) = 0, \quad J_{3,5}^{1,1}(4) = 0, \quad J_{4,5}^{1,1}(4) = 0.$$

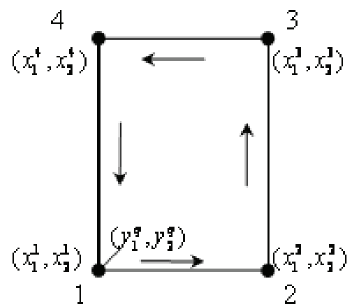

Figure 3:

We have taken into account that integration in (9.11) – (9.22) has to be done in the way as it is shown by arrows in Fig. 3. All integrals of the type $I_{p,l}$ in (9.14) - (9.19) are represented by formulae (8.18).

Finally sums of the integrals in (9.5) have the form

$$J_{q,p}^{l,m} = \sum_{k=1}^{4} J_{q,p}^{l,m}(k). \tag{9.23}$$

All integrals of the type $J_{q,p}^{l,m}(k)$have already calculated above.

In the Table 3 are presented results of the divergent and regular integrals calculations for the square of a unit side.

In order to check validation of the above regularized equations we compare results with the ones presented in the Table 1 and for weakly singular and regular integrals with results obtained using regular 2-D numerical calculation. Our calculations show that results of calculation obtained using presented here regularized equations agree with ones obtained by other methods. As in the case of piecewise constant approximation our calculations show that with analytical formulas (8.17) results are more accurate and time of calculation is 5-7 times faster in comparison with numerical formulas (8.17) and 8-12 times faster then obtained with 2-D numerical integration.

Substituting all obtained for each side results in (9.1) and take into account (9.23) finally we have

Table 3: Divergent integrals calculated for unit square at collocation points: 1 -$y_1^0 = -0.5 y_2^0 = -0.5$, 2-$y_1^0 = 0.0 y_2^0 = 0.0$, 3 -$y_1^0 = 1.0 y_1^0 = 1.0$.

Points	$J_{1,1}^{0,0}$	$J_{1,3}^{2,0}$	$J_{1,3}^{0,2}$	$J_{1,3}^{1,1}$	$J_{1,3}^{1,0}$	$J_{1,3}^{0,1}$	$J_{1,3}^{0,0}$	$J_{1,5}^{2,0}$	$J_{1,5}^{0,2}$	$J_{1,5}^{1,1}$
1	0.743	0.371	0.371	0.234	0.885	0.885	-1.065	-0.532	-0.532	-0.431
2	0.881	0.440	0.440	0.058	-0.881	-0.881	-2.828	-1.414	-1.414	0.471
3	0.153	0.076	0.076	0.072	-0.067	-0.067	0.062	0.031	0.031	0.029
Points	$J_{2,1}^{0,0}$	$J_{2,3}^{2,0}$	$J_{2,3}^{0,2}$	$J_{2,3}^{1,1}$	$J_{2,3}^{1,0}$	$J_{2,3}^{0,1}$	$J_{2,3}^{0,0}$	$J_{2,5}^{2,0}$	$J_{2,5}^{0,2}$	$J_{2,5}^{1,1}$
1	0.505	0.276	0.095	0.117	0.647	0.352	-0.467	0.217	-0.684	-0.049
2	0.881	0.440	0.440	-0.058	0.881	-0.881	-2.828	-1.414	-1.414	-0.471
3	0.177	0.061	0.115	0.079	-0.075	-0.103	0.098	0.035	0.063	0.044
Points	$J_{3,1}^{0,0}$	$J_{3,3}^{2,0}$	$J_{3,3}^{0,2}$	$J_{3,3}^{1,1}$	$J_{3,3}^{1,0}$	$J_{3,3}^{0,1}$	$J_{3,3}^{0,0}$	$J_{3,5}^{2,0}$	$J_{3,5}^{0,2}$	$J_{3,5}^{1,1}$
1	0.276	0.138	0.138	0.116	0.233	0.233	0.585	0.292	0.292	0.116
2	0.881	0.440	0.440	0.058	0.881	0.881	-2.828	-1.414	-1.414	0.471
3	0.217	0.108	0.108	0.101	-0.137	-0.137	0.098	0.092	0.092	0.088
Points	$J_{4,1}^{0,0}$	$J_{4,3}^{2,0}$	$J_{4,3}^{0,2}$	$J_{4,3}^{1,1}$	$J_{4,3}^{1,0}$	$J_{4,3}^{0,1}$	$J_{4,3}^{0,0}$	$J_{4,5}^{2,0}$	$J_{4,5}^{0,2}$	$J_{4,5}^{1,1}$
1	0.371	0.095	0.276	0.117	0.352	0.647	-0.467	-0.684	0.217	-0.049
2	0.881	0.440	0.440	-0.058	-0.881	0.881	-2.828	-1.414	-1.414	-0.471
4	0.177	0.115	0.061	0.079	-0.103	-0.753	0.098	0.063	0.035	0.044
Points	$J_{1}^{0,0}$	$J_{3}^{2,0}$	$J_{3}^{0,2}$	$J_{3}^{1,1}$	$J_{3}^{1,0}$	$J_{3}^{0,1}$	$J_{3}^{0,0}$	$J_{5}^{2,0}$	$J_{5}^{0,2}$	$J_{5}^{1,1}$
1	2.258	0.881	0.881	0.585	2.118	2.118	-1.414	-0.707	-0.707	-0.415
2	3.525	1.762	1.762	0.0	0.0	0.0	-11.31	-5.656	-5.656	0.0
4	0.724	0.362	0.362	0.333	-0.383	-0.383	0.445	0.222	0.222	0.206

$$U_{11}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}\left((3-4\upsilon)\sum_{k=1}^{4}J_{q,1}^{0,0}(k)+\sum_{k=1}^{4}J_{q,3}^{2,0}(k)\right)\quad ,$$

$$U_{22}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}\left((3-4\upsilon)\sum_{k=1}^{4}J_{q,1}^{0,0}(k)+\sum_{k=1}^{4}J_{q,3}^{0,2}(k)\right),$$

$$U_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}\sum_{k=1}^{4}J_{q,3}^{1,1}(k),$$

$$U_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{(3-4\upsilon)}{16\pi\mu(1-\upsilon)}\sum_{k=1}^{4}J_{q,1}^{0,0}(k),$$

$$W_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\sum_{k=1}^{4}J_{q,3}^{1,0}(k),$$

$$W_{23}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{23}^{n,q}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\sum_{k=1}^{4}J_{q,3}^{0,1}(k),$$

$$F_{11}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{4}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{4}J_{q,5}^{2,0}(k)\right],$$

$$F_{22}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{4}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{4}J_{q,5}^{0,2}(k)\right],\tag{9.24}$$

$$F_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{\mu}{4\pi(1-\upsilon)}\sum_{k=1}^{4}J_{q,3}^{0,0}(k),$$

$$F_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}\sum_{k=1}^{4}J_{q,5}^{1,1}(k).$$

It is important to mention that here all calculations can be done analytically, no numerical integration is needed.

9.2 Triangular boundary elements

Let us consider a triangular BE that is shown in Fig. 4. In order to simplify situation we transform global system of coordinates such that the origins of global and local systems of coordinates coincide. The coordinate axes x_1 and x_2 are located in the plane of the element, while the axis x_3 is perpendicular to that plane. In this case $x_3 = 0$ and $n_1 = 0$, $n_2 = 0$, $n_3 = 1$. The axes of local coordinates ξ_1 and ξ_2 coincide with sides of the triangular BE that joint in the nodal point 3.

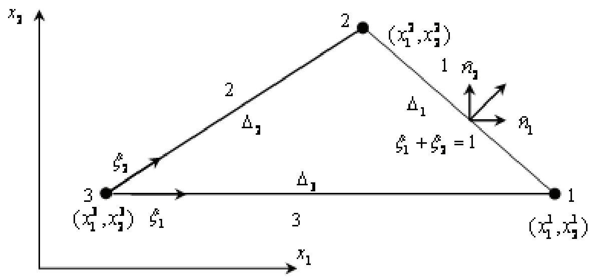

Figure 4:

The triangular BE is defined by its angular nodes and its shape functions are

$$\varphi_1(\xi_1,\xi_2) = \xi_1, \quad \varphi_2(\xi_1,\xi_2) = \xi_2, \quad \varphi_3(\xi_1,\xi_2) = (1-\xi_1-\xi_2)\,, \tag{9.25}$$

$\xi_1 \in [0,1], \quad \xi_2 \in [0,1]$

Then global coordinates can be expressed as functions of local ones in the form

$$x_i(\xi_1,\xi_2) = \sum_{q=1}^{3} x_i^q \varphi_q(\xi_1,\xi_2), \quad y_i(\xi_1,\xi_2) = \sum_{q=1}^{3} y_i^q \varphi_q(\xi_1,\xi_2), \tag{9.26}$$

or

$$x_1 = x_1^3 + \Delta x_1^1 \xi_1 - \Delta x_1^2 \xi_2, \quad x_2 = x_2^3 + \Delta x_2^1 \xi_1 - \Delta x_2^2 \xi_2, \tag{9.27}$$

where $\Delta x_i^1 = (x_i^1 - x_i^3)$ and $\Delta x_i^2 = -(x_i^2 - x_i^3)$.
Derivatives of the shape functions are

$$\frac{\partial \varphi_1(\boldsymbol{\xi})}{\partial \xi_1} = 1, \quad \frac{\partial \varphi_1(\boldsymbol{\xi})}{\partial \xi_2} = 0, \quad \frac{\partial \varphi_2(\boldsymbol{\xi})}{\partial \xi_1} = 0, \tag{9.28}$$

$$\frac{\partial \varphi_2(\boldsymbol{\xi})}{\partial \xi_2} = 1, \quad \frac{\partial \varphi_3(\boldsymbol{\xi})}{\partial \xi_1} = -1, \quad \frac{\partial \varphi_3(\boldsymbol{\xi})}{\partial \xi_2} = -1.$$

Taking into account that the coordinates ξ_1 and ξ_2 are oblique, normal derivative have to be calculated using the following formula

$$\partial_n = \hat{n}_1 \frac{\partial}{\partial x_1} + \hat{n}_2 \frac{\partial}{\partial x_2} = \left(\frac{\hat{n}_1 \Delta_2}{\Delta} \hat{n}_\alpha(2) + \frac{\hat{n}_2 \Delta_3}{\Delta} \hat{n}_\alpha(3) \right) \frac{\partial}{\partial \xi_\alpha} \quad , \tag{9.29}$$

where $\Delta = \Delta x_1^1 \Delta x_2^2 - \Delta x_1^2 \Delta x_2^1$.
Normal derivatives of the shape functions are

$$\partial_n \varphi_1(k) = \frac{\hat{n}_1(k)\hat{n}_1(2)\Delta_2}{\Delta} + \frac{\hat{n}_2(k)\hat{n}_1(3)\Delta_3}{\Delta},$$

$$\partial_n \varphi_2(k) = \frac{\hat{n}_1(k)\hat{n}_2(2)\Delta_2}{\Delta} + \frac{\hat{n}_2(k)\hat{n}_2(3)\Delta_3}{\Delta},$$

$$\partial_n \varphi_3(k) = -\frac{\hat{n}_1(k)\left(\hat{n}_1(2)+\hat{n}_2(2)\right)\Delta_2 + \hat{n}_2(k)\left(\hat{n}_1(3)+\hat{n}_2(3)\right)\Delta_3}{\Delta} \tag{9.30}$$

Coordinates of the nodal points are point 1-(x_1^1, x_2^1) , point 2-(x_1^2, x_2^2) and point 3-(x_1^3, x_2^3). Length of the triangle sides and radius are

$$\Delta_k = \sqrt{(x_1^{k+1} - x_1^k)^2 + (x_2^{k+1} - x_2^k)^2}, \quad r(\boldsymbol{\xi}, \mathbf{y}^q) = \sqrt{(x_1 - y_1^q)^2 + (x_2 - y_2^q)^2} \tag{9.31}$$

Taking into account (9.6) calculation of divergent integrals will be done side by side using the formula

$$J_{q,p}^{l,m}(k) = \int\limits_{l_k} \varphi_q(\xi) \frac{x_1^l(\boldsymbol{\xi}) x_2^m(\boldsymbol{\xi})}{r^p(\xi)} dl(\boldsymbol{\xi}) \tag{9.32}$$

Details of the calculations are presented in the Appendix B. Final results side by side of the calculations are presented bellow.

Side 1-2. In this case the sums of the integrals (9.32) are

Weakly singular

$$J_{1,1}^{0,0}(1) = r_n(I_{1,0} - I_{1,1})/2 + \Delta_1 \hat{n}_1 \hat{n}_2 (I_{1,1} - I_{1,2}) - \partial_n \varphi_1(1) I_{-1,0},$$

$$J_{2,1}^{0,0}(1) = r_n I_{1,1}/2 + \Delta_1 \hat{n}_1 \hat{n}_2 I_{2,2} - \partial_n \varphi_2(1) I_{-1,0}, \quad J_{3,3}^{0,0}(1) = \partial_n \varphi_3(1) I_{1,0},$$

$$J_{1,3}^{2,0}(1)=\frac{1}{3}\left((x_1^1)^2 r_n(I_{3,0}-I_{3,1})+2\Delta_1\hat{n}_2 r_n x_1^1(I_{3,1}-I_{3,2})+2(x_1^1)^2\Delta_1\hat{n}_1\hat{n}_2(I_{3,1}-I_{3,2})\right.$$
$$+r_n(\Delta_1\hat{n}_2)^2(I_{3,2}-I_{3,3})++4\Delta_1^2\hat{n}_1\hat{n}_2^2(I_{3,2}-I_{3,3})$$
$$\left.+2\Delta_1^3\hat{n}_1\hat{n}_2^3(I_{3,3}-I_{3,4})+2(r_n-x_1^1\hat{n}_1)(I_{1,0}-I_{1,1})-\Delta_1\hat{n}_1^2(I_{1,1}-I_{1,2})\right)+$$
$$+\frac{1}{3}\partial_n\varphi_1(1)\left((x_1^1)^2 I_{1,0}+2\Delta_1\hat{n}_2 x_1^1 I_{1,1}+\Delta_1^2\hat{n}_2^2 I_{1,2}-2I_{-1,0}\right),$$

$$J_{2,3}^{2,0}(1)=\frac{1}{3}\left((x_1^1)^2 r_n I_{3,1}+2\Delta_1\hat{n}_2 r_n x_1^1 I_{3,2}+2(x_1^1)^2\Delta_1\hat{n}_1\hat{n}_2^2 I_{3,2}+r_n(\Delta_1\hat{n}_2)^2 I_{3,3}+\right.$$
$$+4\Delta_1^2\hat{n}_1\hat{n}_2^2 I_{3,3}+2\Delta_1^3\hat{n}_1\hat{n}_2^3 I_{3,4}$$
$$\left.+(2r_n-x_1^1\hat{n}_1 I_{1,1})-2\Delta_1\hat{n}_1^2 I_{1,2}\right)$$
$$+\frac{1}{3}\partial_n\varphi_2(1)\left((x_1^1)^2 I_{1,0}+2\Delta_1\hat{n}_2 x_1^1 I_{1,1}+\Delta_1^2\hat{n}_2^2 I_{1,2}-2I_{-1,0}\right),$$

$$J_{3,3}^{2,0}(1)=\frac{1}{3}\partial_n\varphi_3(1)\left((x_1^1)^2 I_{1,0}+2\Delta_1\hat{n}_2 x_1^1 I_{1,1}+\Delta_1^2\hat{n}_2^2 I_{1,2}-2I_{-1,0}\right),$$

$$J_{1,3}^{0,2}(1)=\frac{1}{3}\left((x_2^1)^2 r_n(I_{3,0}-I_{3,1})+2\Delta_1\hat{n}_1 r_n x_1^1(I_{3,1}-I_{3,2})+2(x_2^1)^2\Delta_1\hat{n}_1^2\hat{n}_2(I_{3,1}-I_{3,2})\right.$$
$$+r_n(\Delta_1\hat{n}_1)^2(I_{3,2}-I_{3,3})+4\Delta_1^2\hat{n}_1^2\hat{n}_2(I_{3,2}-I_{3,3})$$
$$\left.+2\Delta_1^3\hat{n}_1^3\hat{n}_2(I_{3,3}-I_{3,4})+2(r_n-x_2^1\hat{n}_2)(I_{1,0}-I_{1,1})-\Delta_1\hat{n}_1^2(I_{1,1}-I_{1,2})\right)+$$
$$+\frac{1}{3}\partial_n\varphi_1(1)\left((x_2^1)^2 I_{1,0}+2\Delta_1\hat{n}_1 x_1^1 I_{1,1}+\Delta_1^2\hat{n}_1^2 I_{1,2}-2I_{-1,0}\right),$$

$$J_{2,3}^{0,2}(1)=\frac{1}{3}\left((x_2^1)^2 r_n I_{3,1}+2\Delta_1\hat{n}_1 r_n x_1^1 I_{3,2}+2(x_2^1)^2\Delta_1\hat{n}_1^2\hat{n}_2 I_{3,2}+r_n(\Delta_1\hat{n}_1)^2 I_{3,3}+\right.$$
$$\left.4\Delta_1^2\hat{n}_1^2\hat{n}_2 I_{3,3}+2\Delta_1^3\hat{n}_1^3\hat{n}_2 I_{3,4}++2(r_n-x_2^1\hat{n}_2)I_{1,1}-\Delta_1\hat{n}_1^2 I_{1,2}\right)$$
$$+\frac{1}{3}\partial_n\varphi_2(1)\left((x_2^1)^2 I_{1,0}+2\Delta_1\hat{n}_1 x_1^1 I_{1,1}+\Delta_1^2\hat{n}_1^2 I_{1,2}-2I_{-1,0}\right), \tag{9.33}$$

$$J^{0,1}_{3,3}(1) = \frac{1}{3}\partial_n\varphi_3(1)\left((x^1_2)^2 I_{1,0} + 2\Delta_1\hat{n}_1 x^1_1 I_{1,1} + \Delta^2_1\hat{n}^2_1 I_{1,2} - 2I_{-1,0}\right),$$

$$J^{1,1}_{1,3}(1) = x^1_1 x^1_2 r_n(I_{3,0} - I_{3,1}) + x^1_2\Delta_1\hat{n}_2 r_n(I_{3,1} - I_{3,2}) + x^1_1\Delta_1\hat{n}_1 r_n(I_{3,1} - I_{3,2})$$
$$+2\Delta_1\hat{n}_1\hat{n}_2 x^1_1 x^1_2(I_{3,1} - I_{3,2}) + \Delta^2_1\hat{n}_1\hat{n}_2 r_n(I_{3,1} - I_{3,3})$$
$$+2\Delta^2_1\hat{n}_1\hat{n}^2_2 x^1_2(I_{3,2} - I_{3,3}) + 2\Delta^2_1\hat{n}^2_1\hat{n}_2 x^1_1(I_{3,2} - I_{3,3}) + 2\Delta^4_1\hat{n}^2_1\hat{n}_2(I_{3,3} - I_{3,4}) -$$
$$-\frac{1}{3}\left(r_+(I_{1,0} - I_{1,1}) + \xi_2\Delta_k(I_{1,1} - I_{1,2})\right)$$
$$+\partial_n\varphi_1(1)\left(x^1_1 x^1_2 I_{1,0} + \Delta_1(\hat{n}_2 x^1_1 + \hat{n}_1 x^1_2)I_{1,1} + \Delta^2_1\hat{n}_1\hat{n}_2 I_{1,2}\right).$$

$$J^{1,1}_{2,3}(1) = x^1_1 x^1_2 r_n I_{3,1} + x^1_2\Delta_1\hat{n}_2 r_n I_{3,2} + x^1_1\Delta_1\hat{n}_1 r_n I_{3,2} + 2\Delta_1\hat{n}_1\hat{n}_2 x^1_1 x^1_2 I_{3,2} + \Delta^2_1\hat{n}_1\hat{n}_2 r_n I_{3,3}$$
$$+2\Delta^2_1\hat{n}_1\hat{n}^2_2 x^1_2 I_{3,3} + 2\Delta^2_1\hat{n}^2_1\hat{n}_2 x^1_1 I_{3,3} + 2\Delta^4_1\hat{n}^2_1\hat{n}_2 I_{3,4}$$
$$-\frac{1}{3}\left(r_+ I_{1,1} + \xi_2\Delta_k I_{1,2}\right) + \partial_n\varphi_2(1)\left(x^1_1 x^1_2 I_{1,0} + \Delta_1(\hat{n}_2 x^1_1 + \hat{n}_1 x^1_2)I_{1,1} + \Delta^2_1\hat{n}_1\hat{n}_2 I_{1,2}\right),$$

$$J^{1,1}_{3,3}(1) = \partial_n\varphi_3(1)\left(x^1_1 x^1_2 I_{1,0} + \Delta_1(\hat{n}_2 x^1_1 + \hat{n}_1 x^1_2)I_{1,1} + \Delta^2_1\hat{n}_1\hat{n}_2 I_{1,2}\right).$$

Singular

$$J^{1,0}_{1,3}(1) = \left(x^1_1 r_n(I_{3,0} - I_{3,1}) + (\Delta_1\hat{n}_2 r_n + 2x^1_1\Delta_1\hat{n}_1\hat{n}_2)(I_{3,1} - I_{3,2}) + \Delta_1\hat{n}_1\hat{n}^2_2(I_{3,2} - I_{3,3})\right)$$
$$+\partial_n\varphi_1(1)\left(x^1_1 I_{1,0} + \Delta_1\hat{n}_2 I_{1,1}\right),$$

$$J^{1,0}_{2,3}(1) = \left(x^1_1 r_n I_{3,1} + (\Delta_1\hat{n}_2 r_n + 2x^1_1\Delta_1\hat{n}_1\hat{n}_2)I_{3,2} + \Delta_1\hat{n}_1\hat{n}^2_2 I_{3,3}\right)$$
$$+\partial_n\varphi_2(1)\left(x^1_1 I_{1,0} + \Delta_1\hat{n}_2 I_{1,1}\right),$$
$$J^{1,0}_{3,3}(1) = \partial_n\varphi_3(1)\left(x^1_1 I_{1,0} + \Delta_1\hat{n}_2 I_{1,1}\right) \tag{9.34}$$

$$J^{0,1}_{1,3}(1) = \left(x^1_2 r_n (I_{3,0} - I_{3,1}) + (\Delta_1 \hat{n}_1 r_n + 2x^1_2 \Delta_1 \hat{n}_1 \hat{n}_2)(I_{3,1} - I_{3,2}) + \Delta_1 \hat{n}_1 \hat{n}^2_2 (I_{3,2} - I_{3,3})\right)$$
$$+\partial_n \varphi_1(1) \left(x^1_2 I_{1,0} + \Delta_1 \hat{n}_1 I_{1,1}\right),$$

$$J^{0,1}_{2,3}(1) = \left(x^1_2 r_n I_{3,1} + (\Delta_1 \hat{n}_1 r_n + 2x^1_2 \Delta_1 \hat{n}_1 \hat{n}_2) I_{3,2} + \Delta_1 \hat{n}_1 \hat{n}^2_2 I_{3,3}\right)$$
$$+\partial_n \varphi_2(1) \left(x^1_2 I_{1,0} + \Delta_1 \hat{n}_1 I_{1,1}\right),$$

$$J^{0,1}_{3,3}(1) = \partial_n \varphi_3(1) \left(x^1_2 I_{1,0} + \Delta_1 \hat{n}_1 I_{1,1} + I_{-1,0}\right)$$

Hypersingular

$$J^{0,0}_{1,3}(1) = r_n (I_{3,0} + I_{3,1})/2 + \Delta_1 \hat{n}_1 \hat{n}_2 (I_{3,1} + I_{3,2}) + \partial_n \varphi_1(1) I_{1,0}.$$

$$J^{0,0}_{2,3}(1) = r_n I_{3,1}/2 + \Delta_1 \hat{n}_1 \hat{n}_2 I_{3,2} + \partial_n \varphi_2(1) I_{1,0}, J^{0,0}_{3,3}(1) = \partial_n \varphi_3(1) I_{1,0},$$

$$J^{2,0}_{1,5}(1) = \frac{1}{3}\left(r_n (I_{3,0} - I_{3,1}) + 2\Delta_1 \hat{n}_1 \hat{n}_2 (I_{3,1} - I_{3,2})\right) - (x^1_2)^2 r_n (I_{5,0} - I_{5,1})$$
$$-2\Delta_1 \hat{n}_1 r_n x^1_2 (I_{5,1} - I_{5,2}) - 2(x^1_2)^2 \Delta_1 \hat{n}_1 \hat{n}_2 (I_{5,1} - I_{5,2}) - r_n (\Delta_1 \hat{n}_1)^2 (I_{5,2} - I_{5,3})$$
$$-4\Delta^2_1 \hat{n}^2_1 \hat{n}^2_2 (I_{5,2} - I_{5,3}) - 2\Delta^3_1 \hat{n}^3_1 \hat{n}_2 (I_{5,3} - I_{5,4}) -$$
$$-\frac{2}{3}\left(x^1_1 (I_{3,0} - I_{3,1}) + \Delta_1 \hat{n}_2 (I_{3,1} - I_{3,2})\right) \hat{n}_1$$
$$+\frac{1}{3} \partial_n \varphi_1(1) \left((x^1_1)^2 I_{3,0} + 2\Delta_1 \hat{n}_2 x^1_1 I_{3,1} + \Delta^2_1 \hat{n}^2_2 I_{3,2} - 2I_{1,0}\right),$$

$$J^{2,0}_{2,5}(1) = \frac{1}{3}\left(r_n I_{3,1} + 2\Delta_1 \hat{n}_1 \hat{n}_2 I_{3,2}\right) - (x^1_2)^2 r_n I_{5,1} - 2\Delta_1 \hat{n}_1 r_n x^1_2 I_{5,2}$$
$$-2(x^1_2)^2 \Delta_1 \hat{n}_1 \hat{n}_2 I_{5,2} - r_n (\Delta_1 \hat{n}_1)^2 I_{5,3} - 4\Delta^2_1 \hat{n}^2_1 \hat{n}^2_2 I_{5,3} - 2\Delta^3_1 \hat{n}^3_1 \hat{n}_2 I_{5,4}$$

$$-\frac{2}{3}\left(x_1^1 I_{3,1}+\Delta_1 \hat{n}_2 I_{3,2}\right)\hat{n}_1+\frac{1}{3}\partial_n\varphi_2(1)\left((x_1^1)^2 I_{3,0}+2\Delta_1\hat{n}_2 x_1^1 I_{3,1}+\Delta_1^2\hat{n}_2^2 I_{3,2}-2I_{1,0}\right),$$

$$J_{3,5}^{2,0}(1)=\frac{1}{3}\partial_n\varphi_3(1)\left((x_1^1)^2 I_{3,0}+2\Delta_1\hat{n}_2 x_1^1 I_{3,1}+\Delta_1^2\hat{n}_2^2 I_{3,2}-2I_{1,0}\right), \tag{9.35}$$

$$J_{1,5}^{0,2}(1)=\frac{1}{3}\left(r_n(I_{3,0}-I_{3,1})+2\Delta_1\hat{n}_1\hat{n}_2(I_{3,1}-I_{3,2})\right)-(x_1^1)^2 r_n(I_{5,0}-I_{5,1})$$

$$-2\Delta_1\hat{n}_2 r_n x_1^1(I_{5,1}-I_{5,2})-2(x_1^1)^2\Delta_1\hat{n}_1\hat{n}_2(I_{5,1}-I_{5,2})-r_n(\Delta_1\hat{n}_2)^2(I_{5,2}-I_{5,3})$$

$$-4\Delta_1^2\hat{n}_1^2\hat{n}_2^2(I_{5,2}-I_{5,3})-2\Delta_1^3\hat{n}_1\hat{n}_2^3(I_{5,3}-I_{5,4})-\frac{2}{3}\left(x_2^1(I_{3,0}-I_{3,1})+\Delta_1\hat{n}_1(I_{3,1}-I_{3,2})\right)\hat{n}_2$$

$$+\frac{1}{3}\partial_n\varphi_1(1)\left((x_2^1)^2 I_{3,0}+2\Delta_1\hat{n}_1 x_2^1 I_{3,1}+\Delta_1^2\hat{n}_1^2 I_{3,2}-2I_{1,0}\right),$$

$$J_{2,5}^{0,2}(1)=\frac{1}{3}\left(r_n I_{3,1}+2\Delta_1\hat{n}_1\hat{n}_2 I_{3,2}\right)-(x_1^1)^2 r_n I_{5,1}-2\Delta_1\hat{n}_2 r_n x_1^1 I_{5,2}-2(x_1^1)^2\Delta_1\hat{n}_1\hat{n}_2 I_{5,2}$$

$$-r_n(\Delta_1\hat{n}_2)^2 I_{5,3}-4\Delta_1^2\hat{n}_1^2\hat{n}_2^2 I_{5,3}-2\Delta_1^3\hat{n}_1\hat{n}_2^3 I_{5,4}-\frac{2}{3}\left(x_2^1 I_{3,1}+\Delta_1\hat{n}_1 I_{3,2}\right)\hat{n}_2$$

$$+\frac{1}{3}\partial_n\varphi_2(1)\left((x_2^1)^2 I_{3,0}+2\Delta_1\hat{n}_1 x_2^1 I_{3,1}+\Delta_1^2\hat{n}_1^2 I_{3,2}-2I_{1,0}\right),$$

$$J_{3,5}^{0,2}(1)=\frac{1}{3}\partial_n\varphi_2(1)\left((x_2^1)^2 I_{3,0}+2\Delta_1\hat{n}_1 x_2^1 I_{3,1}+(\Delta_1^2\hat{n}_1^2 I_{3,2}-2I_{1,0}\right),$$

$$J_{1,5}^{1,1}(1)=x_1^1 x_2^1 r_n(I_{5,0}-I_{5,1})+x_2^1\Delta_1\hat{n}_2 r_n(I_{5,1}-I_{5,2})+x_1^1\Delta_1\hat{n}_1 r_n(I_{5,1}-I_{5,2})$$

$$+2\Delta_1\hat{n}_1\hat{n}_2 x_1^1 x_2^1(I_{5,1}-I_{5,2})+\Delta_1^2\hat{n}_1\hat{n}_2 r_n(I_{5,1}-I_{5,3})+2\Delta_1^2\hat{n}_1\hat{n}_2^2 x_2^1(I_{5,2}-I_{5,3})$$

$$+2\Delta_1^2\hat{n}_1^2\hat{n}_2 x_1^1(I_{5,2}-I_{5,3})+2\Delta_1^4\hat{n}_1^2\hat{n}_2(I_{5,3}-I_{5,4})-\frac{1}{3}\left(r_+(I_{3,0}-I_{3,1})+\xi_2\Delta_k(I_{3,1}-I_{3,2})\right)$$

$$+\frac{1}{3}\partial_n\varphi_1(1)\left(x_1^1 x_2^1 I_{3,0}+\Delta_1(\hat{n}_2 x_1^1+\hat{n}_1 x_2^1)I_{3,1}+\Delta_1^2\hat{n}_1\hat{n}_2 I_{3,2}\right).$$

$$J_{2,5}^{1,1}(1)=x_1^1 x_2^1 r_n I_{5,1}+x_2^1\Delta_1\hat{n}_2 r_n I_{5,2}+x_1^1\Delta_1\hat{n}_1 r_n I_{5,2}+2\Delta_1\hat{n}_1\hat{n}_2 x_1^1 x_2^1 I_{5,2}+\Delta_1^2\hat{n}_1\hat{n}_2 r_n I_{5,3}$$

$$+2\Delta_1^2\hat{n}_1\hat{n}_2^2x_2^1I_{5,3}+2\Delta_1^2\hat{n}_1^2\hat{n}_2x_1^1I_{5,3}+2\Delta_1^4\hat{n}_1^2\hat{n}_2I_{5,4}-\frac{1}{3}\left(r_+I_{3,1}+\xi_2\Delta_kI_{3,2}\right)$$
$$+\frac{1}{3}\partial_n\varphi_2(1)\left(x_1^1x_2^1I_{3,0}+\Delta_1(\hat{n}_2x_1^1+\hat{n}_1x_2^1)I_{3,1}+\Delta_1^2\hat{n}_1\hat{n}_2I_{3,2}\right),$$

$$J_{3,5}^{1,1}(1)=\frac{1}{3}\partial_n\varphi_3(1)\left(x_1^1x_2^1I_{3,0}+\Delta_1(\hat{n}_2x_1^1+\hat{n}_1x_2^1)I_{3,1}+\Delta_1^2\hat{n}_1\hat{n}_2I_{3,2}\right).$$

Side 2-3. In this case the sums of the integrals (9.21) are

Weakly singular

$$J_{1,1}^{0,0}(2)=-\frac{\Delta_2^2}{\Delta}J_{2,1}^{0,0}(2)=\frac{\Delta_2^2}{2\Delta},\quad J_{3,1}^{0,0}(2)=\frac{\Delta_2^2}{2\Delta}. \tag{9.36}$$

$$J_{1,3}^{2,0}(2)=-\frac{2\Delta_2^2}{3\Delta},J_{2,3}^{2,0}(2)=\frac{\Delta_2^2}{3\Delta},J_{3,3}^{2,0}(2)=\frac{\Delta_2^2}{3\Delta}.$$

$$J_{1,2}^{0,2}(2)=-\frac{\Delta_2^2}{3\Delta},J_{2,3}^{0,2}(2)=\frac{\Delta_2^2}{6\Delta},J_{3,3}^{0,2}(2)=\frac{\Delta_2^2}{6\Delta}.$$

$$J_{1,3}^{1,1}(2)=0\quad,\quad J_{2,3}^{1,1}(2)=\frac{\Delta_2}{3},\quad J_{3,3}^{1,1}(2)=\frac{\Delta_2}{3}.$$

Singular

$$J_{1,3}^{1,0}(2)=0,\quad J_{2,3}^{1,0}(2)=2,\quad J_{3,3}^{1,0}(2)=2,$$
$$J_{1,3}^{0,1}(2)=\frac{2\Delta_2}{\Delta},\quad J_{2,3}^{0,1}(2)=-\frac{\Delta_2}{\Delta},\quad J_{3,3}^{0,1}(2)=-\frac{\Delta_2}{\Delta}. \tag{9.37}$$

Hypersingular

$$J_{1,3}^{0,0}(2)=0,J_{2,3}^{0,0}(2)=0,J_{3,3}^{0,0}(2)=0,$$

$$J_{1,5}^{2,0}(2)=0,J_{2,5}^{2,0}(2)=0,J_{3,5}^{2,0}(2)=0,$$

$$J_{1,5}^{0,2}(2) = 0, J_{2,5}^{0,2}(2) = 0, J_{3,5}^{0,2}(2) = 0,$$

$$J_{1,5}^{1,1}(2) = 0, \quad J_{2,5}^{1,1}(2) = 0, J_{3,5}^{1,1}(2) = -\frac{2}{3\Delta_2}. \tag{9.38}$$

Side 3-1. In this case the sums of the integrals (9.21) are

Weakly singular

$$J_{1,1}^{0,0}(3) = \frac{\Delta_3^2}{\Delta}, \quad J_{2,1}^{0,0}(3) = -\frac{\Delta_3^2}{\Delta}, J_{3,1}^{0,0}(3) = \frac{\Delta_3^2}{\Delta},$$

$$J_{1,3}^{2,0}(3) = \frac{\Delta_3^2}{6\Delta}, J_{2,3}^{2,0}(3) = -\frac{\Delta_3^2}{3\Delta}, J_{3,2}^{2,0}(3) = \frac{\Delta_3^2}{6\Delta},$$

$$J_{1,3}^{0,2}(3) = \frac{\Delta_3^2}{3\Delta}, J_{2,3}^{0,2}(3) = -\frac{2\Delta_3^2}{3\Delta}, J_{3,2}^{0,2}(3) = \frac{\Delta_3^2}{3\Delta},$$

$$J_{1,3}^{1,1}(3) = \frac{\Delta_3}{3}, J_{2,3}^{1,1}(3) = 0, \quad J_{3,3}^{1,1}(3) = \frac{\Delta_3}{3} \tag{9.39}$$

Singular

$$J_{1,3}^{1,0}(3) = -\frac{\Delta_3}{\Delta}, \quad J_{2,3}^{1,0}(3) = \frac{2\Delta_3}{\Delta}, \quad J_{3,3}^{1,0}(1) = \frac{\Delta_3}{\Delta},$$

$$J_{1,3}^{0,1}(3) = 2, \quad J_{2,3}^{0,1}(3) = 0, \quad J_{3,3}^{0,1}(3) = 2. \tag{9.40}$$

Hypersingular

$$J_{1,3}^{0,0}(3) = 0, J_{2,3}^{0,0}(3) = 0, J_{3,3}^{0,0}(3) = 0,$$

$$J_{1,5}^{2,0}(3) = 0, J_{2,5}^{2,0}(3) = 0, J_{3,5}^{2,0}(3) = 0,$$

$$J_{1,5}^{0,2}(3) = 0, J_{2,5}^{0,2}(3) = 0, J_{3,5}^{0,2}(3) = 0,$$

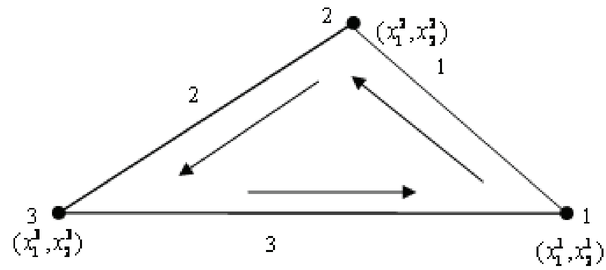

Figure 5:

$$J_{1,5}^{1,1}(3) = 0, \quad J_{2,5}^{1,1}(3) = 0, J_{3,5}^{1,1}(3) = -\frac{2}{3\Delta_2}. \tag{9.41}$$

We have taken into account that integration in (9.22) – (9.30) has to be done in the way as it is shown by arrows in Fig. 5.

Finally sums of the integrals in (9.21) have the form

$$J_p^{l,m} = \sum_{k=1}^{3} J_p^{l,m}(k). \tag{9.42}$$

All integrals of the type $J_p^{l,m}(k)$ have already calculated above.

In the Table 4 are presented results of the divergent and regular integrals calculations for the triangle of a unit side.

In order to check validation of the above regularized equations we compare results with the ones presented in the Table 2 and for weakly singular and regular integrals with results obtained using regular 2-D numerical calculation. Our calculations show that results of calculation obtained using presented here regularized equations agree with ones obtained by other methods. As in the case of piecewise constant approximation our calculations show that with analytical formulas (8.17) results are more accurate and time of calculation is 5-7 times faster in comparison with numerical formulas (8.17) and 8-12 times faster then obtained with 2-D numerical integration.

Substituting in (9.1) all obtained for each side results and take into account (9.31) finally we have

$$U_{11}^{n}(\mathbf{y}_r, \mathbf{x}_q) = \frac{1}{16\pi\mu(1-\upsilon)} \left((3-4\upsilon) \sum_{k=1}^{3} J_{q,1}^{0,0}(k) + \sum_{k=1}^{3} J_{q,3}^{2,0}(k) \right),$$

Table 4: Divergent integrals calculated for unit square at collocation points: 1 -$y_1^0 = 0.0 y_2^0 = 0.0$, 2 -$y_1^0 = 0.5 y_2^0 =$ 0.288, 3 -$y_1^0 = 1.25 y_2^0 =$ 0.721

	$J_{1,1}^{0,0}$	$J_{1,3}^{2,0}$	$J_{1,3}^{0,2}$	$J_{1,3}^{1,1}$	$J_{1,3}^{1,0}$	$J_{1,3}^{0,1}$	$J_{1,3}^{0,0}$	$J_{1,5}^{2,0}$	$J_{1,5}^{0,2}$	$J_{1,5}^{1,1}$
1	0.237	0.199	0.038	0.066	1.053	1.598	-1.211	-0.494	-0.716	0.091
2	0.760	0.397	0.362	-0.030	1.140	-0.658	-6.0	-2.880	-3.119	-0.207
3	0.180	0.104	0.075	0.077	-0.172	-0.141	0.305	0.177	0.127	0.129
	$J_{2,1}^{0,0}$	$J_{2,3}^{2,0}$	$J_{2,3}^{0,2}$	$J_{2,3}^{1,1}$	$J_{2,3}^{1,0}$	$J_{2,3}^{0,1}$	$J_{2,3}^{0,0}$	$J_{2,5}^{2,0}$	$J_{2,5}^{0,2}$	$J_{2,5}^{1,1}$
1	0.030	0.135	0.102	0.103	1.235	0.901	-1.211	-0.582	-0.629	0.142
2	0.760	0.345	0.415	0.0	0.0	1.316	-6.0	-3.239	-2.760	0.0
3	0.180	0.149	0.030	0.051	-0.209	-0.078	0.305	0.252	0.052	0.086
	$J_{3,1}^{0,0}$	$J_{3,3}^{2,0}$	$J_{3,3}^{0,2}$	$J_{3,3}^{1,1}$	$J_{3,3}^{1,0}$	$J_{3,3}^{0,1}$	$J_{3,3}^{0,0}$	$J_{3,5}^{2,0}$	$J_{3,5}^{0,2}$	$J_{3,5}^{1,1}$
1	0.475	0.335	0.140	0.169	-2.374	-1.971	1.268	0.547	0.721	-2.816
2	0.760	0.397	0.362	0.030	-1.140	-0.658	-6.0	-2.880	-3.119	0.207
3	0.145	0.106	0.039	0.058	-0.128	0.074	0.164	0.119	0.044	0.064
	$J_1^{0,0}$	$J_3^{2,0}$	$J_3^{0,2}$	$J_3^{1,1}$	$J_3^{1,0}$	$J_3^{0,1}$	$J_3^{0,0}$	$J_5^{2,0}$	$J_5^{0,2}$	$J_5^{1,1}$
1	0.744	0.670	0.280	0.338	-0.085	0.528	-1.154	-0.529	-0.625	-2.583
2	2.281	1.140	1.140	0.0	0.0	0.0	-18.0	-9.0	-9.0	0.0
3	0.505	0.361	0.144	0.187	-0.510	-0.294	0.774	0.549	0.225	0.280

$$U_{22}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}\left((3-4\upsilon)\sum_{k=1}^{3}J_{q,1}^{0,0}(k)+\sum_{k=1}^{3}J_{q,3}^{0,2}(k)\right),$$

$$U_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{1}{16\pi\mu(1-\upsilon)}\sum_{k=1}^{3}J_{q,3}^{1,1}(k),$$

$$U_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{(3-4\upsilon)}{16\pi\mu(1-\upsilon)}\sum_{k=1}^{3}J_{q,1}^{0,0}(k),$$

$$W_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{13}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,3}^{1,0}(k),$$

$$W_{23}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-K_{23}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{(1-2\upsilon)}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,3}^{0,1}(k),$$

$$F_{11}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{3}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{3}J_{q,5}^{2,0}(k)\right],$$

$$F_{22}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{3}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{3}J_{q,5}^{0,2}(k)\right], \tag{9.43}$$

$$F_{33}^{n}(\mathbf{y}_r,\mathbf{x}_q)=-\frac{\mu}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,3}^{0,0}(k),$$

$$F_{12}^{n}(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,5}^{1,1}(k).$$

All calculations can be done analytically, no numerical integration is needed.

It is important to mention that obtained here formulas can be easy applied for regularization of the divergent integrals in elastodynamics and with small modification for the regular integrals calculation. Also developed methodology easy can be extended to calculation divergent integrals in the case of quadratic and higher BE: for flat elements directly and for curvilinear ones in combination with equations (7.10).

10 Conclusions

Based on the theory of distribution and Green theorems the approach for the divergent hypersingular integrals regularization is developed here and applied for the BIE methods of the 3-D elastostatic problems solution. We consider the 2-D weakly singular, singular and hypersingular integrals over arbitrary convex polygon for piecewise constant approximation and over rectangular and triangular BE for piecewise linear approximation. The divergent integrals over the BE have been transformed to regular ones over contour of the BE. Convenient for their calculation regular formulae have been obtained. In the presented equations all calculations can be done analytically, no numerical integration is needed. It is important to mention that proposed methodology easy can be applied for regularization of the divergent integrals in elastodynamics and for calculation regular integrals when collocation point situated outside BE. Also developed here methodology can applied to regularization of the divergent integrals in the case of quadratic and higher BE.

Calculations of the divergent and regular integrals for square and triangle of the unit side are presented. Our calculations show that results obtained with regularized formulas and analytical representations (8.17) results are more accurate and time of calculation is 5-7 times faster in comparison with numerical formulas (8.17) and 8-12 times faster then obtained with 2-D numerical integration.

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Appendix A.

Side 1-2. In this case $n_1 = 0$, $n_2 = -1$, $\xi_2 = -1$. The main parameters defined by (9.7)-(9.10) are

$$x_1 = \Delta_1(1+\xi_1), \quad x_2 = -\Delta_2, \quad dl = \Delta_1 d\xi_1,$$

$$r(\xi_1) = \Delta_1(1+\xi_1), \quad r_+(\xi_1) = -\Delta_1(1+\xi_1), \quad r_-(\xi_1) = \Delta_1(1+\xi_1), \quad r_n(\xi_1) = 0,$$

$$\varphi_1 = \frac{1-\xi_1}{2}, \quad \varphi_2 = \frac{1+\xi_1}{2}, \quad \varphi_3 = 0, \quad \varphi_4 = 0,$$

$$\partial_n \varphi_1(\boldsymbol{\xi}) = \frac{1-\xi_1}{4}, \quad \partial_n \varphi_2(\boldsymbol{\xi}) = \frac{1+\xi_1}{4},$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = -\frac{1+\xi_1}{4}, \quad \partial_n\varphi_4(\boldsymbol{\xi}) = -\frac{1-\xi_1}{4}.$$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$,

$$J_{1,1}^{0,0}(1) = -\frac{\Delta_1}{4}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = -\frac{\Delta_1}{3},$$

$$J_{2,1}^{0,0}(1) = -\frac{\Delta_1}{4}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = -\frac{2\Delta_1}{3},$$

$$J_{3,1}^{0,0}(1) = \frac{\Delta_1}{4}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = \frac{2\Delta_1}{3}, \quad J_{4,1}^{0,0}(1) = \frac{\Delta_1}{4}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = \frac{\Delta_1}{3}.$$

Integrals of the type $J_{q,3}^{2,0}$,

$$J_{1,3}^{2,0}(1) = -\frac{\Delta_1}{12}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = -\frac{\Delta_1}{9}, \quad J_{2,3}^{2,0}(1) = -\frac{\Delta_1}{12}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = -\frac{2\Delta_1}{9},$$

$$J_{3,3}^{2,0}(1) = \frac{\Delta_1}{12}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = \frac{2\Delta_1}{9}, \quad J_{4,3}^{2,0}(1) = \frac{\Delta_1}{12}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = \frac{\Delta_1}{9}.$$

Integrals of the type $J_{q,3}^{0,2}$,

$$J_{1,3}^{0,2}(1) = -\frac{\Delta_1}{6}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = -\frac{2\Delta_1}{9}, \quad J_{2,3}^{0,2}(1) = -\frac{\Delta_1}{6}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = -\frac{4\Delta_1}{9},$$

$$J_{3,3}^{0,2}(1) = \frac{\Delta_1}{6}\int_{-1}^{1}(1+\xi_1)^2d\xi_1 = \frac{4\Delta_1}{9}, \quad J_{4,3}^{0,2}(1) = \frac{\Delta_1}{6}\int_{-1}^{1}(1-\xi_1^2)d\xi_1 = \frac{2\Delta_1}{9}.$$

Integrals of the type $J_{q,3}^{1,1}$,

$$J_{1,3}^{1,1}(1) = \frac{\Delta_1}{6}\int_{-1}^{1}(1-\xi_1)d\xi_1 = \frac{\Delta_1}{3}, \quad J_{2,3}^{1,1}(1) = \frac{\Delta_1}{6}\int_{-1}^{1}(1+\xi_1)d\xi_1 = \frac{\Delta_1}{3},$$

$$J_{3,3}^{1,1}(1) = 0, \quad J_{4,3}^{1,1}(1) = 0.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$,

$$J_{1,3}^{1,0}(1) = \frac{1}{4}\int_{-1}^{1}(1-\xi_1)d\xi_1 = \frac{1}{2},$$

$$J_{2,3}^{1,0}(1) = \frac{1}{4}\int_{-1}^{1}(1+\xi_1)d\xi_1 = \frac{1}{2},$$

$$J_{3,3}^{1,0}(1) = -\frac{1}{4}\int_{-1}^{1}(1+\xi_1)d\xi_1 = -\frac{1}{2},$$

$$J_{4,3}^{1,0}(1) = -\frac{1}{4}\int_{-1}^{1}(1-\xi_1)d\xi_1 = -\frac{1}{2}.$$

Integrals of the type $J_{q,3}^{0,1}$,

$$J_{1,3}^{0,1}(1) = \frac{1}{2}\int_{-1}^{1}\frac{1-\xi_1}{1+\xi_1}d\xi_1 = 1,$$

$$J_{2,3}^{0,1}(1) = 1, \quad J_{3,3}^{0,1}(1) = 0, \quad J_{4,3}^{0,1}(1) = 0.$$

Hypersingular

Integrals of the type $J_{q,3}^{0,0}$,

$$J_{1,3}^{0,0}(1) = -\frac{1}{4\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = -\frac{1}{2\Delta_1}, J_{2,3}^{0,0}(1) = -\frac{1}{2\Delta_1},$$

$$J_{3,3}^{0,0}(1) = \frac{1}{2\Delta_1},$$

$$J_{4,3}^{0,0}(1) = \frac{1}{4\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = \frac{1}{2\Delta_1}.$$

Integrals of the type $J_{q,5}^{2,0}$,

$$J_{1,5}^{2,0}(1) = -\frac{1}{12\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = -\frac{1}{6\Delta_1},$$

$$J_{2,5}^{2,0}(1) = -\frac{1}{6\Delta_1}, \quad J_{3,5}^{2,0}(1) = \frac{1}{6\Delta_1},$$

$$J_{4,5}^{2,0}(1) = \frac{1}{12\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = \frac{1}{6\Delta_1}.$$

Integrals of the type $J_{q,5}^{0,2}$,

$$J_{1,5}^{0,2}(1) = -\frac{1}{6\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = -\frac{1}{3\Delta_1},$$

$$J_{2,5}^{0,2}(1) = -\frac{1}{3\Delta_1}, \quad J_{3,5}^{0,2}(1) = \frac{1}{3\Delta_1},$$

$$J_{4,5}^{0,2}(1) = \frac{1}{6\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)}d\xi_1 = \frac{1}{3\Delta_1}.$$

Integrals of the type $J_{q,5}^{1,1}$,

$$J_{1,5}^{1,1}(1) = \frac{1}{6\Delta_1}\int_{-1}^{1}\frac{(1-\xi_1)}{(1+\xi_1)^2}d\xi_1 = -\frac{1}{12\Delta_1},$$

$$J_{2,5}^{1,1}(1) = \frac{1}{6\Delta_1}\int_{-1}^{1}\frac{1}{1+\xi_1}d\xi_1 = 0,$$

$$J_{3,5}^{1,1}(1) = 0, \quad J_{4,5}^{1,1}(1) = 0.$$

Side 2-3. In this case $n_1 = 1$, $n_2 = 0$, $\xi_1 = 1$, The main parameters defined by (9.7)-(9.10) are

$$x_1 = \Delta_1, \quad x_2 = \Delta_2(1+\xi_2), \quad dl = \Delta_2 d\xi_2,$$

$$r(\xi_2) = \sqrt{\Delta_k^2\xi_2^2 + 2\xi_2\Delta_k r_+(2) + r^2(2)}, \quad r^2(2) = 4\Delta_1^2 + \Delta_2^2, \quad r_+(2) = \Delta_2,$$

$$r_n(\xi_2) = 2\Delta_1, \quad r_+(\xi_2) = \Delta_2(1+\xi_2), \quad r_-(\xi_2) = \Delta_2(1+\xi_2),$$

$$\varphi_1 = 0, \quad \varphi_2 = \frac{1-\xi_2}{2}, \quad \varphi_3 = \frac{1+\xi_2}{2},$$

$$\varphi_4 = 0, \quad \partial_n\varphi_1(\boldsymbol{\xi}) = 0, \quad \partial_n\varphi_2(\boldsymbol{\xi}) = -\frac{1}{2},$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = \frac{1}{2}, \quad \partial_n\varphi_4(\boldsymbol{\xi}) = 0$$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$,

$$J_{1,1}^{0,0}(2) = \frac{1}{4}\int_{-1}^{1} r(\xi_2)(1-\xi_2)d\xi_2,$$

$$J_{2,1}^{0,0}(2) = -\frac{1}{4}\int_{-1}^{1}\left(r(\xi_2) - \frac{4\Delta_1\Delta_2}{r(\xi_2)}\right)(1-\xi_2)d\xi_2,$$

$$J_{3,1}^{0,0}(2) = -\frac{1}{4}\int_{-1}^{1}\left(\frac{4\Delta_1\Delta_2}{r(\xi_2)} - r(\xi_2)\right)(1+\xi_2)d\xi_2,$$

$$J_{4,1}^{0,0}(2) = \frac{1}{4}\int_{-1}^{1} r(\xi_2)(1+\xi_2)d\xi_2.$$

Integrals of the type $J_{q,3}^{2,0}$,

$$J_{1,3}^{2,0}(2)=\frac{1}{6}\int_{-1}^{1}\left(r(\xi_2)-\frac{2\Delta_1^2}{r(\xi_2)}\right)(1-\xi_2)d\xi_2,$$

$$J_{2,3}^{2,0}(2)=\frac{1}{6}\int_{-1}^{1}\left(\frac{8\Delta_1^3\Delta_2}{r(\xi_2)^3}-\frac{2\Delta_1^2}{r(\xi_2)}-r(\xi_2)\right)(1-\xi_2)d\xi_2,$$

$$J_{3,3}^{2,0}(2)=\frac{1}{6}\int_{-1}^{1}\left(\frac{8\Delta_1^3\Delta_2}{r(\xi_2)^3}-\frac{2\Delta_1^2}{r(\xi_2)}-r(\xi_2)\right)(1+\xi_2)d\xi_2,$$

$$J_{4,3}^{2,0}(2)=\frac{1}{6}\int_{-1}^{1}\left(r(\xi_2)-\frac{2\Delta_1^2}{r(\xi_2)}\right)(1+\xi_2)d\xi_2.$$

Integrals of the type $J_{q,3}^{0,2}$,

$$J_{1,3}^{0,2}(2)=\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_2)+\frac{(1+\xi_2)^2\Delta_2^2}{r(\xi_2)}\right)(1-\xi_2)d\xi_2,$$

$$J_{2,3}^{0,2}(2)=-\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_2)-\frac{(1+\xi_2)^2\Delta_2^2+8\Delta_1\Delta_2}{r(\xi_2)}-\frac{4\Delta_1(1+\xi_2)^2\Delta_2^3)}{r(\xi_2)^3}\right)(1-\xi_2)d\xi_2,$$

$$J_{3,3}^{0,2}(2)=-\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_2)-\frac{(1+\xi_2)^2\Delta_2^2+8\Delta_1\Delta_2}{r(\xi_2)}-\frac{4\Delta_1(1+\xi_2)^2\Delta_2^3)}{r(\xi_2)^3}\right)(1+\xi_2)d\xi_2,$$

$$J_{4,3}^{0,2}(2)=\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_2)+\frac{(1+\xi_2)^2\Delta_2^2}{r(\xi_2)}\right)(1+\xi_2)d\xi_2.$$

Integrals of the type $J_{q,3}^{1,1}$,

$$J_{1,3}^{1,1}(2)=-\int_{-1}^{1}\frac{\Delta_1\Delta_2(1-\xi_2^2)}{6r(\xi_2)}d\xi_2,$$

$$J_{2,3}^{1,1}(2) = \frac{1}{6}\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2 - \Delta_2^2}{r(\xi_2)} + \frac{4\Delta_1^2\Delta_2^2}{r(\xi_2)^3}\right)(1-\xi_2^2)d\xi_2,$$

$$J_{3,3}^{1,1}(2) = \frac{1}{6}\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2 - \Delta_2^2}{r(\xi_2)} + \frac{4\Delta_1^2\Delta_2^2}{r(\xi_2)^3}\right)(1+\xi_2)^2d\xi_2,$$

$$J_{4,3}^{1,1}(2) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2(1+\xi_2)^2}{6r(\xi_2)}d\xi_2.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$,

$$J_{1,3}^{1,0}(2) = -\int_{-1}^{1}\frac{\Delta_1(1-\xi_2)}{2r(\xi_2)}d\xi_2,$$

$$J_{2,3}^{1,0}(2) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2}{r(\xi_2)^3} - \frac{\Delta_1 - \Delta_2}{2r(\xi_2)}\right)(1-\xi_2)\,d\xi_2,$$

$$J_{3,3}^{1,0}(2) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2}{r(\xi_2)^3} - \frac{\Delta_1 - \Delta_2}{2r(\xi_2)}\right)(1+\xi_2)\,d\xi_2,$$

$$J_{4,3}^{1,0}(2) = -\int_{-1}^{1}\frac{\Delta_1(1+\xi_2)}{2r(\xi_2)}d\xi_2.$$

Integrals of the type $J_{q,3}^{0,1}$,

$$J_{1,3}^{0,1}(2) = -\int_{-1}^{1}\frac{\Delta_2(1-\xi_2^2)}{4r(\xi_2)}d\xi_2,$$

$$J_{2,3}^{0,1}(2) = \int_{-1}^{1}\left(\frac{\Delta_2}{4r(\xi_2)} + \frac{\Delta_1\Delta_2^2}{r(\xi_2)^3}\right)(1-\xi_2^2)d\xi_2,$$

$$J_{3,3}^{0,1}(2) = \int_{-1}^{1}\left(\frac{\Delta_2}{4r(\xi_2)} + \frac{\Delta_1\Delta_2^2}{r(\xi_2)^3}\right)(1+\xi_2)^2d\xi_2,$$

$$J_{4,3}^{0,1}(2) = -\int_{-1}^{1} \frac{\Delta_2(1+\xi_2)^2}{4r(\xi_2)} d\xi_2.$$

Hypersingular

Integrals of the type $J_{q,3}^{0,0}$,

$$J_{1,3}^{0,0}(2) = \int_{-1}^{1} \frac{1-\xi_2}{4r(\xi_2)} d\xi_2,$$

$$J_{2,3}^{0,0}(2) = -\int_{-1}^{1} \left(\frac{\Delta_1\Delta_2}{r(\xi_2)^3} + \frac{1}{4r(\xi_2)} \right)(1-\xi_2) d\xi_2,$$

$$J_{3,3}^{0,0}(2) = -\int_{-1}^{1} \left(\frac{\Delta_1\Delta_2}{r(\xi_2)^3} + \frac{1}{4r(\xi_2)} \right)(1+\xi_2) d\xi_2,$$

$$J_{4,3}^{0,0}(2) = \int_{-1}^{1} \frac{1+\xi_2}{4r(\xi_2)} d\xi_2.$$

Integrals of the type $J_{q,5}^{2,0}$,

$$J_{1,5}^{2,0}(2) = \frac{1}{6}\int_{-1}^{1} \left(\frac{1}{r(\xi_2)} - \frac{2\Delta_1^2}{r(\xi_2)^3} \right)(1-\xi_2) d\xi_2,$$

$$J_{2,5}^{2,0}(2) = \frac{1}{6}\int_{-1}^{1} \left(\frac{24\Delta_1^3\Delta_2}{r(\xi_2)^5} - \frac{2\Delta_1^2 - 8\Delta_1\Delta_2}{r(\xi_2)^3} - \frac{1}{r(\xi_2)} \right)(1-\xi_2) d\xi_2,$$

$$J_{3,5}^{2,0}(2) = \frac{1}{6}\int_{-1}^{1} \left(\frac{24\Delta_1^3\Delta_2}{r(\xi_2)^5} - \frac{2\Delta_1^2 - 8\Delta_1\Delta_2}{r(\xi_2)^3} - \frac{1}{r(\xi_2)} \right)(1+\xi_2) d\xi_2,$$

$$J_{4,5}^{2,0}(2) = \frac{1}{6}\int_{-1}^{1} \left(\frac{1}{r(\xi_2)} - \frac{2\Delta_1^2}{r(\xi_2)^3} \right)(1+\xi_2) d\xi_2.$$

Integrals of the type $J_{q,5}^{0,2}$,

$$J_{1,5}^{0,2}(2) = -\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_2)} - \frac{(1+\xi_2)^2\Delta_2^2}{r(\xi_2)^3}\right)(1-\xi_2)d\xi_2,$$

$$J_{2,5}^{0,2}(2) = -\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_2)} - \frac{(1+\xi_2)^2\Delta_2^2 - 8\Delta_1\Delta_2}{r(\xi_2)^3} - \frac{12\Delta_1\Delta_2^3(1+\xi_2)^2)}{r(\xi_2)^5}\right)(1-\xi_2)d\xi_2,$$

$$J_{3,5}^{0,2}(2) = -\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_2)} - \frac{(1+\xi_2)^2\Delta_2^2 - 8\Delta_1\Delta_2}{r(\xi_2)^3} - \frac{12\Delta_1\Delta_2^3(1+\xi_2)^2)}{r(\xi_2)^5}\right)(1+\xi_2)d\xi_2,$$

$$J_{4,5}^{0,2}(2) = -\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_2)} - \frac{(1+\xi_2)^2\Delta_2^2}{r(\xi_2)^3}\right)(1+\xi_2)d\xi_2.$$

Integrals of the type $J_{q,5}^{1,1}$,

$$J_{1,5}^{1,1}(2) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2((1-\xi_2^2)}{6r(\xi_2)^3}d\xi_2,$$

$$J_{2,5}^{1,1}(2) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2^2}{r(\xi_2)^5} - \frac{\Delta_1\Delta_2+\Delta_2^2}{6r(\xi_2)^3}\right)(1-\xi_2^2)d\xi_2,$$

$$J_{3,5}^{1,1}(2) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2^2}{r(\xi_2)^5} - \frac{\Delta_1\Delta_2+\Delta_2^2}{6r(\xi_2)^3}\right)(1+\xi_2)^2d\xi_2,$$

$$J_{4,5}^{1,1}(2) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2(1+\xi_2)^2}{2r(\xi_2)^3}d\xi_2.$$

Side 3-4. In this case $n_1 = 0$, $n_2 = 1$, $\xi_2 = 1$. The main parameters defined by (9.7)-(9.10) are

$$x_1 = \Delta_1(1+\xi_1), \quad x_2 = \Delta_2, \quad dl = \Delta_1 d\xi_1,$$

$$r(\xi_1) = \sqrt{\Delta_k^2\xi_1^2 + 2\xi_1\Delta_k r_+(3) + r^2(3)}, \quad r^2(3) = 4\Delta_2^2 + \Delta_1^2, \quad r_+(3) = \Delta_1,$$

$$r_n(\xi_1) = 2\Delta_2, \quad r_+(\xi_1) = \Delta_1(1+\xi_1), \quad r_- = -\Delta_1(1+\xi_1),$$

$$\varphi_1 = 0, \quad \varphi_2 = 0, \quad \varphi_3 = \frac{1+\xi_1}{2},$$

$$\varphi_4 = \frac{1-\xi_1}{2}, \quad \partial_n\varphi_1(\boldsymbol{\xi}) = 0, \quad \partial_n\varphi_2(\boldsymbol{\xi}) = 0,$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = -\frac{1}{2}, \quad \partial_n\varphi_4(\boldsymbol{\xi}) = \frac{1}{2}.$$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$,

$$J_{1,1}^{0,0}(3) = \frac{1}{4}\int_{-1}^{1} r(\xi_1)(1-\xi_1)d\xi_1,$$

$$J_{2,1}^{0,0}(3) = \frac{1}{4}\int_{-1}^{1} r(\xi_1)(1+\xi_1)d\xi_1,$$

$$J_{3,1}^{0,0}(3) = -\frac{1}{4}\int_{-1}^{1}\left(\frac{4\Delta_1\Delta_2}{r(\xi_1)} - r(\xi_1)\right)(1+\xi_1)d\xi_1,$$

$$J_{4,1}^{0,0}(3) = -\frac{1}{4}\int_{-1}^{1}\left(r(\xi_1) - \frac{4\Delta_1\Delta_2}{r(\xi_1)}\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,3}^{2,0}$,

$$J_{1,3}^{0,2}(3) = \frac{1}{12}\int_{-1}^{1}\left(2r(\xi_1) + \frac{(1+\xi_1)^2\Delta_1^2}{r(\xi_1)}\right)(1-\xi_1)d\xi_1,$$

$$J_{2,3}^{0,2}(3) = \frac{1}{12}\int_{-1}^{1}\left(2r(\xi_1) + \frac{(1+\xi_1)^2\Delta_1^2}{r(\xi_1)}\right)(1+\xi_1)d\xi_1,$$

$$J_{3,3}^{0,2}(3) = -\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_1) - \frac{(1+\xi_1)^2\Delta_1^2 + 8\Delta_1\Delta_2}{r(\xi_1)} - \frac{4(1+\xi_1)^2\Delta_1^3\Delta_2)}{r(\xi_1)^3}\right)(1+\xi_1)d\xi_1,$$

$$J_{4,3}^{0,2}(3) = -\frac{1}{12}\int_{-1}^{1}\left(2r(\xi_1) - \frac{(1+\xi_1)^2\Delta_1^2 + +8\Delta_1\Delta_2}{r(\xi_1)} - \frac{4(1+\xi_1)^2\Delta_1^3\Delta_2)}{r(\xi_1)^3}\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,3}^{0,2}$,

$$J_{1,3}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(r(\xi_1) - \frac{2\Delta_2^2}{r(\xi_1)}\right)(1-\xi_1)d\xi_1,$$

$$J_{2,3}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(r(\xi_1) - \frac{2\Delta_1^2}{r(\xi_1)}\right)(1+\xi_1)d\xi_1,$$

$$J_{3,3}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{8\Delta_2^3\Delta_1}{r(\xi_1)^3} - \frac{2\Delta_2^2}{r(\xi_1)} - r(\xi_1)\right)(1+\xi_1)d\xi_1,$$

$$J_{4,3}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{8\Delta_2^3\Delta_1}{r(\xi_1)^3} - \frac{2\Delta_2^2}{r(\xi_1)} - r(\xi_1)\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,3}^{1,1}$,

$$J_{1,3}^{1,1}(3) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2(1-\xi_1^2)}{6r(\xi_1)}d\xi_1,$$

$$J_{2,3}^{1,1}(3) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2(1+\xi_1^2)}{6r(\xi_1)}d\xi_1,$$

$$J_{3,3}^{1,1}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2 - \Delta_1^2}{r(\xi_1)} + \frac{4\Delta_1^2\Delta_2^2}{r(\xi_1)^3}\right)(1+\xi_1^2)d\xi_1,$$

$$J_{4,3}^{1,1}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2 - \Delta_1^2}{r(\xi_1)} + \frac{4\Delta_1^2\Delta_2^2}{r(\xi_1)^3}\right)(1-\xi_1^2)d\xi_1.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$,

$$J_{1,3}^{0,1}(3) = -\int_{-1}^{1}\frac{\Delta_1(1-\xi_1^2)}{4r(\xi_1)}d\xi_1, \quad J_{2,3}^{0,1}(3) = -\int_{-1}^{1}\frac{\Delta_2(1+\xi_1)^2}{4r(\xi_1)}d\xi_1,$$

$$J_{3,3}^{0,1}(3) = \int_{-1}^{1}\left(\frac{\Delta_1}{4r(\xi_1)} + \frac{\Delta_1^2\Delta_2}{r(\xi_1)^3}\right)(1+\xi_1)^2 d\xi_1,$$

$$J_{4,3}^{0,1}(3) = \int_{-1}^{1}\left(\frac{\Delta_1}{4r(\xi_1)} + \frac{\Delta_1^2\Delta_2}{r(\xi_1)^3}\right)(1-\xi_1^2)d\xi_1.$$

Integrals of the type $J_{q,3}^{0,1}$,

$$J_{1,3}^{1,0}(3) = -\int_{-1}^{1}\frac{\Delta_2(1-\xi_1)}{2r(\xi_1)}d\xi_1, \quad J_{2,3}^{1,0}(3) = -\int_{-1}^{1}\frac{\Delta_1(1+\xi_1)}{2r(\xi_1)}d\xi_1,$$

$$J_{3,3}^{1,0}(3) = \int_{-1}^{1}\left(\frac{2\Delta_2^2\Delta_1}{r(\xi_1)^3} - \frac{\Delta_1 - \Delta_2}{2r(\xi_1)}\right)(1+\xi_1)\, d\xi_1,$$

$$J_{4,3}^{1,0}(3) = \int_{-1}^{1}\left(\frac{2\Delta_2^2\Delta_1}{r(\xi_1)^3} - \frac{\Delta_1 - \Delta_2}{2r(\xi_1)}\right)(1-\xi_1)\, d\xi_1.$$

Hypersingular

Integrals of the type $J_{q,3}^{0,0}$,

$$J_{1,3}^{0,0}(3) = \int_{-1}^{1}\frac{1-\xi_1}{4r(\xi_1)}d\xi_1,$$

$$J_{2,3}^{0,0}(3)=\int_{-1}^{1}\frac{1+\xi_1}{4r(\xi_1)}d\xi_1,$$

$$J_{3,3}^{0,0}(3)=-\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2}{r(\xi_1)^3}+\frac{1}{4r(\xi_1)}\right)(1+\xi_1)d\xi_1,$$

$$J_{4,3}^{0,0}(3)=-\int_{-1}^{1}\left(\frac{\Delta_1\Delta_2}{r(\xi_1)^3}+\frac{1}{4r(\xi_1)}\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,5}^{2,0}$,

$$J_{1,5}^{0,2}(3)=-\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_1)}-\frac{(1+\xi_1)^2\Delta_1^2}{r(\xi_1)^3}\right)(1-\xi_1)d\xi_1,$$

$$J_{2,5}^{0,2}(3)=-\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_1)}-\frac{(1+\xi_1)^2\Delta_2^2}{r(\xi_1)^3}\right)(1+\xi_1)d\xi_1,$$

$$J_{3,5}^{0,2}(3)=-\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_1)}-\frac{(1+\xi_1)^2\Delta_1^2-8\Delta_1\Delta_2}{r(\xi_1)^3}-\frac{12\Delta_1^3\Delta_2(1+\xi_1)^2)}{r(\xi_1)^5}\right)(1+\xi_1)d\xi_1,$$

$$J_{4,5}^{0,2}(3)=-\frac{1}{12}\int_{-1}^{1}\left(\frac{2}{r(\xi_1)}-\frac{(1+\xi_1)^2\Delta_2^2-8\Delta_1\Delta_2}{r(\xi_1)^3}-\frac{12\Delta_1^3\Delta_2(1+\xi_1)^2)}{r(\xi_1)^5}\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,5}^{0,2}$,

$$J_{1,5}^{2,0}(3)=\frac{1}{6}\int_{-1}^{1}\left(\frac{1}{r(\xi_1)}-\frac{2\Delta_2^2}{r(\xi_1)^3}\right)(1-\xi_1)d\xi_1,$$

$$J_{2,5}^{2,0}(3)=\frac{1}{6}\int_{-1}^{1}\left(\frac{1}{r(\xi_1)}-\frac{2\Delta_1^2}{r(\xi_1)^3}\right)(1+\xi_1)d\xi_1,$$

$$J_{3,5}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{24\Delta_2^3\Delta_1}{r(\xi_1)^5} - \frac{2\Delta_2^2 - 8\Delta_1\Delta_2}{r(\xi_1)^3} - \frac{1}{r(\xi_1)}\right)(1+\xi_1)d\xi_1,$$

$$J_{2,5}^{2,0}(3) = \frac{1}{6}\int_{-1}^{1}\left(\frac{24\Delta_2^3\Delta_1}{r(\xi_1)^5} - \frac{2\Delta_2^2 - 8\Delta_1\Delta_2}{r(\xi_1)^3} - \frac{1}{r(\xi_1)}\right)(1-\xi_1)d\xi_1.$$

Integrals of the type $J_{q,5}^{1,1}$,

$$J_{1,5}^{1,1}(3) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2((1-\xi_1^2)}{6r(\xi_1)^3}d\xi_1,$$

$$J_{2,5}^{1,1}(3) = -\int_{-1}^{1}\frac{\Delta_1\Delta_2(1+\xi_1)^2}{2r(\xi_1)^3}d\xi_1,$$

$$J_{3,5}^{1,1}(3) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2^2}{r(\xi_1)^5} - \frac{\Delta_1\Delta_2+\Delta_1^2}{6r(\xi_1)^3}\right)(1+\xi_1)^2 d\xi_1,$$

$$J_{4,5}^{1,1}(3) = \int_{-1}^{1}\left(\frac{2\Delta_1^2\Delta_2^2}{r(\xi_1)^5} - \frac{\Delta_1\Delta_2+\Delta_1^2}{6r(\xi_1)^3}\right)(1-\xi_1^2)d\xi_1.$$

Side 4-1. In this case $n_1 = -1$, $n_2 = 0$, $\xi_1 = -1$. The main parameters defined by (9.7)-(9.10) are

$$x_1 = -\Delta_1, \quad x_2 = \Delta_2(1+\xi_2), \quad dl = \Delta_2 d\xi_2 r(\xi_2) = \Delta_2(1+\xi_2), \quad r_n(\xi_2) = 0,$$

$$r_+(\xi_2) = -\Delta_2(1+\xi_2), \quad r_-(\xi_2) = 0, \quad \varphi_1 = \frac{1-\xi_2}{2}, \quad \varphi_2 = 0, \quad \varphi_3 = 0,$$

$$\varphi_4 = \frac{1+\xi_2}{2}, \quad \partial_n\varphi_1(\boldsymbol{\xi}) = \frac{1}{2}, \quad \partial_n\varphi_2(\boldsymbol{\xi}) = 0, \quad \partial_n\varphi_3(\boldsymbol{\xi}) = 0, \quad \partial_n\varphi_4(\boldsymbol{\xi}) = \frac{1}{2}$$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$,

$$J_{1,1}^{0,0}(4) = -\frac{\Delta_2}{4}\int_{-1}^{1}(1-\xi_2^2)d\xi_2 = -\frac{\Delta_2}{3},$$

$$J_{2,1}^{0,0}(4) = \frac{\Delta_2}{4}\int_{-1}^{1}(1-\xi_2^2)d\xi_2 = \frac{\Delta_2}{3}$$

$$J_{3,1}^{0,0}(4) = \frac{\Delta_2}{4}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = \frac{2\Delta_2}{3}, \quad J_{4,1}^{0,0}(4) = -\frac{\Delta_2}{4}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = -\frac{2\Delta_2}{3}.$$

Integrals of the type $J_{q,3}^{2,0}$,

$$J_{1,3}^{2,0}(4) = -\frac{\Delta_2}{6}\int_{-1}^{1}(1-\xi_2^2)d\xi_2 = -\frac{2\Delta_2}{9}, \quad J_{2,3}^{2,0}(4) = \frac{\Delta_2}{6}\int_{-1}^{1}(1-\xi_2)^2 d\xi_2 = \frac{2\Delta_2}{9},$$

$$J_{3,3}^{2,0}(4) = \frac{\Delta_2}{6}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = \frac{4\Delta_2}{9}, \quad J_{4,3}^{2,0}(4) = -\frac{\Delta_2}{6}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = -\frac{4\Delta_2}{9}.$$

Integrals of the type $J_{q,3}^{0,2}$,

$$J_{1,3}^{0,2}(4) = -\frac{\Delta_2}{12}\int_{-1}^{1}(1-\xi_2^2)d\xi_2 = -\frac{2\Delta_2}{9}, \quad J_{2,3}^{0,2}(4) = \frac{\Delta_2}{12}\int_{-1}^{1}(1-\xi_2)^2 d\xi_2 = \frac{\Delta_2}{9},$$

$$J_{3,3}^{0,2}(4) = \frac{\Delta_2}{12}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = \frac{2\Delta_2}{9}, \quad J_{4,3}^{0,2}(4) = -\frac{\Delta_2}{12}\int_{-1}^{1}(1+\xi_2)^2 d\xi_2 = -\frac{2\Delta_2}{9}.$$

Integrals of the type $J_{q,3}^{1,1}$,

$$J_{1,3}^{1,1}(4) = \frac{\Delta_2}{6}\int_{-1}^{1}(1-\xi_2)d\xi_2 = \frac{\Delta_2}{3}, \quad J_{2,3}^{1,1}(4) = 0, \quad J_{3,3}^{1,1}(4) = 0,$$

$$J_{4,3}^{1,1}(4) = \frac{\Delta_1}{6}\int_{-1}^{1}(1+\xi_2)d\xi_2 = \frac{\Delta_2}{3}.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$,

$$J_{1,3}^{1,0}(4) = \frac{1}{2}\int_{-1}^{1}\frac{1-\xi_1}{1+\xi_1}d\xi_1 = 1, \quad J_{2,3}^{1,0}(4) = 0, \quad J_{3,3}^{1,0}(4) = 0, \quad J_{4,3}^{1,0}(4) = 1.$$

Integrals of the type $J_{q,3}^{0,1}$,

$$J_{1,3}^{0,1}(4) = \frac{1}{4}\int_{-1}^{1}(1-\xi_2)d\xi_2 = \frac{1}{2}, \quad J_{2,3}^{0,1}(4) = -\frac{1}{4}\int_{-1}^{1}(1-\xi_2)d\xi_2 = -\frac{1}{2},$$

$$J_{3,3}^{0,1}(4) = -\frac{1}{4}\int_{-1}^{1}(1+\xi_1)d\xi_1 = -\frac{1}{2}, \quad J_{4,3}^{0,1}(4) = -\frac{1}{4}\int_{-1}^{1}(1+\xi_2)d\xi_2 = \frac{1}{2}.$$

Hyperingular

Integrals of the type $J_{q,3}^{0,0}$,

$$J_{1,3}^{0,0}(4) = -\frac{1}{4\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = -\frac{1}{2\Delta_2},$$

$$J_{2,3}^{0,0}(4) = \frac{1}{4\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = \frac{1}{2\Delta_2},$$

$$J_{3,3}^{0,0}(4) = \frac{1}{2\Delta_2}, \quad J_{4,3}^{0,0}(4) = -\frac{1}{2\Delta_2}$$

Integrals of the type $J_{q,5}^{2,0}$,

$$J_{1,5}^{2,0}(4) = -\frac{1}{6\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = -\frac{1}{3\Delta_2},$$

$$J_{2,5}^{2,0}(4) = \frac{1}{6\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = \frac{1}{3\Delta_2},$$

$$J_{3,5}^{2,0}(4) = \frac{2}{3\Delta_2}, \quad J_{4,5}^{2,0}(4) = -\frac{1}{3\Delta_2}$$

Integrals of the type $J_{q,5}^{0,2}$,

$$J_{1,5}^{0,2}(4) = -\frac{1}{12\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = -\frac{1}{6\Delta_2},$$

$$J_{2,5}^{0,2}(4) = \frac{1}{12\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)}d\xi_2 = \frac{1}{9\Delta_2},$$

$$J_{3,5}^{0,2}(4) = \frac{1}{6\Delta_2}, \quad J_{4,5}^{0,2}(4) = -\frac{1}{6\Delta_2}.$$

Integrals of the type $J_{q,5}^{1,1}$,

$$J_{1,5}^{1,1}(4) = \frac{1}{6\Delta_2}\int_{-1}^{1}\frac{(1-\xi_2)}{(1+\xi_2)^2}d\xi_2 = -\frac{1}{12\Delta_2},$$

$$J_{2,5}^{1,1}(4) = 0, \quad J_{3,5}^{1,1}(4) = 0,$$

$$J_{4,5}^{1,1}(4) = \frac{1}{6\Delta_2}\int_{-1}^{1}\frac{1}{1+\xi_2}d\xi_2 = 0.$$

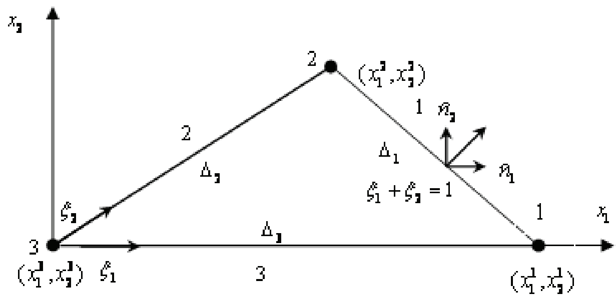

Figure 6:

Appendix B.

Side 1-2. From Fig. 6 follows that in this case $\xi_1 = 1 - \xi_2$.
The main parameters defined by (9.25)-(9.29) are

$$x_1 = x_1^1 + \Delta_1 \hat{n}_2 \xi_2, \quad x_2 = x_2^1 + \Delta_1 \hat{n}_1 \xi_2, \quad dl = \Delta_1 d\xi_2,$$

$$r(\xi_2) = \sqrt{\Delta_1^2 \xi_2^2 + 2\xi \Delta_1 r_+ + r^2}, \quad r = \sqrt{(x_1^1)^2 + (x_2^1)^2},$$

$$r_n = x_\alpha^1 \hat{n}_\alpha, \quad r_+ = x_2^1 \hat{n}_1 + x_1^1 \hat{n}_2,$$

$$r_- = x_2^1 \hat{n}_1 - x_1^1 \hat{n}_2, \quad r_n(\xi_2) = r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2, \quad r_+(\xi_2) = r_+ + \xi_2 \Delta_1,$$

$$\varphi_1(\xi_1, \xi_2) = 1 - \xi_2, \quad \varphi_2(\xi_1, \xi_2) = \xi_2, \quad \varphi_1(\xi_1, \xi_2) = 0,$$

$$\partial_n \varphi_1(1) = \frac{(\hat{n}_1(1)\hat{n}_1(2) + \hat{n}_2(1)\hat{n}_2(2))\Delta_2}{\Delta},$$

$$\partial_n \varphi_2(1) = \frac{(\hat{n}_1(1)\hat{n}_1(3) + \hat{n}_2(1)\hat{n}_2(3))\Delta_3}{\Delta},$$

$$\partial_n \varphi_3(1) = -\frac{(\hat{n}_1(1)\hat{n}_1(2) + \hat{n}_2(1)\hat{n}_2(2))\Delta_2}{\Delta} - \frac{(\hat{n}_1(1)\hat{n}_1(3) + \hat{n}_2(1)\hat{n}_2(3))\Delta_3}{\Delta}.$$

Weakly singular
Integrals of the type $J_{q,1}^{0,0}$.

$$J_{1,1}^{0,0}(1)=\int_0^1\left((1-\xi_2)\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{2r(\xi_2)}-\partial_n\varphi_1(1)r(\xi_2)\right)\Delta_1 d\xi_2,$$

$$J_{2,1}^{0,0}(1)=\int_0^1\left(\xi_2\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{2r(\xi_2)}-\partial_n\varphi_2(1)r(\xi_2)\right)\Delta_1 d\xi_2,$$

$$J_{3,1}^{0,0}(1)=\int_0^1\partial_n\varphi_3(1)r(\xi_2)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,3}^{2,0}$.

$$J_{1,3}^{2,0}(1)=$$

$$\frac{1}{3}\int_0^1\left((1-\xi_2)\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^3(\xi_2)}+\frac{2(r_n-\hat{n}_1(x_1^1+\Delta_1\hat{n}_1\xi_2))}{r(\xi_2)}\right)+\right.$$

$$\left.+\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_1(1)\right)\Delta_1 d\xi_2,$$

$$J_{2,3}^{2,0}(1)=$$

$$\frac{1}{3}\int_0^1\left(\xi_2\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^3(\xi_2)}+\frac{2(r_n-\hat{n}_1(x_1^1+\Delta_1\hat{n}_1\xi_2))}{r(\xi_2)}\right)+\right.$$

$$\left.+\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_2(1)\right)\Delta_1 d\xi_2,$$

$$J_{3,3}^{2,0}(1)=\frac{1}{3}\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,3}^{0,2}$.

$$J_{1,3}^{0,2}(1) =$$

$$\frac{1}{3}\int_0^1 \left((1-\xi_2)\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r(\xi_2)^3}+\frac{2(r_n-\hat{n}_2(x_2^1+\Delta_1\hat{n}_1\xi_2))_1}{r(\xi_2)}\right)+\right.$$

$$\left.+\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_1(1)\right)\Delta_1 d\xi_2,$$

$$J_{2,3}^{0,2}(1) =$$

$$\frac{1}{3}\int_0^1 \left(\xi_2\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r(\xi_2)^3}+\frac{2(r_n-\hat{n}_2(x_2^1+\Delta_1\hat{n}_1\xi_2))_1}{r(\xi_2)}\right)+\right.$$

$$\left.+\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_2(1)\right)\Delta_1 d\xi_2,$$

$$J_{3,3}^{0,2}(1) = \frac{1}{3}\int_0^1 \left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r(\xi_2)}-2r(\xi_2)\right)\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,3}^{1,1}$.

$$J_{1,3}^{1,1}(1) = \int_0^1 (1-\xi_2)$$

$$\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^3(\xi_2)}-\frac{r_+ +\xi_2\Delta_k}{3r(\xi_2)}\right)\Delta_1 d\xi_2+$$

$$+\int_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r(\xi_2)}\partial_n\varphi_1(1)\Delta_1 d\xi_2,$$

$$J_{2,3}^{1,1}(1) = \int_0^1 \xi_2$$

$$\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^3(\xi_2)}-\frac{r_+ +\xi_2\Delta_k}{3r(\xi_2)}\right)\Delta_1 d\xi_2-$$

$$-\int\limits_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r(\xi_2)}\partial_n\varphi_2(1)\Delta_1 d\xi_2,$$

$$J_{3,3}^{1,1}(1)=\int\limits_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r(\xi_2)}\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Singular.

Integrals of the type $J_{q,3}^{1,0}$.

$$J_{1,3}^{1,0}(1)=\int\limits_0^1\left((1-\xi_2)\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r(\xi_2)^3}-\frac{\hat{n}_1}{r(\xi_2)}\right)\right.$$

$$\left.+\frac{x_1^1+\Delta_1\hat{n}_2\xi_2}{r(\xi_2)}\partial_n\varphi_1(1)\right)\Delta_1 d\xi_2,$$

$$J_{2,3}^{1,0}(1)=\int\limits_0^1\left(\xi_2\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r(\xi_2)^3}-\frac{\hat{n}_1}{r(\xi_2)}\right)\right.$$

$$\left.+\frac{x_1^1+\Delta_1\hat{n}_2\xi_2}{r(\xi_2)}\partial_n\varphi_1(1)\right)\Delta_1 d\xi_2,$$

$$J_{3,3}^{1,0}(1)=\int\limits_0^1 \frac{x_1^1+\Delta_1\hat{n}_2\xi_2}{r(\xi_2)}\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,3}^{0,1}$.

$$J_{1,3}^{0,1}(1)=\int\limits_0^1\left((1-\xi_2)\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r(\xi_2)^3}-\frac{\hat{n}_2}{r(\xi_2)}\right)\right.$$

$$\left.+\frac{x_2^1+\Delta_1\hat{n}_1\xi_2}{r(\xi_2)}\partial_n\varphi_1(1)\right)\Delta_1 d\xi_2,$$

$$J_{2,3}^{0,1}(1) = \int_0^1 \left(\xi_2 \left(\frac{(x_2^1 + \Delta_1 \hat{n}_1 \xi_2)(r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2)}{r(\xi_2)^3} - \frac{\hat{n}_2}{r(\xi_2)} \right) \right.$$

$$\left. + \frac{x_2^1 + \Delta_1 \hat{n}_1 \xi_2}{r(\xi_2)} \partial_n \varphi_1(1) \right) \Delta_1 d\xi_2,$$

$$J_{3,3}^{0,1}(1) = \int_0^1 \frac{x_2^1 + \Delta_1 \hat{n}_1 \xi_2}{r(\xi_2)} \partial_n \varphi_1(1) \Delta_1 d\xi_2.$$

Hyperingular.

Integrals of the type $J_{q,3}^{0,0}$.

$$J_{1,1}^{0,0}(1) = -\int_0^1 \left((1-\xi_2) \frac{r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2}{2r^3(\xi_2)} + \frac{\partial_n \varphi_1(1)}{r(\xi_2)} \right) \Delta_1 d\xi_2,$$

$$J_{2,1}^{0,0}(1) = \int_0^1 \left(\xi_2 \frac{r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2}{2r^3(\xi_2)} - \frac{\partial_n \varphi_2(1)}{r(\xi_2)} \right) \Delta_1 d\xi_2,$$

$$J_{3,1}^{0,0}(1) = \int_0^1 \frac{\partial_n \varphi_3(1)}{r(\xi_2)} \Delta_1 d\xi_2.$$

Integrals of the type $J_{q,5}^{2,0}$.

$$J_{1,5}^{2,0}(1) = \int_0^1 (1-\xi_2) \left(\frac{(x_2^1 + \Delta_1 \hat{n}_1 \xi_2)^2 (r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2)}{r^5(\xi_2)} - \frac{2(r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2)}{3r^3(\xi_2)} - \right.$$

$$\left. - \frac{2(x_1^1 + \Delta_1 \hat{n}_2 \xi_2)\hat{n}_1}{3r^3(\xi_2)} \right) \Delta_1 d\xi_2 + \int_0^1 \left(\frac{(x_1^1 + \Delta_1 \hat{n}_2 \xi_2)^2}{3r^3(\xi_2)} - \frac{2}{3r(\xi_2)} \right) \partial_n \varphi_1(1) \Delta_1 d\xi_2,$$

$$J_{2,5}^{2,0}(1) = \int_0^1 \xi_2 \left(\frac{(x_2^1 + \Delta_1 \hat{n}_1 \xi_2)^2 (r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2)}{r^5(\xi_2)} - \frac{2(r_n + 2\Delta_1 \hat{n}_1 \hat{n}_2 \xi_2)}{3r^3(\xi_2)} - \right.$$

$$-\frac{2(x_1^1+\Delta_1\hat{n}_2\xi_2)\hat{n}_1}{3r^3(\xi_2)}\Bigg)\Delta_1 d\xi_2+\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{3r^3(\xi_2)}-\frac{2}{3r(\xi_2)}\right)\partial_n\varphi_2(1)\Delta_1 d\xi_2,$$

$$J_{3,5}^{2,0}(1)=\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{3r^3(\xi_2)}-\frac{2}{3r(\xi_2)}\right)\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,5}^{0,2}$.

$$J_{1,5}^{0,2}(1)=\int_0^1(1-\xi_2)\Bigg(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}-\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}-$$

$$-\frac{2(x_2^1+\Delta_1\hat{n}_1\xi_2)\hat{n}_2}{3r^3(\xi_2)}\Bigg)\Delta_1 d\xi_2+\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{3r^3(\xi_2)}-\frac{2}{3r(\xi_2)}\right)\partial_n\varphi_1(1)\Delta_1 d\xi_2,$$

$$J_{2,5}^{0,2}(1)=\int_0^1\xi_2\Bigg(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}-\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}-$$

$$-\frac{2(x_2^1+\Delta_1\hat{n}_1\xi_2)\hat{n}_2}{3r^3(\xi_2)}\Bigg)\Delta_1 d\xi_2+\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{3r^3(\xi_2)}-\frac{2}{3r(\xi_2)}\right)\partial_n\varphi_2(1)\Delta_1 d\xi_2,$$

$$J_{3,5}^{0,2}(1)=\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{3r^3(\xi_2)}-\frac{2}{3r(\xi_2)}\right)\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Integrals of the type $J_{q,5}^{1,1}$.

$$J_{1,5}^{1,1}(1)=$$

$$\int_0^1(1-\xi_2)\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}-\frac{r_++\xi_2\Delta_k}{3r^3(\xi_2)}\right)\Delta_1 d\xi_2+$$

$$+\int_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{3r^3(\xi_2)}\partial_n\varphi_1(1)\Delta_1 d\xi_2,$$

$$J_{2,5}^{1,1}(1) =$$

$$\int_0^1 \xi_2\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}-\frac{r_+ +\xi_2\Delta_k}{3r^3(\xi_2)}\right)\Delta_1 d\xi_2+$$

$$+\int_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{3r^3(\xi_2)}\partial_n\varphi_2(1)\Delta_1 d\xi_2,$$

$$J_{3,5}^{1,1}(1) = \int_0^1 \frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{3r^3(\xi_2)}\partial_n\varphi_3(1)\Delta_1 d\xi_2.$$

Side 2-3. From Fig. 7 follows that in this case $\xi_1 = 0$, $\hat{n}_1 = 1$, $\hat{n}_2 = 0$.

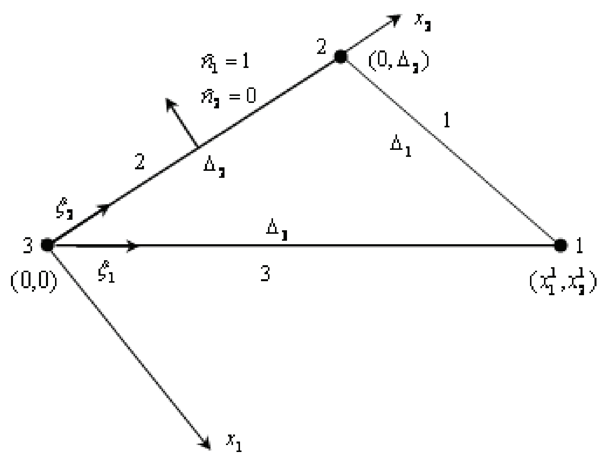

Figure 7:

The main parameters defined by ((9.25)-(9.29) are

$$x_2 = \Delta_2\xi_2, \quad x_1 = 0, \quad dl = \Delta_2 d\xi_2,$$

$r(\xi_2) = \Delta_2 \xi_2, \quad r_n(\xi_2) = 0, \quad r_n = 0,$

$r_+ = 0, \quad r_- = 0, \quad r_+(\xi_2) = \Delta_2 \xi_2, \quad r_-(\xi_2) = -\Delta_2 \hat{n}_1^2 \xi_2,$

$\varphi_1(\xi_2) = 0, \quad \varphi_2(\xi_2) = \xi_2, \quad \varphi_3(\xi_2) = 1 - \xi_2,$

$\partial_n \varphi_1(2) = \Delta_2/\Delta, \quad \partial_n \varphi_2(2) = 0, \quad \partial_n \varphi_3(2) = -\Delta_2/\Delta.$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$

$$J_{1,1}^{0,0}(2) = -\frac{2\Delta_2^2}{\Delta} \int_0^1 \xi_2 d\xi_2 = -\frac{\Delta_2^2}{\Delta},$$

$$J_{2,1}^{0,0}(2) = \frac{\Delta_2^2}{\Delta} \int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{2\Delta},$$

$$J_{3,1}^{0,0}(2) = \frac{\Delta_2^2}{\Delta} \int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{2\Delta}.$$

Integrals of the type $J_{q,3}^{2,0}$

$$J_{1,3}^{2,0}(2) = -\frac{4\Delta_2^2}{3\Delta} \int_0^1 \xi_2 d\xi_2 = -\frac{2\Delta_2^2}{3\Delta},$$

$$J_{2,3}^{2,0}(2) = \frac{2\Delta_2^2}{3\Delta} \int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{3\Delta},$$

$$J_{3,3}^{2,0}(2) = \frac{2\Delta_2^2}{3\Delta} \int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{3\Delta}.$$

Integrals of the type $J_{q,3}^{0,2}$

$$J_{1,2}^{0,2}(2) = -\frac{2\Delta_2^2}{3\Delta} \int_0^1 \xi_2 d\xi_2 = -\frac{\Delta_2^2}{3\Delta},$$

$$J_{2,3}^{0,2}(2) = \frac{\Delta_2^2}{3\Delta}\int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{6\Delta},$$

$$J_{3,3}^{0,2}(2) = \frac{\Delta_2^2}{3\Delta}\int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2^2}{6\Delta}.$$

Integrals of the type $J_{q,3}^{1,1}$

$$J_{1,3}^{1,1}(2) = 0,$$

$$J_{2,3}^{1,1}(2) = \frac{2\Delta_2}{3}\int_0^1 \xi_2 d\xi_2 = \frac{\Delta_2}{3},$$

$$J_{3,3}^{1,1}(2) = -\frac{2\Delta_2}{3}\int_0^1 (\xi_2 - 1) d\xi_2 = \frac{\Delta_2}{3}.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$

$$J_{1,3}^{1,0}(2) = 0, \quad J_{2,3}^{1,0}(2) = 2\int_0^1 d\xi_2 = 2,$$

$$J_{3,3}^{1,0}(2) = 2\int_0^1 \frac{1-\xi_2}{\xi_2} d\xi_2 = -2.$$

Integrals of the type $J_{q,3}^{0,1}$

$$J_{1,3}^{0,1}(2) = \frac{2\Delta_2}{\Delta}\int_0^1 d\xi_2 = \frac{2\Delta_2}{\Delta},$$

$$J_{2,3}^{0,1}(2) = -\frac{\Delta_2}{\Delta}\int_0^1 d\xi_2 = -\frac{\Delta_2}{\Delta},$$

$$J_{3,3}^{0,1}(2) = -\frac{\Delta_2}{\Delta}\int_0^1 d\xi_2 = -\frac{\Delta_2}{\Delta}.$$

Hyperingular

Integrals of the type $J_{q,3}^{0,0}$

$$J_{1,3}^{0,0}(2) = -\frac{2}{\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{2,3}^{0,0}(2) = \frac{1}{\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{3,3}^{0,0}(2) = \frac{1}{\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0.$$

Integrals of the type $J_{q,5}^{2,0}$

$$J_{1,5}^{2,0}(2) = -\frac{4}{3\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{2,5}^{2,0}(2) = \frac{2}{3\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{3,5}^{2,0}(2) = \frac{2}{3\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0.$$

Integrals of the type $J_{q,5}^{0,2}$

$$J_{1,5}^{0,2}(2) = -\frac{2}{3\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{2,5}^{0,2}(2) = \frac{1}{3\Delta}\int_0^1 \frac{1}{\xi_2}d\xi_2 = 0,$$

$$J_{3,5}^{0,2}(2) = \frac{}{3\Delta}\int_0^1 \frac{1}{\xi_2} d\xi_2 = 0.$$

Integrals of the type $J_{q,5}^{1,1}$

$$J_{1,5}^{1,1}(2) = 0,$$

$$J_{2,5}^{1,1}(2) = \frac{2}{3\Delta_2}\int_0^1 \frac{1}{\xi_2} d\xi_2 = 0\ ,$$

$$J_{3,5}^{1,1}(2) = \frac{2}{3\Delta_2}\left(\int_0^1 \frac{1}{\xi_2^2} d\xi_2 - \int_0^1 \frac{1}{\xi_2} d\xi_2\right) = -\frac{2}{3\Delta_2}.$$

Side 3-1. From Fig. 8 follows that in this case $\xi_2 = 0$, $\hat{n}_1 = 0$, $\hat{n}_2 = -1$.

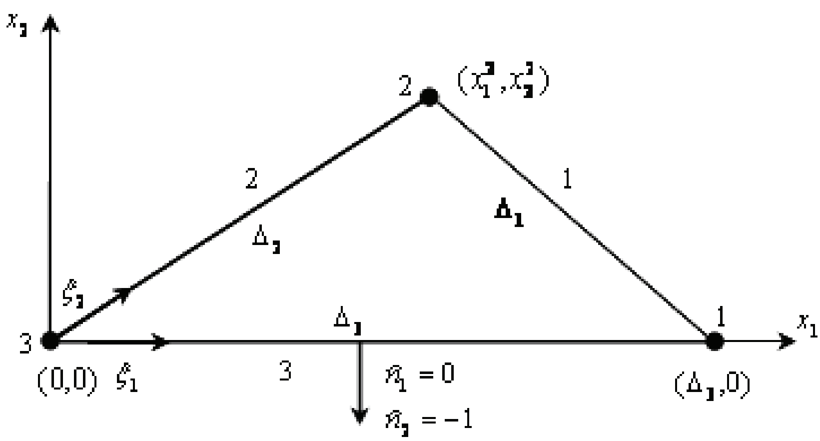

Figure 8:

The main parameters defined by (9.25)-(9.29 are

$$x_1 = \Delta_3\xi_1, \quad x_2 = 0, \quad dl = \Delta_3 d\xi_1,$$

$$r(\xi_1) = \Delta_3\xi_1, \quad r_n(\xi_1) = 0, \quad r_n = 0, \quad r_+ = 0, \quad r_- = 0,$$

$r_+(\xi_1) = \Delta_3\xi_1, \quad r_-(\xi_1) = \Delta_3\hat{n}_2^2\xi_1, \quad \varphi_1(\xi_1) = \xi_1, \quad \varphi_2(\xi_1) = 0,$
$\varphi_3(\xi_1) = 1-\xi_1, \quad \partial_n\varphi_1(3) = 0, \quad \partial_n\varphi_2(3) = \Delta_3/\Delta, \quad \partial_n\varphi_3(3) = -\Delta_3/\Delta.$

Weakly singular

Integrals of the type $J_{q,1}^{0,0}$

$$J_{1,1}^{0,0}(3) = \frac{\Delta_3^2}{\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{2\Delta},$$

$$J_{2,1}^{0,0}(3) = -\frac{2\Delta_3^2}{\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{\Delta},$$

$$J_{3,1}^{0,0}(3) = \frac{\Delta_3^2}{\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{2\Delta}.$$

Integrals of the type $J_{q,3}^{2,0}$

$$J_{1,3}^{2,0}(3) = \frac{\Delta_3^2}{3\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{6\Delta},$$

$$J_{2,3}^{2,0}(3) = -\frac{2\Delta_3^2}{3\Delta}\int_0^1 \xi_1 d\xi_1 = -\frac{\Delta_3^2}{3\Delta},$$

$$J_{3,2}^{2,0}(3) = \frac{\Delta_3^2}{3\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{6\Delta}.$$

Integrals of the type $J_{q,3}^{0,2}$

$$J_{1,3}^{0,2}(3) = \frac{2\Delta_3^2}{3\Delta}\int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{3\Delta},$$

$$J_{2,3}^{0,2}(3) = -\frac{4\Delta_3^2}{3\Delta}\int_0^1 \xi_1 d\xi_1 = -\frac{2\Delta_3^2}{3\Delta},$$

$$J_{3,2}^{0,2}(3) = \frac{2\Delta_3^2}{3\Delta} \int_0^1 \xi_1 d\xi_1 = \frac{\Delta_3^2}{3\Delta}.$$

Integrals of the type $J_{q,3}^{1,1}$

$$J_{1,3}^{1,1}(3) = \frac{2\Delta_3}{3} \int_0^1 \xi_1 d\xi_1 \ = \frac{\Delta_3}{3},$$

$$J_{2,3}^{1,1}(3) = 0,$$

$$J_{3,3}^{1,1}(3) = -\frac{2\Delta_3}{3} \int_0^1 (\xi_1 - 1) d\xi_1 = \frac{\Delta_3}{3}.$$

Singular

Integrals of the type $J_{q,3}^{1,0}$

$$J_{1,3}^{1,0}(3) = -\frac{\Delta_3}{\Delta} \int_0^1 d\xi_2 = -\frac{\Delta_3}{\Delta},$$

$$J_{2,3}^{1,0}(3) = \frac{2\Delta_3}{\Delta} \int_0^1 d\xi_2 = \frac{2\Delta_3}{\Delta},$$

$$J_{3,3}^{1,0}(3) = -\frac{\Delta_3}{\Delta} \int_0^1 d\xi_2 = -\frac{\Delta_3}{\Delta}.$$

Integrals of the type $J_{q,3}^{0,1}$

$$J_{1,3}^{0,1}(3) = 2 \int_0^1 d\xi_2 = 2, \quad J_{2,3}^{0,1}(3) = 0,$$

$$J_{3,3}^{0,1}(3) = 2 \int_0^1 \frac{1 - \xi_2}{\xi_2} d\xi_2 = -2.$$

Hyperingular

Integrals of the type $J_{q,3}^{0,0}$

$$J_{1,3}^{0,0}(3) = \frac{1}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{2,3}^{0,0}(3) = -\frac{2}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{3,3}^{0,0}(3) = \frac{1}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0.$$

Integrals of the type $J_{q,5}^{2,0}$

$$J_{1,5}^{2,0}(3) = \frac{1}{3\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{2,5}^{2,0}(3) = -\frac{2}{3\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{3,5}^{2,0}(3) = \frac{1}{3\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0.$$

Integrals of the type $J_{q,5}^{0,2}$

$$J_{1,5}^{0,2}(3) = \frac{2}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{2,5}^{0,2}(3) = -\frac{4}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0,$$

$$J_{3,5}^{0,2}(3) = \frac{2}{\Delta}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0.$$

Integrals of the type $J_{q,5}^{1,1}$

$$J_{1,5}^{1,1}(3) = \frac{2}{3\Delta_3}\int_0^1 \frac{1}{\xi_1}d\xi_1 = 0\ , \quad J_{2,5}^{1,1}(3) = 0,$$

$$J_{3,5}^{1,1}(3) = \frac{2}{3\Delta_3}\left(\int_0^1 \frac{1}{\xi_1^2}d\xi_1 - \int_0^1 \frac{1}{\xi_1}d\xi_1\right) = -\frac{2}{3\Delta_3}.$$

11 Appendix C.

On the sides that contain nodal point $\mathbf{y}^q$, appear 1-D divergent integrals. For rectangular element they are sides 1-2 and 4-1, and for triangular element they are sides 3-1 and 2-3.

For rectangular element divergent integrals have the form

$$\int_{-1}^{1} \frac{1}{1+\xi}d\xi, \quad \int_{-1}^{1} \frac{1-\xi}{1+\xi}d\xi, \quad \int_{-1}^{1} \frac{1}{(1+\xi)^2}d\xi, \quad \int_{-1}^{1} \frac{1-\xi}{(1+\xi)^2}d\xi.$$

For their calculation let us consider adjacent elements, as it is follows from Fig.9.

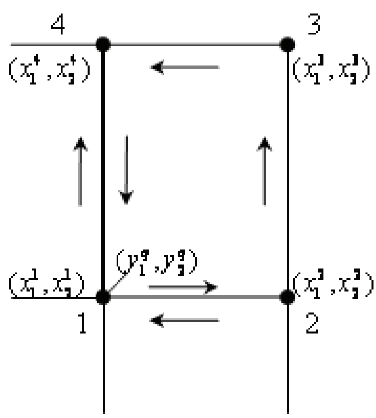

Figure 9:

Because in final BE equations have to be taken into account influence of all BEs adjacent to nodal point 1 we can consider the following sum of integrals

$\int_{-1}^{1} \frac{1}{1+\xi} d\xi + \int_{1}^{-1} \frac{1}{1+\xi} d\xi$, $\int_{-1}^{1} \frac{1-\xi}{1+\xi} d\xi + \int_{1}^{-1} \frac{1-\xi}{1+\xi} d\xi$ and $\int_{-1}^{1} \frac{1-\xi}{(1+\xi)^2} d\xi + \int_{1}^{-1} \frac{1-\xi}{(1+\xi)^2} d\xi$

We have taken into account that integration over side associated with adjacent BEs has to be done in opposite directions as it is shown on Fig. 9.

Easy calculations with considering Cauchy principal value and Hadamard's finite part integrals lead to the following result

$$\int_{-1}^{1} \frac{1}{1+\xi} d\xi + \int_{1}^{-1} \frac{1}{1+\xi} d\xi = 0,$$

$$\int_{-1}^{1} \frac{1-\xi}{1+\xi} d\xi + \int_{1}^{-1} \frac{1-\xi}{1+\xi} d\xi = 4,$$

$$\int_{-1}^{1} \frac{1-\xi}{(1+\xi)^2} d\xi + \int_{1}^{-1} \frac{1-\xi}{(1+\xi)^2} d\xi = -1$$

For triangular element divergent integrals have the form

$$\int_{0}^{1} \frac{1}{\xi} d\xi, \quad \int_{0}^{1} \frac{1}{\xi^2} d\xi.$$

In the same way as in the case of rectangular element for their calculation let us consider adjacent elements, as it is follows from Fig.10.

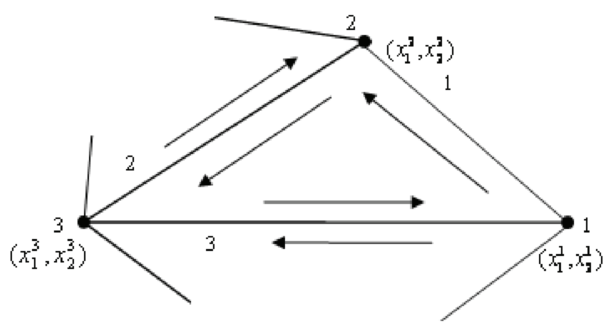

Figure 10:

Because in final BE equations have to be taken into account influence of all BEs adjacent to nodal point 3 we can consider the following sum of integrals

$$\int_0^1 \frac{1}{\xi} d\xi + \int_1^0 \frac{1}{\xi} d\xi, \quad \int_0^1 \frac{1}{\xi^2} d\xi + \int_1^0 \frac{1}{\xi^2} d\xi.$$

Easy calculations with considering Cauchy principal value and Hadamard's finite part integrals lead to the following result

$$\int_0^1 \frac{1}{\xi} d\xi + \int_1^0 \frac{1}{\xi} d\xi = 0, \quad \int_0^1 \frac{1}{\xi^2} d\xi + \int_1^0 \frac{1}{\xi^2} d\xi = -2.$$

We have taken into account that integration over side associated with adjacent BEs has to be done in opposite directions as it is shown on Fig. 10.

We obtain the same result if consider integral $\int_0^1 \frac{1}{\xi} d\xi$ and $\int_0^1 \frac{1}{\xi^2} d\xi$ in the sense of Hadamard's finite part

$$F.P. \int_0^1 \frac{1}{\xi} d\xi = \lim_{\xi \to 0} \left(\int_\xi^1 \frac{1}{\xi} d\xi - \ln \frac{1}{\xi} \right) = 0,$$

$$F.P. \int_0^1 \frac{1}{\xi^2} d\xi = \lim_{\xi \to 0} \left(\int_\xi^1 \frac{1}{\xi^2} d\xi - \frac{1}{\xi} \right) = -1.$$