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The Influence of the Imperfectness of the Interface Conditions on the Dispersion of the Axisymmetric Longitudinal Waves in the Pre-Strained Compound Cylinder

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Abstract: This paper studies the influence of the imperfectness of the interface conditions on the dispersion of the axisymmetric longitudinal waves in the pre-strained compound cylinder. The investigations are made within the framework of the piecewise homogeneous body model by utilizing the 3D linearized theory of elastic waves in elastic bodies with initial stresses. It is assumed that the layers of the compound cylinder are made from high elastic compressible materials and their elasticity relations are given through the harmonic potential. The shear spring type imperfectness of the interface conditions is considered and the degree of this imperfectness is estimated by the shear-spring parameter. Numerical results on the influence of this parameter on the behavior of the dispersion curves related to the fundamental mode are presented and discussed. In particular, it is established that as a result of the aforementioned imperfectness of the interface conditions, the dispersion curve related to the fundamental mode has two branches: the first disappears, but the second approaches the dispersion curve obtained for the perfect interface case by decreasing the shear-spring parameter.

Keywords: Compound cylinder, imperfect interface conditions, initial strains, longitudinal axisymmetric waves, wave dispersion.

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1 Introduction

Initial strains (or stresses) in construction elements are one of the reference details or factors which must be taken into account not only because of their static behavior but particularly for their dynamic behavior. It is known that these initial strains occur in structural elements during their manufacture and assembly, in the Earth's crust under the action of geostatic forces, in composite materials, etc. Moreover, construction elements are loaded by external forces, as well as by additional forces acting on the external forces, during the construction process. When it is necessary to identify the mechanical problems caused by these additional forces, then the stresses caused by the working load can be taken as the initial stresses.

Thus, the scope of the problem regarding the initially stressed body is significantly wide and it is of utmost importance to study it in both the practical as well as theoretical sense.

Wave propagation in pre-strained bodies has been studied by many researchers, the systematic analyses of which are given in the monographs by Biot (1965), Guz (2004) and Eringen and Suhubi (1975). A review of the recent investigations is given in a paper by Akbarov (2007). A considerable part of these studies relates to wave propagation in pre-strained cylinders and plates, for which a brief review is given below.

We begin this review by considering investigations related to harmonic wave propagation in a pre-strained layered medium. It is most likely that the study of these problems began with the paper by Hayes and Rivline (1961). In this paper the Rayleigh surface waves in a pre-stressed half-space were studied. It was assumed that the wave propagates in one of the principal directions of the initial strains and it was established that Rayleigh surface waves in a pre-strained half-space are not dispersive, just as in the classical linear theory of elastodynamics. After that, a large number of investigations were done in this field. We now consider some which were carried out in the last twenty years.

In a paper by Dowaik and Ogden (1991), an explicit form of the surface wave secular equation was obtained and analyzed for arbitrary strain-energy function and propagation along one principal pre-strain axis. The propagation of elastic interfacial waves (Stoneley waves) along the plane boundary separating two pre-strained compressible half-spaces was studied by Sotiropoulos (1998). It was assumed that the half-spaces were subjected to pure homogeneous finite strains, of which the principal directions are aligned, one direction being normal to the interface. A detailed analysis of the secular equation was performed for the case in which one of the media was a stress-free

A paper by Rogerson and Fu (1995) concerns an asymptotic analysis of dispersion relations for wave propagation in a pre-strained incompressible elastic plate. Asymptotic expansions for the wave speed as a function of wave number and pre-stress were obtained. The concrete numerical results were obtained for the case where the plate material constants satisfied special conditions. It was shown that these conditions satisfy the neo-Hookean and Mooney-Rivlin strain-energy functions.

Sandiford and Rogerson (2000) extended the investigation carried out by Rogerson and Fu (1995) for a slightly compressible pre-deformed elastic plate. From numerical investigations, the elasticity relations were given through the Varga strain energy function.

The problem considered in the paper by Sandiford and Rogerson (2000) was also considered by Nolde et al. (2004) for the cases where the elasticity relations of the plate material are described by the most general appropriate strain energy function. It utilized particular strain energy functions, such as compressible neo-Hookean, Varga and Blatz-Ko strain energy functions, to demonstrate different types of behavior.

In a paper by Rogerson and Sandiford (2000), the problem of extensional wave propagation in a pre-stressed incompressible 4-ply symmetrically-layered structure was considered. It was assumed that a 4-ply laminated plate which is symmetrical about its mid-plane consists of two identical outer layers. In fact, this paper studies the extensional wave propagation in a three-layered (sandwich) plate made from an incompressible material. Numerical investigations were made for the Mooney-Rivlin and Varga potentials.

The dispersion behavior of time-harmonic waves propagating along a principal direction in a perfectly bonded pre-stressed compressible elastic bi-material (bi-layered) plate was studied by Kayestha et al. (2010). From the numerical results obtained, either a two-parameter compressible neo-Hookean material or two-parameter compressible Varga material was used.

In all papers reviewed above, which are related to the wave propagation of pre-stressed plates, it has been assumed that the outer plane-surfaces are free of incremental surface traction. In other words, it was assumed that there is no external constraint which acts on the dispersion behavior of the time-harmonic waves which were considered. This was the basis of the study by Wijeyewickrema et al. (2008) in which the time-harmonic wave propagation in a pre-strained and constrained homogeneous compressible high elastic layer was investigated. Moreover, in the same paper the influence of the degree of this constraint on the dispersion relations and of the initial strains on the asymptotic behavior of these dispersion curves were studied.

It should be noted that all the foregoing investigations were made within the scope of the perfect contact conditions satisfied between the layers. The influence of the imperfectness of the mentioned contact conditions on the wave propagation in the pre-strained layered plates was the subject of the investigations by Leungvichcharoen and Wijeyewickrema (2003), Leungvichcharoen et al. (2004) and Wijeyewickrema and Leungvichcharoen (2009). Note that in the papers by Leungvichcharoen and Wijeyewickrema (2003) and Leungvichcharoen et al. (2004) the dispersive behavior of the symmetric and anti-symmetric waves, respectively in a pre-strained incompressible symmetric sandwich plate with imperfect interface conditions was analyzed. In the paper by Wijeyewickrema and Leungvichcharoen (2009) the investigations carried out in the papers by Leungvichcharoen and Wijeyewickrema (2003), Leungvichcharoen et al. (2004) were developed for the pre-strained compressible symmetric sandwich plate. Note that in these papers the plane-strain state was considered and for the numerical examples, either neo-Hookean material or Varga material was assumed.

Now we consider a review of the investigations related to time-harmonic wave dispersion in the pre-strained homogeneous and layered cylinders with circular cross sections which are directly relevant to the present paper. Note that the pioneer work in this field was made by Green (1961) in which the torsional wave propagation in the pre-stretched homogeneous cylinder was studied. The paper by Demiray and Suhubi (1970) analyzed the axisymmetric wave propagation in an initially twisted circular cylinder. It was established that the initial twisting of the circular cylinder causes the coupled wave propagation field between the axisymmetric torsional and longitudinal waves to occur. In other words, it was established that in the initially twisted circular cylinder the axisymmetric torsional and longitudinal waves cannot be propagated separately. However, in the paper by Demiray and Suhubi (1970), as an example of numerical results, only the approximate analytical expression for the perturbation of the torsional oscillation frequency caused by the initial twisting is given.

In a paper by Belward (1976), the wave propagation in a pre-strained cylinder made from an incompressible material was studied. Initial strains in the cylinder were determined within the framework of the non-linear theory of elasticity. The study of the longitudinal wave propagation in the homogeneous cylinder was also a subject of papers which are listed and discussed in a monograph by Guz (2004).

Note that in the foregoing papers the subject of the study was the homogeneous circular cylinder. Before the beginning of the 21^{st} century there was no investigation of the wave propagation problems in a pre-stressed bi-material compounded cylinder. The first attempt in this field was made by Akbarov and Guz (2004) in which it was assumed that the initial stretching is small and the initial stress state in the compound cylinder is calculated within the scope of the first version of the small initial deformation theory, the meaning of which was described in the monographs by Guz (1999, 2004).

A paper by Akbarov and Guliev (2009) extended the work by Akbarov and Guz (2004) to the case where the initial strains are finite and the mechanical relations of their materials are assumed to be compressible and are given by the harmonic-type potential. Within the same assumptions the influence of the finite initial strains on the axisymmetric wave dispersion in a circular cylinder embedded in a compressible elastic medium was studied in a work by Akbarov and Guliev (2010).

It should be noted that in the investigations reviewed above it was assumed that the initial strains are caused by the uni-axial stretching or compression along the wave propagation direction, i.e. along the cylinders. The dispersion of the axisymmetric longitudinal wave in the initially twisted compound cylinder was a subject of a paper by Akbarov et al. (2011). It was assumed that in the initial state the cylinders are twisted and each of them has a constant twist per unit length and this initial stress-strain state is determined within the scope of the classical linear theory of elasticity. The materials of the constituents are isotropic and homogeneous.

In all the foregoing papers it is assumed that the contact condition on the interface between the inner and outer cylinders is a perfect one; i.e., it is assumed that the forces and displacements are continuous functions across the interface surface. However, in many cases (for an example, in the case where the reinforced cables are modeled as bi-material compounded cylinders), it is unrealistic to assume a perfectly bounded interface. Consequently, in order to apply the results of the theoretical investigations to the indicated cases, it is necessary to take the imperfectness of the contact conditions into account in the study of the wave propagation in the bi-material compounded circular cylinders. Note that the study of the torsional wave propagation in the bi-material compounded cylinder (without initial stresses) with an imperfect interface is studied in the paper by Berger et al. (2000) in which the imperfection of the contact condition is presented according to the model proposed by Jones and Whitter (1967). Investigations of a similar type for the initially stressed bi-material compound cylinder have been carried out in a paper by Kepceler (2010). It is assumed that the elasticity relations of the cylinders' materials are given through the Murnaghan potential.

To the best of the authors' knowledge, up to now there has not been any investigation related to the study of the influence of the imperfectness of the contact conditions on the axisymmetric longitudinal wave propagation either in the compound cylinder with initial strains or in the compound cylinder without initial strains. Taking this statement into account, in the present paper the effect of the specified imperfectness of the contact conditions on the dispersion of the longitudinal axisymmetric waves in the pre-strained bi-material compound cylinder is studied. It is assumed that the materials of the constituents are high elastic compressible ones and the elasticity relations of those are described by the harmonic potential. The numerical results of this effect are presented and discussed.

2 Formulation of the problem

We consider the bi-material compounded cylinder which consists of a solid inner cylinder and hollow surrounding cylinder. The geometry of the cylinders is shown schematically in Fig. 1.

In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the Cartesian system of coordinates $Oy_1y_2y_3$ as well as in the cylindrical system of coordinates $Or\theta y_3$. Assume that in the natural state, the radius of the solid cylinder is R and the thickness of the external surrounding cylinder is h. Moreover, assume that the cylinders have infinite length in the direction of the Oy_3 axis and the initial stress state in each component of the considered body is axisymmetric with respect to this axis and that it is homogeneous. Such a stress field may be present with stretching of the considered body along the Oy_3 axis.

The stretching may be conducted for the all constituents of the cylinder separately before they are compounded. However, the results which will be discussed below can also be related to the case where the solid inner and hollow external cylinders are stretched together after the compounding. In this case, as a result of the difference in "Poisson's coefficients" of the inner and external cylinders' materials, the inhomogeneous initial stresses acting on the areas which are parallel to the Oy_3 axis arise. Nevertheless, according to well known mechanical considerations, the mentioned inhomogeneous initial stresses under consideration can be neglected because these stresses are less significant than those acting on the areas which are perpendicular to the Oy_3 axis.

With the initial state of the cylinders we associate the Lagrangian cylindrical system of coordinates $O'r'\theta' y'_3$ and the Cartesian system of coordinates $O'y'_1y'_2y'_3$. The values related to the solid inner cylinder and hollow external cylinder will be denoted by the upper indices (2) and (1), respectively. Furthermore, we denote the values related to the initial state by an additional upper index, 0. Thus, the initial strain state in the solid inner cylinder and hollow external cylinder can be determined as follows:

$$u_m^{(k),0} = (\lambda_m^{(k)} - 1)y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = const, \quad m = 1,2,3; \quad k = 1,2, \tag{1}$$

where $u_m^{(k),0}$ is the displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m axis. We introduce the following notation:

$$y_i' = \lambda_i^{(k)} y_i, \quad r' = \lambda_1^{(k)} r, \quad R' = \lambda_1^{(2)} R. \tag{2}$$

The values related to the system of the coordinates associated with the initial state below, i.e. with $O'y_1'y'_2y_3'$, will be denoted by an upper prime.

Within this framework, let us investigate the axisymmetric wave propagation along the $O'y_3'$ axis in the considered body. We do this investigation with the use of coordinates r' and y_3' in the framework of the TLTEWISB. We will follow the style and notation used in the paper by Akbarov and Guliev (2008). Thus, we write the basic relations of the TLTEWISB for the compressible body under an axisymmetrical state. These relations are satisfied within each constituent of the considered body because we use the piecewise homogeneous body model.

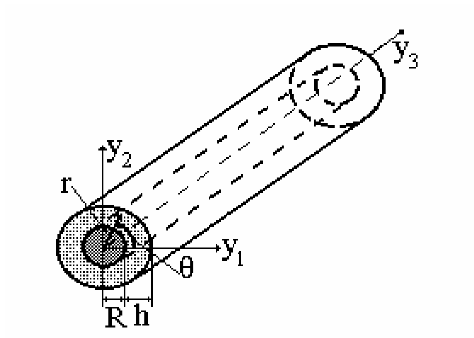

Figure 1: The geometry of the compound cylinder.

The equations of motion are:

$$\frac{\partial}{\partial r'} Q'^{(k)}_{r'r'} + \frac{\partial}{\partial y_3'} Q'^{(k)}_{r'3} + \frac{1}{r'}\left(Q'^{(k)}_{r'r'} - Q'^{(k)}_{\theta'\theta'}\right) = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'} Q'^{(k)}_{3r'} + \frac{\partial}{\partial y_3'} Q'^{(k)}_{33} + \frac{1}{r'} Q'^{(k)}_{3r'} = \rho'^{(k)} \frac{\partial^2}{\partial t^2} u'^{(k)}_{3}. \tag{3}$$

The mechanical relations are:

$$Q'^{(k)}_{r'r'} = \omega'^{(k)}_{1111} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{1122} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{1133} \frac{\partial u'^{(k)}_{3}}{\partial y_3'},$$

$$Q'^{(k)}_{\theta'\theta'} = \omega'^{(k)}_{2211}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{2222}\frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{2233}\frac{\partial u'^{(k)}_{3}}{\partial y'_3},$$

$$Q'^{(k)}_{33} = \omega'^{(k)}_{3311}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{3322}\frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{3333}\frac{\partial u'^{(k)}_{3}}{\partial y'_3},$$

$$Q'^{(k)}_{r'3} = \omega'^{(k)}_{1313}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \omega'^{(k)}_{1331}\frac{\partial u'^{(k)}_{3}}{\partial r'},$$

$$Q'^{(k)}_{3r'} = \omega'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \omega'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}. \tag{4}$$

In (3) and (4) through $Q'^{(k)}_{r'r'}, \ldots, Q'^{(k)}_{3r'}$, the perturbation of the components of the Kirchoff stress tensor are denoted. The notation $u'^{(k)}_{r'}$, $u'^{(k)}_{3}$ shows the perturbations of the components of the displacement vector. The constants $\omega'^{(k)}_{1111}, \ldots, \omega'^{(k)}_{3333}$ in (3) and (4) are determined through the mechanical constants of the inner and outer cylinders' materials and through the initial stress state. $\rho'^{(k)}$ is the density of the k-th material.

As noted above, in the present investigation we assume that the elasticity relations of the cylinders' materials are described by harmonic potential. This potential is given as follows:

$$\Phi = \frac{1}{2}\lambda s_1^2 + \mu s_2 \tag{5}$$

where:

$$s_1 = \sqrt{1+2\varepsilon_1} + \sqrt{1+2\varepsilon_2} + \sqrt{1+2\varepsilon_3} - 3,$$

$$s_2 = \left(\sqrt{1+2\varepsilon_1} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_2} - 1\right)^2 + \left(\sqrt{1+2\varepsilon_3} - 1\right)^2. \tag{6}$$

In relations (5) and (6), λ, μ are material constants, $\varepsilon_i (i = 1,2,3)$ are the principal values of Green's strain tensor. The expressions (5) and (6) are supplied by the corresponding indices under the solution procedure. Within the scope of the non-linear theory of elasticity for the considered axisymmetric case the components of Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{r3} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial y_3} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right),$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2. \tag{7}$$

In this case, the components S_{ij} of the Lagrange stress tensor are calculated as follows:

$$S_{rr} = \frac{\partial \Phi}{\partial \varepsilon_{rr}}, \quad S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}}, \quad S_{33} = \frac{\partial \Phi}{\partial \varepsilon_{33}}, \quad S_{r3} = \frac{1}{2}\left(\frac{\partial \Phi}{\partial \varepsilon_{r3}} + \frac{\partial \Phi}{\partial \varepsilon_{3r}}\right), \quad S_{r3} = S_{3r}. \tag{8}$$

Note that the expressions (6)-(8) are written in the arbitrary cylindrical coordinate system without any restriction related to the association of this system to the natural or initial state of the considered compound cylinders.

For the considered case, the relations between the perturbation of the Kirchoff stress tensor and the perturbation of the components of the Lagrange stress tensor are obtained as follows:

$$Q'^{(k)}_{r'r'} = \lambda_1^{(k)} S^{(k)}_{r'r'}, Q'^{(k)}_{\theta'\theta'} = \lambda_1^{(k)} S^{(k)}_{\theta'\theta'}, \quad Q'^{(k)}_{33} = \left(\lambda_3^{(k)}\right)^2 S^{(k)}_{33} + \lambda_3^{(k)} S^{(k),0}_{33} \frac{\partial u_3^{(k)}}{\partial y_3},$$

$$Q'^{(k)}_{r'3} = \left(\lambda_1^{(k)}\right)^{-1} S^{(k)}_{r'3}, \quad Q'^{(k)}_{3r'} = \left(\lambda_1^{(k)}\right)^{-1} S^{(k)}_{3r'} + \lambda_3^{(k)} S^{(k),0}_{33} \frac{\partial u_r^{(k)}}{\partial y_3}. \tag{9}$$

According to Guz (2004), by linearization of equations (6)-(8) and taking (9) and (1) into account, we obtain the following expressions for the stress $S_{33}^{(k),0}$ and for the constants $\lambda_2^{(k)}$, $\lambda_1^{(k)}$, $\omega'^{(k)}_{1111}, \ldots, \omega'^{(k)}_{3333}$ in (4) for the potential (5):

$$S_{33}^{(k),0} = \left[\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right) + 2\mu^{(k)}\left(\lambda_3^{(k)} - 1\right)\right]\left(\lambda_3^{(k)}\right)^{-1},$$

$$\lambda_2^{(k)} = \lambda_1^{(k)} = \left[2 - \frac{\lambda^{(k)}}{\mu^{(k)}}\left(\lambda_3^{(k)} - 3\right)\right]\left[2\left(\frac{\lambda^{(k)}}{\mu^{(k)}} + 1\right)\right]^{-1},$$

$$\omega'^{(k)}_{1111} = \left(\lambda_3^{(k)}\right)^{-1}\left(\lambda^{(k)} + 2\mu^{(k)}\right),$$

$$\omega'^{(k)}_{3333} = \left(\frac{\lambda_3^{(k)}}{\lambda_1^{(2)}}\right)^2\left(\lambda^{(k)} + 2\mu^{(k)}\right), \quad \omega'^{(k)}_{1122} = \left(\lambda_3^{(k)}\right)^{-1}\lambda^{(k)},$$

$$\omega'^{(k)}_{1133} = \left(\lambda_1^{(k)}\right)^{-1} \lambda^{(k)}, \quad \omega'^{(k)}_{1221} = \left(\lambda_3^{(k)}\right)^{-1} \mu^{(k)},$$

$$\omega'^{(k)}_{1313} = 2\mu^{(k)} \left(\lambda_1^{(k)} + \lambda_3^{(k)}\right)^{-1}, \quad \omega'^{(k)}_{3113} = 2\mu^{(k)} \left(\lambda_1^{(k)}\right)^{-2} \left(\lambda_3^{(k)}\right)^{2} \left(\lambda_1^{(k)} + \lambda_3^{(k)}\right)^{-1}. \tag{10}$$

Thus, propagation of the longitudinal axisymmetric wave in the considered system will be investigated by the use of Eqs. (3), (4), (9) and (10). These equations must be supplied with the corresponding boundary and contact conditions. First, we consider the boundary conditions which can be written as follows:

$$\left. Q'^{(1)}_{r'r'} \right|_{r'=R'+h'^{(1)}} = 0, \quad \left. Q'^{(1)}_{r'z'} \right|_{r'=R'+h'^{(1)}} = 0. \tag{11}$$

Now we consider the formulation of the incomplete contact conditions on the interface surface between the inner and outer cylinders. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities in the displacements and forces across the interface mentioned. A review of the mathematical modeling of the various type incomplete contact conditions for elastodynamic problems has been detailed in a paper by Martin (1992). It follows from this paper that for most models the discontinuity of the displacement $\mathbf{u}^+$ and force $\mathbf{f}^+$ vectors on one side of the interface are assumed to be linearly related to the displacement $\mathbf{u}^-$ and force $\mathbf{f}^-$ vectors on the other side of the interface. This statement, as in the paper by Rokhlin and Wang (1991), can be presented as follows:

$$[\mathbf{f}] = \mathbf{C}\mathbf{u}^- + \mathbf{D}\mathbf{f}^-, \quad [\mathbf{u}] = \mathbf{G}\mathbf{u}^- + \mathbf{F}\mathbf{f}^-, \tag{12}$$

where **C**, **D**, **G** and **F** are three-dimensional (3×3) matrices and the square brackets indicate a jump in the corresponding quantity across the interface. Consequently, if the interface is at $r' = R'$, then:

$$[\mathbf{u}] = \mathbf{u}|_{r'=R'+0} - \mathbf{u}|_{r'=R'-0} \quad , \quad [\mathbf{f}] = \mathbf{f}|_{r'=R'+0} - \mathbf{f}|_{r'=R'-0}. \tag{13}$$

It follows from (12) that we can write imperfect contact conditions for various particular cases by the selection of the matrices **C**, **D**, **G** and **F**. One of these selections was made in the paper by Jones and Whitter (1967), according to which, it is assumed that $\mathbf{C} = \mathbf{D} = \mathbf{G} = \mathbf{0}$. In this case it is obtained from (13) that:

$$[\mathbf{f}] = \mathbf{0}, \quad [\mathbf{u}] = \mathbf{F}\mathbf{f}^-, \tag{14}$$

where **F** is a constant diagonal matrix. The model (14) significantly simplifies the solution procedure of the corresponding problems and is sufficiently adequate with many real cases. Therefore, this model (i.e. the model (14)) is called a shear-spring type resistance model and has been used in many investigations carried out within the framework of the classical elastodynamics model by Jones and Wittier (1967), Mul and Xu (1989), Pilarski and Rose (1988) and Berger, Martin and McCaffery (2000). Moreover, this model has also been used in the papers by Leungvichcharoen and Wijeyewickrema (2003), Leungvichcharoen et al. (2004), Wijeyewickrema and Leungvichcharoen (2009) and Kepceler (2010) for the dynamics of pre-strained systems. Thus, we also use the model (14) for the mathematical formulation of the imperfectness of contact conditions. For the problems under consideration these conditions can be written as follows:

$$Q'^{(1)}_{r'r'}\Big|_{r'=R'} = Q'^{(2)}_{r'r'}\Big|_{r'=R'},\ Q'^{(1)}_{r'z'}\Big|_{r'=R'} = Q'^{(2)}_{r'z'}\Big|_{r'=R'},\quad u'^{(1)}_{r'}\Big|_{r'=R'} = u'^{(2)}_{r'}\Big|_{r'=R'},$$

$$u'^{(1)}_{z'}\Big|_{r'=R'} - u'^{(2)}_{z'}\Big|_{r'=R'} = F\frac{R}{\mu^{(2)}}Q'^{(2)}_{r'z'}, \tag{15}$$

where F is the non-dimensional shear-spring parameter. The case where $F=0$ corresponds with the perfect contact condition, but the case where $F=\infty$ corresponds with the fully slipping imperfectness of the contact condition.

With this we exhaust the formulation of the problem. It should be noted that in the case where $\lambda_3^{(k)} = \lambda_1^{(k)} = 1.0$, $(k=1,2)$, the above described formulation transforms to the corresponding one of the classical linear theory of elastodynamics for the compressible body.

3 Solution procedure and obtaining the dispersion equation

Substituting (4) in (3) we obtain the equation of motion in displacement terms. For solution to this equation, according to the monograph by Guz (2004), we use the following representation for the displacement:

$$u'^{(k)}_{r'} = -\frac{\partial^2}{\partial r'\partial y'_3}\mathrm{X}^{(k)},$$

$$u'^{(k)}_{3} = \frac{1}{\omega'^{(k)}_{1133}+\omega'^{(k)}_{1313}}\left(\omega'^{(k)}_{1111}\Delta'_1 + \omega'^{(k)}_{3113}\frac{\partial^2}{\partial y'^2_3} - \rho'^{(k)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(k)}, \tag{16}$$

where $X^{(k)}$ satisfies the following equation:

$$\left[\left(\Delta_1' + \left(\xi'^{(k)}_2\right)^2 \frac{\partial^2}{\partial y_3'^2}\right)\left(\Delta_1' + \left(\xi'^{(k)}_3\right)^2 \frac{\partial^2}{\partial y_3'^2}\right) - \right.$$
$$-\rho'^{(k)}\left(\frac{\omega'^{(k)}_{1111} + \omega'^{(k)}_{1331}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\Delta_1' + \frac{\omega'^{(k)}_{3333} + \omega'^{(k)}_{3113}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\frac{\partial^2}{\partial y_3'^2}\right)\frac{\partial^2}{\partial t^2} +$$
$$\left. + \frac{\rho'^{(k)}}{\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}}\frac{\partial^4}{\partial t^4}\right] X^{(k)} = 0. \quad (17)$$

In (16) and (17) the following notation is used:

$$\Delta_1' = \frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'},$$

$$\left(\xi'^{(k)}_{2,3}\right)^2 = d^{(k)} \pm \left[\left(d^{(k)}\right)^2 - \omega'^{(k)}_{3333}\omega'^{(k)}_{3113}\left(\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\right]^{\frac{1}{2}},$$

$$d^{(k)} = \left(2\omega'^{(k)}_{1111}\omega'^{(k)}_{1331}\right)^{-1}\left[\omega'^{(k)}_{1111}\omega'^{(k)}_{3333} + \omega'^{(k)}_{1331}\omega'^{(k)}_{3113} - \left(\omega'^{(k)}_{1133} + \omega'^{(k)}_{1313}\right)\right]. \quad (18)$$

We represent the function $X^{(m)} = X^{(m)}(r', y'_3, t)$ as:

$$X^{(m)} = X_1^{(m)}(r')\cos\left(ky'_3 - \omega t\right), \quad m = 1, 2. \quad (19)$$

Substituting (19) in (17) and by doing some manipulations we obtain the following equation for $X_1^{(m)}(r')$:

$$\left(\Delta_1' + \left(\zeta'^{(m)}_2\right)^2\right)\left(\Delta_1' + \left(\zeta'^{(m)}_3\right)^2\right) X_1^{(m)}(r') = 0. \quad (20)$$

The constants $\zeta'^{(k)}_{2,3}$ are determined from the following equation:

$$\omega'^{(m)}_{1111}\omega'^{(m)}_{1331}\left(\zeta'^{(m)}\right)^4 -$$
$$-k^2\left(\zeta'^{(m)}\right)^2\left[\omega'^{(m)}_{1111}\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3333}\right) + \omega'^{(m)}_{1331}\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3113}\right) + \right.$$
$$\left. + \left(\omega'^{(m)}_{1133} + \omega'^{(m)}_{1313}\right)^2\right] + k^4\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3333}\right)\left(\rho^{(m)}c^2 - \omega'^{(m)}_{3113}\right) = 0, \quad (21)$$

where $c = \omega/k$, i.e. c is the phase velocity of the propagating wave. We determine the following expression for $X_1^{(m)}(r')$ from equations (20) and (21):

$$\begin{aligned} X_1^{(1)}(r') = & A_2^{(1)} E_0^{(1)}\left(kr'\zeta'^{(1)}_2\right) + A_3^{(1)} E_0^{(1)}\left(kr'\zeta'^{(1)}_3\right) + \\ & + B_2^{(1)} G_0^{(1)}\left(kr'\zeta'^{(1)}_2\right) + B_3^{(1)} G_0^{(1)}\left(kr'\zeta'^{(1)}_3\right), \end{aligned}$$

$$X_1^{(2)}(r') = A_2^{(2)} E_0^{(2)}\left(kr'\zeta'^{(2)}_2\right) + A_3^{(2)} E_0^{(2)}\left(kr'\zeta'^{(2)}_3\right). \qquad (22)$$

where:

$$E_0^{(k)}\left(kr'\zeta'^{(k)}_m\right) = \begin{cases} J_0\left(kr'\zeta'^{(k)}_m\right) & if \quad \left(\zeta_m^{(k)}\right)^2 > 0, \\ I_0\left(kr'\left|\zeta'^{(k)}_m\right|\right) & if \quad \left(\zeta_m^{(k)}\right)^2 < 0, \end{cases}$$

$$G_0^{(1)}\left(kr'\zeta'^{(k)}_m\right) = \begin{cases} Y_0\left(kr'\zeta'^{(1)}_m\right) & if \quad \left(\zeta_m^{(1)}\right)^2 > 0, \\ K_0\left(kr'\left|\zeta'^{(1)}_m\right|\right) & if \quad \left(\zeta_m^{(1)}\right)^2 < 0. \end{cases} \qquad (23)$$

In (23) $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind of order zero; $I_0(x)$ and $K_0(x)$ are respectively, Bessel functions of a purely imaginary argument of order zero and Macdonald functions of order zero.

Thus, using the equations (4), (16), (19), (22), and (23) we obtain the following dispersion equation from (11) and (15):

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i; j = 1,2,3,4,5,6, \qquad (24)$$

where:

$$\alpha_{ij} = \alpha_{ij}\left(c/c_2^{(2)}, kR, F, \mu^{(2)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \lambda^{(1)}/\mu^{(1)}, \lambda_3^{(2)}, \lambda_3^{(1)}\right). \qquad (25)$$

To reduce the size of the article we do not give here the explicit expressions of α_{ij}. Thus, the dispersion equation is obtained in the forms (24) and (25).

4 Numerical results and discussions

Assume that $\rho^{(2)}/\rho^{(1)} = 1.0$, $\lambda^{(2)}/\mu^{(2)} = \lambda^{(1)}/\mu^{(1)} = 1.5$ and consider the dispersion curves $c = c(kR)$ and analyze the influence of the non-dimensional shear-spring parameter F on these curves for various values of elongation parameters

$\lambda_3^{(2)}$ and $\lambda_3^{(1)}$. To simplify the following discussions we introduce the following notation:

$$c_{20}^{(k)} = \sqrt{\frac{\mu^{(k)}}{\rho^{(k)}}}, \quad c_{10}^{(k)} = \sqrt{\frac{\lambda^{(k)} + 2\mu^{(k)}}{\rho^{(k)}}},$$

$$c_2^{(k)}(\lambda_3^{(k)}) = \sqrt{\frac{\omega'^{(k)}_{1313}}{\rho'^{(k)}}}, \quad c_1^{(k)}(\lambda_3^{(k)}) = \sqrt{\frac{\omega'^{(k)}_{3333}}{\rho'^{(k)}}} \tag{26}$$

where $c_{20}^{(k)} = c_2^{(k)}(1.0)$, $c_{10}^{(k)} = c_1^{(k)}(1.0)$. Moreover, through $c_{2R}^{(1)}$ we denote the Rayleigh wave velocity in the pre-strained outer cylinder material.

4.1 On the calculation algorithm

The numerical results of the dispersion of the considered wave propagation problem are obtained from the numerical solution to equation (25) which is solved by utilizing the well known “bisection method”. In this case, for fixed values of the problem parameters for each value of kR, the roots of the dispersion equation with respect to $c/c_2^{(2)}$ are found.

In the present paper the main purpose of the numerical investigations is the study of the influence of shear-spring type imperfectness on the contact conditions between the inner and outer cylinders of the pre-strained compound cylinder on the fundamental modes. However, for construction of the dispersion curves, corresponding to these modes it is necessary to use the certain N number roots of equation (25). In this case, the graphs of the dependencies among the roots $\left(c/c_2^{(2)}\right)_1$, $\left(c/c_2^{(2)}\right)_2$, $\left(c/c_2^{(2)}\right)_N$ and kR create the net on the plane $\left\{c/c_2^{(2)}, kR\right\}$. Note that, in general, the graph corresponding to the dependence between $\left(c/c_2^{(2)}\right)_n$ and kR, contains dispersive and non-dispersive parts related to various dispersion modes. Below, under construction of the dispersion curves, we mainly use the dispersive parts of these graphs.

Now we return to the analysis of the numerical results on the influences of the non-dimensional shear-spring parameter F. First we consider the case where the initial strains are absent in the constituents of the compound cylinder.

4.2 Numerical results related the case where $\lambda_3^{(2)} = \lambda_3^{(1)} = 1.0$

We differentiate between two cases with respect to the ratio of the material stiffness of the outer and inner cylinders. In the first (the second) case we assume that the inner cylinder material is stiffer (softer) than that of the outer cylinder material.

According to this assumption in the first case (the second case) the numerical results are obtained for $\mu^{(2)}/\mu^{(1)}$ =2.0 ($\mu^{(2)}/\mu^{(1)}$ =0.5).

Thus, consider these numerical results which are given in Fig. 2 (in Fig. 3) obtained for the case where $\mu^{(2)}/\mu^{(1)}$ =2.0 ($\mu^{(2)}/\mu^{(1)}$ =0.5). These results (dispersion curves) are given for the cases where $h/R = 1.0$ (Figs.2a, 3a), 0.5 (Figs.2b, 3b), 0.3 (Figs.2c, 3c) and 0.1 (Figs.2d, 3d). Note that in each of the foregoing figures the dispersion curves are constructed for the various values of the dimensionless shear-spring parameter F which characterizes the degree of the imperfectness of the contact condition.

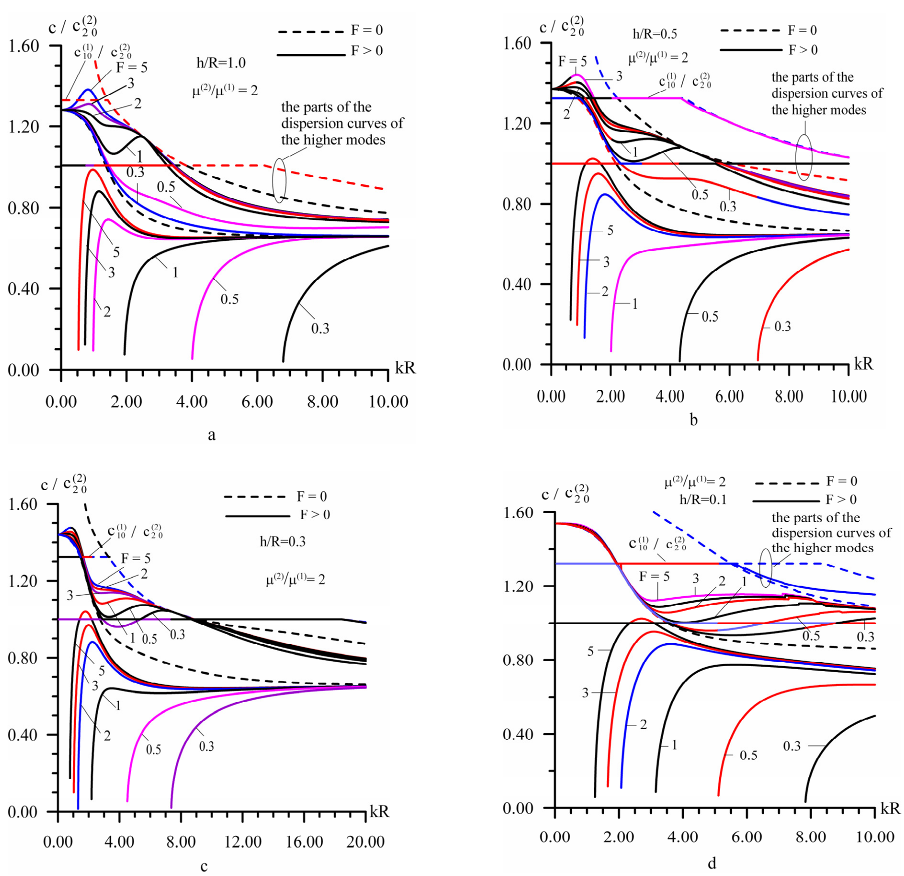

Figure 2: The influence of the shear-spring parameter F on the dispersion curves of the first mode constructed where $\mu^{(2)}/\mu^{(1)} = 2.0$, $\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0$ for the cases where $h/R = 1.0$(a); 0.5 (b); 0.3 (c) and 0.1(d).

The analyses of the foregoing numerical results show that as a result of the shear-spring type imperfectness of the contact conditions, instead of the dispersion curve corresponding to the fundamental dispersive mode constructed under satisfaction of perfect contact conditions, i.e. for $F = 0$, two types of mode arise. The first (the second) appears below (over) the dispersion curve corresponding to the first fundamental mode constricted by $F = 0$. Throughout the discussion below the aforementioned first (second) type dispersion curves will be called the first (second) branch of the fundamental mode obtained under imperfect contact conditions, i.e. for $F > 0$.

We denote the velocity of the wave propagation for $F = 0$ with c, but the wave propagation velocity of the first (second) branch for $F > 0$we denote with c_{IF} (c_{IIF}). It follows from the numerical results given in Figs. 2 and 3 that the following relation takes place:

$$c_{IF} < c < c_{IIF}. \tag{27}$$

Consider the low wave number limit as $kR \to 0$ for both the first and the second branches of the dispersion curves. The numerical results show that the first branch of the dispersion curves has "cut off" values for kR(denoted by $(kR)_{cf.}$), i.e. the dispersion curves related to this branch appear after certain values of kR. In this case the values of $(kR)_{cf.}$ depend on the non-dimensional shear-spring parameter F. According to the numerical results, we can conclude that:

$$(kR)_{cf.} \to \infty \text{ as } F \to 0. \tag{28}$$

The wave propagation velocity in the second branch of the fundamental mode has a finite limit as $kR \to 0$ and this limit coincides with that obtained for the case where the contact conditions are perfect, i.e. for the case where $F = 0$. As shown in papers by Lai et al. (1971) and Akbarov and Guliev (2009), the specified limit is determined by the following expression:

$$\frac{c_{IIF}}{c_{20}^{(2)}} = \left[\frac{e^{(2)}\eta^{(2)} + e^{(1)}\eta^{(1)}\mu^{(1)}/\mu^{(2)}}{\eta^{(2)} + \eta^{(1)}\rho^{(1)}/\rho^{(2)}}\right]^{\frac{1}{2}} \text{ as } kR \to 0, \tag{29}$$

where:

$$e^{(k)} = 2\left(1 + \frac{\lambda^{(k)}}{2\left(\lambda^{(k)} + \mu^{(k)}\right)}\right),$$

$$\eta^{(2)} = \left(1 + \frac{h}{R}\right)^{-2}, \quad \eta^{(1)} = \left(2\frac{h}{R} + \left(\frac{h}{R}\right)^2\right)\left(1 + \frac{h}{R}\right)^{-2}. \tag{30}$$

This statement, i.e. the independence of the low wave number limit as $kR \rightarrow 0$ on the imperfectness of the contact conditions agrees with the physical considerations and has been also

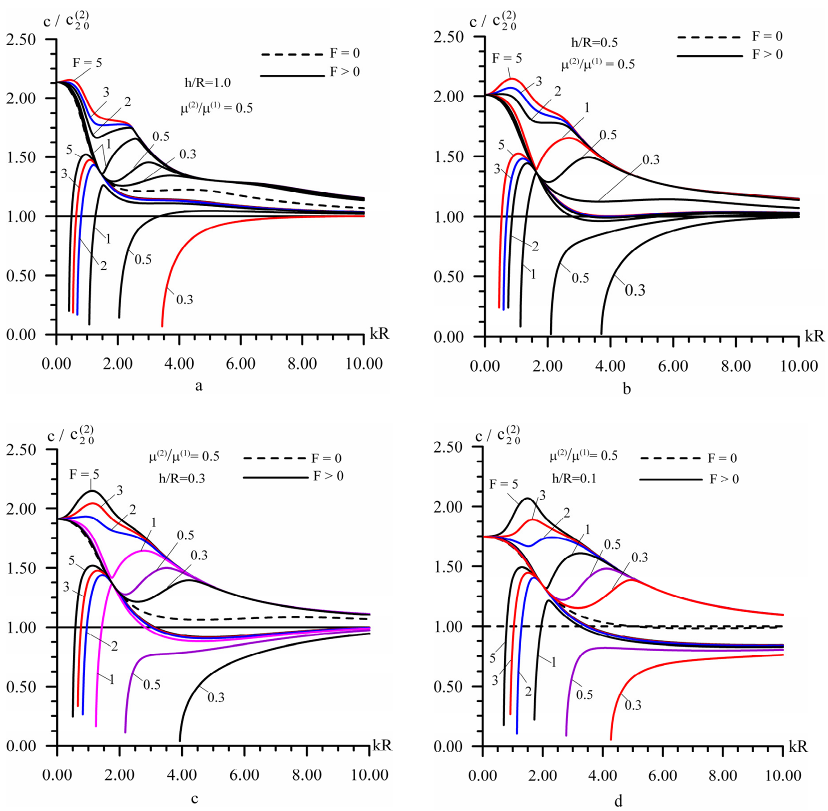

Figure 3: The influence of the shear-spring parameter F on the dispersion curves of the first mode constructed where $\mu^{(2)}/\mu^{(1)} = 0.5$, $\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0$ for the cases where $h/R = 1.0$ (a); 0.5 (b); 0.3 (c) and 0.1(d).

pointed out in papers by Berger et al. (2000) and Kepceler (2010) for torsional wave propagation in a compounded cylinder.

Consider the high wave number limit values as $kR \rightarrow \infty$. It follows from Figs. 2 and 3 and other results obtained (which are not given here) that for the values of $kR >> 10$, the following high wave number limit values occur:

For the case where $\mu^{(2)}/\mu^{(1)} = 2.0$

$$c_{IF} \rightarrow c_R^{(1)} - 0, \quad c_{IIF} \rightarrow c_R^{(1)} + 0 \text{ as } kR \rightarrow \infty. \tag{31}$$

For the case where $\mu^{(2)}/\mu^{(1)}$ =0.5

$$c_{IF} \to c_2^{(2)} - 0, \quad c_{IIF} \to c_2^{(2)} + 0 \text{ as } kR \to \infty. \tag{32}$$

In expressions (31) and (32), $c_R^{(1)}$ ($c_2^{(2)}$) is the Rayleigh (shear) wave velocity in the outer (inner) cylinder material. Note that these expressions can be generalized for all possible values of $\mu^{(2)}/\mu^{(1)}$ as follows:

$$c_{IF} \to \min\left\{c_R^{(1)} - 0,\ c_2^{(2)} - 0,\ c_S - 0\right\},$$

$$c_{IIF} \to \min\left\{c_R^{(1)} + 0,\ c_2^{(2)} + 0,\ c_S + 0\right\} \text{ as } kR \to \infty, \tag{33}$$

where c_S is the Stoneley Wave velocity. It is known that the Stoneley waves exist only for certain pairs of materials. As for the pair of materials considered in the present investigation, the Stoneley waves do not exist, therefore they are not observed in the results given in Figs. 2 and 3.

Now we analyze the character of the dispersion curves. The dispersion curves obtained for the first branch show that after a certain value (denoted by F_{I*}) of the shear-spring parameter F, the dependence between c_{IF} and kR becomes non-monotonic. In other words, in the cases where $F > F_{I*}$ in the near right vicinity of $(kR)_{cf.}$, the values of c_{IF} (denoted by $(kR)_{Icr.}$) increase sharply with kR and have their maximum at a certain value of kR. So, we can write:

$$\left.\frac{dc_{IF}}{d(kR)}\right|_{kR=(kR)_{Icr.}} = 0. \tag{34}$$

The results show that in the case where $\mu^{(2)}/\mu^{(1)} = 2$ for $kR > (kR)_{Icr.}$the values of c_{IF} decrease monotonically with kR. However, in the case where $\mu^{(2)}/\mu^{(1)} = 0.5$, the character of the dependence between c_{IF} and kR for $kR > (kR)_{Icr.}$ depends on the values of h/R and of F. For instance, for $F \geq 1$ where $h/R = 0.3$ the mentioned dependence is non-monotonic, but where $h/R = 0.1$, it is monotonic. Remember that the foregoing results occur for the case where $F > F_{I*}$ and the values of F_{I*} depend mainly on the values of $\mu^{(2)}/\mu^{(1)}$. The results show that for the case where $\mu^{(2)}/\mu^{(1)} = 2$ ($\mu^{(2)}/\mu^{(1)} = 0.5$) it can be approximately assumed that $F_{I*} = 2$ ($F_{I*} = 1$). The results also show that in the cases where $F < F_{I*}$, the values of c_{IF} increase monotonically with kR.

Consider the behavior of the second branch of the dispersion curves. The analyses of these curves show that for each selected value of the shear-spring parameter F there exists such values of kR(denoted by $(kR)_S$) after which (i.e. where $kR >$

$(kR)_S$) the second branch of the dispersion curves constructed separates from the dispersion curves corresponding to the fundamental mode constructed for $F = 0$. In this case the following estimation occurs:

$$(kR)_S \to 0 \text{ as } F \to \infty \tag{35}$$

The results also show that the values of c_{IIF} increase with F. However, the character of the second branch of the dispersion curves becomes more complicated with F. There exists such values of F(denoted by F_{II*}) after which,(i.e. where $F > F_{II*}$) the local maximum or minimum for the wave propagation velocity c_{IIF} appears. The values of kR (denoted by $(kR)_{IIcr.}$) which correspond to these local maximums or minimums are determined from the following relation:

$$\left.\frac{dc_{IIF}}{d(kR)}\right|_{kR=(kR)_{IIcr.}} = 0. \tag{36}$$

According to the numerical results it can be concluded that the values of F_{II*} depend not only on the values of $\mu^{(2)}/\mu^{(1)}$, but also on the values of h/R.

Note that the existence of (34) and (36) type relations means that where $kR = (kR)_{Icr.}$ or where $kR = (kR)_{IIcr.}$ the group velocity of the wave propagation is equal to its phase velocity. Consequently, the point $kR = (kR)_{Icr.}$ or the point $kR = (kR)_{IIcr.}$ separates the parts of the dispersion curves which correspond to the anomalous and normal dispersions. This is clearly imaginable from Fig. 4 which shows the dispersion diagrams constructed for the case where $\mu^{(2)}/\mu^{(1)} = 0.5$ and $h/R = 0.3$. Note that similar type dispersion diagrams are also obtained for other values of the problem parameters.

As noted above, the case where $F = 0$ corresponds to the perfect interface conditions but the case where $F = \infty$ corresponds to the fully slipping interface conditions. It follows from the results discussed above that:

$$c_{IIF} \to c + 0 \text{ as } F \to 0. \tag{37}$$

This statement agrees with well known mechanical considerations.

Consider the behavior of the dispersion curves as $F \to \infty$. For this purpose analyze Fig. 5 which shows the dispersion curves constructed for the cases where $0 \le F \le 1000$ when $\mu^{(2)}/\mu^{(1)} = 2$, $h/R = 1.0$ (Fig. 5a) and when $\mu^{(2)}/\mu^{(1)} = 0.5$, $h/R = 0.3$(Fig. 5b). It follows from these results that the curves $c_{IF} = c_{IF}(kR)$ and $c_{IIF} = c_{IIF}(kR)$ approach their limits as F increases. In this case for each fixed kR, the velocities c_{IF} and c_{IIF} increase monotonically with F and the difference between the values of $c_{IF}/c_{20}^{(2)}$ (or $c_{IIF}/c_{20}^{(2)}$) obtained in the cases where $F = 500$

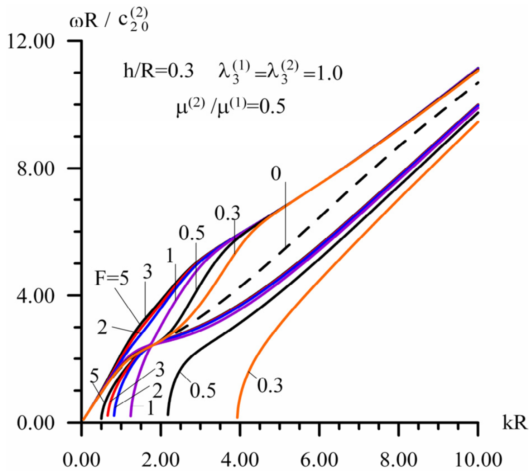

Figure 4: Dispersion diagrams for the first mode constructed for various values of the shear-spring parameter F for the case where $\mu^{(2)}/\mu^{(1)} = 0.5$, $\lambda_3^{(1)} = \lambda_3^{(2)}$ =1.0, $h/R = 0.3$

and $F = 1000$, is less than 10^{-5}. Consequently, the results obtained when $F =$ 1000, can be taken with very high accuracy as results corresponding to the fully slipping interface conditions.

With this we exhaust the discussions of the numerical results related to the case where the initial strains are absent in the layers of the compounded cylinder.

4.3 Numerical esults related to the pre-strained case

The perfect contact condition, i.e. for the case where $F = 0$, was studied by Akbarov and Guliev (2009) and the numerical results obtained for the imperfect interface conditions will be compared below with the corresponding ones given in this paper. Thus, consider the numerical results which illustrate the influence of the initial stretching or compression of the compound cylinder in the wave propagation direction. These numerical results are given in Figs. 6 and 7 for the second and first branches respectively of the fundamental mode for the cases where $\mu^{(2)}/\mu^{(1)} = 2$, $h/R = 1.0$. Note that the first and second branches are given in separate figures to improve the illustrations and in each figure the dispersion curves are given for the following selected pair of values of $\lambda_3 \left(= \lambda_3^{(2)} = \lambda_3^{(1)} \right)$: $\{1.0; 1.2\}$ (Fig. 6a and Fig. 7a), $\{1.2; 1.5\}$ (Fig. 6b and Fig. 7b), $\{1.0; 0.9\}$ (Fig. 6c and Fig. 7c) and $\{0.9; 0.8\}$ (Fig. 6d and Fig. 7d). Not that in Figs. 6 and 7 the graphs constructed

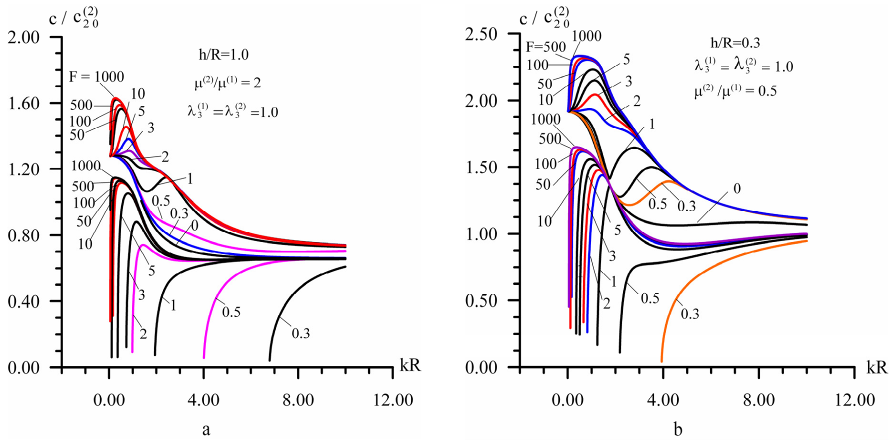

Figure 5: Limit dispersion curves obtained by increasing the shear-spring parameter F in the case where $\mu^{(2)}/\mu^{(1)} = 2.0$, $h/R = 1.0$(a) and $\mu^{(2)}/\mu^{(1)} = 0.5$, $h/R = 0.3$(b) when $\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0$.

for $F = 0$ coincide with those given in a paper by Akbarov and Guliev (2009).

It follows from the foregoing numerical results that the considered type initial strains in the compound cylinder do not change (in the qualitative sense) the character of the influence of the imperfectness of the interface conditions on the character of the dispersion curves. Consequently, the considered type initial strains cause an increase (under initial stretching) or a decrease (under initial compression). However, in this case the relation (27) must be rewritten as follows:

$$c_{IF}\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right) < c\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right) < c_{IIF}\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right) \quad (38)$$

Here $c_{IF}\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right)$, $c_{IIF}\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right)$ and $c\left(\lambda_3^{(1)}, \lambda_3^{(2)}\right)$ are the values of c_{IF}, c_{IIF} and c respectively in the pre-strained case.

At the same time, the numerical results show that the relation (28) also holds for the pre-strained case. Nevertheless, the initial strains significantly change the values of $(kR)_{cf.}$ so that under initial stretching, the values of $(kR)_{cf.}$ decrease with $\lambda_3^{(1)} \left(= \lambda_3^{(2)}\right)$ and under initial compression the values of $(kR)_{cf.}$ increase with a decrease in the parameter $\lambda_3^{(1)} \left(= \lambda_3^{(2)}\right)$.

The graphs given in Figs. 6 and 7 illustrate that in the pre-strained case, the low wave number limit as $kR \to 0$ of the wave propagation velocity related to the second branch of the fundamental mode does not also depend on the shear-spring parameter F. However, in the pre-strained case this limit value is determined by the following expression which is also given in a paper by Akbarov and Guliev (2009).

$$\frac{c_{IIF}\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)}{c_{20}^{(2)}}=\left[\frac{e^{(2)}\left(\lambda_3^{(2)}\right)^2\eta^{(2)}+e^{(1)}\left(\lambda_3^{(1)}\right)^2\eta^{(1)}\mu^{(1)}/\mu^{(2)}}{\eta^{(2)}+\eta^{(1)}\rho^{(1)}/\rho^{(2)}}\right]^{\frac{1}{2}} \quad \text{as } kR\to 0 \qquad (39)$$

From the foregoing numerical results it also follows that in the pre-strained case the expression (35) for the high wave number limit for the wave propagation velocities and the expression (39) must be changed with the expressions (42) and (43) respectively which are given below:

$$c_{IF}\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)\to\min\left\{c_R^{(1)}\left(\lambda_3^{(1)}\right)-0,\ c_2^{(2)}\left(\lambda_3^{(2)}\right)-0,\ c_S\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)-0\right\},$$

$$c_{IIF}\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)\to\min\left\{c_R^{(1)}\left(\lambda_3^{(1)}\right)+0,\ c_2^{(2)}\left(\lambda_3^{(2)}\right)+0,\ c_S\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)+0\right\} \quad \text{as } kR\to\infty, \qquad (40)$$

$$c_{IIF}\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)\to c\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)+0 \text{ as } F\to 0, \qquad (41)$$

where $c_S\left(\lambda_3^{(1)},\lambda_3^{(2)}\right)$ in (40) is the Stoneley Wave velocity in the pre-strained case.

The detailed analyses of the graphs given in Figs. 6 and 7 show that the critical values of the wave number parameter, i.e. the values of $(kR)_{Icr.}$ and $(kR)_{IIcr.}$ decrease with the parameter $\lambda_3^{(1)}\left(=\lambda_3^{(2)}\right)$.

Although the foregoing conclusions are based on the results obtained for the case where the stiffness of the material of the inner cylinder is greater than that for the outer cylinder (i.e. for the case where $\mu^{(2)}/\mu^{(1)}=2$), they hold also for the case where the stiffness of the material of the inner cylinder is less than that for the outer cylinder (i.e. for the case where $\mu^{(2)}/\mu^{(1)}=0.5$). This state is proven with the graphs given in Figs. 8 and 9 which show the dispersion curves constructed for the second and first branches, respectively in the pre-strained state for the case where $\mu^{(2)}/\mu^{(1)}=0.5$, $h/R=0.3$.

5 Conclusions

Thus, in the present paper within the scope of the piecewise homogeneous body model utilizing the 3D linearized theory of elastic waves in initially stressed bodies, the effect of the imperfectness of the interface conditions on the dispersion

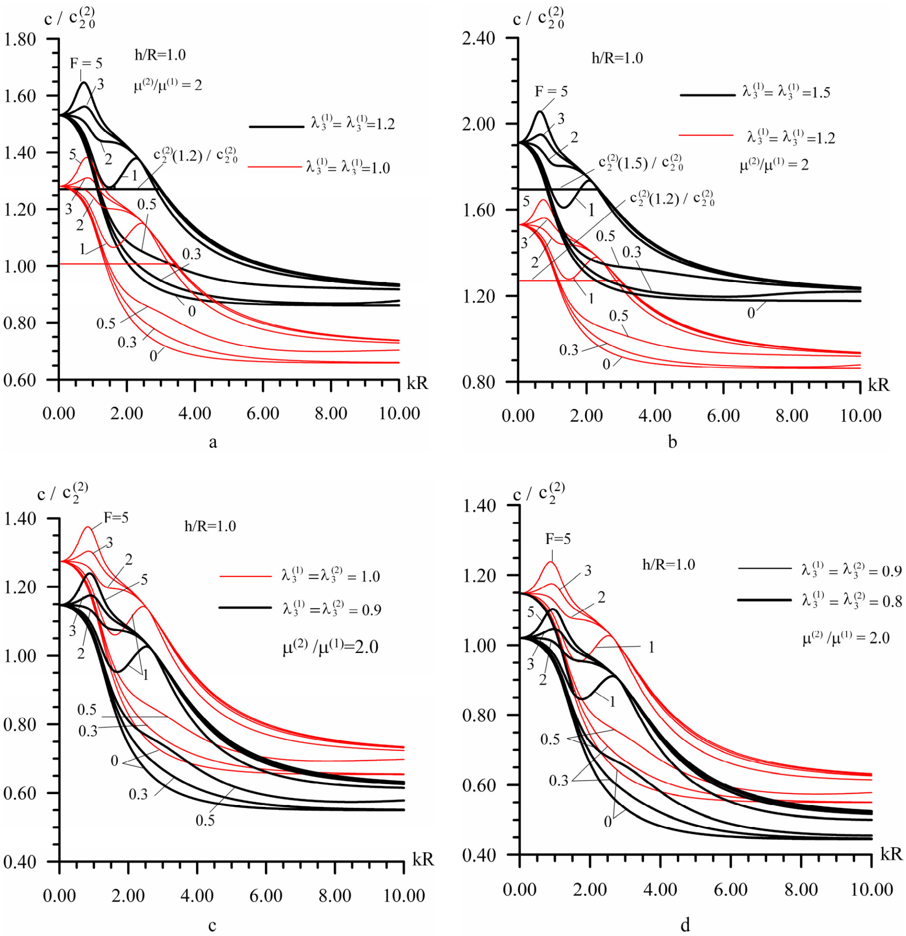

Figure 6: The influence of the initial strains on the second branch of the dispersion curves of the fundamental mode where $\mu^{(2)}/\mu^{(1)} = 2.0$, $h/R = 1.0$ for the pairs $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 1.2\}$ (a); $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.2;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 1.5\}$ (b); $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 0.9\}$ (c) and $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 0.9;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 0.8\}$ (d).

of the longitudinal axisymmetric waves in the pre-strained bi-material compound cylinder, is studied. It is assumed that the materials of the constituents are high elastic compressible ones and the elasticity relations of these are described by the harmonic potential. The shear-spring type imperfectness of the interface conditions is considered and the degree of this imperfectness is estimated through the

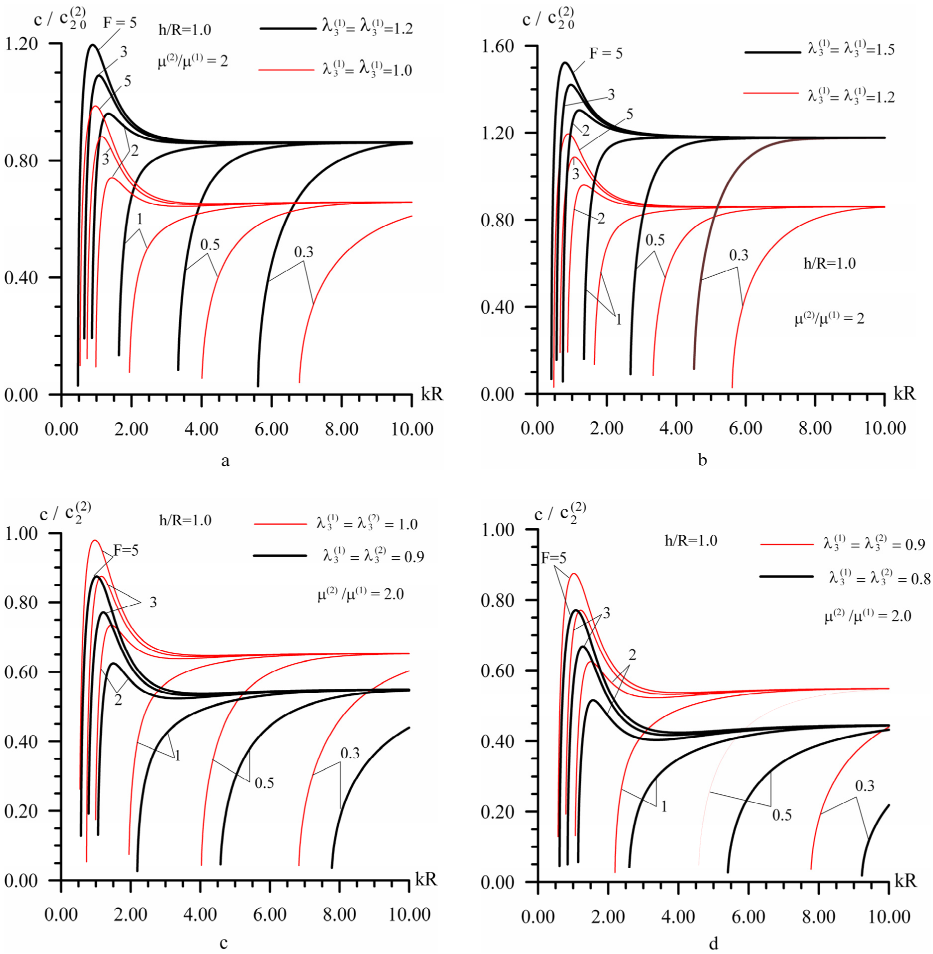

Figure 7: The influence of the initial strains on the first branch of the dispersion curves of the fundamental mode where $\mu^{(2)}/\mu^{(1)} = 2.0$, $h/R = 1.0$ for the pairs $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 1.2\}$ (a); $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.2;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 1.5\}$ (b); $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 1.0;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 0.9\}$ (c) and $\{\lambda_3^{(1)} = \lambda_3^{(2)} = 0.9;\ \lambda_3^{(1)} = \lambda_3^{(2)} = 0.8\}$ (d).

shear-spring parameter$F$. The cases where $F = 0$ and $F = \infty$ correspond to the perfect and the fully slipping interface conditions, respectively. The solution method for the formulated corresponding eigen-value problem and the algorithm for constructing the dispersion curves are developed. Numerical results are presented for the fundamental mode and are discussed. According to these numerical results the

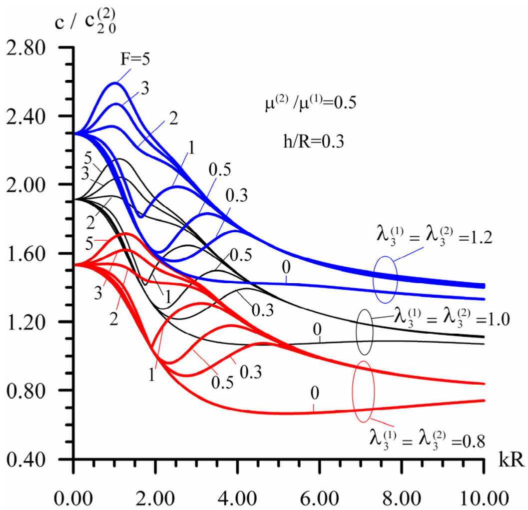

Figure 8: The influence of the initial strains on the second branch of the dispersion curves of the fundamental mode where $\mu^{(2)}/\mu^{(1)} = 0.5$, $h/R = 0.3$.

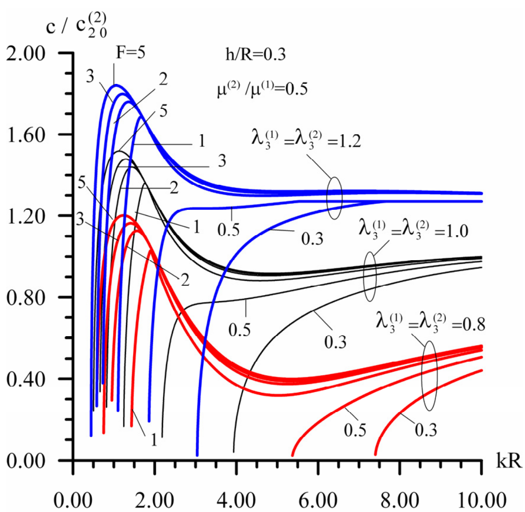

Figure 9: The influence of the initial strains on the first branch of the dispersion curves of the fundamental mode where $\mu^{(2)}/\mu^{(1)} = 0.5$, $h/R = 0.3$.

following conclusions are reached:

1. the shear-spring type imperfectness of the interface conditions causes two branches of the dispersion curve related to the fundamental mode to appear, the first of which disappears, but with the second approach the dispersion curve obtained for the perfect interface case **as** $F \to 0$;

2. the dispersion curves of the foregoing two branches of the fundamental mode approach the corresponding limit dispersion curves related to the fully slipping interface conditions as F increases;

3. the shear-spring type imperfectness of the contact conditions does not change the low and high wave number limits;

4. the wave propagation velocity in the first (second) branch of the fundamental dispersion mode is less (greater) than that obtained in the perfect interface case;

5. there exists "cut off" values for kR(denoted by $(kR)_{cf}$) for the first branch of the dispersion curve of the fundamental mode and $(kR)_{cf} \to 0$ as $F \to \infty$; as well as $(kR)_{cf} \to \infty$ as $F \to 0$;

6. the initial strains of the layers of the compound cylinder qualitatively change only the influence of the considered imperfectness of the interface conditions on the behavior of the dispersion curves;

7. the numerical results obtained in the pre-strained case approach the corresponding ones obtained in the paper by Akbarov and Guliev (2009) as$F \to 0$.

Although the discussed numerical results are obtained for the selected cases, they also have a general meaning for the estimation of the influence of the imperfectness of the interface conditions on the wave dispersion in many-layered compound cylinders.

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