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Explicit Solutions of Stresses for a Three-Phase Elliptic Inclusion Problem Subject to a Remote Uniform Load

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Abstract: A general solution to a three-phase elliptic inclusion problem subjected to a remote uniform load is provided in this paper. Analysis of the present elasticity problem is rather tedious due to the presence of material inhomogeneities and complex geometric configurations. Based on the technique of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the general expressions of the displacement and stresses in each layer medium are derived explicitly in a series form. The effects of the material combinations and geometric configurations on the interfacial stresses are discussed in detail and shown in graphic form.

Keywords: conformal mapping, analytical continuation, alternating technique, a reinforced elliptic hole

Nomenclature

a_1, a_2	semimajor of the two confocal ellipses
b_1, b_2	semiminor of the two confocal ellipses
L_1, L_2	boundaries of the coated layer in the ζ-plane
l	$\sqrt{a_2^2 - b_2^2}$
R	$\sqrt{\frac{a_2+b_2}{a_2-b_2}}$
S_1	the matrix in the ζ-plane
S_2	the intermediate layer in the ζ-plane
T	magnitude of a remote uniform load
U_{21}	$2G_1(G_1+G_2)^{-1}$
V_{21}	$(G_1-G_2)(G_1+G_2)^{-1}$

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z	Cartesian coordinates

Greek symbols

Ω_1	the matrix in the z-plane
Ω_2	the intermediate layer in the z-plane
Γ_1, Γ_2	boundaries of the coated layer in the z-plane
Π_{12}	$\frac{G_1-G_2}{\kappa_2 G_1+G_2}$
Λ_{12}	$\frac{G_1\kappa_2-G_2\kappa_1}{G_1+G_2\kappa_1}$
ζ	polar coordinates
ρ_1, ρ_2	$\rho_i = \frac{a_i+b_i}{a_2+b_2}, \quad i=1,2$
$\phi(\zeta), \omega(\zeta)$	the stress functions

1 Introduction

The determination of the elastic field in a heterogeneous material is an important topic in solid mechanics. The exact solution of the three-dimensional elasticity problem of multiple elastic inclusions embedded in an infinite elastic matrix appears unobtainable. However, some insight into the response of the composite solid may be gained by idealizing the medium as two-dimensional and the various phases as homogeneous. The solution of this related two-dimensional elasticity problem would become useful in engineering applications. The elastic interaction of dislocations with inhomogeneities is important in studying the mechanical behaviour of many materials. [Dundurs and Mura (1964)] solved the problem of an edge dislocation in an infinite elastic medium containing a circular inclusion with different elastic properties. Airy stress functions were presented for Burger's vectors in both the $x-$and y-directions. The limiting cases of a void, a rigid inclusion and identical materials were also considered. Additionally, the degenerate case of two joined elastic half-planes was examined both with the dislocation at the interface, and with the dislocation near the interface. The Airy stress functions for both radial and tangential Burger's vectors were presented by [Dundurs and Sendeckyj (1965)], for the case of an edge dislocation inside a circular elastic inclusion embedded within an unbounded elastic matrix. In the last of the series of related problems, [Dundurs and Gangadharan (1969)] considered the interaction between an edge dislocation and a circular inclusion with a slipping interface. In their paper, the slipping interface was modelled by requiring continuity of normal displacements and normal tractions across the interface, while allowing no transmission of tangential traction between the matrix and interface. The above mentioned works are related to a circular shaped inclusion. For the problem with an elliptical shaped inclusion, the details of the stress field around the elliptical cavity under uniform loading for an isotropic and homogeneous material was given by [Muskhelishvili (1953)]. The problem of an infinite elastic region containing an elliptical shaped inclusion with different elastic properties, and an edge dislocation at an arbitrary point within the inclusion was solved by [Warren (1983)], through conformal mapping, and the use of Muskhelishvili's complex potentials taken in the form of Laurent's series expansions. [Stagni (1982)] and [Stagni and Lizzio (1983)] solved the problem for the interaction of an edge dislocation outside of an elliptical inhomogeneity, where the properties of the inhomogeneity range from those of a void to those of a rigid inclusion. The solution of a rigid elliptical inhomogeneity was also considered by [Santare and Keer (1986)]. Special care was taken in their work to include the rigid-body rotation of the inhomogeneity with respect to the dislocation.

The above work on interaction between dislocations and inhomogeneities involves an isolated inhomogeneity only. For multiple-phase materials, the dislocations interact not only with the nearest inclusion but also with the surrounding ones. [Luo and Chen (1991)] studied the problem of an edge dislocation located in the intermediate matrix phase based on the three-phase composite cylinder model. An exact solution of heat conduction problem (or antiplane elasticity problem) for a three-phase elliptical composite has been provided by [Chao, Chen and Chen (2009)]. Based on the method of conformal mapping and the method of analytical continuation, an analytical solution for a reinforced elliptical hole embedded in an infinite matrix subjected to a point heat source was provided by [Chao, Chen and Chen (2010)]. In this work, we consider a three-phase elliptical inclusion in an infinite plate subject to a remote uniform load. The proposed method is based on the method of conformal mapping and the technique of analytical continuation that is alternatively applied across two concentric circles. The plan of this paper is as follows. The general formulation for plane elasticity and the method of conformal mapping are provided in Section 2. The series form solutions of the complex potentials of the stresses are given in Section 3. Some numerical examples are solved in Section 4. Finally, Section 5 concludes the article.

2 Problem Formulation

Consider a reinforced elliptical hole in an unbounded matrix subjected to a remote uniform load (see Fig. 1). Let Ω_1 denote the matrix, Ω_2 denote the interface layer and Ω_3 denote the inner core respectively. The boundaries of the reinforced layer are two confocal ellipses Γ_1, Γ_2 with a_1, a_2 and b_1, $b_2$being the semimajor and semiminor, respectively.

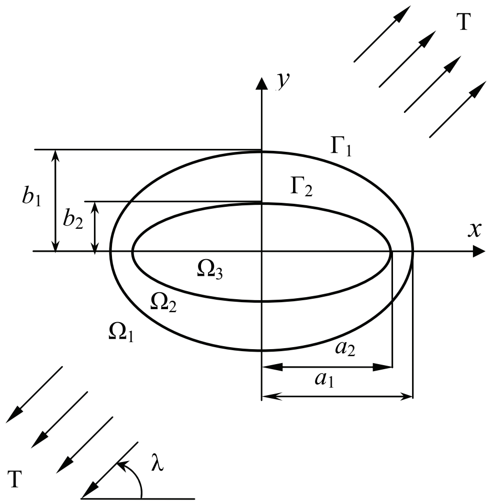

Figure 1: A coated elliptic inclusion in an infinite plate subjected to a remote uniform stress.

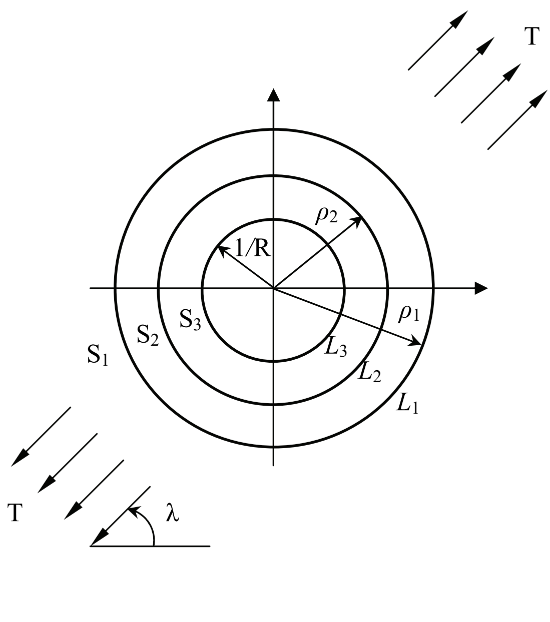

Figure 2: The problem in ζ-plane.

It is well known that for plane deformations, the displacement components (u_x, u_y), stress components $(\sigma_{xx,} \sigma_{yy,} \sigma_{xy})$ and the components of the resultant force (X, Y) are given in terms of two holomorphic functions $\phi(z)$ and $\psi(z)$by [Muskhelishvili (1963)]

$$2G(u_x + iu_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{1}$$

$$\sigma_{xx} + \sigma_{yy} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] \tag{2}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right] \tag{3}$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{4}$$

where G is the shear modulus, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$for plane stress with ν being the Poisson's ratio. Here a superimposed bar represents the complex conjugate.

The boundary stresses are written in normal-tangential ((n,t)-) coordinates as:

$$\sigma_{nn} + i\sigma_{nt} = \phi'(z) + \overline{\phi'(z)} - \left[\bar{z}\overline{\phi''(z)} + \overline{\psi'(z)}\right] e^{-2i\alpha(z)} \tag{5}$$

where n is the outward unit normal at the boundary which is also represented, in complex form, by $e^{i\alpha(z)}$ (where α defines the angle between the normal direction n and the positive x-axis).

Now we introduce the following mapping function

$$z = m(\zeta) = \frac{l}{2}\left[R\zeta + \frac{1}{R\zeta}\right], \quad R\zeta = \frac{z}{l}\left\{1 + \left[1 - (\frac{l}{z})^2\right]^{1/2}\right\}, \quad \zeta = \xi + i\eta = re^{i\theta} \tag{6}$$

where $R = \sqrt{\frac{a_2+b_2}{a_2-b_2}} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$, $\varepsilon = \frac{b_2}{a_2}$ and $l = \sqrt{a_2^2 - b_2^2}$.

This mapping function maps the confocal ellipses Γ_1, Γ_2 in the z-plane onto the concentric circles L_1, L_2 in theζ-plane with radii ρ_1, ρ_2 (see Fig.2)

$$\rho_i = \frac{a_i + b_i}{a_2 + b_2} \qquad i = 1, 2$$

For convenience of calculation, we write $\phi(\zeta) = \phi(m(\zeta))$ and $\psi(\zeta) = \psi(m(\zeta))$ so that in the mapped $\zeta-$plane, the displacements, stresses and resultant forces take the form

$$2G(u_x + iu_y) = \kappa\phi(\zeta) - \frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\psi(\zeta)} \tag{7}$$

$$\sigma_{xx}+\sigma_{yy}=2\left\{\frac{\phi'(\zeta)}{m'(\zeta)}+\overline{\frac{\phi'(\zeta)}{m'(\zeta)}}\right\} \tag{8}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2\left[\frac{\overline{m(\zeta)}}{m'(\zeta)}\frac{d}{d\zeta}\left\{\frac{\phi'(\zeta)}{m'(\zeta)}\right\}+\frac{\psi'(\zeta)}{m'(\zeta)}\right] \tag{9}$$

$$-Y+iX=\phi(\zeta)+\frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)}+\overline{\psi(\zeta)} \tag{10}$$

In the mapped plane, noting the following relation from [England (1971)],

$$e^{i2\alpha(z)}=\frac{\zeta}{\overline{\zeta}}\frac{m(\zeta)}{\overline{m'(\zeta)}}$$

Eq. (5) becomes

$$\sigma_{nn}+i\sigma_{nt}=\left\{\frac{\phi'(\zeta)}{m'(\zeta)}+\overline{\frac{\phi'(\zeta)}{m'(\zeta)}}\right\}-\left[\frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\frac{d}{d\zeta}\left\{\frac{\phi'(\zeta)}{m'(\zeta)}\right\}}+\overline{\frac{\psi'(\zeta)}{m'(\zeta)}}\right]\frac{\overline{\zeta m'(\zeta)}}{\zeta m'(\zeta)} \tag{11}$$

3 Stress field

In this section we will derive the stress fields for a reinforced elliptic hole in an infinite plate subjected to a remote uniform tension. The solution for a homogeneous infinite plate subjected to a remote uniform tension T acting with an angle λ to the x-axis can be trivially given as

$$\phi_0(\zeta)=\frac{T}{4}\zeta \tag{12}$$

$$\psi_0(\zeta)=-\frac{Te^{-2i\lambda}\zeta}{2} \tag{13}$$

For a region bounded by a circle, say $c=|\zeta|$, we introduce an auxiliary stress function $\omega(\zeta)$ such that

$$\omega(\zeta)=\frac{\overline{m}(\frac{c^2}{\zeta})}{m'(\zeta)}\phi'(\zeta)+\psi(\zeta) \tag{14}$$

Unlike the standard Muskhelishvili complex functions $\phi(\zeta)$ and $\psi(\zeta)$, the function $\omega(z)$ is dependent on the radius of any circular interface.

The stress functions can be assumed as

$$\phi(\zeta)=\begin{cases}\phi_0(\zeta)+\sum_{n=1}^{\infty}\phi_{bn}^{(1)}(\zeta) & z\in S_1\\ \sum_{n=1}^{\infty}\phi_{an}^{(2)}(\zeta)+\sum_{n=1}^{\infty}\phi_{bn}^{(2)}(\zeta) & z\in S_2\\ \sum_{n=1}^{\infty}\phi_{an}^{(3)}(\zeta)+\sum_{n=1}^{\infty}\phi_{bn}^{(3)}(\zeta) & z\in S_3\end{cases} \tag{15}$$

$$\omega(\zeta)=\begin{cases}\omega_0(\zeta)+\sum_{n=1}^{\infty}\omega_{bn}^{(1)}(\zeta) & z\in S_1\\ \sum_{n=1}^{\infty}\omega_{an}^{(2)}(\zeta)+\sum_{n=1}^{\infty}\omega_{bn}^{(2)}(\zeta) & z\in S_2\\ \sum_{n=1}^{\infty}\omega_{an}^{(3)}(\zeta)+\sum_{n=1}^{\infty}\omega_{bn}^{(3)}(\zeta) & z\in S_3\end{cases} \tag{16}$$

The alternating technique and the analytical continuation method are applied to derive the unknown stress functions as follows.

Step 1: Analytical continuation acrossL_1

Two pairs of stress functions $\varphi_{a1}^{(2)}(\zeta)$, $\omega_{a1}^{(2)}(\zeta)$ holomorphic in $|\zeta|\le\rho_1$ and $\phi_{b1}^{(1)}(\zeta)$, $\omega_{b1}^{(1)}(\zeta)$ holomorphic in $|\zeta|\ge\rho_1$ are introduced to satisfy the continuity conditions along L_1 that

$$\phi_{b1}^{(1)}(\sigma)+\overline{\omega_{b1}^{(1)}(\sigma)}+\phi_0(\sigma)+\overline{\omega_0(\sigma)}=\phi_{a1}^{(2)}(\sigma)+\overline{\omega_{a1}^{(2)}(\sigma)}\quad \sigma\in L_1 \tag{17}$$

$$\frac{\kappa_1}{G_1}\left[\phi_{b1}^{(1)}(\sigma)+\phi_0(\sigma)\right]-\frac{1}{G_1}\left[\overline{\omega_{b1}^{(1)}(\sigma)}+\overline{\omega_0(\sigma)}\right]=\frac{\kappa_2}{G_2}\phi_{a1}^{(2)}(\sigma)-\frac{1}{G_2}\overline{\omega_{a1}^{(2)}(\sigma)} \tag{18}$$

By the standard analytical continuation arguments, it follows that

$$\varphi_{b1}^{(1)}(\zeta)+\overline{\omega_0}(\frac{\rho_1^2}{\zeta})-\overline{\omega_{a1}^{(2)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}(c_{a1}^{(2)}-c_0)=0\quad |\zeta|\ge\rho_1 \tag{19}$$

$$\varphi_{a1}^{(2)}(\zeta)-\varphi_0(\zeta)-\overline{\omega_{b1}^{(1)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}(c_{a1}^{(2)}-c_0)=0\quad |\zeta|\le\rho_1 \tag{20}$$

$$\frac{\kappa_1}{G_1}\varphi_{b1}^{(1)}(\zeta)-\frac{1}{G_1}\overline{\omega_0}(\frac{\rho_1^2}{\zeta})+\frac{1}{G_2}\overline{\omega_{a1}^{(2)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}(\frac{c_0}{G_1}-\frac{c_{a1}^{(2)}}{G_2})=0\quad |\zeta|\ge\rho_1 \tag{21}$$

$$\frac{\kappa_2}{G_2}\varphi_{a1}^{(2)}(\zeta)-\frac{\kappa_1}{G_1}\varphi_0(\zeta)+\frac{1}{G_1}\overline{\omega_{b1}^{(1)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}(\frac{c_0}{G_1}-\frac{c_{a1}^{(2)}}{G_2})=0 \quad |\zeta|\le\rho_1 \tag{22}$$

where $c_0=\overline{\varphi_0'}(\frac{1}{R})$, $c_{a1}^{(2)}=\overline{\phi_{a1}^{(2)'}}(\frac{1}{R})$.
Solve Eqs. (19) - (22) to yield

$$\varphi_{b1}^{(1)}(\zeta)=\Pi_{21}\overline{\omega_0}(\frac{\rho_1^2}{\zeta})-\Pi_{21}\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}c_0 \quad \zeta\in S_1 \tag{23}$$

$$\phi_{a1}^{(2)}(\zeta)=(1+\Lambda_{21})\phi_0(\zeta)+\Pi_{12}\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}c_{a1}^{(2)} \quad \zeta\in S_2 \tag{24}$$

$$\omega_{a1}^{(2)}(\zeta)=(1+\Pi_{21})\omega_0(\zeta)+\frac{(\frac{1}{R^4\rho_1^4}+1)\frac{\rho_1^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\left(\overline{c_{a1}^{(2)}}-(1+\Pi_{21})\overline{c_0}\right) \quad \zeta\in S_2 \tag{25}$$

$$\omega_{b1}^{(1)}(\zeta)=\Lambda_{21}\overline{\varphi_0}(\frac{\rho_1^2}{\zeta})+\left((1+\Pi_{12})\overline{c_{a1}^{(2)}}-\overline{c_0}\right)\frac{(\frac{1}{R^4\rho_1^4}+1)\frac{\rho_1^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}} \quad \zeta\in S_1 \tag{26}$$

where $\Pi_{12}=\frac{G_1-G_2}{\kappa_2 G_1+G_2}$ and $\Lambda_{12}=\frac{G_1\kappa_2-G_2\kappa_1}{G_1+G_2\kappa_1}$.
Step 2: Analytical continuation across L_2
Since $\phi_{a1}^{(2)}(\zeta)$ and $\omega_{a1}^{(2)}(\zeta)$ can not satisfy the continuity conditions at L_2, two pairs of stress functions $\phi_{b1}^{(2)}(\zeta)$, $\omega_{b1}^{(2)}(\zeta)$and $\phi_{a1}^{(3)}(\zeta)$, $\omega_{a1}^{(3)}(\zeta)$ respectively holomorphic in $|\zeta|\ge\rho_2$ and $|\zeta|\le\rho_2$ are introduced that

$$\phi_{a1}^{(2)}(\sigma)+\overline{\omega_{a1}^{(2)*}(\sigma)}+\phi_{b1}^{(2)}(\sigma)+\overline{\omega_{b1}^{(2)}(\sigma)}=\phi_{a1}^{(3)}(\sigma)+\overline{\omega_{a1}^{(3)}(\sigma)} \quad \sigma\in L_2 \tag{27}$$

$$\frac{\kappa_2}{G_2}\left[\phi_{a1}^{(2)}(\sigma)+\phi_{b1}^{(2)}(\sigma)\right]-\frac{1}{G_2}\left[\overline{\omega_{a1}^{(2)*}(\sigma)}+\overline{\omega_{b1}^{(2)}(\sigma)}\right]=\frac{\kappa_3}{G_3}\phi_{a1}^{(3)}(\sigma)-\frac{1}{G_3}\overline{\omega_{a1}^{(3)}(\sigma)} \quad \sigma\in L_2 \tag{28}$$

where

$$\omega_{a1}^{(2)*}(\zeta)=\frac{\frac{R}{\zeta}(\rho_2^2-\rho_1^2)+\frac{\zeta}{R}(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2})}{R-\frac{1}{R\zeta^2}}\phi_{a1}^{(2)'}(\zeta)+\omega_{a1}^{(2)}(\zeta)$$

By the analytical continuation method, we have

$$\phi_{a1}^{(2)}(\zeta)-\phi_{a1}^{(3)}(\zeta)+\overline{\omega_{b1}^{(2)}}(\frac{\rho_2^2}{\zeta})+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a1}^{(2)}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a1}^{(3)}=0 \quad |\zeta|\le\rho_2 \tag{29}$$

$$\overline{\omega_{a1}^{(3)}}(\frac{\rho_2^2}{\zeta})-\overline{\omega_{a1}^{(2)*}}(\frac{\rho_2^2}{\zeta})-\varphi_{b1}^{(2)}(\zeta)+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a1}^{(2)}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a1}^{(3)}=0 \quad |\zeta|\ge\rho_2 \tag{30}$$

$$\frac{\kappa_2}{G_2}\phi_{a1}^{(2)}(\zeta)-\frac{\kappa_3}{G_3}\phi_{a1}^{(3)}(\zeta)-\frac{1}{G_2}\overline{\omega_{b1}^{(2)}}(\frac{\rho_2^2}{\zeta})-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a1}^{(2)}}{G_2}+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a1}^{(3)}}{G_3}=0 \quad |\zeta|\le\rho_2 \tag{31}$$

$$\frac{1}{G_2}\overline{\omega_{a1}^{(2)*}}(\frac{\rho_2^2}{\zeta})-\frac{1}{G_3}\overline{\omega_{a1}^{(3)}}(\frac{\rho_2^2}{\zeta})-\frac{\kappa_2\varphi_{b1}^{(2)}(\zeta)}{G_2}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a1}^{(2)}}{G_2}+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a1}^{(3)}}{G_3}=0 \quad |\zeta|\ge\rho_2 \tag{32}$$

Solve Eqs. (29) - (32) to yield

$$\varphi_{a1}^{(3)}(\zeta)=(1+\Lambda_{32})\varphi_{a1}^{(2)}(\zeta)+\Pi_{23}\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a1}^{(3)} \quad \zeta\in S_3 \tag{33}$$

$$\omega_{b1}^{(2)}(\zeta)=\Lambda_{32}\overline{\varphi_{a1}^{(2)}}(\frac{\rho_2^2}{\zeta})-\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a1}^{(2)}}+(1+\Pi_{23})\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a1}^{(3)}} \quad \zeta\in S_2 \tag{34}$$

$$\phi_{b1}^{(2)}(\zeta)=\Pi_{32}\overline{\omega_{a1}^{(2)*}}(\frac{\rho_2^2}{\zeta})-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\Pi_{32}c_{a1}^{(2)} \quad \zeta\in S_2 \tag{35}$$

$$\omega_{a1}^{(3)}(\zeta)=(1+\Pi_{32})\omega_{a1}^{(2)*}(\zeta)-\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}(1+\Pi_{32})\overline{c_{a1}^{(2)}}+\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a1}^{(3)}} \quad \zeta\in S_2 \tag{36}$$

where $c_{a1}^{(3)}=\overline{\phi_{a1}^{(3)'}}(\frac{1}{R})$.

Step 3: Analytical continuation across L_3

Since the points $\sigma=\frac{1}{R}e^{i\theta}$ and $\bar{\sigma}=\frac{1}{R}e^{-i\theta}$ correspond to the same points of the segment from $-(a_2^2-b_2^2)^{\frac{1}{2}}$ to $(a_2^2-b_2^2)^{\frac{1}{2}}$ in the z-plane. The following conditions must be satisfied

$$\phi^{(3)}(\sigma)=\phi^{(3)}(\bar{\sigma}) \tag{37}$$

$$\psi^{(3)}(\sigma)=\psi^{(3)}(\bar{\sigma}) \tag{38}$$

The functions $\phi_{b1}^{(3)}(\zeta)$ and $\omega_{b1}^{(3)}(\zeta)$ holomorphic in $|\zeta|\geq\frac{1}{R}$ is introduced to satisfy this conditions that

$$\phi_{a1}^{(3)}(\sigma)+\phi_{b1}^{(3)}(\sigma)=\phi_{a1}^{(3)}(\overline{\sigma})+\phi_{b1}^{(3)}(\overline{\sigma}) \quad \sigma\in L_3 \tag{39}$$

$$\omega_{a1}^{(3)*}(\sigma)-(\frac{\frac{1}{R\sigma}+R\sigma}{R-\frac{1}{R\sigma^2}})\phi_{a1}^{(3)'}(\sigma)+\omega_{b1}^{(3)}(\sigma)-(\frac{\frac{1}{R\sigma}+R\sigma}{R-\frac{1}{R\sigma^2}})\phi_{b1}^{(3)'}(\sigma)$$

$$=\omega_{a1}^{(3)*}(\overline{\sigma})-(\frac{\frac{1}{R\overline{\sigma}}+R\overline{\sigma}}{R-\frac{1}{R\overline{\sigma}^2}})\varphi_{a1}^{(3)'}(\overline{\sigma})+\omega_{b1}^{(3)}(\overline{\sigma})-(\frac{\frac{1}{R\overline{\sigma}}+R\overline{\sigma}}{R-\frac{1}{R\overline{\sigma}^2}})\varphi_{b1}^{(3)'}(\overline{\sigma}) \quad \sigma\in L_3 \tag{40}$$

where $\omega_{a1}^{(3)*}(\zeta)=\omega_{a1}^{(3)}(\zeta)+\frac{\frac{R}{\zeta}(\frac{1}{R^2}-\rho_2^2)+\frac{\zeta}{R}(R^2-\frac{1}{\rho_2^2})}{R-\frac{1}{R\zeta^2}}\phi_{a1}^{(3)'}(\zeta)$

By the analytical continuation method, we have

$$\phi_{b1}^{(3)}(\zeta)=\phi_{a1}^{(3)}(\frac{1}{R^2\zeta}) \quad \zeta\in S_3 \tag{41}$$

$$\omega_{b1}^{(3)}(\zeta)=\omega_{a1}^{(3)*}(\frac{1}{R^2\zeta}) \quad \zeta\in S_3 \tag{42}$$

Step 4: Analytical continuation across L_1

Since $\phi_{b1}^{(2)}(\zeta)$,$\omega_{b1}^{(2)}(\zeta)$ can not satisfy the continuity condition at L_1, two pairs of stress functions $\phi_{b2}^{(1)}(\zeta)$, $\omega_{b1}^{(1)}(\zeta)$ and $\phi_{a2}^{(2)}(\zeta)$, $\omega_{a2}^{(2)}(\zeta)$ respectively holomorphic in $|\zeta|\geq\rho_1$ and $|\zeta|\leq\rho_1$ are introduced that

$$\varphi_{b2}^{(1)}(\sigma)+\overline{\omega_{b2}^{(1)}(\sigma)}=\varphi_{b1}^{(2)}(\sigma)+\overline{\omega_{b1}^{(2)*}(\sigma)}+\varphi_{a2}^{(2)}(\sigma)+\overline{\omega_{a2}^{(2)}(\sigma)} \quad \sigma\in L_1 \tag{43}$$

$$\frac{\kappa_1}{G_1}\phi_{b2}^{(1)}(\sigma)-\frac{1}{G_1}\overline{\omega_{b2}^{(1)}(\sigma)}=\frac{\kappa_2}{G_2}\phi_{b1}^{(2)}(\sigma)-\frac{1}{G_2}\overline{\omega_{b1}^{(2)*}(\sigma)}+\frac{\kappa_2}{G_2}\phi_{a2}^{(2)}(\sigma)-\frac{1}{G_2}\overline{\omega_{a2}^{(2)}(\sigma)}\quad \sigma\in L_1 \quad (44)$$

where

$$\omega_{b1}^{(2)*}(\zeta)=\omega_{b1}^{(2)}(\zeta)+\frac{\frac{R(\rho_1^2-\rho_2^2)}{\zeta}+\frac{\zeta}{R}(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2})}{R-\frac{1}{R\zeta^2}}\phi_{b1}^{(2)'}(\zeta)$$

By the standard analytical continuation arguments, it follows that

$$\phi_{b2}^{(1)}(\zeta)-\phi_{b1}^{(2)}(\zeta)-\overline{\omega_{a2}^{(2)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}c_{a2}^{(2)}=0\quad \zeta\in S_1 \quad (45)$$

$$\phi_{a2}^{(2)}(\zeta)+\overline{\omega_{b1}^{(2)*}}(\frac{\rho_1^2}{\zeta})-\overline{\omega_{b2}^{(1)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}c_{a2}^{(2)}=0\quad \zeta\in S_2 \quad (46)$$

$$\frac{\kappa_1}{G_1}\phi_{b2}^{(1)}(\zeta)-\frac{\kappa_2}{G_2}\phi_{b1}^{(2)}(\zeta)+\frac{1}{G_2}\overline{\omega_{a2}^{(2)}}(\frac{\rho_1^2}{\zeta})-\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}\frac{c_{a2}^{(2)}}{G_2}=0\quad \zeta\in S_1 \quad (47)$$

$$\frac{\kappa_2}{G_2}\phi_{a2}^{(2)}(\zeta)-\frac{1}{G_2}\overline{\omega_{b1}^{(2)*}}(\frac{\rho_1^2}{\zeta})+\frac{1}{G_1}\overline{\omega_{b2}^{(1)}}(\frac{\rho_1^2}{\zeta})-\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}\frac{c_{a2}^{(2)}}{G_2}=0\quad \zeta\in S_2 \quad (48)$$

where $c_{a2}^{(2)}=\overline{\varphi_{a2}^{(2)'}}(\frac{1}{R})$.
Solve Eqs. (45)-(48) to yield

$$\phi_{b2}^{(1)}(\zeta)=(1+\Lambda_{12})\phi_{b1}^{(2)}(\zeta)\quad \zeta\in S_1 \quad (49)$$

$$\omega_{b2}^{(1)}(\zeta)=(1+\Pi_{12})\omega_{b1}^{(2)*}(\zeta)+(1+\Pi_{12})\frac{(\frac{1}{R^4\rho_1^4}+1)\frac{\rho_1^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(2)}}\quad \zeta\in S_1 \quad (50)$$

$$\phi_{a2}^{(2)}(\zeta)=\Pi_{12}\overline{\omega_{b1}^{(2)*}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_1^4}}\Pi_{12}c_{a2}^{(2)}\quad \zeta\in S_2 \quad (51)$$

$$\omega_{a2}^{(2)}(\zeta)=\Lambda_{12}\overline{\phi_{b1}^{(2)}}(\frac{\rho_1^2}{\zeta})+\frac{(\frac{1}{R^4\rho_1^4}+1)\frac{\rho_1^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(2)}}\quad \zeta\in S_2 \quad (52)$$

Step 5: Analytical continuation across L_2

Since $\phi_{a2}^{(2)}(\zeta)$ and $\omega_{a2}^{(2)}(\zeta)$ can not satisfy the continuity conditions at L_2, two pairs of stress functions $\phi_{b2}^{(2)}(\zeta)$, $\omega_{b2}^{(2)}(\zeta)$and$\phi_{a2}^{(3)}(\zeta)$, $\omega_{a2}^{(3)}(\zeta)$ respectively holomorphic in $|\zeta| \geq \rho_2$ and $|\zeta| \leq \rho_2$ are introduced that

$$\phi_{a2}^{(2)}(\sigma)+\overline{\omega_{a2}^{(2)*}(\sigma)}+\phi_{b2}^{(2)}(\sigma)+\overline{\omega_{b2}^{(2)}(\sigma)}=\phi_{a2}^{(3)}(\sigma)+\overline{\omega_{a2}^{(3)}(\sigma)}+\phi_{b1}^{(3)}(\sigma)+\overline{\omega_{b1}^{(3)}(\sigma)} \quad \sigma\in L_2 \quad (53)$$

$$\frac{\kappa_2}{G_2}\left[\phi_{a2}^{(2)}(\sigma)+\phi_{b2}^{(2)}(\sigma)\right]-\frac{1}{G_2}\left[\overline{\omega_{a2}^{(2)*}(\sigma)}+\overline{\omega_{b2}^{(2)}(\sigma)}\right]=\frac{\kappa_3}{G_3}\phi_{a2}^{(3)}(\sigma)-\frac{1}{G_3}\overline{\omega_{a2}^{(3)}(\sigma)}$$
$$+\frac{\kappa_3}{G_3}\phi_{b1}^{(3)}(\sigma)-\frac{1}{G_3}\overline{\omega_{b1}^{(3)*}(\sigma)} \quad \sigma\in L_2 \quad (54)$$

where $\omega_{b1}^{(3)*}(\zeta)=\omega_{b1}^{(3)}(\zeta)+\dfrac{\frac{R(\rho_2^2-\frac{1}{R^2})}{\zeta}+\frac{\zeta}{R}(\frac{1}{\rho_2^2}-R^2)}{R-\frac{1}{R\zeta^2}}\phi_{b1}^{(3)\prime}(\zeta)$

$$\omega_{a2}^{(2)*}(\zeta)=\frac{\frac{R}{\zeta}(\rho_2^2-\rho_1^2)+\frac{\zeta}{R}(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2})}{R-\frac{1}{R\zeta^2}}\phi_{a2}^{(2)\prime}(\zeta)+\omega_{a2}^{(2)}(\zeta)$$

By the standard analytical continuation arguments, it follows that

$$\varphi_{a2}^{(2)}(\zeta)+\overline{\omega_{b2}^{(2)}}(\frac{\rho_2^2}{\zeta})-\varphi_{a2}^{(3)}(\zeta)-\overline{\omega_{b1}^{(3)*}}(\frac{\rho_2^2}{\zeta})+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(2)}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(3)}=0 \quad |\zeta|\leq\rho_2 \quad (55)$$

$$\overline{\omega_{a2}^{(3)}}(\frac{\rho_2^2}{\zeta})-\overline{\omega_{a2}^{(2)*}}(\frac{\rho_2^2}{\zeta})+\phi_{b1}^{(3)}(\zeta)-\phi_{b2}^{(2)}(\zeta)+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(2)}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(3)}=0 \quad |\zeta|\geq\rho_2 \quad (56)$$

$$\frac{\kappa_2\varphi_{a2}^{(2)}(\zeta)}{G_2}-\frac{\overline{\omega_{b2}^{(2)}}(\frac{\rho_2^2}{\zeta})}{G_2}-\frac{\kappa_3\varphi_{a2}^{(3)}(\zeta)}{G_3}+\frac{\overline{\omega_{b1}^{(3)*}}(\frac{\rho_2^2}{\zeta})}{G_3}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a2}^{(2)}}{G_2}$$
$$+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a2}^{(3)}}{G_3}=0 \quad |\zeta|\leq\rho_2 \quad (57)$$

$$\frac{\kappa_3\phi_{b1}^{(3)}(\zeta)}{G_3}-\frac{\kappa_2\phi_{b2}^{(2)}(\zeta)}{G_2}-\frac{\overline{\omega_{a2}^{(3)}}(\frac{\rho_2^2}{\zeta})}{G_3}+\frac{\overline{\omega_{a2}^{(2)*}}(\frac{\rho_2^2}{\zeta})}{G_2}-\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a2}^{(2)}}{G_2}$$
$$+\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}\frac{c_{a2}^{(3)}}{G_3}=0 \quad |\zeta|\geq\rho_2 \quad (58)$$

Solve Eqs. (55)-(58) to yield

$$\varphi_{a2}^{(3)}(\zeta)=(1+\Lambda_{32})\varphi_{a2}^{(2)}(\zeta)+\Pi_{23}\overline{\omega_{b1}^{(3)*}}(\frac{\rho_2^2}{\zeta})+\Pi_{23}\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(3)} \ \zeta\in S_3 \quad (59)$$

$$\omega_{b2}^{(2)}(\zeta)=\Lambda_{32}\overline{\varphi_{a2}^{(2)}}(\frac{\rho_2^2}{\zeta})+(1+\Pi_{23})\omega_{b1}^{(3)*}(\zeta)-\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(2)}}$$
$$+\frac{(1+\Pi_{23})(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(3)}} \ \zeta\in S_2 \quad (60)$$

$$\varphi_{b2}^{(2)}(\zeta)=(1+\Lambda_{23})\varphi_{b1}^{(3)}(\zeta)+\Pi_{32}\overline{\omega_{a2}^{(2)*}}(\frac{\rho_2^2}{\zeta})-\Pi_{32}\frac{(\frac{1}{R^4\rho_2^4}+1)\zeta}{1-\frac{\zeta^2}{R^2\rho_2^4}}c_{a2}^{(2)} \ \zeta\in S_2 \quad (61)$$

$$\omega_{a2}^{(3)}(\zeta)=(1+\Pi_{32})\omega_{a2}^{(2)*}(\zeta)+\Lambda_{23}\overline{\varphi_{b1}^{(3)}}(\frac{\rho_2^2}{\zeta})-(1+\Pi_{32})\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(2)}}$$
$$+\frac{(\frac{1}{R^4\rho_2^4}+1)\frac{\rho_2^2}{\zeta}}{1-\frac{1}{R^2\zeta^2}}\overline{c_{a2}^{(3)}} \ \zeta\in S_3 \quad (62)$$

By repetitions of the previous three steps to get the results of which the continuity condition and the boundary condition are satisfied, one can obtain the full field stress functions as

$$\begin{cases}\phi_{b1}^{(1)}(\zeta)=\frac{\Pi_{21}T}{4}(\frac{1}{R^2\zeta}-2e^{2i\lambda}\frac{\rho_1^2}{\zeta})\\ \omega_{b1}^{(1)}(\zeta)=\Lambda_{21}\overline{\phi_0}(\frac{\rho_1^2}{\zeta})-\frac{TR}{4}(\frac{\rho_1^2+\frac{1}{R^4\rho_1^2}}{R\zeta-\frac{1}{R\zeta}})+(1+\Pi_{12})\overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{a1}^{(2)}}\end{cases} \quad (63)$$

$$\begin{cases} \varphi_{a1}^{(2)}(\zeta) = (1+\Lambda_{21})\varphi_0(\zeta) + \Pi_{12} d_1(\zeta) c_{a1}^{(2)} \\ \omega_{a1}^{(2)}(\zeta) = \frac{(1+\Pi_{21})T}{4}\left(\frac{\zeta}{R^2\rho_1^2} - 2e^{-2i\lambda}\zeta\right) + \overline{d_1}\left(\frac{\rho_1^2}{\zeta}\right)\overline{c_{a1}^{(2)}} \\ \varphi_{b1}^{(2)}(\zeta) = \Pi_{32}\left[\overline{\omega_{a1}^{(2)}}\left(\frac{\rho_2^2}{\zeta}\right) + \overline{t_{12}}\left(\frac{\rho_2^2}{\zeta}\right)\overline{\varphi_{a1}^{(2)\prime}}\left(\frac{\rho_2^2}{\zeta}\right)\right] - \Pi_{32} d_2(\zeta) c_{a1}^{(2)} \\ \omega_{b1}^{(2)}(\zeta) = \Lambda_{32}\overline{\varphi_{a1}^{(2)}}\left(\frac{\rho_2^2}{\zeta}\right) - \overline{d_2}\left(\frac{\rho_1^2}{\zeta}\right)\overline{c_{a1}^{(2)}} + (1+\Pi_{23})\overline{d_2}\left(\frac{\rho_1^2}{\zeta}\right)\overline{c_{a1}^{(3)}} \end{cases} \tag{64}$$

$$\begin{cases} \varphi_{a1}^{(3)}(\zeta) = (1+\Lambda_{32})\varphi_{a1}^{(2)}(\zeta) + \Pi_{23} d_2(\zeta) c_{a1}^{(3)} \\ \omega_{a1}^{(3)}(\zeta) = (1+\Pi_{32})\left[\omega_{a1}^{(2)}(\zeta) + t_{12}(\zeta)\varphi_{a1}^{(2)\prime}(\zeta)\right] \\ \qquad -(1+\Pi_{32})\overline{d_2}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{a1}^{(2)}} + \overline{d_2}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{a1}^{(3)}} \\ \varphi_{b1}^{(3)}(\zeta) = (1+\Lambda_{32})\varphi_{a1}^{(2)}\left(\frac{1}{R^2\zeta}\right) + \Pi_{23} d_2\left(\frac{1}{R^2\zeta}\right) c_{a1}^{(3)} \\ \omega_{b1}^{(3)}(\zeta) = \omega_{a1}^{(3)}\left(\frac{1}{R^2\zeta}\right) + t_{23}\left(\frac{1}{R^2\zeta}\right)\varphi_{a1}^{(3)\prime}\left(\frac{1}{R^2\zeta}\right) \end{cases} \tag{65}$$

$$\begin{cases} \phi_{bn}^{(1)}(\zeta) = (1+\Lambda_{12})\phi_{b(n-1)}^{(2)}(\zeta) \\ \qquad n = 2,3,4... \\ \omega_{bn}^{(1)}(\zeta) = (1+\Pi_{12})\left[\omega_{b(n-1)}^{(2)}(\zeta) + t_{21}(\zeta)\phi_{b(n-1)}^{(2)\prime}(\zeta)\right] + (1+\Pi_{12})\overline{d_1}\left(\frac{\rho_1^2}{\zeta}\right)\overline{c_{an}^{(2)}} \\ \qquad n = 2,3,4... \end{cases} \tag{66}$$

$$\begin{cases} \varphi_{an}^{(2)}(\zeta) = \Pi_{12}\left[\overline{\omega_{b(n-1)}^{(2)}}\left(\frac{\rho_1^2}{\zeta}\right) + \overline{t_{21}}\left(\frac{\rho_1^2}{\zeta}\right)\overline{\varphi_{b(n-1)}^{(2)\prime}}\left(\frac{\rho_1^2}{\zeta}\right)\right] + d_1(\zeta)\Pi_{12} c_{an}^{(2)} \\ \qquad n = 2,3,4... \\ \omega_{an}^{(2)}(\zeta) = \Lambda_{12}\overline{\varphi_{b(n-1)}^{(2)}}\left(\frac{\rho_1^2}{\zeta}\right) + \overline{d_1}\left(\frac{\rho_1^2}{\zeta}\right)\overline{c_{an}^{(2)}} \\ \qquad n = 2,3,4... \\ \varphi_{bn}^{(2)}(\zeta) = (1+\Lambda_{23})\varphi_{b(n-1)}^{(3)}(\zeta) + \Pi_{32}\Lambda_{12}\varphi_{b(n-1)}^{(2)}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right) + \Pi_{32} d_1\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right) c_{an}^{(2)} \\ -\Pi_{12}\Pi_{32}\frac{\rho_1^2\zeta^2}{\rho_2^4}\overline{t_{12}}\left(\frac{\rho_2^2}{\zeta}\right)\left[\omega_{b(n-1)}^{(2)\prime}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right) + t_{21}'\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right)\varphi_{b(n-1)}^{(2)\prime}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right) + t_{21}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right)\varphi_{b(n-1)}^{(2)\prime\prime}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right)\right] \\ +\Pi_{12}\Pi_{32}\overline{t_{12}}\left(\frac{\rho_2^2}{\zeta}\right)\overline{d_1'}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{an}^{(2)}} - \Pi_{32} d_2(\zeta) c_{an}^{(2)} \\ \qquad n = 2,3,4... \\ \omega_{bn}^{(2)}(\zeta) = \Lambda_{32}\Pi_{12}\left[\omega_{b(n-1)}^{(2)}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right) + t_{21}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right)\varphi_{b(n-1)}^{(2)\prime}\left(\frac{\rho_1^2}{\rho_2^2}\zeta\right)\right] + \Lambda_{32}\Pi_{12}\overline{d_1}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{an}^{(2)}} \\ +(1+\Pi_{23})\left[\omega_{b(n-1)}^{(3)}(\zeta) + t_{32}(\zeta)\varphi_{b(n-1)}^{(3)\prime}(\zeta)\right] \\ -\overline{d_2}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{an}^{(2)}} + (1+\Pi_{23})\overline{d_2}\left(\frac{\rho_2^2}{\zeta}\right)\overline{c_{an}^{(3)}} \\ \qquad n = 2,3,4... \end{cases}$$

(67)

$$\begin{cases} \varphi_{an}^{(3)}(\zeta) = (1+\Lambda_{32})\Pi_{12}\left[\overline{\omega_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \overline{t_{21}}(\frac{\rho_1^2}{\zeta})\overline{\varphi_{b(n-1)}^{(2)'}}(\frac{\rho_1^2}{\zeta})\right] \\ +(1+\Lambda_{32})\Pi_{12}d_1(\zeta)c_{an}^{(2)} \\ +\Pi_{23}\left[\overline{\omega_{b(n-1)}^{(3)}}(\frac{\rho_2^2}{\zeta}) + \overline{t_{32}}(\frac{\rho_2^2}{\zeta})\overline{\varphi_{b(n-1)}^{(3)'}}(\frac{\rho_2^2}{\zeta})\right] + \Pi_{23}d_2(\zeta)c_{an}^{(3)} \\ \quad n = 2,3,4... \\ \omega_{an}^{(3)}(\zeta) = (1+\Pi_{32})\left[\Lambda_{12}\overline{\varphi_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{an}^{(2)}} + t_{12}(\zeta)d_1'(\zeta)\Pi_{12}c_{an}^{(2)}\right] \\ -(1+\Pi_{32})\Pi_{12}\frac{\rho_1^2}{\zeta^2}t_{12}(\zeta)\left[\overline{\omega_{b(n-1)}^{(2)'}}(\frac{\rho_1^2}{\zeta}) + \overline{t_{21}'}(\frac{\rho_1^2}{\zeta})\overline{\varphi_{b(n-1)}^{(2)'}}(\frac{\rho_1^2}{\zeta}) + \overline{t_{21}}(\frac{\rho_1^2}{\zeta})\overline{\varphi_{b(n-1)}^{(2)''}}(\frac{\rho_1^2}{\zeta})\right] \\ +\Lambda_{23}\overline{\varphi_{b(n-1)}^{(3)}}(\frac{\rho_2^2}{\zeta}) - (1+\Pi_{32})\overline{d_2}(\frac{\rho_2^2}{\zeta})\overline{c_{an}^{(2)}} + \overline{d_2}(\frac{\rho_2^2}{\zeta})\overline{c_{an}^{(3)}} \\ \quad n = 2,3,4.... \\ \varphi_{bn}^{(3)}(\zeta) = \varphi_{an}^{(3)}(\frac{1}{R^2\zeta}) \\ \quad n = 2,3,4.... \\ \omega_{bn}^{(3)}(\zeta) = \omega_{an}^{(3)}(\frac{1}{R^2\zeta}) + t_{23}(\frac{1}{R^2\zeta})\varphi_{an}^{(3)'}(\frac{1}{R^2\zeta}) \\ \quad n = 2,3,4... \end{cases}$$

(68)

where

$$t_{21}(\zeta) = -t_{12}(\zeta) = \frac{\frac{R(\rho_1^2-\rho_2^2)}{\zeta} + \frac{\zeta}{R}(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2})}{R - \frac{1}{R\zeta^2}}$$

$$t_{23}(\zeta) = -t_{32}(\zeta) = \frac{\frac{R}{\zeta}(\frac{1}{R^2} - \rho_2^2) + \frac{\zeta}{R}(R^2 - \frac{1}{\rho_2^2})}{R - \frac{1}{R\zeta^2}}$$

$$d_i(\zeta) = \frac{(\frac{1}{R^4\rho_i^4}+1)\zeta}{1 - \frac{\zeta^2}{R^2\rho_i^4}} \quad i = 1,2$$

$$c_{an} = \overline{\phi_{an}'}(\frac{1}{R}).$$

4 Some specific examples

4.1 Three-phase circularly cylindrical layered media

For a limiting case that the aspect ratio b_i/a_i equals one ($1/R = 0$), Eqs. (63)∼(64) reduce to

$$\varphi(\zeta) = \begin{cases} \varphi_0(\zeta) + \sum\limits_{n=1}^{\infty} \varphi_{bn}^{(1)}(\zeta) & z \in S_1 \\ \sum\limits_{n=1}^{\infty} \varphi_{an}^{(2)}(\zeta) + \sum\limits_{n=1}^{\infty} \varphi_{bn}^{(2)}(\zeta) & z \in S_2 \\ \sum\limits_{n=1}^{\infty} \varphi_{an}^{(3)}(\zeta) & z \in S_3 \end{cases} \tag{69}$$

$$\omega(\zeta) = \begin{cases} \omega_0(\zeta) + \sum\limits_{n=1}^{\infty} \omega_{bn}^{(1)}(\zeta) & z \in S_1 \\ \sum\limits_{n=1}^{\infty} \omega_{an}^{(2)}(\zeta) + \sum\limits_{n=1}^{\infty} \omega_{bn}^{(2)}(\zeta) & z \in S_2 \\ \sum\limits_{n=1}^{\infty} \omega_{an}^{(3)}(\zeta) & z \in S_3 \end{cases} \tag{70}$$

where $\omega_{0a}(\zeta) = -\frac{T}{2}e^{-2i\lambda}\zeta$ and $\omega_{0b}(\zeta) = \frac{T}{4}\frac{\rho_1^2}{\zeta}$ which are holomorphic in $|\zeta| \le \rho_1$ and $|\zeta| \ge \rho_1$, respectively.

$$\begin{cases} \varphi_{b1}^{(1)}(\zeta) = \Pi_{21}\overline{\omega_{0a}}(\frac{\rho_1^2}{\zeta}) & \zeta \in S_1 \\ \omega_{b1}^{(1)}(\zeta) = \Lambda_{21}\overline{\varphi_0}(\frac{\rho_1^2}{\zeta}) - \omega_{0b}(\zeta) + (1+\Pi_{12})\frac{\rho_1^2}{\zeta}\overline{c_{a1}^{(2)}} & \zeta \in S_1 \\ \varphi_{a1}^{(2)}(\zeta) = (1+\Lambda_{21})\varphi_0(\zeta) + \Pi_{12}\zeta c_{a1}^{(2)} & \zeta \in S_2 \\ \omega_{a1}^{(2)}(\zeta) = (1+\Pi_{21})\omega_{0a}(\zeta) + \frac{\rho_1^2}{\zeta}\overline{c_{a1}^{(2)}}\zeta \in S_2 \end{cases} \tag{71}$$

$$\begin{cases} \varphi_{a1}^{(3)}(\zeta) = (1+\Lambda_{32})\varphi_{a1}^{(2)}(\zeta) + \Pi_{23}\zeta c_{a1}^{(3)} \\ \qquad \zeta \in S_3 \\ \omega_{a1}^{(3)}(\zeta) = (1+\Pi_{32})\left[\omega_{a1}^{(2)}(\zeta) + t_{12}(\zeta)\varphi_{a1}^{(2)\prime}(\zeta)\right] - (1+\Pi_{32})\frac{\rho_2^2}{\zeta}\overline{c_{a1}^{(2)}} + \frac{\rho_2^2}{\zeta}\overline{c_{a1}^{(3)}} \\ \qquad \zeta \in S_2 \\ \varphi_{b1}^{(2)}(\zeta) = \Pi_{32}\left[\overline{\omega_{a1}^{(2)}}(\frac{\rho_2^2}{\zeta}) + \overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{\varphi_{a1}^{(2)\prime}}(\frac{\rho_2^2}{\zeta})\right] - \Pi_{32}\zeta c_{a1}^{(2)} \\ \qquad \zeta \in S_2 \\ \omega_{b1}^{(2)}(\zeta) = \Lambda_{32}\overline{\varphi_{a1}^{(2)}}(\frac{\rho_2^2}{\zeta}) - \frac{\rho_2^2}{\zeta}\overline{c_{a1}^{(2)}} + (1+\Pi_{23})\frac{\rho_2^2}{\zeta}\overline{c_{a1}^{(3)}} \\ \qquad \zeta \in S_2 \end{cases} \tag{72}$$

The recurrence formulae are

$$\begin{cases}
\varphi_{bn}^{(1)}(\zeta) = (1+\Lambda_{12})\varphi_{b(n-1)}^{(2)}(\zeta) \\
\qquad \zeta \in S_1 \\
\omega_{bn}^{(1)}(\zeta) = (1+\Pi_{12})\left[\omega_{b(n-1)}^{(2)}(\zeta) + t_{21}(\zeta)\varphi_{b(n-1)}^{(2)'}(\zeta)\right] + (1+\Pi_{12})\frac{\rho_1^2}{\zeta}\overline{c_{an}^{(2)}} \\
\qquad \zeta \in S_1 \\
\varphi_{an}^{(2)}(\zeta) = \Pi_{12}\left[\overline{\omega_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \overline{t_{21}}(\frac{\rho_1^2}{\zeta})\overline{\varphi_{b(n-1)}^{(2)'}}(\frac{\rho_1^2}{\zeta})\right] + \Pi_{12}\zeta c_{an}^{(2)} \\
\qquad \zeta \in S_2 \\
\omega_{an}^{(2)}(\zeta) = \Lambda_{12}\overline{\varphi_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \frac{\rho_1^2}{\zeta}\overline{c_{an}^{(2)}} \\
\qquad \zeta \in S_2 \\
\\
\varphi_{an}^{(3)}(\zeta) = (1+\Lambda_{32})\varphi_{an}^{(2)}(\zeta) + \Pi_{23}\zeta c_{an}^{(3)} \\
\qquad \zeta \in S_3 \\
\omega_{an}^{(3)}(\zeta) = (1+\Pi_{32})\left[\omega_{an}^{(2)}(\zeta) + t_{12}(\zeta)\varphi_{an}^{(2)'}(\zeta)\right] - (1+\Pi_{32})\frac{\rho_2^2}{\zeta}\overline{c_{an}^{(2)}} + \frac{\rho_2^2}{\zeta}\overline{c_{an}^{(3)}} \\
\qquad \zeta \in S_3 \\
\varphi_{bn}^{(2)}(\zeta) = \Pi_{32}\left[\overline{\omega_{an}^{(2)}}(\frac{\rho_2^2}{\zeta}) + \overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{\varphi_{an}^{(2)'}}(\frac{\rho_2^2}{\zeta})\right] - \Pi_{32}\zeta c_{an}^{(2)} \\
\qquad \zeta \in S_2 \\
\omega_{bn}^{(2)}(\zeta) = \Lambda_{32}\overline{\varphi_{an}^{(2)}}(\frac{\rho_2^2}{\zeta}) - \frac{\rho_2^2}{\zeta}\overline{c_{an}^{(2)}} + (1+\Pi_{23})\frac{\rho_2^2}{\zeta}\overline{c_{an}^{(3)}} \\
\qquad \zeta \in S_2
\end{cases} \tag{73}$$

for $n = 2, 3, 4, \ldots$

$$t_{21}(\zeta) = -t_{12}(\zeta) = \frac{\rho_1^2 - \rho_2^2}{\zeta}$$

$$t_{23}(\zeta) = -t_{32}(\zeta) = \frac{\zeta^2 - \rho_2^2}{\zeta}$$

The results presented in Eqs. (67)∼(69) are the same as those provided by [Chao, Chen and Shen (2006)].

4.2 A reinforcement layer with an elliptic hole

For the second special case when the material S_3 becomes a hole ($\Lambda_{32} = \Pi_{32} = -1$), Eqs. (63) and (64) reduce to

$$\varphi(\zeta) = \begin{cases} \varphi_0(\zeta) + \sum\limits_{n=1}^{\infty} \varphi_{bn}^{(1)}(\zeta) & z \in S_1 \\ \sum\limits_{n=1}^{\infty} \varphi_{an}^{(2)}(\zeta) + \sum\limits_{n=1}^{\infty} \varphi_{bn}^{(2)}(\zeta) & z \in S_2 \end{cases} \tag{74}$$

$$\omega(\zeta) = \begin{cases} \omega_0(\zeta) + \sum\limits_{n=1}^{\infty} \omega_{bn}^{(1)}(\zeta) & z \in S_1 \\ \sum\limits_{n=1}^{\infty} \omega_{an}^{(2)}(\zeta) + \sum\limits_{n=1}^{\infty} \omega_{bn}^{(2)}(\zeta) & z \in S_2 \end{cases} \tag{75}$$

where

$$\begin{cases} \varphi_{b1}^{(1)}(\zeta) = \Pi_{21}\overline{\omega_{0a}}(\frac{\rho_1^2}{\zeta}) & \zeta \in S_1 \\ \omega_{b1}^{(1)}(\zeta) = \Lambda_{21}\overline{\varphi_0}(\frac{\rho_1^2}{\zeta}) - \omega_{0b}(\zeta) + (1+\Pi_{12})\overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{a1}^{(2)}} & \zeta \in S_1 \\ \varphi_{a1}^{(2)}(\zeta) = (1+\Lambda_{21})\varphi_0(\zeta) + \Pi_{12}d_1(\zeta)c_{a1}^{(2)} & \zeta \in S_2 \\ \omega_{a1}^{(2)}(\zeta) = (1+\Pi_{21})\omega_{0a}(\zeta) + \overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{a1}^{(2)}} & \zeta \in S_2 \\ \varphi_{b1}^{(2)}(\zeta) = -\left[\overline{\omega_{a1}^{(2)}}(\frac{\rho_2^2}{\zeta}) + \overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{\varphi_{a1}^{(2)\prime}}(\frac{\rho_2^2}{\zeta})\right] + d_1(\zeta)c_{a1}^{(2)} & \zeta \in S_2 \\ \omega_{b1}^{(2)}(\zeta) = -\overline{\varphi_{a1}^{(2)}}(\frac{\rho_2^2}{\zeta}) - \overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{a1}^{(2)}} & \zeta \in S_2 \end{cases} \tag{76}$$

The recurrence formulae are

$$\begin{cases} \phi_{bn}^{(1)}(\zeta) = (1+\Lambda_{12})\phi_{b(n-1)}^{(2)}(\zeta) \\ \qquad n = 2,3,4... \\ \omega_{bn}^{(1)}(\zeta) = (1+\Pi_{12})\left[\omega_{b(n-1)}^{(2)}(\zeta) + t_{21}(\zeta)\phi_{b(n-1)}^{(2)\prime}(\zeta)\right] + (1+\Pi_{12})\overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{an}^{(2)}} \\ \qquad n = 2,3,4... \end{cases} \tag{77}$$

$$\begin{cases} \varphi_{an}^{(2)}(\zeta) = \Pi_{12}\left[\overline{\omega_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \overline{t_{21}}(\frac{\rho_1^2}{\zeta})\overline{\varphi_{b(n-1)}^{(2)'}}(\frac{\rho_1^2}{\zeta})\right] + d_1(\zeta)\Pi_{12}c_{an}^{(2)} \\ \qquad n = 2,3,4... \\ \omega_{an}^{(2)}(\zeta) = \Lambda_{12}\overline{\varphi_{b(n-1)}^{(2)}}(\frac{\rho_1^2}{\zeta}) + \overline{d_1}(\frac{\rho_1^2}{\zeta})\overline{c_{an}^{(2)}} \\ \qquad n = 2,3,4... \\ \varphi_{bn}^{(2)}(\zeta) = -\left[\overline{\omega_{an}^{(2)}}(\frac{\rho_2^2}{\zeta}) + \overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{\varphi_{an}^{(2)'}}(\frac{\rho_2^2}{\zeta})\right] + d_2(\zeta)c_{an}^{(2)} \\ \qquad n = 2,3,4... \\ \omega_{bn}^{(2)}(\zeta) = -\overline{\varphi_{an}^{(2)}}(\frac{\rho_2^2}{\zeta}) - \overline{d_2}(\frac{\rho_2^2}{\zeta})\overline{c_{an}^{(2)}} \\ \qquad n = 2,3,4... \end{cases} \tag{78}$$

which are found to agree with the results given by [Chao, Chen and Chen (2009)].

4.3 A single elliptical hole under tension

When the regions S_1 and S_2 are made of the same material for the corresponding an elliptical hole problem, Eqs. (73)∼(75) can be simplified to an exact form

$$\varphi(\zeta) = \frac{T}{4}\zeta + \frac{T}{4}\left[\frac{2e^{2i\gamma}}{\zeta} - \frac{u}{\zeta}\right] \tag{79}$$

$$\psi(\zeta) = -\frac{T}{2}e^{-2i\gamma}\zeta - \frac{T}{4}\frac{1}{\zeta} - \frac{(u^2+1)\zeta}{\zeta^2-u}\frac{T}{4} - \frac{\zeta+u\zeta^3}{\zeta^2-u}\left(\frac{u}{\zeta^2}\frac{T}{4} - \frac{T}{2}e^{2i\gamma}\frac{1}{\zeta^2}\right) \tag{80}$$

where $u = \frac{a_2-b_2}{a_2+b_2}$, which are the same as the results provided by [England (1971)].

4.4 A rigid elliptical inhomogeneity

For the corresponding rigid elliptical inclusion problem, i.e., $G_3 = G_2$, and $\Pi_{12} = \Lambda_{12} = -1$, Eqs. (63) and (64) reduce to

$$\varphi(\zeta) = \varphi_0(\zeta) + \varphi_{b1}^{(1)}(\zeta) = \frac{T}{4}\zeta + \frac{1}{\kappa_1\zeta}\left(\frac{1}{R^2}\frac{T}{4} - \frac{T}{2}e^{2i\lambda}\right) \tag{81}$$

$$\psi(\zeta) = \psi_0(\zeta) + \omega_{b1}^{(1)}(\zeta) - \frac{\overline{m}(\frac{1}{\zeta})}{m'(\zeta)}\varphi_{b1}^{(1)'}(\zeta)$$

$$= -\frac{T\zeta}{2}e^{-2i\lambda} + \frac{T}{4}\left(\frac{\kappa_1}{\zeta} - \frac{R^2\zeta + R^{-2}\zeta}{R^2\zeta^2-1}\right) + \left(\frac{T}{4R^2} - \frac{T}{2}e^{2i\lambda}\right)\frac{R^2+\zeta^2}{R^2\zeta^2-1}\frac{1}{\kappa_1\zeta} \tag{82}$$

which are consistent with the solutions givens by [Muskhelishvili (1953)].

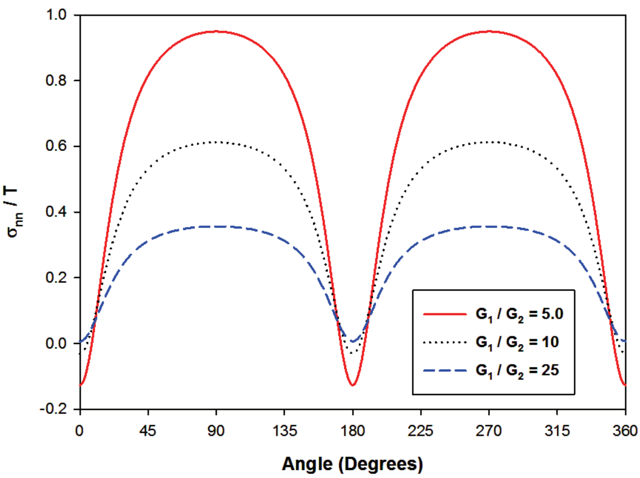

Figure 3: Angular variations of the interfacial normal stress along Γ_1 for different shear modulus ratios$(a_1/b_1 = 1.5, a_2/a_1 = 0.9, G_3/G_2 = 5, \upsilon_1 = \upsilon_2 = \upsilon_3 = 0.3, \lambda = \pi/2)$.

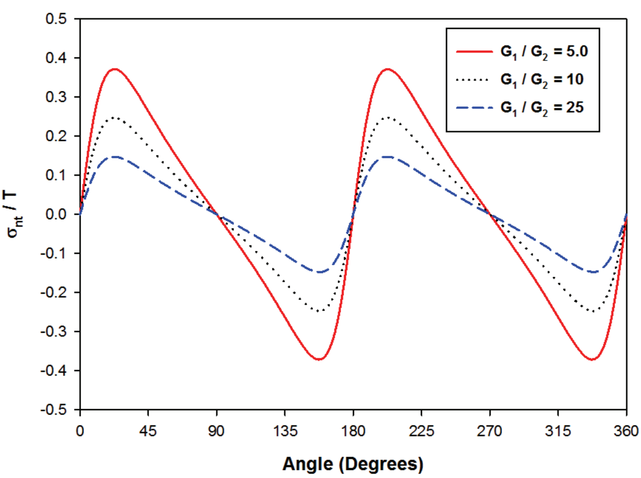

Figure 4: Angular variations of the interfacial shear stress along Γ_1 for different shear modulus ratios$(a_1/b_1 = 1.5, a_2/a_1 = 0.9, G_3/G_2 = 5, \upsilon_1 = \upsilon_2 = \upsilon_3 = 0.3, \lambda = \pi/2)$.

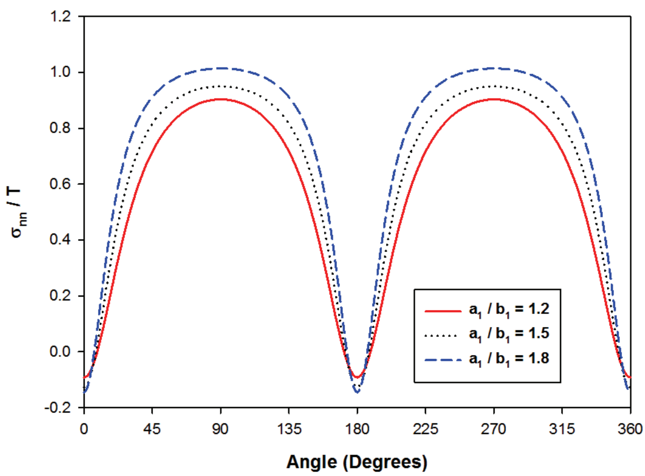

Figure 5: Angular variations of the interfacial normal stress along Γ_1 for different shear modulus ratios$(a_2/a_1 = 0.9, G_3/G_2 = G_1/G_2 = 5, \upsilon_1 = \upsilon_2 = \upsilon_3 = 0.3, \lambda = \pi/2)$.

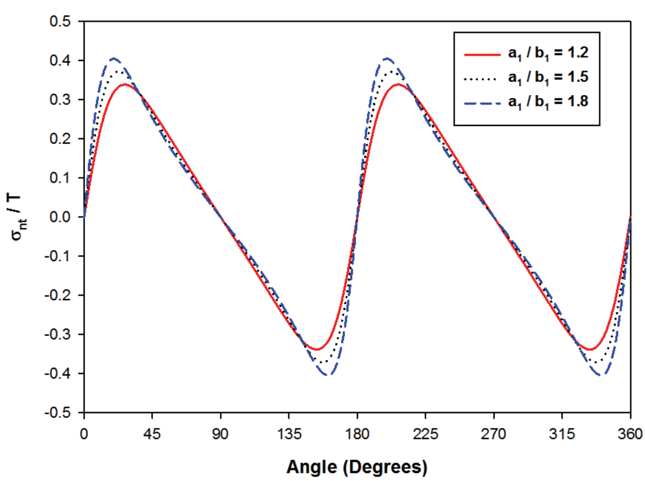

Figure 6: Angular variations of the interfacial shear stress along Γ_1 for different shear modulus ratios$(a_2/a_1 = 0.9, G_3/G_2 = G_1/G_2 = 5, \upsilon_1 = \upsilon_2 = \upsilon_3 = 0.3, \lambda = \pi/2)$.

5 Results and discussion

All the stress functions $\varphi_{an}^{(2)}(\zeta)$, $\varphi_{an}^{(3)}(\zeta)$, $\varphi_{bn}^{(1)}(\zeta)$, $\varphi_{bn}^{(2)}(\zeta)$, $\varphi_{bn}^{(3)}(\zeta)$, $\omega_{an}^{(2)}(\zeta)$, $\omega_{an}^{(3)}(\zeta)$, $\omega_{bn}^{(1)}(\zeta)$, $\omega_{bn}^{(2)}(\zeta)$, $\omega_{bn}^{(3)}(\zeta)$ in Eqs. (63) and (64) can be calculated from $\varphi_0(z)$ and $\varphi_0(z)$. The rate of the convergence depends on the two non-dimensional bimaterial constants Λ_{12} and Π_{12}. Since Λ_{12} and Π_{12} are less than 1 and 0.5, respectively, which guarantees rapid convergence. The angular variations of the interfacial normal stress and interfacial shear stress between the reinforcement layer and the matrix, under the condition that a uniform tensile load is applied along the y axis, are shown in Figures 3 and 4 respectively. As expected, the tangential normal stress is symmetric about the y-axis while the tangential shear stress is asymmetric about the y-axis. The magnitudes of both the normal stress and the shear stress decrease with an increasing ratio of G_1/G_2. This is simply because that the interfacial stresses can be further intensified (or diminished) by the adjacent material having a higher (or lower) stiffness. The effects of the geometric configuration on the interfacial stresses are displayed in Figures 5 and 6. It is clear to see that the magnitudes of the interfacial stresses increase with the ratio a_1/b_1. Based on the above findings, it allows us to find an optimum design of the intermediate layer such that the magnitude of both stress concentration and the interfacial stresses could be fairly reduced. Note that all these calculated results have been determined by summing up the first ten terms in Eq. (63) and Eq. (64). A good accuracy for the current problem can be demonstrated by the contribution of the leading terms appearing in Eq. (15) and Eq. (16). The contribution of the stresses for the leading terms of a series solution is 26.13% (first term), 11.24% (third term), 1.39% (fifth term) and 0.57% (tenth term), respectively. The contribution accounts for the ratio of each term to the summation of the first ten terms of a series solution. The leading ten terms have over 99% contribution, making the series solution rapidly convergent. This demonstrates the accuracy and the efficiency of our proposed method. Note that the convergence rate depends on the combinations of material properties and geometric configurations. In general, the convergence rate becomes more rapid if the differences of the elastic constants of the neighboring materials get smaller and the ratio a_1/b_1 (or a_2/b_2) approaches one.

6 Conclusion

The explicit solutions for the three-phase elliptic inclusion problem subjected to a remote uniform load are provided in this paper. Based on the method of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the elastic fields of the present problem are expressed in terms of the solution to the corresponding homogeneous solution. The present proposed method can also be extended to the problem with any number of layers. Practically, a graded interface can be achieved by multilayered materials with stepwise homogeneous elastic properties. Consequently, the problem with functionally graded materials can be solved using the present proposed method.

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