 CMES, vol.69, no.1, pp.43-60, 2010

Numerical Solution of Dual Phase Lag Model of Bioheat Transfer Using the General Boundary Element Method

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Abstract: Heat transfer processes proceeding in domain of heating tissue are discussed. The typical model of bioheat transfer bases, as a rule, on the well known Pennes equation, this means the heat diffusion equation with additional terms corresponding to the perfusion and metabolic heat sources. Here, the other approach basing on the dual-phase-lag equation (DPLE) is considered in which two time delays τ_q, τ_T (phase lags) appear. The DPL equation contains a second order time derivative and higher order mixed derivative in both time and space. This equation is supplemented by the adequate boundary and initial conditions. To solve the problem the general boundary element method is adapted. The examples of computations for 2D problem are presented in the final part of the paper. The efficiency and exactness of the algorithm proposed are also discussed.

Keywords: bioheat transfer, dual-phase-lag model, general boundary element method

1 Introduction

Heat transfer processes proceeding in the living tissues are described, as a rule, by the well known Pennes equation (heat diffusion equation supplemented by the source functions called the perfusion heat source and the metabolic heat source) [Pennes (1948); Liu and Xu (2000); Majchrzak (1998); Erhart, Divo and Kassab (2008)]. In the special cases e.g. tissue freezing, the additional source function appears [Comini and Del Giudice (1976); Majchrzak and Dziewonski (2000); Mochnacki and Dziewonski (2004)]. The Pennes bioheat equation bases on the classical Fourier's law with its assumption of instantaneous thermal propagation. Cattaneo and Vernotte [Cattaneo (1958); Vernotte (1958)] proposed a thermal wave model with a relaxation time τ_q that is required for a heat flux to respond to the temperature gradient and then the thermal wave propagates through the medium with

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a finite speed. Because the living tissues are highly nonhomogeneous and accumulating energy to transfer to the nearest elements would take time, therefore the Cattaneo-Vernotte equation in comparison with the Pennes one describes more exactly the heat transfer proceeding in the biological tissue [Kaminski (1990); Özisik and Tzou (1994); Tamma and Zhou (1998)]. Although the thermal wave model takes into account the microscale response in time, the wave concept does not explain the microscale response in space. Additionally, the thermal wave model gives sometimes unusual physical solutions. To consider the effect of microstructural interactions in the fast transient process of heat transport, a phase lag τ_T for temperature gradient is introduced. In this way the dual-phase lag heat conduction model (DPLM) is considered in which two time delays τ_q, τ_T (phase lags) appear [Özisik and Tzou (1994); Xu, Seffen and Lu (2008); Zhou, Chen and Zhang (2009); Majchrzak, Mochnacki, Greer and Suchy (2009)].

In this work, the general boundary element method proposed by Liao [Liao (1997); Liao (2002); Liao and Chwang (1999)] is extended and adapted to solve the dual-phase lag equation. It should be pointed out that for the DPL equation the corresponding fundamental solution is either unknown or very difficult to obtain. So, the concept for solving the unsteady non-linear differential equation using a direct boundary element approach based on homotopy analysis method is applied [Liao (1997)]. To verify the algorithm proposed, the results of computations under the assumption that $\tau_q = \tau_T$=0 have been compared with the 1st scheme of the BEM [Brebbia, Telles and Wrobel (1984); Majchrzak (2001)], while in the final part of the paper the examples of DPLM solutions are presented.

2 Bioheat transfer models

The general bioheat transfer equation is given as

$$c\frac{\partial T(x,t)}{\partial t} = -\nabla \mathbf{q}(x,t) + Q(x,t) \tag{1}$$

where c is the volumetric specific heat of tissue, $\mathbf{q}(x, t)$ is the heat flux, $Q(x, t)$ is the source term due to metabolism and blood perfusion, T is the temperature, x are the spatial co-ordinates and t is the time.

Heat transfer in biological systems is usually described by the equation basing on the classical Fourier law

$$\mathbf{q}(x,t) = -\lambda \nabla T(x,t) \tag{2}$$

where λ is the thermal conductivity of tissue and $\nabla T(x, t)$ is the temperature gradient.

Introducing (2) into (1) one obtains the traditional Pennes bioheat transfer equation (for constant λ) [Pennes (1948)]

$$c\frac{\partial T(x,t)}{\partial t} = \lambda \nabla^2 T(x,t) + Q(x,t) \tag{3}$$

The source term $Q(x,t)$ is as follows

$$Q(x,t) = G_B c_B [T_B - T(x,t)] + Q_m \tag{4}$$

where G_B is the blood perfusion rate, c_B is the volumetric specific heat of blood, T_B is the blood temperature and Q_m is the metabolic heat source. The Pennes equation constitutes, as yet, the most popular model of thermal processes proceeding in domain of living tissue (e.g. [Torvi and Dale (1994); Liu and Xu (2000); Lv, Deng and Liu (2005); Majchrzak and Mochnacki (2006); Erhart, Divo and Kassab (2008)]).

Because the biological tissue is the material with particular nonhomogeneous inner structure therefore the others, modified heat conduction equations, should be taken into account. One of them is the Cattaneo-Vernotte model [Cattaneo (1958); Vernotte (1958); Lu, Liu and Zeng (1998)] with corresponding equation

$$\mathbf{q}(x,\tau_q) = -\lambda \nabla T(x,t) \tag{5}$$

where $\tau_q = a/C^2$ is defined as the relaxation time, $a = \lambda/c$ is the thermal diffussivity of tissue and C is the speed of thermal wave in the medium.

According to this equation τ_q is the phase-lag in establishing the heat flux and associated conduction through the medium.

Taking into account the first order Taylor expansion for $\mathbf{q}$

$$\mathbf{q}(x,t) + \tau_q \frac{\partial \mathbf{q}(x,t)}{\partial t} = -\lambda \nabla T(x,t) \tag{6}$$

one obtains the Cattaneo-Vernotte equation in the following form

$$c\left[\frac{\partial T(x,t)}{\partial t} + \tau_q \frac{\partial^2 T(x,t)}{\partial t^2}\right] = \lambda \nabla^2 T(x,t) + Q(x,t) + \tau_q \frac{\partial Q(x,t)}{\partial t} \tag{7}$$

According to the newest publications, e.g. [Xu, Seffen and Lu (2008); Zhou, Chen and Zhang (2009)] the heat transfer in biological tissues should be described by dual-phase-lag model basing on the assumption that

$$\mathbf{q}(x,t+\tau_q) = -\lambda \nabla T(x,t+\tau_T) \tag{8}$$

where τ_q is the relaxation time, τ_T is the thermalization time, τ_q is the phase-lag in establishing the heat flux and associated conduction through the medium, τ_T is the phase-lag in establishing the temperature gradient across the medium during which conduction occurs through its small-scale structures [Xu, Seffen and Lu (2008)].

Taking into account the first order Taylor expansions for $\mathbf{q}$ and T

$$q(x,t)+\tau_q\frac{\partial \mathbf{q}(x,t)}{\partial t}=-\lambda\nabla T(x,t)-\lambda\tau_T\frac{\partial \nabla T(x,t)}{\partial t} \tag{9}$$

one obtains the following form of bioheat transfer equation

$$c\left[\frac{\partial T(x,t)}{\partial t}+\tau_q\frac{\partial^2 T(x,t)}{\partial t^2}\right]=\lambda\nabla^2 T(x,t)+\lambda\tau_T\frac{\partial \nabla^2 T(x,t)}{\partial t}+Q(x,t)+\tau_q\frac{\partial Q(x,t)}{\partial t} \tag{10}$$

It should be pointed out that for τ_T=0 the dual-phase-lag (DPL) equation (10) reduces to the Cattaneo-Vernotte equation (7), while for $\tau_q=\tau_T$=0 reduces to the Pennes one (3).

3 General boundary element method for DPL equation

From the mathematical point of view the Pennes equation (3) is the parabolic one, the Cattaneo-Vernotte equation (7) is the hyperbolic one, while the DPL equation (10) contains a second order time derivative and higher order mixed derivative in both time and space.

To solve the Pennes equation by means of the boundary element method the several variants basing on a time marching technique have been applied, this means the 1st scheme of the BEM e.g. [Majchrzak (1998); Majchrzak and Dziewonski (2000); Majchrzak and Jasinski (2003); Mochnacki and Majchrzak (2003)], the BEM using discretization in time e.g. [Majchrzak and Kaluza (2006)] and the dual reciprocity BEM [Liu and Xu (2000); Zhou, Zhang and Chen (2008); Erhart, Divo and Kassab (2008)]. The other approaches can be also taken into account e.g. [Sladek, Sladek and Atluri (2004); Sladek, Sladek, Tan and Atluri (2008)].

Application of the traditional BEM for solving the Cattaneo-Vernotte equation (7) or DPL equation (10) is more complicated. The fundamental solution is difficult to obtain, thus the equivalent boundary integral equation is not available. It should be pointed out that in literature [Lu, Liu and Zeng (1998)] one can find the solutions of Cattaneo-Vernotte equation obtained by means of the DRBEM but the DPL equation, up to the present, is not solved using the BEM.

In this work, the general boundary element method (GBEM) for hyperbolic heat conduction equation proposed by Liao [Liao (1997); Liao (2002)] is adapted to solve the dual-phase lag equation.

Taking into account the form (4) of source function the equation (10) can be expressed as follows (2D problem is considered)

$$(x,y)\in\Omega:\ c\left(\frac{\partial T}{\partial t}+\tau_q\frac{\partial^2 T}{\partial t^2}\right)=\lambda\nabla^2 T+\lambda\tau_T\frac{\partial}{\partial t}(\nabla^2 T)+k(T_B-T)+Q_m-k\tau_q\frac{\partial T}{\partial t} \tag{11}$$

where $k=G_B c_B$ and $T=T(x,\,y,\,t)$.

This equation is supplemented by the boundary conditions

$$(x,y)\in\Gamma_1:\quad T(x,y,t)=T_b(x,y,t)\quad (x,y)\in\Gamma_2:\quad q_n(x,y,t)=q_b(x,y,t) \tag{12}$$

and initial ones

$$t=0:\quad T(x,y,t)=T_{p,}\quad \left.\frac{\partial T(x,y,t)}{\partial t}\right|_{t=0}=0 \tag{13}$$

where $T_b(x,\,y,\,t)$ is known boundary temperature, $q_b(x,\,y,\,t)$ is known boundary heat flux and T_p is the initial tissue temperature.

DPL model requires the adequate transformation of boundary conditions which appear in the typical macro heat conduction models. In the case considered one has (c.f. equation (9))

$$\begin{array}{l}(x,y)\in\Gamma_2:\quad q_b(x,y,t)+\tau_q\frac{\partial q_b(x,y,t)}{\partial t}=\\ -\lambda\left[\frac{\partial T(x,y,t)}{\partial n}+\tau_T\frac{\partial}{\partial t}\left(\frac{\partial T(x,y,t)}{\partial n}\right)\right]\end{array} \tag{14}$$

where n is the normal outward vector and $\partial(\cdot)/\partial n$ is the normal derivative.

Let $\beta=1/\Delta t$ and $T^f=T(x,\,y,\,f\Delta t)$, where Δt is the time step. Then, for time $t^f=f\Delta t$ ($f\geq 2$) the following approximate form of equation (11) can be taken into account

$$\begin{aligned}c\,[\beta(T^f-T^{f-1})+\tau_q\beta^2(T^f-2T^{f-1}+T^{f-2})]&=\lambda\nabla^2 T^f\\ &+\lambda\tau_T\beta\,(\nabla^2 T^f-\nabla^2 T^{f-1})+k(T_B-T^f)+Q_m-k\tau_q\beta(T^f-T^{f-1})\end{aligned} \tag{15}$$

or

$$\nabla^2 T^f-BT^f+C\nabla^2 T^{f-1}+DT^{f-1}+ET^{f-2}+F=0 \tag{16}$$

where

$$
\begin{aligned}
B &= \frac{(c\beta + k)(1 + \tau_q \beta)}{\lambda(1 + \tau_T \beta)} \\
C &= -\frac{\tau_T \beta}{1 + \tau_T \beta} \\
D &= \frac{c\beta(1 + 2\tau_q \beta) + k\tau_q \beta}{\lambda(1 + \tau_T \beta)} \\
E &= -\frac{c\tau_q \beta^2}{\lambda(1 + \tau_T \beta)} \\
F &= kT_b + Q_m
\end{aligned}
\tag{17}
$$

The boundary conditions take a form

$$
(x,y) \in \Gamma_1 : \quad T(x,y,t^f) = T_b(x,y,t^f) \tag{18}
$$

and (c.f. equation (14))

$$
\begin{aligned}
(x,y) \in \Gamma_2 : \quad & q_b(x,y,t^f) + \tau_q \frac{\partial q_b(x,y,t)}{\partial t}\bigg|_{t=t^f} = \\
& -\lambda \left[\frac{\partial T(x,y,t^f)}{\partial n} + \frac{\tau_T}{\Delta t} \left(\frac{\partial T(x,y,t^f)}{\partial n} - \frac{\partial T(x,y,t^{f-1})}{\partial n} \right) \right]
\end{aligned}
\tag{19}
$$

or

$$
(x,y) \in \Gamma_2 : \quad -\lambda \frac{\partial T(x,y,t^f)}{\partial n} = w_b(x,y,t^f) \tag{20}
$$

where

$$
w_b(x,y,t^f) = \frac{\Delta t}{\Delta t + \tau_T} \left[q_b(x,y,t^f) + \tau_q \frac{\partial q_b(x,y,t)}{\partial t}\bigg|_{t=t^f} \right] - \frac{\tau_T}{\Delta t + \tau_T} \lambda \frac{\partial T(x,y,t^{f-1})}{\partial n} \tag{21}
$$

From initial conditions (13) results that $T^0 = T(x, y, 0)$=T_p and
$T^1 = T(x, y, \Delta t) = T_p$.

It should be pointed out that when the equations (16), (18), (20) are solved at the f-th time step $t^f = f\Delta t$ then the temperature distributions T^{f-1} at time t^{f-1} and T^{f-2} at time t^{f-2} are known.

At first, a family of partial differential equations for $\Phi(x, y; p)$ is constructed [Liao (1997); Liao (2002); Liao and Chwang (1999)]

$$(1-p)L[\Phi(x,y;p)-U(x,y)] = -p\mathrm{A}[\Phi(x,y;p)] \tag{22}$$

with boundary conditions

$$\begin{aligned} &(x,y)\in\Gamma_1: \quad \Phi(x,y;p) = pT_b(x,y,t^f) + (1-p)\,U(x,y) \\ &(x,y)\in\Gamma_2: \quad -\lambda\frac{\partial\,\Phi(x,y;p)}{\partial\,n} = pw_b(x,y,t^f) + (1-p)\left[-\lambda\frac{\partial\,U(x,y)}{\partial\,n}\right] \end{aligned} \tag{23}$$

where $p \in [0, 1]$ is an parameter, $U(x, y)$ is an initial approximation of temperature distribution T^f, L is an 2D linear operator whose fundamental solution is known and A is an non-linear operator.

The linear operator is the following

$$L(u) = \nabla^2 u - Bu = \frac{\partial^2 u}{\partial\,x^2} + \frac{\partial^2 u}{\partial\,y^2} - Bu \tag{24}$$

while the non-linear operator takes a form (c.f. equation (16))

$$A[\Phi(x,y;p)] = \nabla^2\Phi - B\,\Phi + C\nabla^2 T^{f-1} + DT^{f-1} + ET^{f-2} + F \tag{25}$$

For p =0 one obtains (c.f. equation (22))

$$\mathrm{L}[\Phi(x,y;0) - U(x,y)] = 0 \tag{26}$$

this means

$$\mathrm{L}[\Phi(x,y;0)] = \mathrm{L}[U(x,y)] \tag{27}$$

and (c.f. conditions (23))

$$\begin{aligned} &(x,y)\in\Gamma_1: \Phi(x,y;0) = U(x,y) \\ &(x,y)\in\Gamma_2: -\lambda\frac{\partial\,\Phi(x,y;0)}{\partial\,n} = -\lambda\frac{\partial\,U(x,y)}{\partial\,n} \end{aligned} \tag{28}$$

The solution of (27), (28) is obvious

$$\Phi(x,y;0) = U(x,y) \tag{29}$$

On the other hand, for p=1 one has

$$A[\Phi(x,y;1)] = 0 \tag{30}$$

and

$$\begin{aligned}(x,y) \in \Gamma_1 &: \Phi(x,y;1) = T_b(x,y,t^f) \\ (x,y) \in \Gamma_2 &: -\lambda \frac{\partial \Phi(x,y;1)}{\partial n} = w_b(x,y,t^f)\end{aligned} \tag{31}$$

Taking into account the form of operator A (equation (25)) it is visible that

$$\Phi(x,y;1) = T^f = T(x,y,t^f) \tag{32}$$

Thus, if p=0 then $\Phi(x, y; p)$ corresponds to the initial approximation $U(x, y)$, while if p=1 then $\Phi(x, y; p)$ corresponds to the unknown temperature $T^f = T\ (x, y, t^f)$. So, the equations (22), (23) form a family of equations in parameter $p \in [0, 1]$ and the process of continuous change of the parameter p from 0 to 1 is the process of continuous variation of solution $\Phi(x, y; p)$ from $U(x, y)$ to $T^f = T\ (x, y, t^f)$.

Function $\Phi(x, y; p)$ is expanded into a Taylor series about value p=0 taking into account the first derivative

$$\Phi(x,y;p) = \Phi(x,y;0) + \left.\frac{\partial\, \Phi(x,y;p)}{\partial\, p}\right|_{p=0} (p-0) \tag{33}$$

or

$$\Phi(x,y;p) = \Phi(x,y;0) + U^{[1]}(x,y)p \tag{34}$$

where

$$U^{[1]}(x,y) = \left.\frac{\partial\, \Phi(x,y;p)}{\partial\, p}\right|_{p=0} \tag{35}$$

For p=1 one obtains

$$\Phi(x,y;1) = \Phi(x,y;0) + U^{[1]}(x,y) \tag{36}$$

this means

$$T^f = U(x,y) + U^{[1]}(x,y) \tag{37}$$

Under the assumption that $U(x, y) = T^{f-1}$ one has

$$T^f = T^{f-1} + U^{[1]}(x,y) \tag{38}$$

Liao and Chwang [Liao and Chwang (1999)]] report that better than initial approximation $U(x, y) = T^{f-1}$ is to use the iterative formula

$$T_k^f = T_{k-1}^f + mU^{[1]}(x,y), \quad k = 1,2,3,...,K \tag{39}$$

where $T_0^f = T^{f-1}$, m is an iterative parameter and K is the number of iterations. Differentiation of equations (22), (23) with respect to parameter p gives

$$-L[\Phi(x,y;p) - U(x,y)] + (1-p)L\left[\frac{\partial \Phi(x,y;p)}{\partial p} - \frac{\partial U(x,y)}{\partial p}\right] = \\ -A[\Phi(x,y;p)] - p\frac{\partial A[\Phi(x,y;p)]}{\partial p} \tag{40}$$

and

$$(x,y) \in \Gamma_1 : \frac{\partial \Phi(x,y;p)}{\partial p} = T_b(x,y,t^f) + p\frac{\partial T_b(x,y,t^f)}{\partial p} - U(x,y) + (1-p)\frac{\partial U(x,y)}{\partial p}$$
$$(x,y) \in \Gamma_2 : -\lambda\frac{\partial}{\partial n}\left(\frac{\partial \Phi(x,y;p)}{\partial p}\right) = w_b(x,y,t^f) + p\frac{\partial w_b(x,y,t^f)}{\partial p} + \lambda\frac{\partial U(x,y)}{\partial n} \\ -(1-p)\lambda\frac{\partial}{\partial n}\left[\frac{\partial U(x,y)}{\partial p}\right] \tag{41}$$

For p=0 one has

$$L[U^{[1]}(x,y)] = -A[U(x,y)] \tag{42}$$

and

$$(x,y) \in \Gamma_1 : \; U^{[1]}(x,y) = T_b(x,y,t^f) - U(x,y)$$
$$(x,y) \in \Gamma_2 : \; -\lambda\left(\frac{\partial U^{[1]}(x,y)}{\partial n}\right) = w_b(x,y,t^f) + \lambda\frac{\partial U(x,y)}{\partial n} \tag{43}$$

Taking into account the form of operators L and A (equations (24), (25)) the equation (42) can be written as follows

$$\nabla^2 U^{[1]} - BU^{[1]} + R(U) = 0 \tag{44}$$

where

$$R(U) = \nabla^2 U - BU + C\nabla^2 T^{f-1} + DT^{f-1} + ET^{f-2} + F \tag{45}$$

To solve the equation (44) supplemented by boundary conditions (43) the traditional BEM for steady-state problem [Brebbia, Telles and Wrobel (1984)] can be applied.

So, the boundary integral equation corresponding to the equation (44) has the following form

$$B(\xi,\eta)U^{[1]}(\xi,\eta)-\int_{\Gamma}U^{*}(\xi,\eta,x,y)\frac{\partial U^{[1]}(x,y)}{\partial n}d\Gamma= -\int_{\Gamma}\frac{\partial U^{*}(\xi,\eta,x,y)}{\partial n}U^{[1]}(x,y)d\Gamma+\iint_{\Omega}R(U)U^{*}(\xi,\eta,x,y)d\Omega \quad (46)$$

where (ξ, η) is the observation point, $B(\xi, \eta) \in (0, 1]$ is the coefficient dependent on the location of point (ξ, η), $U^*(\xi, \eta, x, y)$ is the fundamental solution

$$U^{*}=\frac{1}{2\pi}K_0(r\sqrt{B}) \quad (47)$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero and r is the distance between source point (ξ, η) and field point (x, y).

The equation (46) can be written in the form

$$B(\xi,\eta)U^{[1]}(\xi,\eta)+\frac{1}{\lambda}\int_{\Gamma}U^{*}(\xi,\eta,x,y)\,W^{[1]}(x,y)d\Gamma= \frac{1}{\lambda}\int_{\Gamma}W^{*}(\xi,\eta,x,y)\,U^{[1]}(x,y)d\Gamma+\iint_{\Omega}R(U)\,U^{*}(\xi,\eta,x,y)d\Omega \quad (48)$$

where

$$W^{[1]}(x,y)=-\lambda\frac{\partial U^{[1]}(x,y)}{\partial n} \quad W^{*}(\xi,\eta,x,y)=-\lambda\frac{\partial U^{*}(\xi,\eta,x,y)}{\partial n} \quad (49)$$

Function $W^*(\xi, \eta, x, y)$ can be calculated in analytical way and then

$$W^{*}(\xi,\eta,x,y)=\frac{\lambda d\sqrt{B}}{2\pi r}K_1(r\sqrt{B}) \quad (50)$$

where $K_1(\cdot)$ is the modified Bessel function of the second kind of order one and

$$d=(x-\xi)\cos\alpha+(y-\eta)\cos\beta \quad (51)$$

while $\cos\alpha$, $\cos\beta$ are directional cosines of normal vector n.

In numerical realization of BEM the boundary Γ is divided into N boundary elements, while the interior Ω is divided into L internal cells. For constant boundary elements and constant internal cells one obtains the following approximation of equation (48)

$$\sum_{j=1}^{N} G_{ij} W_j^{[1]} = \sum_{j=1}^{N} H_{ij} U_j^{[1]} + \sum_{j=1}^{L} P_{il} R(U_l) \tag{52}$$

where

$$G_{ij} = \frac{1}{\lambda} \int_{\Gamma_j} U^*(\xi_i, \eta_i, x, y) d\Gamma_j \tag{53}$$

and

$$H_{ij} = \begin{cases} \int_{\Gamma_j} W^*(\xi_i, \eta_i, x, y) d\Gamma_j, & i \neq j \\ -0.5, & i = j \end{cases} \tag{54}$$

while

$$P_{il} = \iint_{\Omega_l} U^*(\xi_i, \eta_i, x, y)\, \mathrm{d}\Gamma_l \tag{55}$$

Introducing the boundary conditions (43) into the linear algebraic equations (52) one obtains the equations for the unknown $W^{[1]}$ on the boundary Γ_1 and unknown $U^{[1]}$ on the boundary Γ_2. After solving the system of equations (52), the values $U^{[1]}$ at the internal points (ξ_i, η_i) are calculated using the formula

$$U_i^{[1]} = \sum_{j=1}^{N} H_{ij} U_j^{[1]} - \sum_{j=1}^{N} G_{ij} W_j^{[1]} + \sum_{j=1}^{L} P_{il} R(U_l) \tag{56}$$

Summing up, numerical solution of DPL equation by means of the GBEM is connected with the determination of function $U^{[1]}$ and next for transition $t^{f-1} \rightarrow t^f$ the temperature T^f is calculated using the iterative formula (39). It should be pointed out that for each iteration the problem described by equation (44) and boundary conditions (43) should be solved and then in the place of function U (c.f. equation (45)) the value T_{k-1}^f is introduced.

4 Results of computations

The biological tissue domain of dimensions 0.02m×0.02m (square) is considered. The following values of parameters are assumed: volumetric specific heat of tissue c =3 MW/(m^3K), thermal conductivity of tissue λ=0.5W/(mK), blood perfusion rate G_B=0.002 1/s, volumetric specific heat of blood c_B=3.9962 MW/(m^3K), blood temperature T_B=37^{o}C, metabolic heat source Q_m=245 W/m^3, relaxation time τ_q=15s, thermalization time τ_T=10s. Initial temperature of tissue equals T_p=37^{o}C.

Along the boundary Γ_1 ($0 \leq x \leq 0.02$m, y=0 - c.f. Figure 1) the Dirichlet condition T_b=37^{o}C is assumed, on the external surface of tissue (Γ_2: $0 \leq x \leq 0.02$m, y=0.02m) the Neumann condition in the form

$$q_b(x,0.02,t) = q_0 \frac{t}{t_e}\left(1-\frac{t}{t_e}\right)\exp\left(-\frac{x^2}{d^2}\right) \tag{57}$$

is accepted (Figure 2), where q_0=20 kW/m^2 is the maximum heat flux, t_e=20s is the exposure time and d=0.01m. On the remaining boundaries (Γ_3) the no-flux condition is assumed.

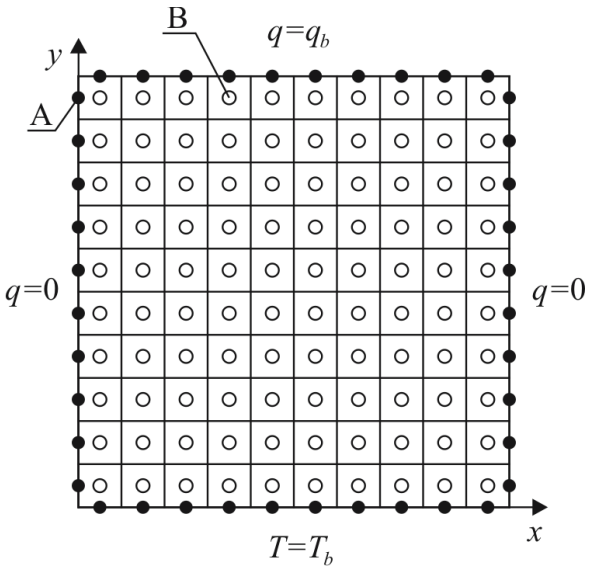

Figure 1: Boundary conditions and discretization

The problem is solved by means of the GBEM. At first, for transition $t^{f-1} \rightarrow t^f$ ($f \geq 2$) the equation (44) with corresponding boundary conditions (c.f. equations

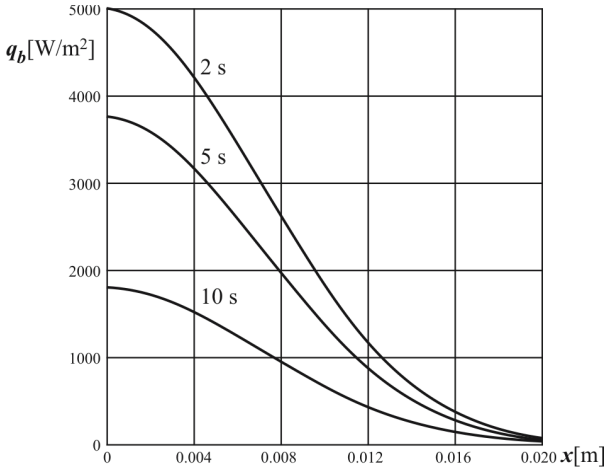

Figure 2: Boundary heat flux

(43), (21))

$$
\begin{aligned}
&(x,y)\in\Gamma_1:\ U^{[1]}(x,y)=0\\
&(x,y)\in\Gamma_2:\ -\lambda\frac{\partial U^{[1]}(x,y)}{\partial y}=\frac{\Delta t}{\Delta t+\tau_T}\cdot q_0\left[\frac{t^f}{t_e}\left(1-\frac{t^f}{t_e}\right)+\frac{\tau_q}{t_e}\left(1-\frac{2t^f}{t_e}\right)\right]\\
&\qquad\exp\left(-\frac{x^2}{d^2}\right)-\frac{\tau_T}{\Delta t+\tau_T}\lambda\frac{\partial T^{f-1}}{\partial y}+\lambda\frac{\partial U(x,y)}{\partial y}\\
&(x,y)\in\Gamma_3:\ -\lambda\frac{\partial U^{[1]}(x,y)}{\partial n}=-\frac{\tau_T}{\Delta t+\tau_T}\lambda\frac{\partial T^{f-1}}{\partial n}+\lambda\frac{\partial U(x,y)}{\partial n}
\end{aligned}
\tag{58}
$$

is considered. In formulas (58) $U(x,\,y)=T^f_{k-1}$ ($T^f_0=T^{f-1}$, of course), where k is the number of iteration (c.f. equation (39)). Next, for iteration k this problem is solved by traditional BEM under the assumption that the time step equals Δt=1s, number of constant boundary elements N=40, number of internal cells L=100 (Figure 1) and iterative parameter m=0.9. For each iteration k the error of numerical solution is calculated (c.f. equation (44))

$$
Er_k=\sqrt{\frac{1}{L^2}\sum_{l=1}^{L}\left[\nabla^2U_l^{[1]}-BU_l^{[1]}+R(U_l)\right]_k^2}
\tag{59}
$$

If $Er_k\le10^{-4}$ then the values $U^{[1]}(x_l,\,y_l)$ at the internal nodes $(x_l,\,y_l)$, $l=N$+1, ..., $N+L$ are accepted and the temperatures $T^f=T(x_l,\,y_l,\,t^f)=T^f_k$ (c.f. equation (39)) constitute the pseudo-initial condition for the next transition $t^f\to t^{f+1}$.

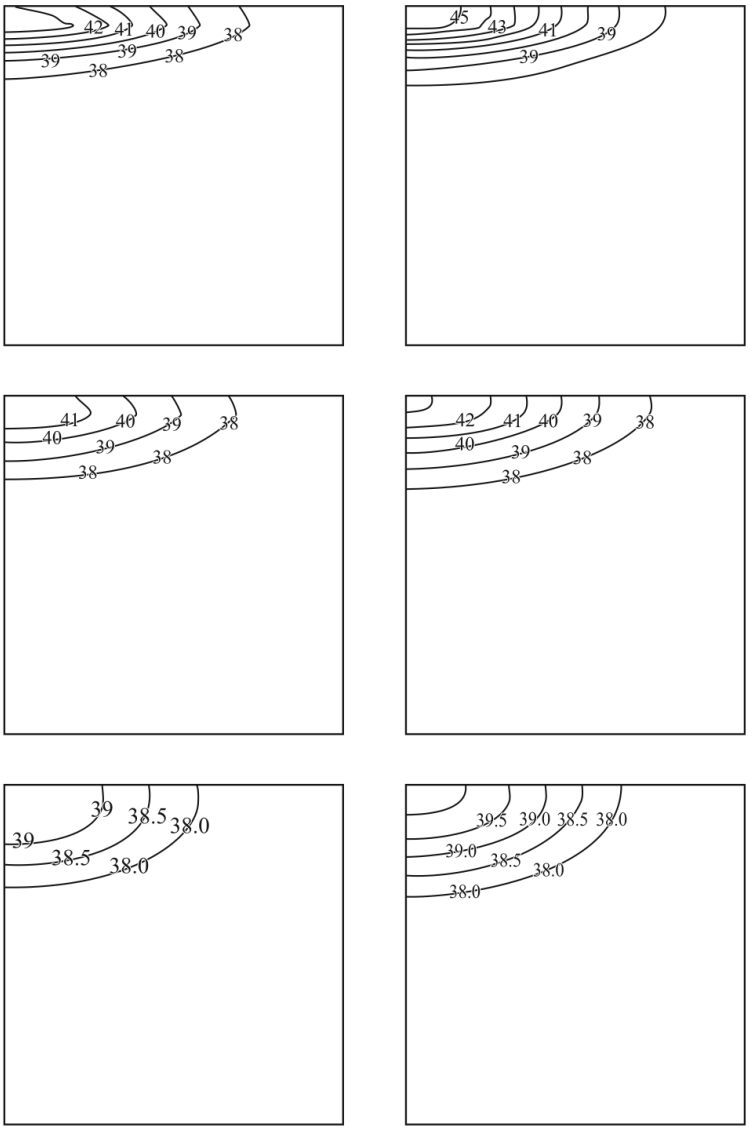

Figure 3: Temperature distribution for time 20, 30 and 60 s - DPL equation (left-hand side), Pennes equation (right-hand side)

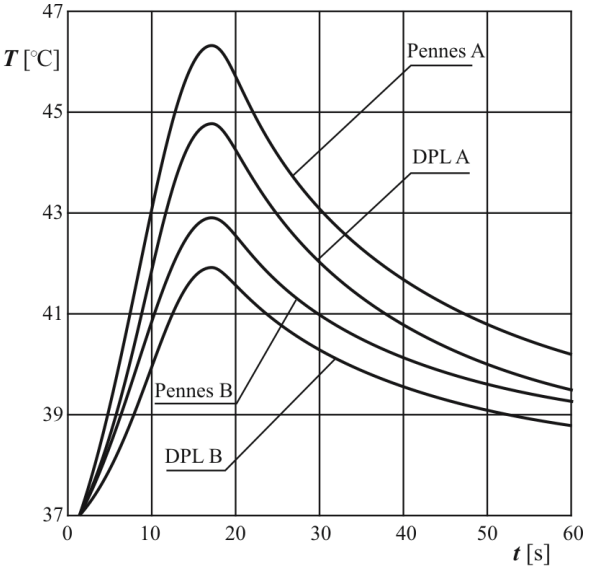

Figure 4: Comparison of DPL and Pennes models - temperature history at points A and B

On the left-hand side of Figure 3 the temperature distribution for times 20, 30 and 60 s for DPL model is shown, while on the right-hand side the temperature distribution obtained by means of the Pennes model ($\tau_q = \tau_T$=0) is presented. Figure 4 illustrates the temperature history at the points A and B marked in Figure 1 both for the DPL equation as well as the Pennes one. The differences between these solutions are visible, and in the case of Pennes equation application the temperatures are greater in comparison with the DPL equation.

The computations have been also done for others time steps, for example Δt=0.5s and Δt=1.5s, respectively, and also for other number of boundary elements (N=80). It turned out that the number of iterations assuring the assumed exactness increases when the size of time step is reduced and decreases when the number of boundary elements increases. Very important is also the proper choice of iteration parameter m. In the case considered the optimum value of this parameter is about 0.9.

To verify the algorithm proposed the results obtained using the GBEM have been compared with the results obtained by means of the 1^{st} scheme of the BEM [Brebbia, Telles and Wrobel (1984); Majchrzak (2001)] and that the differences between these solutions were very small. This comparison is possible only in case when $\tau_q = \tau_T = 0$, of course.

5 Conclusions

The general boundary element method is adapted to solve the DPL equation supplemented by the Dirichlet and Neumann boundary conditions. In this method the solution obtained for time t^{f-1} is a starting point to get the solution for the time t^f. For each transition $t^{f-1} \rightarrow t^f$ the 1st-order deformation derivative $U^{[1]}(x, y)$ is determined from linear equation which can be solved by the traditional BEM and next the temperature T^f is calculated using the iterative procedure. The exactness of the method proposed is connected with the proper choice of time step Δt, discretization of the domain considered and the value of iterative parameter m.

Acknowledgement: This work was supported by Grant No N N501 3667 34 sponsored by the Polish Ministry of Science and Higher Education.

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