Optimum Design of a Thin Elastic Rod Using a Genetic Algorithm

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Abstract: The best methods of the genetic algorithms (GA) are obtained in order to optimize the shape of a thin elastic rod subjected to spatial bending and torsion. The optimal cross-section is determined from the minimum volume condition, against the three modal bucklings.

Keywords: thin elastic rod, optimum design, genetic algorithm.

1   Introduction

The Lagrange problem [Cox (1992) and Seyranian and Privalova (2003)] of determining the shape of a column of given volume from the condition of the maximum strength against buckling, was formulated by Lagrange in 1777 [Lagrange (1868)], and solved by Clausen in 1851. The bimodal optimization of a column on elastic foundation was treated by Shin *et al*. (1988a), Shin *et al*. (1988b), Atanackovic and Novakovic (2006) for a variety of boundary conditions. The shape design for beams and plates on an elastic foundation was treated by Shin *et al*. (1988a), Shin *et al*. (1988b) and Plaut *et al*. (1986). Recently, with the increasing interest in the optimum design of structures, various models and methods are explored to solve the optimum design problems, with goals to improve the computational efficiency, see for example 3D finite element analysis [Zhou and Wang (2006); Wang and Wang (2006); Wang *et al*. (2007a, b)], the finite volume meshshless local Petrov-Galerkin method [Zheng *et al*. (2009)], and evolutionary structural optimization approach [Zhou and Rozvany (2001)].

Structural optimization techniques based on mathematical programming and optimality criteria approach are presented in the book [Popescu and Chiroiu 1980]. Recent techniques developed by imitating the design methods existing in the nature are referred to as genetic algorithms (GAs). These evolutionary structural optimization methods have attracted attention and have been applied to optimum

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design of different types of structural problems. The current trend indicates that GA based structural optimization techniques are quite promising and they are to be used as a standard solution algorithm for the design problem of structures, where the design variables are to be selected from a discrete set [Khoshravan and Hossein-zadch (2009); Narayana, Gopalakrishnan and Ganguli (2008); Chiroiu and Chiroiu (2003)]. The mixed method has been applied to solve elasto-static problems [Atluri et al. (2004)] and nonlinear problems with large deformations and rotations [Han *et al.* (2005); Chiroiu *et al.* (2005)].

Our goal in this paper is to use a GA to obtain the shape of a thin elastic rod subjected to the spatial bending and torsion, without making any assumptions about the cross section. The most important findings regarding the GAs corresponding to our research are related to the treatment of combinations of continuous and discrete variables associated with the initial conditions and three modal bucklings [Olhoff and Rasmussen (1977) and Seyranian (1984); Cox (1992); Cox and Overton, 1992)].

2   Mathematical formulation

The theory of thin elastic rods is presented in the spirit of [Munteanu and Donescu (2004)]. Let us consider a straight, thin elastic, homogeneous and isotropic rod of length l, having a variable cross section in its natural state. External forces and couples fix the ends of the bar. We assume that the rod deforms in space by bending and torsion. The rod occupies, at time $t = 0$, the region $\Omega_9 \subset R^3$. After motion takes place at time t, the rod occupies the region $\Omega(t)$.

The motion of the rod between $t = 0$ and λ, θ, ψ it is known if and only if we know the mapping [Truesdell and Toupin (1960; Şoós (1974)]

$$S(0,t), \quad \forall t \in [0, t_1], \tag{2.1}$$

which takes a material point in $\xi = s - vt$ at $t = 0$ to a spatial position in $\lambda = \zeta d_3 + (0,0,\lambda_3)$ at $t = \zeta = -\rho v^2$.

The mapping (2.1) is single valued and possesses continuous partial derivatives with respect to their arguments. The position of a material point in Ω_0 may be denoted by a rectangular fixed coordinate system $X \equiv (X,Y,Z)$ and the spatial position of the same point in $\Omega(t)$, by the moving coordinate system $x \equiv (x,y,z)$.

In the following, X and x are referred to as the material or Lagrange coordinates and the spatial or Euler coordinates, respectively. The origin of these coordinate systems is lying on the central axis of the rod. The motion of the rod carries various material points through various spatial positions. This is expressed by $x = f_i(X,t)$, $i = 1, 2, 3$.

We take s to be the coordinate along the central line of the natural state. The orthonormal basis of the Lagrange coordinate system is denoted by (e_1, e_2, e_3), and the orthonormal basis of the Euler coordinate system is denoted by (d_1, d_2, d_3).

The basis $\{d_k\}$, $k = 1, 2, 3$ is related to $\{e_k\}$, $k = 1, 2, 3$ by the Euler angles θ, ψ and φ. These angles determine the orientation of the Euler axes with respect to the Lagrange axes Tsuru (1986, 1987)]

$$d_1 = (-\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta)e_1 + (\cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta)e_2$$
$$- \sin\theta\cos\varphi\, e_3,$$

$$d_2 = (-\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta)e_1 + (\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta)e_2$$
$$+ \sin\theta\sin\varphi e_3,$$

$$d_3 = \sin\theta\cos\psi e_1 + \sin\theta\sin\psi e_2 + \cos\theta\, e_3 \tag{2.2}$$

The Z-axis coincides with the central axis. The plane (xy) intersects the plane (XY) by the nodal line. The motion of the rod is described by three vector functions

$$R \times R(s,t) \rightarrow r(s,t),\, d_1(s,t),\, d_2(s,t) \in E^3.$$

The material sections of the rod are identified by the coordinate s. The position vector $r(s,t)$ can be interpreted as the image of the central axis in the Euler configuration. The functions $d_1(s,t), d_2(s,t)$ can be interpreted as defining the orientation of the material section s in the Euler configuration. The function $d_3(s,t) = d_1(s,t) \times d_2(s,t)$ represents the unit tangential vector along the rod and can be expressed as $d_3(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$.

We introduce the strains y_1, y_2, y_3 by

$$r' = y_k d_k, \tag{2.3}$$

where $(')$ means the partial differentiation with respect to s. Since $\{d_k\}$, $k = 1, 2, 3$, is orthonormal, there is a vector u such as $d'_k = u \times d_k$. The components of u with respect to the basis $\{d_k\}$ are

$$u_k = \frac{1}{2} e_{klm} d'_l \cdot d_m, \tag{2.4}$$

where e_{klm} the components of the alternating tensor. Relation (2.4) becomes

$$u_1 = d_{31} d'_{21} + d_{32} d'_{22} + d_{33} d'_{23},$$

$$u_2 = d_{11}d'_{31} + d_{12}d'_{32} + d_{13}d'_{33},$$

$$u_3 = d_{21}d'_{11} + d_{22}d'_{12} + d_{23}d'_{13},$$

where d_{ij}, $i = j = 1,2,3$, are the components of the vectors d_i, $i = 1,2,3$, given by (2).

Substitution of (2.2) into the above relations gives

$$u_1 = \theta' \sin \varphi - \psi' \sin \theta \cos \varphi, \quad u_2 = \theta' \cos \varphi + \psi' \sin \theta \sin \varphi, \quad u_3 = \varphi' + \psi' \cos \theta. \tag{2.5}$$

These functions measure the bending and torsion of the elastic rod. The functions u_k, $k = 1,2,3$, can be interpreted as the components of the angular velocity vector (the variation with respect to s) for the rotational motion of the moving system of coordinates with respect to the fixed system of coordinates. If we substitute the differentiation with respect to s by the differentiation with respect to time we will obtain the components of the angular velocity vector defined by (2.5). The functions u_1 and u_2 represent the components of the curvature of the central line denoted by κ corresponding to the planes (yz) and (xz)

$$\kappa^2 = u_1^2 + u_2^2 = \theta'^2 + \psi'^2 \sin^2 \theta, \tag{2.6}$$

and u_3 is the torsion of the bar denoted by τ

$$\psi(0) = \psi(l) = \psi_0. \tag{2.7}$$

In this way, we consider the rod is rigid along the tangential direction and the total length of the rod $\theta(0) = \theta(l) = \theta_0$ is invariant, the ends being fixed by external forces.

The full set of strains of the rod is $\tau(0) = \tau(l) = \tau_0$. In the natural state $f(u) = 0$ coincides with $u_3 < u_2 < u_1$, and $f(u) = 0$ are constant functions of s. The values of the strains in the natural state are

$$y_1 = y_2 = 0, \quad y_3 = 1, \quad u_k = 0. \tag{2.8}$$

In the following we assume that extensional and compression strains have the values $m = \frac{u_2 - u_3}{u_1 - u_3}$ and focus only on the bending and torsion of the rod.

The link between the position vector $r = (x,y,z)$ and unit tangential vector d_3 is obtained from the first two relations of (2.8) and (2.3)

$$r' = d_3. \tag{2.9}$$

From (2.9) we obtain $\Pi(x,z,m)$. To characterize the position of the ends of the rod, we introduce the vector D whose components are $x(L), y(L), z(L)$

$$D = \int_0^l d_3 \mathrm{d}s. \tag{2.10}$$

2.1 The equilibrium equations

The elastic energy U of the deformed rod is composed of the bending energy and the torsional energy

$$U = \frac{B}{2} \int_0^l \kappa^2 \mathrm{d}s + \frac{T}{2} \int_0^l \tau^2 \mathrm{d}s, \tag{2.11}$$

Where κ and τ are given by (2.6) and (2.7) [Landau and Lifshitz (1968), Solomon (1968)]. The quantities B and T are the bending stiffness and the torsional stiffness, respectively, which are generally related to the area A of the cross section and the material constants, i.e. the Young's modulus E and the shear modulus μ

$$B = \alpha A^2 E_0, \quad T = \beta A^2 \mu_0, \tag{2.12}$$

with α and β dimensionless parameters, and E_0, μ_0 the referenced values for the elastic constants. On using (2.6) and (2.7), the elastic energy (2.11) can be written in the following form

$$U = \frac{\alpha A^2 E_0}{2} \int_0^l (\theta'^2 + \psi'^2 \sin^2 \theta) \mathrm{d}s + \frac{\beta A^2 \mu_0}{2} \int_0^l (\varphi' + \psi' \cos \theta)^2 \mathrm{d}s.$$

To write the equilibrium equations, the variation of the elastic energy U with respect to θ, φ and ψ is considered.

THEOREM 1 [Munteanu and Donescu (2004)]. *The exact static equilibrium equations of a thin elastic rod with the ends fixed by the external force $F = -\lambda$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are given by*

$$\alpha A^2 E_0(\psi'^2 \sin \theta \cos \theta - \theta'') - \beta A^2 \mu_0(\varphi' + \psi' \cos \theta)\psi' \sin \theta + \lambda_1 \cos \theta \cos \psi +$$
$$+ \lambda_2 \cos \theta \sin \psi - \lambda_3 \sin \theta = 0,$$

$$\frac{\partial}{\partial s}[\alpha A^2 E_0 \psi' \sin^2 \theta + \beta A^2 \mu_0(\varphi' + \psi' \cos \theta) \cos \theta] -$$
$$- \lambda_1 \sin \theta \sin \psi + \lambda_2 \sin \theta \cos \psi = 0,$$

$$\frac{\partial}{\partial s}[-\beta A^2 \mu_0(\varphi' + \psi' \cos\theta)] = 0. \tag{2.13}$$

The end couples at $f(u) = 0$ and $s = l$ are $M_i(0)$, $i = 1,2,3$, and $M_i(l)$, $i = 1,2,3$, respectively, where

$M_1(s) = \alpha A^2 E_0 \theta'$ *are the couples with respect to the nodal line ON,*

$M_2(s) = \alpha A^2 E_0 \psi' \sin^2\theta + \beta A^2 \mu_0(\psi' \cos\theta + \varphi') \cos\theta$ *is the couple with respect to Z-axis,*

$M_3(s) = \beta A^2 \mu_0(\varphi' \cos\theta + \psi')$ *is the couple with respect to z-axis.*

If $f(u) = 0$ is the force applied to the ends of the rod, where F_i, $i = 1,2,3$, are the components of the force with respect to the fixed coordinate system $m = \frac{u_2 - u_3}{u_1 - u_3}$, then this force is related to $w = \sqrt{\frac{|\lambda_3|}{2A}(u_1 - u_3)}$, by $F = \frac{\partial U}{\partial D} = -\lambda$. Therefore, $-\lambda$ represents the external force that fixes the ends of the rod. The couples $M = (M_1, M_2, M_3)$ at the ends of the rod with respect to the line node ON and Z and z axes are given by

$$M_1 = \frac{\partial \Im}{\partial \theta}\big|_{s=0 \, or \, l} = \frac{\partial U}{\partial \theta}\big|_{s=0 \, or \, l} = |\alpha A^2 \theta' E_0|_{s=0 \, or \, l},$$

$$M_2 = \frac{\partial \Im}{\partial \psi}\big|_{s=0 \, or \, l} = \frac{\partial U}{\partial \psi}\big|_{s=0 \, or \, l}$$
$$= |A^2 \left(\alpha E_0 \psi' \sin^2\theta + \beta \mu_0(\psi' \cos\theta + \varphi') \cos\theta\right)|_{s=0 \, or \, l}, \tag{2.14}$$

$$M_3 = \frac{\partial \Im}{\partial \psi}\big|_{s=0 \, or \, l} = \frac{\partial U}{\partial \psi}\big|_{s=0 \, or \, l} = |\beta A^2 \mu_0(\varphi' \cos\theta + \psi')|_{s=0 \, or \, l}.$$

The equilibrium equations (2.13) are coupled nonlinear ordinary differential equations with respect to the unknown Euler angles. Next, we see that equation (2.13)$_3$ can be solved

$$\varphi'(s) + \psi'(s) \cos\theta(s) = c,$$

with c an integration constant. From the definition of the torsion (2.7) we can conclude that $c = \tau$. So, the above relation becomes

$$\varphi'(s) + \psi'(s) \cos\theta(s) = \tau. \tag{2.15}$$

By using equation (2.15), the first two equations (2.13) can be written as

$$\alpha A^2 E_0(\psi'^2 \sin\theta \cos\theta - \theta'') - \beta A^2 \mu_0 \tau \psi' \sin\theta + \lambda_1 \cos\theta \cos\psi$$
$$+ \lambda_2 \cos\theta \sin\psi - \lambda_3 \sin\theta = 0,$$

$$\alpha A^2 E_0(\psi'' \sin\theta + 2\psi'\theta' \cos\theta) - \beta A^2 \mu_0 \tau\theta' + \lambda_1 \sin\psi - \lambda_2 \cos\psi = 0. \qquad (2.16)$$

We introduce $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}$, $\psi_1 = \psi + \pi + \arctan\frac{\lambda_1}{\lambda_2}$, and write

$$u_1 = u_2 = -1, \quad u_3 = 1$$

Adding $(2.16)f(u) = 0$ multiplied by $2\theta'$ and $(2.16)_2$ multiplied by $(-2\psi_1' \sin\theta)$, we obtain

$$-\alpha A^2 \psi_1'^2 (\sin^2\theta)' - \alpha A^2 (\psi_1'^2)' \sin^2\theta - \alpha A^2 (\theta'^2)' + \lambda_3 (\cos\theta)' - $$
$$-2\lambda \sin\psi_1 (\sin\theta)' - 2\lambda \sin\theta (\sin\psi_1)' = 0.$$

Dividing the above relation by $1/2$ we have

$$\left[0.5\alpha A^2 E_0(\theta'^2 + \psi_1'^2 \sin^2\theta) + \lambda \sin\theta \sin\psi_1 - \lambda_3 \cos\theta \right]' = 0.$$

Integrating this expression with respect to s, we find the bending energy density of the thin elastic rod as

$$\alpha A^2 E_0 \kappa^2 = -2\lambda \sin\theta \sin\psi_1 + 2\lambda_3 \cos\theta + C_0,$$

where C_0 is an integration constant.

In the case when $\lambda = (0, 0, \lambda_3)$, the equilibrium equations (2.15) and (2.16) become

$$\varphi'(s) + \psi'(s) \cos\theta(s) = \tau,$$
$$\alpha A^2 E_0(\psi'^2 \sin\theta \cos\theta - \theta'') - \beta A^2 \mu_0 \tau\psi' \sin\theta - \lambda_3 \sin\theta = 0, \qquad (2.17)$$
$$\alpha A^2 E_0(\psi'' \sin\theta + 2\psi'\theta' \cos\theta) - \beta A^2 \mu_0 \tau\theta' = 0.$$

2.2   The equations of motion

To write the equations of motion, let us introduce the inertia of the rod characterized by the functions $Rs \rightarrow (\rho_0 A)(s), (\rho_0 I_1)(s), (\rho_0 I_2(s) \in (0, \infty)$, where A is the cross-sectional area, $(\rho_0 I_1)$ is the principal mass moment of inertia around the axis, which is perpendicular to the central axis, and $(\rho_0 I_2)$ is the principal mass moment of inertia around the central axis, ρ_0 is the mass density per unit volume, and I_1, I_2 are the geometrical moments of inertia around the axis, which are perpendicular to the central axis and around the central axis, respectively.

The cross-sectional area A is related to the moments of inertia I_1 and I_2 by the following relations

$$I_1 = \gamma A^2, \quad I_2 = \delta A^2, \qquad (2.18)$$

where γ, δ are dimensionless constants. The volume of the rod is given by

$$V = \int_0^l A(s)\mathrm{d}s. \tag{2.19}$$

We assume that the cross-sectional area $A(s)$ belongs to the set of *admissible* twice continuously differentiable and positive cross-sectional area functions. We consider to have $0 < A_{\min} \le A(s) \le A_{max}$.

The kinetic energy K of the rod is the sum between the energy of the translational motion K_1, the energy of the rotational motion of the tangential vector K_2 and the energy of the rotational motion around the central axis K_3 [Tsuru (1986, 1987)]

$$K = K_1 + K_2 + K_3, \quad K_1 = \frac{A\rho_0}{2}\int_0^l \dot{r}^2 \mathrm{d}s,$$

$$K_2 = \frac{k_1}{2}\int_0^l \dot{d}_3^2 \mathrm{d}s = \frac{k_1}{2}\int_0^l (\Omega_1^2 + \Omega_2^2)\mathrm{d}s, \quad K_3 = \frac{k_2}{2}\int_0^l \Omega_3^2 \mathrm{d}s, \tag{2.20}$$

where the dot represents differentiation with respect to time,

$$\rho = A\rho_0, \quad k_1 = I_1\rho_0 = \gamma A^2 \rho_0, \quad k_2 = I_2\rho_0 = \delta A^2 \rho_0, \tag{2.21}$$

and $\Omega(\Omega_1, \Omega_2, \Omega_3)$ is the vector of angular velocity of rotation

$$\Omega_1 = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi,$$
$$\Omega_2 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi,$$
$$\Omega_1 = \dot{\psi}\cos\theta + \dot{\varphi}.$$

These relations are analogous to (2.5). On using $d'_k = u \times d_k$, we obtain the following expressions for K_2 and K_3

$$K_2 = \frac{\gamma A^2 \rho_0}{2}\int_0^l (\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2)\mathrm{d}s, \quad K_3 = \frac{\delta A^2 \rho_0}{2}\int_0^l (\dot{\psi}\cos\theta + \dot{\varphi})^2\mathrm{d}s.$$

THEOREM 2 [Munteanu and Donescu (2004)]. *The exact set of equations of motion for a thin elastic rod with the ends fixed by the force* $F = -\lambda$, *where* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, *are the following:*

$$-\rho_0 A\ddot{r} - \lambda' = 0,$$

$$\gamma A^2 \rho_0 (\dot{\psi}^2 \sin\theta\cos\theta - \ddot{\theta})$$
$$- \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta - \alpha A^2 E_0 (\psi'^2 \sin\theta\cos\theta - \theta'') +$$
$$+ \beta A^2 \mu_0 (\varphi' + \psi'\cos\theta)\psi'\sin\theta - \lambda_1 \cos\theta\cos\psi - \lambda_2 \cos\theta\sin\psi + \lambda_3 \sin\theta = 0,$$

$$-\frac{\partial}{\partial t}\left\{ \gamma A^2 \rho_0 \dot{\psi}\sin^2\theta + \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\cos\theta \right\} +$$
$$+ \frac{\partial}{\partial s}\left(\alpha A^2 E_0 \psi'^2 \sin^2\theta + \beta A^2 \mu_0 (\varphi' + \psi'\cos\theta)\cos\theta \right)$$
$$+ \lambda_1 \sin\theta\sin\psi - \lambda_2 \sin\theta\cos\psi = 0,$$

$$-\delta\rho_0\frac{\partial}{\partial t}(\dot{\varphi} + \dot{\psi}\cos\theta) + \beta\mu_0\frac{\partial}{\partial s}(\varphi' + \psi'\cos\theta) = 0. \tag{2.22}$$

The equations of motion (2.22) are coupled nonlinear partial differential equations with respect to the unknown Euler angles and the vector function, which characterizes the external force applied to the ends of the rod in order to maintain it fixed.

We have to add to the equations of motion the following initial conditions

$$\lambda(s,0) = \lambda_0(s) = -\rho_0 A v^2 d_3(s,0) + (\lambda_1, \lambda_2, \lambda_3),$$
$$\theta(s,0) = \theta_0(s), \quad \psi(s,0) = \psi_0(s), \quad \varphi(s,0) = \varphi_0(s). \tag{2.23}$$

In the case when $\lambda = (0,0,\lambda_3)$, the equations of motion (2.22) and (2.23) become

$$-\rho_0 A\ddot{r} - \lambda' = 0,$$

$$\gamma A^2 \rho_0 (\dot{\psi}^2 \sin\theta\cos\theta - \ddot{\theta})$$
$$- \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta - \alpha A^2 E_0 (\psi'^2 \sin\theta\cos\theta - \theta'') +$$
$$+ \beta A^2 \mu_0 (\varphi' + \psi'\cos\theta)\psi'\sin\theta + \lambda_3 \sin\theta = 0,$$

$$-\frac{\partial}{\partial t}\left\{ \gamma A^2 \rho_0 \dot{\psi}\sin^2\theta + \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\cos\theta \right\} +$$
$$+ \frac{\partial}{\partial s}\left\{ \alpha A^2 E_0 \psi'^2 \sin^2\theta + \beta A^2 \mu_0 (\varphi' + \psi'\cos\theta)\cos\theta \right\} = 0,$$

$$-\delta\rho_0\frac{\partial}{\partial t}(\dot{\varphi} + \dot{\psi}\cos\theta) + \beta\mu_0\frac{\partial}{\partial s}(\varphi' + \psi'\cos\theta) = 0. \tag{2.24}$$

and

$$\lambda(s,0) = \lambda_0(s) = -\rho_0 A v^2 d_3(s,0) + (0,0,\lambda_3),$$
$$\theta(s,0) = \theta_0(s), \quad \psi(s,0) = \psi_0(s), \quad \varphi(s,0) = \varphi_0(s). \tag{2.25}$$

3   The optimum design problem

The optimum design problem is based on the relation between the equilibrium equations and the equations of motion for a thin elastic rod [Tsuru (1986, 1987); Munteanu and Donescu (2004)]. The aim of this problem is to determine the conditions when the equations of motion can be reduced to the equilibrium equations.

3.1   The equivalence theorem

THEOREM 3 [Munteanu and Donescu (2004)]. *Let* λ, θ, ψ *and* φ *be given as functions of the variable* $\xi = s - vt$ *only and suppose that*

$$\lambda = \zeta d_3 + (0, 0, \lambda_3), \tag{3.1}$$

where $\zeta = -\rho v^2$ *and* $d_3 = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$. *In this case, the equations of motion* (2.24) *are equivalent to the equilibrium equations* (2.17) *for*

$$\alpha E_0 - \gamma\rho_0 v^2 \rightarrow \alpha E_0, \ \ \beta\mu_0 - \delta\rho_0 v^2 \rightarrow \mu_0\beta, \ \ \xi \rightarrow s. \tag{3.2}$$

Consider now the equilibrium equation (2.17)$_3$. Multiplying both sides of this equation by $\sin\theta$, we get $\alpha(\sin^2\theta\psi')' + \beta\tau(\cos\theta)' = 0$, and integrating with respect to ??, we obtain

$$\alpha A^2 E_0 \sin^2\theta\,\psi' + \beta A^2 \mu_0\tau\cos\theta + b_1 = 0, \tag{3.3}$$

where b_1 is an integration constant. By virtue of (3.3), we obtain

$$\psi' = -\frac{\beta A^2 \mu_0\tau\cos\theta + b_1}{\alpha A^2 E_0 \sin^2\theta}. \tag{3.4}$$

Substituting (3.4) into (2.17)$_2$ and multiplying the resulting equation by $2\theta'$, we obtain by integration with respect to ??

$$\alpha A^2 E_0 \theta'^2 - \frac{(\beta A^2 \mu_0\tau\cos\theta + b_1)^2}{\alpha A^2 E_0 \sin^2\theta} + 2\beta A^2 \mu_0\tau\cos\theta\frac{\beta A^2 \mu_0\tau\cos\theta + b_1}{\alpha A^2 E_0 \sin^2\theta}$$
$$- 2\lambda_3\cos\theta + b_2 = 0,$$

where b_2 is an integration constant. The above equation can be written as

$$\alpha A^2 E_0 \theta'^2 + \frac{\beta^2 A^4 \mu_0\tau^2\cos^2\theta - b_1^2}{\alpha A^2 E_0 \sin^2\theta} - 2\lambda_3\cos\theta + b_2 = 0. \tag{3.5}$$

Substituting $u = \cos\theta$ into (3.5), we obtain the following ordinary differential equation

$$u'^2 = \frac{1-u^2}{\alpha A^2 E_0}\left\{-\frac{\beta^2 A^4 \mu_0 \tau^2 u^2 - b_1^2}{\alpha A^2 E_0 (1-u^2)} + 2\lambda_3 u - b_2\right\},$$

or

$$\frac{1}{2}u'^2 = f(u),\tag{3.6}$$

$$f(u) = -\frac{1}{\alpha A^2 E_0}\left[\lambda_3 u^3 - \frac{1}{2}\left(b_2 - \frac{\beta^2 A^2 \mu_0 \tau^2}{\alpha}\right)u^2 - \lambda_3 u + \frac{1}{2}(b_2 - \frac{b_1^2}{\alpha A^2 E_0})\right].$$

We recognize in (3.6) a Weierstrass equation with a third-order polynomial. The torsion τ and the integration constants b_1 and b_2 are determined from the boundary conditions

$$\psi(0) = \psi(l) = \psi_0, \quad \theta(0) = \theta(l) = \theta_0, \quad \tau(0) = \tau(l) = \tau_0.\tag{3.7}$$

Eq. (3.6) admits closed form solutions [Munteanu and Donescu (2004)]. Starting with the solutions of (3.6), the exact solutions for the equilibrium equations (2.17) and the equation of motion (2.24), respectively, can be easily determined. These solutions are given by two theorems for two cases. In the first case, the equation $f(u) = 0$ has three distinct and real roots.

THEOREM 4 [Munteanu and Donescu (2004)]. *Given $u_3 < u_2 < u_1, u_3 \neq \pm 1$, the distinct and real roots of the cubic equation $f(u) = 0$ given by (3.6), the equilibrium equations (2.17) have a unique solution for the Euler angles*

$$u = u_2 - (u_2 - u_3)\operatorname{cn}^2\left(w(s-s_3), m\right),$$
$$\psi = \Gamma\left\{-\frac{b_1 + \beta A^2 \mu_0 \tau}{1-u_3}\Pi\left[w(s-s_3), \frac{u_2-u_3}{1-u_3}, m\right] - \frac{b_1 - \beta A^2 \mu_0 \tau}{1+u_3}\Pi\left[w(s-s_3), \frac{u_2-u_3}{1+u_3}, m\right]\right\},$$
$$\varphi = -\frac{\tau(\beta\mu_0 - \alpha E_0)}{\alpha E_0}s +$$
$$+\Gamma\left\{\frac{b_1 + \beta A^2 \mu_0 \tau}{1-u_3}\Pi\left[w(s-s_3), \frac{u_2-u_3}{1-u_3}, m\right] - \frac{b_1 - \beta A^2 \mu_0 \tau}{1+u_3}\Pi(w(s-s_3), \frac{u_2-u_3}{1+u_3}, m)\right\},$$

$$(3.8)$$

where $m = \frac{u_2-u_3}{u_1-u_3}, \Gamma = \frac{1}{4\alpha^2 A^4 E_0 w^2}, w = \sqrt{\frac{|\lambda_3|}{2\alpha A^2}(u_1 - u_3)}$, and $\Pi(x,z,m)$ is the normal elliptic integral of the third kind $\Pi(x,z,m) = \int\limits_0^x \frac{dy}{1-z\operatorname{sn}^2(y,m)}$.

In the second case

$$b_1 = \beta A^2 \mu_0 \tau, \quad b_2 = \frac{\beta^2 A^2 \mu_0^2 \tau^2}{\alpha E_0} - 2\lambda_3, \quad \alpha \neq 0,\tag{3.9}$$

$f(u)$ from $(3.6)_2$ becomes

$$f(u) = -\frac{\lambda_3}{\alpha A^2 E_0}(u+1)^2(u-1),$$

with the solutions $u_1 = u_2 = -1, \ u_3 = 1$.

In this case, we have the second theorem:

THEOREM 5 [Munteanu and Donescu (2004)]. *Given* $u_1 = u_2 = -1, \ u_3 = 1$ *the roots of the cubic equation* $f(u) = 0$ *given by* (3.6)*, the Euler angles are uniquely determined from the equilibrium equations* (2.17)

$$\begin{aligned}
u(s) &= -1 + 2\frac{|\lambda_3|}{\alpha A^2 E_0}\operatorname{sech}^2\sqrt{\frac{|\lambda_3|}{\alpha A^2 E_0}}s, \\
\psi &= \frac{-\beta\mu_0\tau s}{2\alpha E_0} + \arctan\left(\frac{4\alpha E_0}{\beta\mu_0\tau}\tanh\left(-\sqrt{\frac{|\lambda_3|}{\alpha A^2 E_0}}s\right)\right), \\
\varphi &= \frac{\tau(2\alpha E_0 - \beta\mu_0)s}{2\alpha E_0} + \arctan\left(\frac{4\alpha}{\phi\tau}\tanh\sqrt{\frac{|\lambda_3|}{\alpha A^2 E_0}}s\right).
\end{aligned} \tag{3.10}$$

3.2 The family of elastica solutions

The stable shape of a long rod compared with the cross section dimensions, into which the central line is deformed, is called *elastica*. These shapes are obtained from (3.8). For a circular cross section of radius r, $r \ll l$, Fig. 1 displays four shapes of elastica for $\tau = 0$ and different set of values. Here we used the notation (3.9). These shapes are similar to the shapes of elastica found by Love in 1926. The case $\tau \neq 0$ is illustrated in Fig. 2.

For $\tau \neq 0$ the rod deviates from a plane and has a 3D structure. This structure is simpler for small values of τ and more complicated when τ increases. The shape of the rod consists of a single loop or a series of loops lying altogether in space.

The family of elastica solutions of (2.17) in the case $\lambda = (0_1, 0, \lambda_3)$ contains a large number of curves. Euler noticed that there exists an infinite variety of such elastic curves, but "it will be worth while to enumerate all the different kinds included in this class of curves. For this way not only will the character of these curves be more profoundly perceived, but also, in any case whatsoever offered, it will be possible to decide from the mere figure into what class the curve formed ought to be put. We shall also list here the different kinds of curves in the same way in which the kinds of algebraic curves included in a given order are commonly enumerated" [Euler (1744)].

3.3 Exact solutions of the equations of motion

We determine the exact solutions of the equations of motion using the equivalence theorem (Theorem 3). Therefore, the theorems demonstrated in the static case are valid also in the dynamic case.
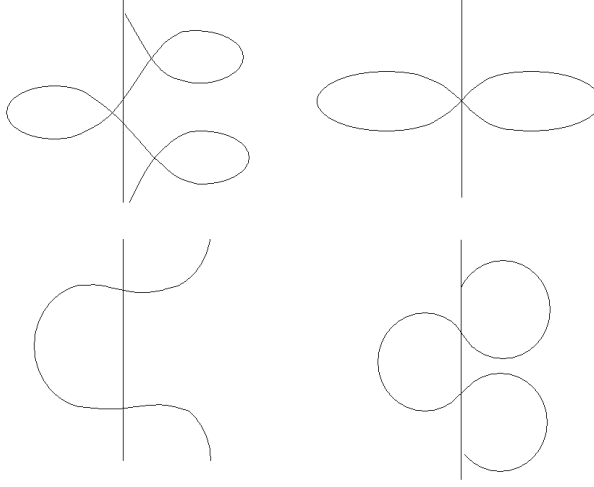
Figure 1: Shapes of *elastica* of Love for $\tau = 0$, and different set of values ($b_1 = 0.3$, $\lambda_3 = 0.4$, $b_2 = 0.2$), ($b_1 = 0.7$, $\lambda_3 = 0.2$, $b_2 = 0.1$), ($b_1 = 0.3$, $\lambda_3 = 0.1$, $b_2 = 0.1$) and ($b_1 = 0.9$, $\lambda_3 = 0.4$, $\gamma = 0.3$) from left to right.

The exact solutions for the equations of motion (2.24) are obtained from the static Theorems 4 and 5, for the same two situations.

In the first case, the following theorem holds:

THEOREM 6 [Munteanu and Donescu (2004)]. *Given $u_3 < u_2 < u_1, u_3 \neq \pm 1$, the distinct and real roots of the cubic equation $f(u) = 0$ given by* (3.6), *the equations of motion* (2.24) *have a unique solution for the Euler angles*

$$
\begin{aligned}
&u = u_2 - (u_2 - u_3)\mathrm{cn}^2[w(\xi - \xi_3), m], \\
&\psi = \bar{\Gamma}\left\{-\frac{b_1 + A^2(\beta\mu_0 - \delta\rho_0 v^2)\tau}{1 - u_3}\Pi(u) - \frac{b_1 - A^2(\beta\mu_0 - \delta\rho_0 v^2)\tau}{1 + u_3}\Pi(v)\right\}, \\
&\varphi = -\frac{\tau[\beta\mu_0 v_0 - \alpha E_0 - (\gamma_2 + \delta)\rho_0 v^2]}{(\alpha E_0 - \gamma\rho_0 v^2)}\xi + \\
&+\bar{\Gamma}\left\{\frac{b_1 + A^2(\beta\mu_0 - \delta\rho_0 v^2)\tau}{1 - u_3}\Pi(u) - \frac{b_1 - A^2(\beta\mu_0 - \delta\rho_0 v^2)\tau}{1 + u_3}\Pi(v)\right\},
\end{aligned}
\tag{3.11}
$$

where $m = \frac{u_2 - u_3}{u_1 - u_3}$, $\bar{\Gamma} = \frac{1}{4A^2(\alpha E_0 - \gamma\rho_0 v^2)^2 w^2}$, $w = \sqrt{\frac{|\lambda_3|}{2A}(u_1 - u_3)}$, $u = \left[w(\xi - \xi_3), \frac{u_2 - u_3}{1 - u_3}, m\right]$, $v = \left[w(\xi - \xi_3), \frac{u_2 - u_3}{1 + u_3}, m\right]$, *and* $\Pi(x, z, m)$ *is the normal elliptic integral of the third kind.*

In the second case (3.9), we the following theorem holds:

THEOREM 7 [Munteanu and Donescu (2004)]. *Given, $u_1 = u_2 = -1$, $u_3 = 1$, the*
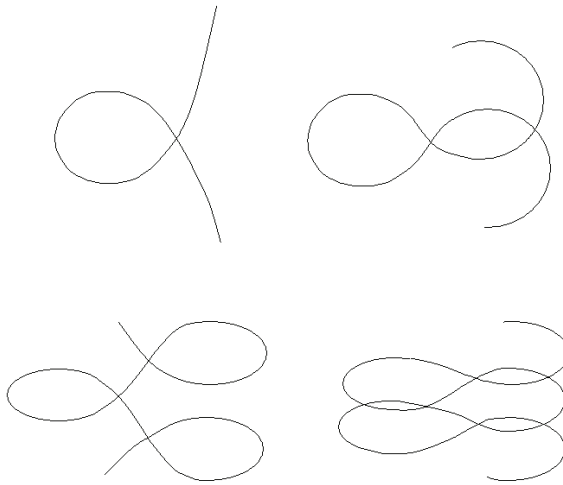
Figure 2: Shapes of *elastica* of Love for $\tau \neq 0$ and $\tau = 0.2, b_1=0.3, \lambda_3= 0.4, b_2=0.2$), ($\tau =0.3$, $b_1=0.7$, $\lambda_3= 0.2$, $b_2=0.1$), ($\tau =0.4$, $b_1=0.3$, $\lambda_3= 0.1$, $b_2=0.1$) and ($\tau =0.5$, $b_1=0.9$, $\lambda_3= 0.4$, $b_2=0.3$) from left to right.

roots of the cubic equation $f(u) = 0$ given by (3.6), the Euler angles are uniquely determined from the equations of motion (24)

$$
\begin{aligned}
u(\xi) &= -1 + 2\frac{|\lambda_3|}{A^2(\alpha E_0 - \gamma \rho_0 v^2)} \operatorname{sech}^2 \sqrt{\frac{|\lambda_3|}{A^2(\alpha E_0 - \gamma \rho_0 v^2)}}\xi \,, \\
\psi &= \frac{-(\beta \mu_0 - \delta \rho_0 v^2)\tau \xi}{2(\alpha E_0 - \gamma \rho_0 v^2)} + \arctan\left[\frac{4(\alpha E_0 - \gamma \rho_0 v^2)}{(\beta \mu_0 - \delta \rho_0 v^2)\tau} \tanh\left(-\sqrt{\frac{|\lambda_3|}{A^2(\alpha E_0 - \gamma \rho_0 v^2)}}\xi\right)\right], \\
\varphi &= \frac{\tau[2\alpha E_0 - \beta \mu_0 + v^2(\delta \rho_0 - 2\gamma \rho_0)]\xi}{2(\alpha E_0 - \gamma \rho_0 v^2)} + \\
&\quad + \arctan\left[\frac{4(\alpha - \gamma \rho_0 v^2)}{(\beta \mu_0 - \delta \rho_0 v^2)\tau} \tanh\sqrt{\frac{|\lambda_3|}{A^2(\alpha E_0 - \gamma \rho_0 v^2)}}\xi\right].
\end{aligned}
\tag{3.12}
$$

3.4 The optimum rod

Let us consider a rod that vibrates around the strained position, which satisfies the static equilibrium equations (2.17). The exact solutions of the equilibrium equations (2.17) are given by (3.8) or (3.9).

Let us suppose that the strained rod has a helical shape. It should be mentioned that there are many 1D media in biology, such as DNA, RNA and α-helix of protein, exhibiting a helical shape.

We start by differentiating the equation $(2.24)_1$ with respect to s

$$-\rho_0 A \ddot{d}_3 - \lambda_3'' = 0. \tag{3.13}$$

The Euler angles are written as

$$
\begin{aligned}
\theta &= \theta_s(s) + \varepsilon \cos(kx - \omega t), \\
\psi &= \psi_s(s) + \varepsilon \sin(ks - \omega t), \\
\varphi &= \varphi_s(s) + \varepsilon \sin(kx - \omega t),
\end{aligned}
\tag{3.14}
$$

where $\theta_s(s)$, $\psi_s(s)$, $\varphi_s(s)$ are given by (3.8) or (3.9), and ε a small parameter. Substituting (3.14) into (3.13), we obtain an equation with respect to the unknown λ_3

$$\lambda_3'' = -\rho_0 A \ddot{d}_3, \tag{3.15}$$

with

$$
\begin{aligned}
\ddot{d}_3 = \\
(\ddot{\theta}\cos\theta\cos\psi - 2\dot{\theta}\dot{\psi}\cos\theta\sin\psi - \dot{\theta}^2\sin\theta\cos\psi - \ddot{\psi}\sin\theta\sin\psi - \dot{\psi}^2\sin\theta\cos\psi, \\
\ddot{\theta}\cos\theta\sin\psi + 2\dot{\theta}\dot{\psi}\cos\theta\cos\psi - \dot{\theta}^2\sin\theta\sin\psi + \ddot{\psi}\sin\theta\cos\psi - \dot{\psi}^2\sin\theta\sin\psi, \\
-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta).
\end{aligned}
$$

Then, we differentiate equations (2.24) with respect to s, where λ_3'' given by (3.15). After some calculations and by neglecting the third-order terms with respect to ε, we obtain the vibrations equation written in a matrix form

$$T\varepsilon = 0, \tag{3.16}$$

where T is a 3×3 symmetric matrix whose components are given by

$$
T_{11} = \frac{\omega^2}{k^2}\left\{ A\rho_0 + \gamma A^2\rho_0 k^2 - \frac{(\psi_s'^4 - 3k^2\psi_s'^2)\cos^2\theta_s}{(k^2 - \psi_s'^2)^2} \right\} +
$$
$$
+ \left\{ (\beta\mu_0 - 2\alpha E_0)\cos^2\theta_s + (\alpha E_0 - \beta\mu_0) \right\} A^2 \psi_0'^2 + \beta\mu_0 A^2 \tau\psi_s\cos\theta_s - \alpha A^2 E_0 k^2,
$$

$$
T_{12} = T_{21} = -\frac{2k\rho_0 A\omega^2}{(k^2 - \psi_s'^2)^2}\psi_0'\cos\theta_s\sin\theta_s
$$
$$
+ kA^2\sin\theta_s\left\{ (\beta\mu_0 - 2\alpha E_0)\cos\theta_s\psi_s' + \beta\mu_0\tau \right\},
$$

$$T_{13} = T_{31} = (2\gamma\rho_0\omega^2 - \beta\mu_0 k^2)A^2\cos\theta_s,$$

$$T_{22} = \omega^2\left\{\frac{A\rho_0}{(k^2 - \psi_s'^2)^2}(\psi_s'^2 + k^2)\sin^2\theta_s + \gamma A^2\rho_0(\cos^2\theta_s + 1)\right\}$$
$$- k^2A^2(\beta\mu_0\cos^2\theta_s + \alpha E_0\sin^2\theta_s),$$

$$T_{23} = M_{32} = (2\gamma\rho_0\omega^2 - \beta\mu_0 k^2)A^2\cos\theta_s,$$

$$T_{33} = A^2(2\gamma\rho_0\omega^2 - \beta\mu_0 k^2), \tag{3.17}$$

and $\varepsilon = (\varepsilon, \varepsilon, \varepsilon)^t$ is a column vector. In (3.17) we have used the relations $k_2 = 2k_1$ and $\varphi_s' = \tau - \psi_s'\cos\theta_s$. The dispersion relations are calculated from $\det T = 0$, where the determinant of T is a cubic polynomial in ω^2.

Consider the case of transverse vibrations. The characteristic equation is given by

$$\rho_0^2[(1 - u_s^2)\psi_s'^2 + k^2(1 + \gamma A^2\rho_0)]\omega^4 - A\rho_0 k^2[\alpha E_0 u_s^2\psi_s'^2(\psi_s'^2 + 13k^2 - \gamma A^2\rho_0(u_s^2 + 1))$$
$$- \beta\mu_0\tau u_s\psi_s'(\psi_s'^2 + 5k^2) + 2\alpha E_0(k^4 - k^2\psi_s'^2 + \gamma A^2\rho_0 k^2 - 2\gamma A^2\rho_0 u_s)]\omega^2$$
$$- k^4(\psi_s'^2 - k^2)[\alpha^2 E_0^2 A^2 u_s^2\psi_s'^2 - 3\alpha\beta\mu_0 E_0 A^2\tau u_s\psi_s'$$
$$+ \alpha E_0(\psi_s'^2 - k^2 + 2\gamma A^2\rho_0 u_s) + \beta^2\mu_0^2 A^2\tau^2] = 0, \tag{3.18}$$

where $u_s = \cos\theta_s \neq \pm 1$.

Equation (3.18) is a polynomial equation with respect to ω^2. The roots of (3.18), $\omega_1 = (\omega^+)^2$ and $\omega_2 = (\omega^-)^2$, are functions of k, α, β, γ and A. The parameters α and β, defined by (2.12), characterize the bending stiffness and torsional stiffness, respectively, for known A. The parameter γ, given by (2.18) and (2.21), characterize the moments of inertia I_1 and I_2, for known $A(k_2 = 2k_1)$.

In the following, we consider that the parameters α, β, γ are known.

We make the assumption that the rod buckles in three modes.

Numerical investigations show that for $\psi_s'^2 < k^2$, the roots $(\omega_1^{}\omega_2^{})$ are positive, and for $\psi_s'^2 \geq k^2$ the root ω_1 is positive and ω_2 is negative. The waves are stable for real values of the angular frequency ω. The initial strain is determined by u_s, ψ_s' and τ. If $A(s)$ is given in (2.17), (2.24)$_{2,3,4}$, (3.15) and the initial conditions are given by (2.25) then the values of (ω_1, ω_2) for which the problem has a nontrivial solution define a set of eigenvalue curves C_j, $j = 1, 2, 3, \ldots$ [Atanackovic and Novakovic (2006)]. Let (ω_1^1, ω_2^1) be a point on the lowest eigenvalue curve corresponding to the first buckling mode, say C_1. For a rod with constant cross-section, the eigenvalue curves can be easily determined. For some value of ω_2, the eigenvalue curves

intersect. This situation is shown on the left part of Fig. 3 for the first and second eigenvalue curves C_1 and C_2. At the point of intersection, bimodal buckling is possible. Let $\bar{\omega}_1^1$ be the value of ω_1^1 at the point of intersection. Then for $\omega_1^1 < \bar{\omega}_1^1$ the rod buckles in the first mode. We refer to the part of the curve C_1 with $0 < \omega_1^1 \leq \bar{\omega}_1^1$ as the *lower part* of C_1. Similarly, let $\bar{\omega}_1^n$ be a point on the ω_1 axis corresponding to the intersection of the nth and $(n+1)$th eigenvalue curve. Then the part of the curve C_n with $\bar{\omega}_1^{n-1} \leq \omega_1^1 \leq \bar{\omega}_1^n$ will be the lower part of C_n (see Fig. 3). For different values of the parameters α, β and γ, we obtain from (3.18) different pairs (ω_1, ω_2). Suppose now that (ω_1^*, ω_2^*) is given. We define *the optimal thin elastic rod* as a rod shaped in such a manner that any other rod of the same length and smaller volume will buckle under the load λ_3 for(ω_1^*, ω_2^*). Thus, the problem of determining the shape of the optimal rod may be mathematically stated as follows:

Given (ω_1^*, ω_2^*), *find* $A^*(s)$ *such that the integral* (2.19) *is a minimum for* $A^*(s)$ *among all* $0 < A_{\min} \leq A(s) \leq A_{max}$, *i.e.*

$$\min_A I_A = \min \int_0^l A(s)\mathrm{d}s = \int_0^l A^*(s)\mathrm{d}s. \tag{3.19}$$

In addition, when $A^*(s)$*is used in* (3.11) *or* (3.12), *the values* (ω_1, ω_2) *determined from* (3.18) *are equal to*(ω_1^*, ω_2^*) *and belong to a point on the lower part of some eigenvalue curve* C_n, $n = 1, 2, ...$

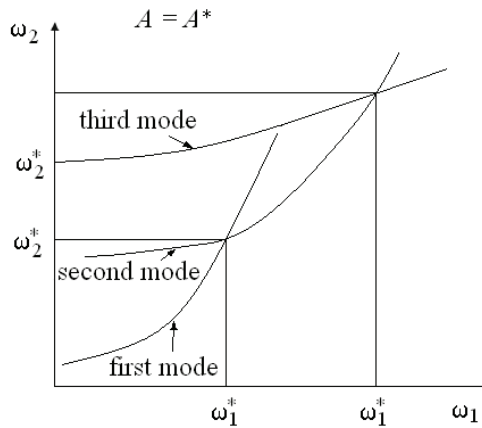This optimization problem is solved by employing a GA.

Figure 3: Eigenvalue curves for the rod of optimal cross sectional area.

4   Genetic algorithm

GAs try to find an optimal answer by evolving a population of trial answers in a way that mimics biological evolution [Goldberg (1989); Gen and Cheng (2000)]. Each answer is called an individual and is coded as a string chromosome. Individual parameters are substrings of characters (genes). From one generation to the next, the strongest genes and chromosomes remain by destroying the weakest ones.

The objective function is built starting with (3.19) and some additional restrictions, which measure the degree of verification of the equations of motion and the initial conditions, as

$$F(p) = I_A + \sum_{j=1}^{8} w_j s_j = \int_0^l A(s)\mathrm{d}s + \sum_{j=1}^{8} w_j s_j. \tag{4.1}$$

The unknown parameters are $p = \{k, A\}$, while the known parameters are $(\alpha, \beta, \gamma, \delta_3)$. Expressions s_j, $j = 1,...,8$, are built by $(2.24)_{2,3,4}$, (3.15) and (2.25). The Euler angles φ, ψ and θ are known analytically, from (3.11) or (3.12). The quantities w_j, $j = 1,...,8$ are selected weights, such that the dimension of $w_j s_j$, $j = 1,...,8$ is m^3. We obtain

$$s_1 = \gamma A^2 \rho_0 (\dot{\psi}^2 \sin\theta \cos\theta - \ddot{\theta})$$
$$- \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta - \alpha E_0 A^2 (\psi'^2 \sin\theta \cos\theta - \theta'')$$
$$+ \beta \mu_0 A^2 (\varphi' + \psi'\cos\theta)\psi'\sin\theta + \lambda_3 \sin\theta = 0,$$

$$s_2 = -\frac{\partial}{\partial t}\{\gamma A^2 \rho_0 \dot{\psi}\sin^2\theta$$
$$+ \delta A^2 \rho_0 (\dot{\varphi} + \dot{\psi}\cos\theta)\cos\theta\} + \frac{\partial}{\partial s}\{\alpha E_0 A^2 \psi'^2 \sin^2\theta$$
$$+ \beta \mu_0 A^2 (\varphi' + \psi'\cos\theta)\cos\theta\} = 0,$$

$$s_3 = -\delta \rho_0 \frac{\partial}{\partial t}(\dot{\varphi} + \dot{\psi}\cos\theta) + \beta \mu_0 \frac{\partial}{\partial s}(\varphi' + \psi'\cos\theta) = 0, \tag{4.2}$$

$$s_4 = \lambda'' + \rho_0 A\ddot{d}_3 = 0,$$

$$s_5 = \lambda(s,0) + \rho_0 A v^2 d_3(s,0) + (0_1, 0, \lambda_3) = 0,$$

$$s_6 = \theta(s,0) - \theta_0(s) = 0,$$

$$s_7 = \psi(s,0) - \psi_0(s) = 0,$$

$$s_8 = \varphi(s,0) - \varphi_0(s) = 0.$$

4.1 Sensitivity analysis of the GA

All steps required when applying a GA to an optimization problem are explained in the literature [Gen and Cheng (2000); Rajasekaran and Vijayalakshmipai (2005)]. The sensitivity analysis of the GA is presented in the spirit of [Khoshravan and Hosseinzadch (2009)].

The successful application of a GA consists of the right choice of its operators: *reproduction, crossover and mutation.* Before choosing these operators, the population size (P), generation size (G), crossover rate (C_r), and mutation rate (M_r) are determined.

For the crossover operator, three methods are applied: *single-point (C_1), two-point (C_2) and scattered (C_3).* In order to prevent large scatter in the responses and obtain uniform responses, the scale factor methods to the chromosomes must to be applied. The scale methods are: *rank (F_1), top (F_2) and uniform (F_3).* For the selection of the chromosomes, three methods are used: *roulette (S_1), tournament (S_2) and uniform (S_3).* For mutation we use the *uniform method (M).*

As already mentioned, we aim to decrease the weight of the rod by decreasing its area and hence to avoid the buckling under load. If a chromosome is not supporting the applied load, the buckling criterion would identify this chromosome. Thus the probability of transferring this chromosome to the next generation would decrease significantly.

We have tested three groups of methods, i.e. (F_1, S_1, C_3), (F_2, S_2, C_1) and (F_3, S_3, C_2), for different values of P (a number from 10 to 100), G (a number from 10 to 100), C_r (a number from 0 to 1) and M_r (a number from 0.05 to 0.5). In the following we report only the optimal results which are acceptable to our problem.

Firstly, to obtain the best crossover rate C_r, we study the variation of the objective function F with respect to this coefficient, for $P = 40$, $G = 30$ and $M_r = 0.15$, and three groups of different methods, i.e. (F_1, S_1, C_3), (F_2, S_2, C_1) and (F_3, S_3, C_2) . From Fig. 4 we see that C_r has its lowest value (for the lowest weight that the rod could have) at $C_r = 0.5$ as given by methods F_1, S_1, C_3. This result was also obtained by [Khoshravan and Hosseinzadch (2009)]. Normally, short chromosomes (0 and 01) do not require a high population size P. A convenient value for P is obtained from Fig. 5, for the same groups of methods, $M_r = 0.15$ and $C_r = 0.5$. Fig. 5 shows that when $P = 35$, there are no significant changes in the values of F. Therefore, $P = 40$ would be acceptable. A similar analysis of the variation of F with respect to G, for different P, has shown that $G = 30$ would be acceptable for all methods. The value $M_r = 0.15$ for the mutation uniform method M and (F_1, S_1, C_3), is chosen from Fig. 6.

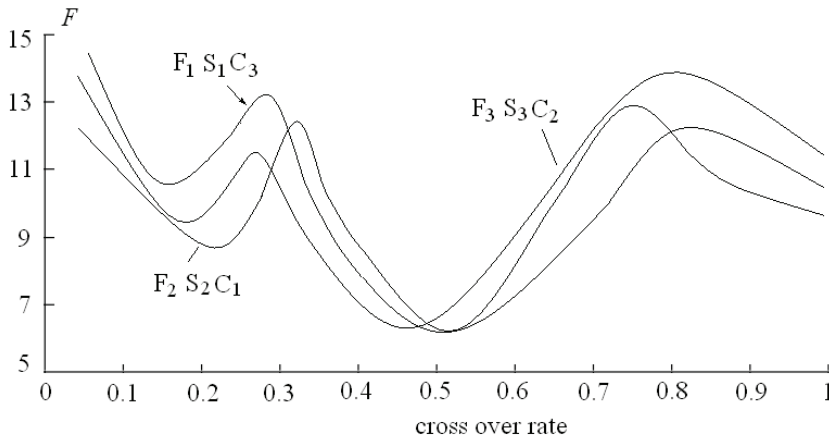In Fig. 7, the scale factor scale methods F_1, F_2 and F_3 are compared for similar

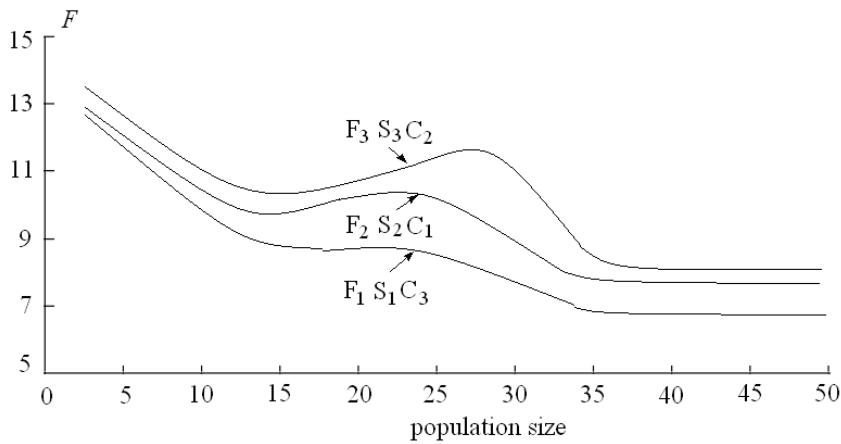Figure 4: Variation of the objective function with respect to the cross over rate.

Figure 5: Variation of the objective function with respect to the population size.

conditions (for S_1 and C_3). It can be seen that the rank method F_1 gives the best response in comparison with F_2 and F_3. In a similar way, the selection methods S_1, S_2 and S_3 are compared for similar F_1 and C_3, in Fig. 8. The method S_1 appears to be the best. In conclusion, we have combined the scale rank method F_1, with the selection roulette method S_1 and the scattered method C_3.
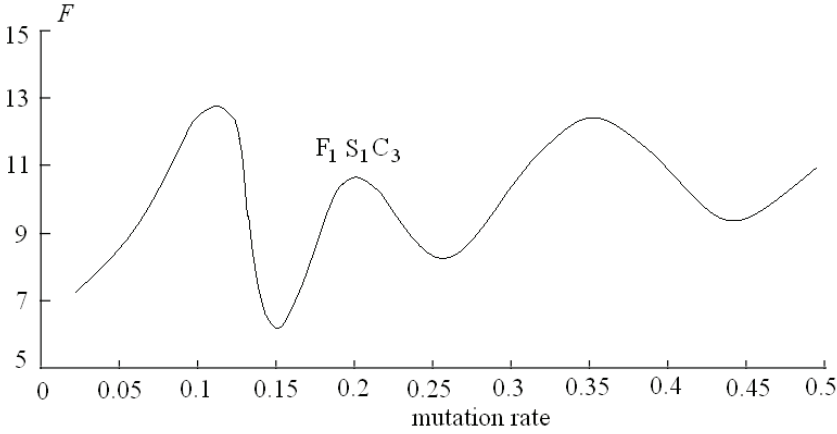
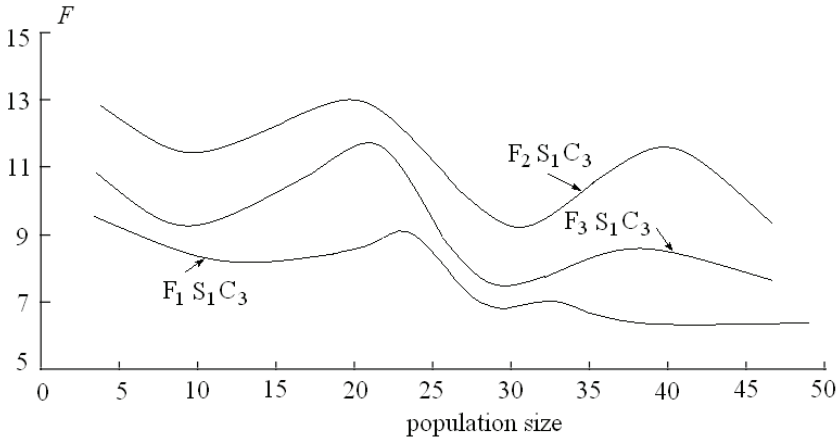Figure 6: Variation of the objective function with respect to the mutation rate.

Figure 7: Scale factor methods testing.

4.2   Optimization results

The GA simulation of the optimization problem is carried out for an elastic heli-
cal rod with $l = 1$m, $E = 194.2$GPa, $E_0 = 109$GPa, $\mu = 75.85$GPa, $\mu_0 = 40.67$GPa,
$\rho_0 = 7876$ kg/m^3, $\alpha = \frac{E}{4\pi E_0} = 0.14$ and $\beta = \frac{\mu}{2\pi\mu_0} = 0.3$. Let us consider that the
rod has a circular cross section, therefore $\gamma = \frac{1}{4\pi}$ and $\delta = \frac{1}{8\pi}$. The results are ob-
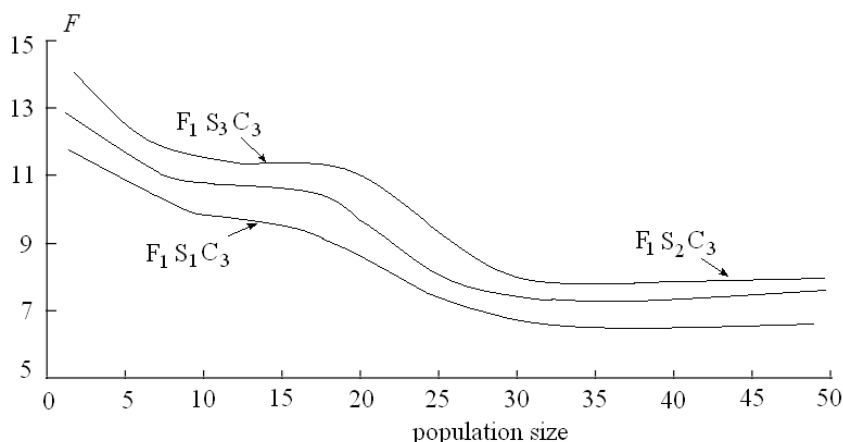
Figure 8: Selection methods testing.

tained after 45 iterations, showing good convergence and an acceptable value for the objective function $F_{min} = 6.18\text{cm}^3$.

For given (ω_1^*, ω_2^*), $A^*(s)$ is found from (4.1). When $A^*(s)$ is used in (3.11) or (3.12), we obtain from (3.18) $(\omega_1, \omega_2) = (\omega_1^*, \omega_2^*)$, belonging to a point on the lower part of the eigenvalue curves C_3, $n = 1, 2, 3$. So, for $\omega_1^* = 50$ and $\omega_1^* = 100$, the first three buckling modes and the corresponding optimal area are shown in Fig. 9. The dimensionless quantities that appear in Fig. 9 are $a = \frac{A}{l^2}$ and $\xi = \frac{s}{l}$.

5 Conclusions

Carrying out a sensitivity analysis for the GA considered herein, we combined the scale rank method F_1 with the selection roulette method S_1 and the scattered method C_3 in order to optimize the shape of a thin elastic rod which vibrates in space by bending and torsion. The vibrations were studied around the strained position of the rod, i.e. the helical form, which satisfies the static equilibrium equations. The optimal cross-section was determined from the minimum weight condition against the three modal buckling.

The basic laws of equilibrium and motion for the rod were studied and solved by using the equivalence theorem, which determines the conditions when the motion equations can be reduced to the equilibrium equations. Some important theorems were also reported. The closed form solutions were expressed using elliptic and hyperbolic functions (or solitons). The rod deviates from a plane and has a 3D
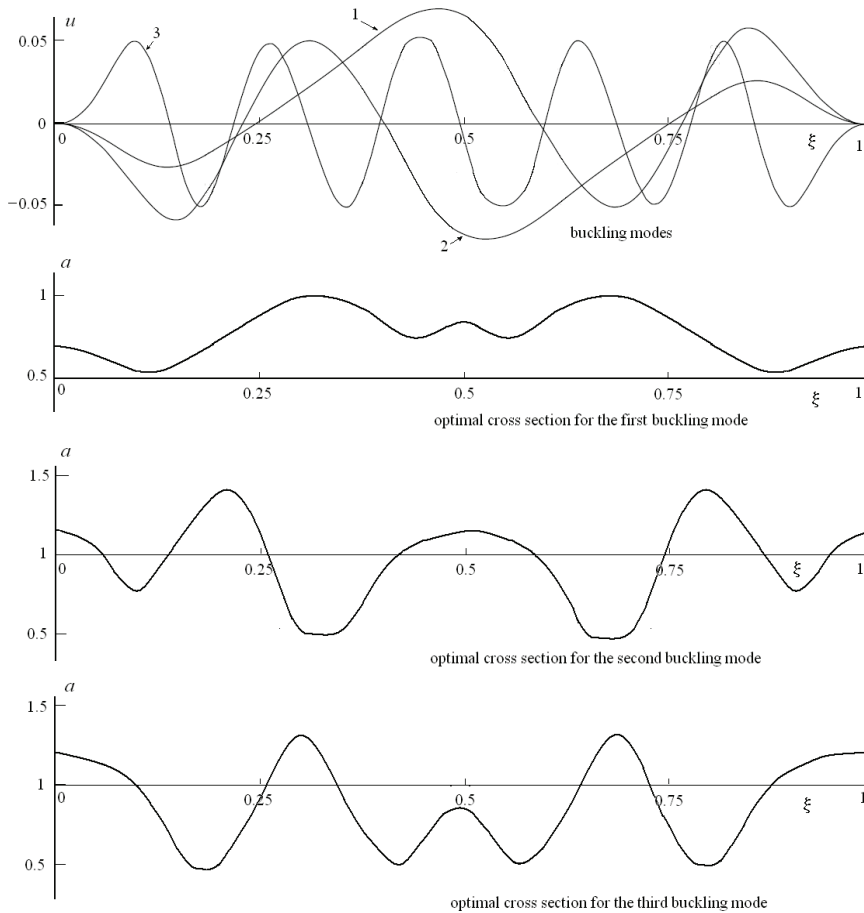
Figure 9: Three buckling modes and optimal cross section corresponding to these modes.

structure, changing its form as the torsion angle τ increases.

The effectiveness of the proposed GA was illustrated by a good convergence and a low computational time for the numerical code employed.

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