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On the Solution of an Inverse Problem for an Integro-differential Transport Equation

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Abstract: In this paper, the solvability conditions for an inverse problem for an integro-differential transport equation are obtained and a numerical approximation method based on the finite difference method is developed. A comparison between the numerical solution and the exact solution of the problem is presented. Experimental results show that proposed method is robust to data noises.

Keywords: Inverse Problem, Scattering Term, Curvature, Transport Equation, Finite Difference Method.

1 Introduction

Inverse problems for differential equations arise in a variety of important applications in science and technology. In these applications the aim is to estimate some unknown attributes of interest, given measurements that are only indirectly related to these attributes. For instance, in medical computerized tomography, one wishes to image structures within the body from measurements of X-rays that have passed through the body. In groundwater flow modelling, one estimates material parameters of an aquifer from measurements of pressure of a fluid that immerses the aquifer. Unfortunately, a small amount of noise in the data can lead to enormous errors in the estimates. This instability phenomenon is called ill-posedness, [Vogel (2002)]. The general theory of ill-posed problems and their applications is developed by A. N. Tikhonov, V. K. Ivanov, M. M. Lavrent'ev and their students, [Ivanov, Vasin, and Tanana (1978), Lavrent'ev, Romanov, and Shishatskii (1986), Tikhonov and Arsenin (1979)].

In this work, we consider the transport equation

$$u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + u_\varphi f\left(x,\varphi\right) + \int_0^{2\pi} K\left(x,\varphi,\varphi'\right) u\left(x,\varphi'\right) d\varphi' = \lambda\left(x\right), \tag{1}$$

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in the domain $\Omega = \{(x,\varphi) : x \in D \subset \mathbb{R}^2,\ \varphi \in (0,2\pi),\ \partial D \in C^3\}$,
where $f(x,\varphi) = f_1(x,\varphi)\cos\varphi + f_2(x,\varphi)\sin\varphi$.

Problem 1 *Find a pair of functions (u,λ) defined in Ω from equation (1), provided that the functions $f_1(x,\varphi)$, $f_2(x,\varphi)$, $K(x,\varphi,\varphi')$ are given, $u(x,\varphi)$ is 2π-periodic in φ and the trace of $u(x,\varphi)$ is known on $\Gamma_1 = \partial D \times (0,2\pi)$.*

Inverse problems for transport equations have a great importance both from theoretical and practical points of view. Some of the application areas of these problems are medical imaging and optical tomography, radiative transfer in the atmosphere and the ocean, neutron transport, as well as the propagation of seismic waves in the earth crust, [Bal (2009)]. Interesting results in this field are presented in [Amirov (1986), Amirov (2001), Anikonov, Kovtanyuk, and Prokhorov (2002), Natterer (1986), Isakov (2006), Klibanov and Yamamoto (2007)].

In this paper, we prove the existence, uniqueness and stability of the solution of Problem 1. Here the main difficulty is overdeterminancy of the problem. In the theory of inverse problems, usually "overdeterminancy" means that the number of free variables in the data exceeds the number of free variables in the unknown coefficient or right-hand side of the equation ($\lambda(x)$), and this is not the case here. In fact, Problem 1 is related to a certain problem of integral geometry (IGP) along regular curves and the underlying operator of this IGP is compact and its inverse operator is unbounded. Therefore, it is impossible to prove general existence results. This is the true reason why for existence of solution to Problem 1 need such special conditions on the data u_0, so we use the term "overdeterminacy" in this sense here.

There is a continuing and increasing interest in investigating the numerical solution of inverse and ill-posed problems. In these studies, the main goal is to improve the convergence and ease of implementation of different numerical algorithms. In this paper, we present a finite difference approach to solve Problem 1 numerically. In literature, there have been many studies devoted to numerical solving of second order partial differential equations using finite difference method. But here, the way of proving the solvability of Problem 1 leads to a Dirichlet type problem for a third order equation. So finite difference scheme is applied to this problem. Here, it is assumed that a family of regular curves $\{\Gamma\}$ passing from each point $x \in D$ and in any direction $\nu = (\cos\varphi, \sin\varphi)$ is given by curvature $f(x,\varphi) = f_1(x,\varphi)\cos\varphi + f_2(x,\varphi)\sin\varphi$, and there exists a curve passing from every $x \in D$ in the arbitrary direction ν, with endpoints on the boundary of D. The functions f_1, f_2 in the statement of the curvature depend on two variables, Problem 1 corresponds to an IGP along geodesics when $K(x,\varphi,\varphi') = 0$. Such problems have not been investigated numerically before. Also the way of specifying dependence of λ upon φ (or determining $\widehat{L}$, see section 2) and the spaces where the problem is investigated are new and original.

In order to obtain more accurate numerical results, several numerical methods have been developed in recent years. For example, Beilina and Klibanov (2008) developed a globally convergent numerical method for a multidimensional coefficient inverse problem for a hyperbolic PDE. On each iterative step, they solve the Dirichlet boundary value problem for a second-order elliptic equation. Ling and Takeuchi (2008) have combined the method of fundamental solutions (MFS) and boundary control technique to solve the inverse Cauchy problem of Laplace equation. Liu and Atluri (2008a) reformulated the inverse Cauchy problem of Laplace equation in a rectangle as an optimization problem, and applied a fictitious time integration method to solve an algebraic equations system to obtain the data on an unspecified portion of boundary. In [Liu and Atluri (2008b)], a novel method was proposed for computing the unknown potential function, the unknown impedance function, or the unknown weighting function in the Sturm-Liouville operator, when the discrete eigenvalues are specified. They employed a $SL(2,\mathbb{R})$ Lie-group shooting method (LGSM), combined with the use of Fictitious Time Integration Method (FTIM), for solving the inverse Sturm-Liouville problems.

2 Solvability of the Problem

In [Amirov (2001)], a general scheme is presented for investigating the solvability of such problems: using some extension of the class of unknown functions λ, overdetermined problem is replaced by a determined one. This is achieved by assuming the unknown function λ depends not only upon the space variable x, but also upon the direction φ in a specific way such that $\lambda(x,\varphi)$ satisfies a certain differential equation ($\widehat{L}\lambda = 0$) with the following properties:

i) Problem 1 with the function $\lambda(x,\varphi)$ becomes a determined one,

ii) The sufficiently smooth functions λ depending only on x satisfy this equation.

Remark 1 *It should be noted the special dependence of $\lambda(x,\varphi)$ upon the direction φ can not be arbitrarily, because in the opposite case the problem would be underdetermined and the nonuniqueness examples of a solution can be easily constructed, [Amirov (2001)].*

Some important definitions and notations which are used in studying the solvability of the problem are presented below. The function spaces $C^k(\Omega)$, $L_2(\Omega)$ and $H^k(\Omega)$ are well known standart spaces and described in detail, for example, in [Lions and Magenes (1972), Mikhailov (1978)].

$C^3_\pi(\Omega)$ denotes the space of all real-valued functions $u(x,\varphi) \in C^3(\Omega)$ which are 2π-periodic with respect to the argument φ in the domain Ω, i.e. the values of the function u and its derivatives up to third order at $\varphi = 0$ are equal to those at $\varphi = 2\pi$. The scalar product:

$$(u,z)_{1,c} = \int_\Omega \left[uz + \left(u_{x_1} + f_1 u_\varphi\right)\left(z_{x_1} + f_1 z_\varphi\right) + \left(u_{x_2} + f_2 u_\varphi\right)\left(z_{x_2} + f_2 z_\varphi\right)\right] d\Omega,$$

is defined in $C^3_\pi(\Omega)$, where $d\Omega = dx_1 dx_2 d\varphi$, and the norm

$$\|u\|_{1,c} = \left[(u,u)_{1,c}\right]^{1/2}$$

is introduced. The completions of the set $C^3_\pi(\Omega)$ with respect to the norms $\|\cdot\|_{1,c}$ and $\|\cdot\|_{H^m(\Omega)}$ $(m = 1,2,3)$ are denoted by $H^\pi_{1,c}(\Omega)$ and $H^\pi_m(\Omega)$, respectively. The set of functions $\psi(x,\varphi) \in C^3_\pi(\Omega)$ such that $\psi = 0$ on Γ_1 is denoted by $C^3_{\pi 0}(\Omega)$. The spaces $\mathring{H}^\pi_{1,c}(\Omega)$ and $\mathring{H}^\pi_m(\Omega)$ are the completions of the set $C^3_{\pi 0}$ with respect to the norm $\|\cdot\|_{1,c}$ and $\|\cdot\|_{H^m(\Omega)}$ $(m = 1,2,3)$, [Amirov (2001)].

Furthermore, we introduce the following notations:

$$Au \equiv \widehat{L}Lu,$$

where

$$\widehat{L}Lu = \frac{\partial^2}{\partial l \partial \varphi}(Lu),$$

$$\frac{\partial}{\partial l} = \sin\varphi\left(\frac{\partial}{\partial x_1} + f_{1\varphi} + f_2\right) - \cos\varphi\left(\frac{\partial}{\partial x_2} - f_1 + f_{2\varphi}\right) + (f_1 \sin\varphi - f_2 \cos\varphi)\frac{\partial}{\partial \varphi}.$$

Here the conjugate of the operator $\frac{\partial}{\partial l}$ in the sense of Lagrange can be obtained as

$$\left(\frac{\partial}{\partial l}\right)^* = -\sin\varphi\left(\frac{\partial}{\partial x_1} + f_1\frac{\partial}{\partial \varphi}\right) + \cos\varphi\left(\frac{\partial}{\partial x_2} + f_2\frac{\partial}{\partial \varphi}\right).$$

$\Gamma''(A)$ is the set of all functions $u(x,\varphi) \in L_2(\Omega)$ with the property that for any $u \in \Gamma''(A)$, there exists a function $y \in L_2(\Omega)$ such that $\forall \eta \in C^\infty_0(\Omega)$, $(u, A^*\eta)_{L_2(\Omega)} = (y,\eta)_{L_2(\Omega)}$ and $y = Au$. Here $(u,v)_{L_2(\Omega)}$ is a scalar product of functions u and v in $L_2(\Omega)$, A^* is the differential expression conjugate to A in the sense of Lagrange, and $C^\infty_0(\Omega)$ is the set of all functions defined in Ω which have continuous partial derivatives of order up to all $k < \infty$, whose supports are compact subsets of Ω. So the equality $y = Au$ is satisfied in the sense of generalized functions.

The subset $\Gamma(A) \subset \Gamma''(A)$ is such that for any $u \in \Gamma(A)$ there is a sequence $\{u_k\} \subset C^3_{\pi 0}$ with the following properties:

i) $u_k \to u$ weakly in $L_2(\Omega)$,
ii) $(Au_k, u_k)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $k \to \infty$.
We now replace Problem 1 by the following determined problem.

Problem 2 *Find a pair of functions (u, λ) defined in Ω that satisfies*

$$Lu \equiv u_{x_1} \cos\varphi + u_{x_2} \sin\varphi + u_\varphi f + \int_0^{2\pi} K\left(x, \varphi, \varphi'\right) u\left(x, \varphi'\right) d\varphi' = \lambda\left(x, \varphi\right), \tag{2}$$

$$u|_{\Gamma_1} = u_0, \quad u(x, \varphi) = u(x, \varphi + 2\pi), \tag{3}$$

$$\widehat{L}\lambda = 0 \tag{4}$$

provided that the functions f and K are known.

Remark 2 *Equation (4) is satisfied in generalized functions sense, i.e. for any function $\eta \in C_0^\infty(\Omega)$, the equality $\left(\lambda, \widehat{L}^*\eta\right)_{L_2(\Omega)} = 0$ is hold. Here $\widehat{L}^*$ is the conjugate operator to $\widehat{L}$ in the Lagrange sense.*

We shall prove the existence of the solution of the problem using Galerkin method. So we need the homogeneous boundary condition. Since $u_0 \in C^3(\Gamma_1)$ and $\partial D \in C^3$ then from by Theorem 2, p. 130 in [Mikhailov (1978)], Problem 2 can be reduced to the following problem with homogeneous data.

Problem 3 *Find a pair of functions (u, λ) defined in Ω that satisfies*

$$Lu = \lambda + G, \tag{5}$$

$$u|_{\Gamma_1} = 0,\ u(x, \varphi) = u(x, \varphi + 2\pi), \tag{6}$$

$$\widehat{L}\lambda = 0 \tag{7}$$

provided that the functions f, K and G are known.

Theorem 1 *If the functions*

$$f_1(x, \varphi), f_2(x, \varphi) \in C^2(\bar{D} \times (0, 2\pi)),\ K\left(x, \varphi, \varphi'\right) \in C^1\left(\bar{D} \times (0, 2\pi) \times (0, 2\pi)\right)$$

are given and the inequality $f_{1x_2} - f_{2x_1} + f_{1\varphi} f_2 - f_1 f_{2\varphi} > 0$ holds for all $x \in \bar{D}$ then Problem 3 has a unique solution (u, λ), such that $u \in \Gamma(A) \cap \mathring{H}_1^\pi(\Omega)$, $\lambda \in L_2(\Omega)$. Also, the inequality

$$\|u\|_{\mathring{H}_1^\pi(\Omega)} + \|\lambda\|_{L_2(\Omega)} \leq C(\|G\|_{L_2(\Omega)} + \left\|G_\varphi\right\|_{L_2(\Omega)}) \tag{8}$$

holds, where $G \in H_2^\pi(\Omega)$, $C > 0$ depends on f_1, f_2 and the Lebesgue measure of D, and $\bar{D}$ is the closure of D.

Proof. We first prove the uniqueness of the solution of Problem 3. Let us assume (u,λ) is a solution to the homogeneous version of Problem 3 ($G=0$) such that $u \in \Gamma(A) \cap \mathring{H}_1^{\pi}(\Omega)$ and $\lambda \in L_2(\Omega)$. Equation (5) and condition (7) imply $Au=0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset C^3_{\pi 0}$ such that $u_k \to u$ weakly in $L_2(\Omega)$ and $(Au_k,u_k)_{L_2(\Omega)} \to 0$ as $k \to \infty$. It can be easily verified that

$$2\left(Au_k,u_k\right)_{L_2(\Omega)} = \int_{\Omega} I(\nabla u_k)d\Omega$$

$$+\int_{\Omega}\left(-\sin\varphi\left(u_{kx_1}+f_1u_{k\varphi}\right)+\cos\varphi\left(u_{kx_2}+f_2u_{k\varphi}\right)\right)\int_0^{2\pi}K_{\varphi}u\left(x,\varphi'\right)d\varphi' d\Omega, \tag{9}$$

where

$$I(\nabla u_k) = (u_{kx_1}+f_1u_{k\varphi})^2+(u_{kx_2}+f_2u_{k\varphi})^2+(f_{1x_2}-f_{2x_1}+f_{1\varphi}f_2-f_1f_{2\varphi})u^2_{k\varphi}.$$

If we take into account the following estimates for $I(\nabla u_k)$,

$$\begin{aligned} 2f_1u_{kx_1}u_{k\varphi} &\geq -\varepsilon u^2_{kx_1}-\varepsilon^{-1}f_1^2u^2_{k\varphi},\ 0<\varepsilon<1 \\ 2f_2u_{kx_2}u_{k\varphi} &\geq -\varepsilon u^2_{kx_2}-\varepsilon^{-1}f_2^2u^2_{k\varphi}, \end{aligned}$$

and the conditions of the theorem, we obtain the following inequalities,

$$\begin{aligned} I(\nabla u_k) &\geq (1-\varepsilon)\left(u^2_{kx_1}+u^2_{kx_2}\right)+\left(1-\varepsilon^{-1}\right)\ell u^2_{k\varphi}+\eta_0u^2_{k\varphi} \\ &\geq (1-\varepsilon)\left|\nabla_x u_k\right|^2+\left(\eta_0+\ell\left(1-\varepsilon^{-1}\right)\right)u^2_{k\varphi}, \end{aligned}$$

where $\eta_0,\ell \in \mathbb{R}$ such that $f_{1x_2}-f_{2x_1}+f_{1\varphi}f_2-f_1f_{2\varphi} \geq \eta_0 > 0$ and $f_1^2+f_2^2 \leq \ell$. For sufficiently close value of ε to 1 we have $\eta_0+\ell\left(1-\varepsilon^{-1}\right) > \dfrac{\eta_0}{2}$, hence

$$I(\nabla u_k) \geq (1-\varepsilon)\left|\nabla_x u_k\right|^2+\frac{\eta_0}{2}u^2_{k\varphi} \geq \gamma_0\left(\left|\nabla_{x,\varphi}u_k\right|^2\right), \tag{10}$$

where $\gamma_0 = \min\left\{(1-\varepsilon),\dfrac{\eta_0}{2}\right\}$, $\left|\nabla_x u_k\right|^2 = u^2_{kx_1}+u^2_{kx_2}$, $\left|\nabla_{x,\varphi}u_k\right|^2 = u^2_{kx_1}+u^2_{kx_2}+u^2_{k\varphi}$. Now we estimate the second term on the right-hand side of (9) using Cauchy-

Schwartz and Steklov inequalities:

$$
\begin{aligned}
& \int_{\Omega}(-\sin\varphi\left(u_{kx_1}+f_1u_{k\varphi}\right)+\cos\varphi\left(u_{kx_2}+f_2u_{k\varphi}\right))\left(\int_0^{2\pi}K_{\varphi}u\left(x,\varphi'\right)d\varphi'\right)d\Omega \\
\geq\; & -\frac{1}{2}\int_{\Omega}\left[-\sin\varphi\left(u_{kx_1}+f_1u_{k\varphi}\right)+\cos\varphi\left(u_{kx_2}+f_2u_{k\varphi}\right)\right]^2 d\Omega \\
& -\frac{1}{2}\int_{\Omega}\left(\int_0^{2\pi}K_{\varphi}\left(x,\varphi,\varphi'\right)u\left(x,\varphi'\right)d\varphi'\right)^2 d\Omega \\
\geq\; & -\int_{\Omega}\left[\left(\sin\varphi\left(u_{kx_1}+f_1u_{k\varphi}\right)\right)^2+\left(\cos\varphi\left(u_{kx_2}+f_2u_{k\varphi}\right)\right)^2\right]d\Omega \\
& -\int_{D}\left(\int_0^{2\pi}u^2\left(x,\varphi'\right)d\varphi'\int_0^{2\pi}\int_0^{2\pi}K_{\varphi}^2\left(x,\varphi,\varphi'\right)d\varphi'd\varphi\right)dx \\
\geq\; & -2\int_{\Omega}\left[u_{kx_1}^2+u_{kx_2}^2+\left(f_1^2+f_2^2\right)u_{k\varphi}^2\right]d\Omega-M_2\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega \\
\geq\; & -2l_1\int_{\Omega}\left(u_{kx_1}^2+u_{kx_2}^2+u_{k\varphi}^2\right)d\Omega-M_2\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega \\
\geq\; & -M\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega,
\end{aligned}
\tag{11}
$$

where $M=\max\left\{l_1,M_2\right\},M_2=M_1l_1,\, M_1=\max\limits_{x\in\bar{D}}\left\{\int_0^{2\pi}\int_0^{2\pi}K_{\varphi}^2\left(x,\varphi,\varphi'\right)d\varphi'd\varphi\right\}$.
Then, from (10) and (11) we obtain

$$
\begin{aligned}
2\left(Au_k,u_k\right)_{L_2(\Omega)} &\geq \int_{\Omega}\gamma_0\left(\left|\nabla_{x,\varphi}u_k\right|^2\right)d\Omega-M\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega \\
&= \left(\gamma_0-M\right)\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega.
\end{aligned}
\tag{12}
$$

From the Steklov inequality, we have $\|u_k\|_{L_2(\Omega)}^2\leq C_0\int_{\Omega}\left|\nabla_{x,\varphi}u_k\right|^2 d\Omega$, so

$$
\|u_k\|_{L_2(\Omega)}^2\leq C_2\left(Au_k,u_k\right)_{L_2(\Omega)}, \tag{13}
$$

where $C_2=2C_0\dfrac{1}{\left(\gamma_0-M\right)}$ and $C_0>0$ is independent of k and depends on Lebesgue measure of D. Consequently, by virtue (13) and the definition of $\Gamma(A)$ we have

$$
\|u\|_{L_2(\Omega)}^2\leq \varliminf_{k\to\infty}\|u_k\|_{L_2(\Omega)}^2\leq C_2\lim_{k\to\infty}\left(Au_k,u_k\right)_{L_2(\Omega)}=0. \tag{14}
$$

From (14), it folllows that $\|u\|_{L_2(\Omega)}^2=0$, i.e. $u=0$ and from (5), $\lambda=0$. Hence, the uniqueness of the solution of the problem is proven.

Now we prove the existence of the solution of the problem in the set:

$(\Gamma(A)\cap \mathring{H}_1^{\pi}(\Omega))\times L_2(\Omega)$.

Select a set $\{e_1,e_2,e_3,...\}\subset C_{\pi 0}^3$ which is complete and orthonormal in $L_2(\Omega)$. We may assume here that the linear span of this set is everywhere dense in $\mathring{H}_{1,c}^{\pi}(\Omega)$. In fact, the space $\mathring{H}_{1,c}^{\pi}(\Omega)\cap \mathring{H}_1(\Omega)$ being seperable, there exists a countable set $\{\varphi_i\}_{i=1}^{\infty}\subset C_{\pi 0}^3$ which is everywhere dense in this space. If necessary, this set up can be extended to a set which is everywhere dense in $L_2(\Omega)$. Orthonormalizing the latter in $L_2\left(\Omega\right)$, we obtain $\{e_1,e_2,e_3,...\}$.We denote the orthogonal projector of $L_2\left(\Omega\right)$ onto $\mathscr{M}_n$ by $\mathscr{P}_n$, where $\mathscr{M}_n$ is the linear span of $\{e_1,e_2,...,e_n\}$.
Using the relations (5)-(7), we obtain the following problem:

$$\begin{aligned} Au &= \widehat{L}G=\mathscr{F}, && (15)\\ u|_{\Gamma_1} &= 0,\quad u(x,\varphi)=u(x,\varphi+2\pi). && (16)\end{aligned}$$

We seek the approximate solution of problem (15)-(16) in the form

$$u_N=\sum_{i=1}^{N}\alpha_{N_i}e_i(x,\varphi);\qquad \alpha_N=(\alpha_{N_1},\alpha_{N_2},...,\alpha_{N_N})\in\mathbb{R}^N$$

with the help of the following relations:

$$\int_{\Omega}\widehat{L}(Lu_N-G)e_j d\Omega=0,\quad j=1,2,...,N,\quad d\Omega=dx_1dx_2d\varphi. \tag{17}$$

The unknown coefficients α_{N_i} are determined from system of linear algebraic equations (17). We now prove that when $G=0$, system (17) has a unique solution for any $G\in H_2^{\pi}(\Omega)$. Let's substitute $\bar{\alpha}_N$ for α_N, multiply the jth equation by $2\bar{\alpha}_{N_j}$ and sum with respect to j from 1 to N, then we obtain

$$2\int_{\Omega}\widehat{L}L\bar{u}_N\bar{u}_N d\Omega=0, \tag{18}$$

where $\bar{u}_N=\sum\limits_{i=1}^{N}\bar{\alpha}_{N_i}e_i$. Then equalities (9) and (18) yield

$$\int_{\Omega}\left[(\bar{u}_{Nx_1}+f_1\bar{u}_{N\varphi})^2+(\bar{u}_{Nx_2}+f_2\bar{u}_{N\varphi})^2+(f_{1x_2}-f_{2x_1}+f_{1\varphi}f_2-f_1f_{2\varphi})\bar{u}_{N\varphi}^2\right]d\Omega$$

$$+\int_{\Omega}\left(-\sin\varphi\left(\bar{u}_{Nx_1}+f_1\bar{u}_{N\varphi}\right)+\cos\varphi\left(\bar{u}_{Nx_2}+f_2\bar{u}_{N\varphi}\right)\right)\int_0^{2\pi}K_{\varphi}\bar{u}_N d\varphi' d\Omega=0. \tag{19}$$

Using inequality (12) and the condition $\bar{u}_N = 0$ on Γ_1, from (19) we have $\bar{u}_N = 0$ in Ω. Since the system $\{e_i\}$ is linearly independent, we get $\bar{\alpha}_{N_i} = 0$, $i = 1,2,...,N$. Thus, the homogeneous version of system (17) has only trivial solution and therefore the original inhomogeneous system (17) has a unique solution α_N for any $G \in H_2^{\pi}(\Omega)$.

Now we estimate the solution u_N of system (17) in terms of the right hand side G. If we multiply the jth equation of (17) by $2\alpha_{N_j}$ and sum from 1 to N with respect to j, then we obtain

$$2\int_{\Omega} u_N \widehat{L} L u_N d\Omega = 2\int_{\Omega} u_N \widehat{L} G d\Omega. \tag{20}$$

The right-hand side of (20) can be obtained as follows:

$$2\left|\int_{\Omega} u_N \widehat{L} G d\Omega\right| \le \alpha_0 \int_{\Omega} G_{\varphi}^2 d\Omega + \alpha_0^{-1} \int_{\Omega} \left(\left(\frac{\partial}{\partial l}\right)^* u_N\right)^2 d\Omega.$$

Since the left hand side of (20) equals $2\left(Au_N, u_N\right)_{L_2(\Omega)}$, from (12) for sufficiently large $\alpha_0 > 0$, we get

$$(\gamma_0 - M)\int_{\Omega} \left|\nabla_{x,\varphi} u_N\right|^2 d\Omega \le \alpha_0 \int_{\Omega} G_{\varphi}^2 d\Omega + \alpha_0^{-1} \int_{\Omega} \left(\left(\frac{\partial}{\partial l}\right)^* u_N\right)^2 d\Omega.$$

Hence, we obtain

$$\int_{\Omega} \left|\nabla_{x,\varphi} u_k\right|^2 d\Omega \le C, \ \int_{\Omega} u_N^2 d\Omega \le C, \tag{21}$$

where the constant C doesn't depend on N. Thus, the set of functions $\{u_N\}$ is bounded in $\mathring{H}_1^{\pi}(\Omega)$. Since $\mathring{H}_1^{\pi}(\Omega)$ is a Hilbert space, the set $\{u_N\}$ is weakly compact in it. Therefore, there exists a subsequence (we again denote it by $\{u_N\}$) such that $u_N \to u$ weakly in $\mathring{H}_1^{\pi}(\Omega)$ as $N \to \infty$, so it follows that

$$\|u\|_{\mathring{H}_1^{\pi}(\Omega)} \le C\left\|G_{\varphi}\right\|_{L_2(\Omega)}. \tag{22}$$

Since $u \in \mathring{H}_1^{\pi}(\Omega)$, by the definition of $\mathring{H}_1^{\pi}(\Omega)$, we have $u|_{\Gamma_1} = 0$. From inequality (22), it can be easily proved that there exists a subsequence of $\{u_N\}$, which is again denoted by $\{u_N\}$, such that u_{Nx_1}, u_{Nx_2} and $u_{N\varphi}$ converge weakly in $L_2(\Omega)$ to u_{x_1}, u_{x_2} and u_{φ}, respectively. Transferring the operator $\widehat{L}$ to e_j in (17) and taking into account the conditions $u_N, e_j \in C_{\pi 0}^3$ and $G \in H_2^{\pi}(\Omega)$, we have

$$\int_{\Omega} (Lu_N - G)(\widehat{L})^* e_j d\Omega = 0, N \ge j.$$

Since the linear span of $\{e_j\}$ is eveywhere dense in the space $\mathring{H}^{\pi}_{1,c}(\Omega)$, passing to the limit as $N \to \infty$ we get

$$\int_{\Omega} (Lu - G)(\widehat{L})^* \zeta d\Omega = 0, \tag{23}$$

for every $\zeta \in \mathring{H}^{\pi}_{1,c}(\Omega)$. If we set $\lambda = Lu - G$, from (23) we see that λ satisfies condition (7) for any $\zeta \in C_0^{\infty}(\Omega) \subset \mathring{H}^{\pi}_{1,c}(\Omega)$, and also by using the inequality $\|u\|_{\mathring{H}^{\pi}_1(\Omega)} \leq C \left\|G_{\varphi}\right\|_{L_2(\Omega)}$, we obtain (8). In the expressions above, C stands for different constants that depend only on the given functions and Lebesgue measure of the domain D. Consequently, we have found a solution (u, λ) to Problem 3, where $u \in \mathring{H}^{\pi}_1(\Omega)$ and $\lambda \in L_2(\Omega)$.

Now it will be proven that $u \in \Gamma(A)$. Since $u \in L_2(\Omega)$ and $G \in H^{\pi}_2(\Omega)$, from (23) it follows that $\mathscr{F} = Au \in L_2(\Omega)$ in the generalized sense.

Finally, let us prove $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$. If we apply $\widehat{L}$ to both sides of $\lambda_N = Lu_N - G$ and bearing (17) in mind, it follows that $\mathscr{P}_N Au_N = \mathscr{P}_N \mathscr{F}$. Since $\mathscr{P}_N$ is an orthogonal projector onto $\mathscr{M}_n$, $\mathscr{P}_N \mathscr{F}$ converges strongly to $\mathscr{F}$ in $L_2(\Omega)$ as $N \to \infty$, i.e. $\mathscr{P}_N Au_N \to \mathscr{F} = Au$ strongly in $L_2(\Omega)$ as $N \to \infty$. Also, since $\{u_N\}$ weakly converges to u in $L_2(\Omega)$ as $N \to \infty$, we have $(\mathscr{P}_N Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$. Hence $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$, which completes the proof. ■

3 A Numerical Method: The Finite Difference Scheme

In this section, we develop a finite difference approximation for the following inverse problem: Determine a solution (u, λ) in

$$\Omega = \{\, (x_1, x_2, \varphi) |\ (x_1, x_2) \in (a,b) \times (c,d)\,,\ a,b,c,d \in \mathbb{R},\ \varphi \in (0, 2\pi)\}$$

from the relations

$$Lu = u_{x_1} \cos\varphi + u_{x_2} \sin\varphi + u_{\varphi}(f_1(x,\varphi)\cos\varphi + f_2(x,\varphi)\sin\varphi) = \lambda(x,\varphi), \tag{24}$$

$$u|_{\Gamma_1} = u_0(x,\varphi), \quad u(x,\varphi) = u(x, \varphi + 2\pi), \tag{25}$$

$$\hat{L}\lambda = 0. \tag{26}$$

If the operator $\widehat{L}$ is applied to both sides of equation (24), then we obtain a Dirichlet type problem for a third order partial differential equation:

$$\begin{aligned}
Au \equiv\;& u_{x_1x_2\varphi}\left(\sin^2\varphi-\cos^2\varphi\right)-u_{x_2x_2\varphi}\sin\varphi\cos\varphi+u_{x_1x_1\varphi}\cos\varphi\sin\varphi-u_{x_1x_1}\sin^2\varphi\\
&-u_{x_2x_2}\cos^2\varphi+2u_{x_1x_2}\sin\varphi\cos\varphi+u_{x_1\varphi\varphi}\left(f_1\sin 2\varphi-f_2\cos 2\varphi\right)\\
&-u_{x_2\varphi\varphi}\left(f_2\sin 2\varphi+f_1\cos 2\varphi\right)+u_{\varphi\varphi\varphi}\left(f_1\cos\varphi+f_2\sin\varphi\right)\left(f_1\sin\varphi-f_2\cos\varphi\right)\\
&+u_{x_1}\left[f_2\cos 2\varphi-f_1\sin 2\varphi-\left(f_{1\varphi}\sin\varphi-f_{2\varphi}\cos\varphi\right)\sin\varphi\right]\\
&+u_{x_2}\left[f_1\cos 2\varphi+f_2\sin 2\varphi+\left(f_{1\varphi}\sin\varphi-f_{2\varphi}\cos\varphi\right)\cos\varphi\right]\\
&+u_{\varphi\varphi}F_1(x,\varphi)+u_{x_1\varphi}F_2(x,\varphi)+u_{\varphi}F_3(x,\varphi)=0,
\end{aligned}\tag{27}$$

$$u|_{\Gamma_1} = u_0(x,\varphi),\; u(x,\varphi)=u(x,\varphi+2\pi),\tag{28}$$

where

$$\begin{aligned}
F_1(x,\varphi)=\;&\left(f_{1x_1}\cos\varphi+f_{2x_1}\sin\varphi\right)\sin\varphi-\left(f_{1x_2}\cos\varphi+f_{2x_2}\sin\varphi\right)\cos\varphi\\
&+2\left(f_{1\varphi}\cos\varphi+f_{2\varphi}\sin\varphi\right)\left(f_1\sin\varphi-f_2\cos\varphi\right)-2\left(f_1\sin\varphi-f_2\cos\varphi\right)^2\\
&+\left(f_1\cos\varphi+f_2\sin\varphi\right)^2+\left(f_{1\varphi}\sin\varphi-f_{2\varphi}\cos\varphi\right)\left(f_1\cos\varphi+f_2\sin\varphi\right),
\end{aligned}$$

$$\begin{aligned}
F_2(x,\varphi)=\;&3\left(f_2\cos\varphi-f_1\sin\varphi\right)\sin\varphi+\left(f_2\sin\varphi+f_1\cos\varphi\right)\cos\varphi\\
&+f_{1\varphi}\sin 2\varphi-f_{2\varphi}\cos 2\varphi]+u_{x_2\varphi}\left[2\cos\varphi\left(f_1\sin\varphi-f_2\cos\varphi\right)\right.\\
&+f_1\sin 2\varphi-f_2\cos 2\varphi-\left(f_{1\varphi}\cos 2\varphi+f_{2\varphi}\sin 2\varphi\right),
\end{aligned}$$

$$\begin{aligned}
F_3(x,\varphi)=\;&\left(\left(f_{1\varphi x_1}+f_{2x_1}\right)\cos\varphi+\left(f_{2\varphi x_1}-f_{1x_1}\right)\sin\varphi\right)\sin\varphi\\
&-(\left(\left(f_{1\varphi x_2}+f_{2x_2}\right)\cos\varphi+\left(f_{2\varphi x_2}-f_{1x_2}\right)\sin\varphi\right)\cos\varphi\\
&+2\left(f_1\cos\varphi+f_2\sin\varphi\right)\left(f_2\cos\varphi-f_1\sin\varphi\right)\\
&+3\left(f_{1\varphi}\sin\varphi-f_{2\varphi}\cos\varphi\right)\left(f_2\cos\varphi-f_1\sin\varphi\right)\\
&+\left(f_{1\varphi}\sin\varphi-f_{2\varphi}\cos\varphi\right)\left(f_{1\varphi}\cos\varphi+f_{2\varphi}\sin\varphi\right)\\
&+\left(f_{1\varphi}\cos\varphi+f_{2\varphi}\sin\varphi\right)\left(f_2\sin\varphi+f_1\cos\varphi\right)\\
&+\left(f_{1\varphi\varphi}\cos\varphi+f_{2\varphi\varphi}\sin\varphi\right)\left(f_1\sin\varphi-f_2\cos\varphi\right)].
\end{aligned}$$

Using the central finite difference formulas in (27), we obtain the following system of simultaneous algebraic nodal equations:

$$
\begin{aligned}
&-a_1^{(k)}\tilde{u}_{i-1,j-1}^{k-1}+a_6^{(k)}\tilde{u}_{i-1,j-1}^{k}+a_1^{(k)}\tilde{u}_{i-1,j-1}^{k+1}-(a_3^{(k)}+b_1^{(i,j,k)}-b_5^{(i,j,k)})\tilde{u}_{i-1,j}^{k-1}\\
&+(a_4^{(k)}+2b_1^{(i,j,k)}-b_7^{(i,j,k)})\tilde{u}_{i-1,j}^{k}+(a_3^{(k)}-b_1^{(i,j,k)}-b_5^{(i,j,k)})\tilde{u}_{i-1,j}^{k+1}+a_1^{(k)}\tilde{u}_{i-1,j+1}^{k-1}\\
&-a_6^{(k)}\tilde{u}_{i-1,j+1}^{k}-a_1^{(k)}\tilde{u}_{i-1,j+1}^{k+1}-(a_2^{(k)}+b_2^{(i,j,k)}-b_6^{(i,j,k)})\tilde{u}_{i,j-1}^{k-1}+(a_5^{(k)}+2b_2^{(i,j,k)}\\
&-b_8^{(i,j,k)})\tilde{u}_{i,j-1}^{k}+(a_2^{(k)}-b_2^{(i,j,k)}-b_6^{(i,j,k)})\tilde{u}_{i,j-1}^{k+1}-b_3^{(i,j,k)}\tilde{u}_{i,j}^{k-2}+(2(a_2^{(k)}+a_3^{(k)}+b_3^{(i,j,k)})\\
&+b_4^{(i,j,k)}-b_9^{(i,j,k)})\tilde{u}_{i,j}^{k-1}-2(a_4^{(k)}+a_5^{(k)}+b_4^{(i,j,k)})\tilde{u}_{i,j}^{k}-(2(a_2^{(k)}+a_3^{(k)}+b_3^{(i,j,k)})-b_4^{(i,j,k)}\\
&-b_9^{(i,j,k)})\tilde{u}_{i,j}^{k+1}+b_3^{(i,j,k)}\tilde{u}_{i,j}^{k+2}-(a_2^{(k)}-b_2^{(i,j,k)}+b_6^{(i,j,k)})\tilde{u}_{i,j+1}^{k-1}+(a_5^{(k)}-2b_2^{(i,j,k)}\\
&+b_8^{(i,j,k)})\tilde{u}_{i,j+1}^{k}+(a_2^{(k)}+b_2^{(i,j,k)}+b_6^{(i,j,k)})\tilde{u}_{i,j+1}^{k+1}+a_1^{(k)}\tilde{u}_{i+1,j-1}^{k-1}-a_6^{(k)}\tilde{u}_{i+1,j-1}^{k}\\
&-a_1^{(k)}\tilde{u}_{i+1,j-1}^{k+1}-(a_3^{(k)}-b_1^{(i,j,k)}+b_5^{(i,j,k)})\tilde{u}_{i+1,j}^{k-1}+(a_4^{(k)}-2b_1^{(i,j,k)}+b_7^{(i,j,k)})\tilde{u}_{i+1,j}^{k}\\
&+(a_3^{(k)}+b_1^{(i,j,k)}+b_5^{(i,j,k)})\tilde{u}_{i+1,j}^{k+1}-a_1^{(k)}\tilde{u}_{i+1,j+1}^{k-1}+a_6^{(k)}\tilde{u}_{i+1,j+1}^{k}+a_1^{(k)}\tilde{u}_{i+1,j+1}^{k+1}\\
&=0,\ (i=1,2,...,I,\quad j=1,2,...,J,\ \ k=1,2,...,K),
\end{aligned}
\tag{29}
$$

where $I,\ J,\ K$ are positive integers, $\Delta x_1=(b-a)/(I+1)$, $\Delta x_2=(d-c)/(J+1)$ and $\Delta\varphi=2\pi/K$ are step sizes in the directions $x_1,\ x_2$ and φ, respectively. $\tilde{u}_{i,j,k}$ is the finite difference approximation to the solution $u(x_{1i},x_{2j},\varphi_k)=u(a+i\Delta x_1,c+j\Delta x_2,k\Delta\varphi)$, and the coefficients in system (29) are defined as follows:

$$a_1^{(k)}=\frac{1}{8\Delta x_1\Delta x_2\Delta\varphi}(\sin^2\varphi_k-\cos^2\varphi_k),\ a_2^{(k)}=-\frac{1}{2\left(\Delta x_2\right)^2\Delta\varphi}\cos\varphi_k\sin\varphi_k,$$

$$a_3^{(k)}=\frac{1}{2\left(\Delta x_1\right)^2\Delta\varphi}\cos\varphi_k\sin\varphi_k,\ a_4^{(k)}=-\frac{1}{\left(\Delta x_1\right)^2}\sin^2\varphi_k,$$

$$a_5^{(k)}=-\frac{1}{\left(\Delta x_2\right)^2}\cos^2\varphi_k,\ a_6^{(k)}=\frac{1}{2\Delta x_1\Delta x_2}\cos\varphi_k\sin\varphi_k,$$

$$b_1^{(i,j,k)}=\frac{1}{2\Delta x_1\left(\Delta\varphi\right)^2}\left(f_1^{i,j,k}\sin 2\varphi_k-\ f_2^{i,j,k}\cos 2\varphi_k\right),$$

$$b_2^{(i,j,k)}=-\frac{1}{2\Delta x_2\left(\Delta\varphi\right)^2}\left(f_2^{i,j,k}\sin 2\varphi_k+\ f_1^{i,j,k}\cos 2\varphi_k\right),$$

$$b_3^{(i,j,k)}=\frac{1}{2\Delta\varphi^3}\left(f_1^{i,j,k}\cos\varphi_k+f_2^{i,j,k}\sin\varphi_k\right)\left(f_1^{i,j,k}\sin\varphi_k-f_2^{i,j,k}\cos\varphi_k\right),$$

$$
\begin{aligned}
b_4^{(i,j,k)} \quad = \quad & \frac{1}{(\Delta\varphi)^2}\Big[\Big(f_{1x_1}^{i,j,k}\cos\varphi_k+f_{2x_1}^{i,j,k}\sin\varphi_k\Big)\sin\varphi_k \\
& -\Big(f_{1x_2}^{i,j,k}\cos\varphi_k+f_{2x_2}^{i,j,k}\sin\varphi_k\Big)\cos\varphi_k \\
& +2\Big(f_{1\varphi}^{i,j,k}\cos\varphi_k+f_{2\varphi}^{i,j,k}\sin\varphi_k\Big)\Big(f_{1}^{i,j,k}\sin\varphi_k-f_{2}^{i,j,k}\cos\varphi_k\Big) \\
& -2(f_{1}^{i,j,k}\sin\varphi_k-f_{2}^{i,j,k}\cos\varphi_k)^2+(f_{1}^{i,j,k}\cos\varphi_k+f_{2}^{i,j,k}\sin\varphi_k)^2 \\
& +\Big(f_{1}^{i,j,k}\cos\varphi_k+f_{2}^{i,j,k}\sin\varphi_k\Big)(f_{1\varphi}^{i,j,k}\sin\varphi_k-f_{2\varphi}^{i,j,k}\cos\varphi_k)\Big],
\end{aligned}
$$

$$
\begin{aligned}
b_5^{(i,j,k)} \quad = \quad & \frac{1}{4\Delta x_1\Delta\varphi}\Big[3\Big(f_{2}^{i,j,k}\cos\varphi_k-f_{1}^{i,j,k}\sin\varphi_k\Big)\sin\varphi_k \\
& +\Big(f_{1}^{i,j,k}\cos\varphi_k+f_{2}^{i,j,k}\sin\varphi_k\Big)\cos\varphi_k+f_{1\varphi}^{i,j,k}\sin 2\varphi_k-f_{2\varphi}^{i,j,k}\cos 2\varphi_k\Big],
\end{aligned}
$$

$$
\begin{aligned}
b_6^{(i,j,k)} \quad = \quad & \frac{1}{4\Delta x_2\Delta\varphi}\Big[\Big(2\cos\varphi_k(f_{1}^{i,j,k}\sin\varphi_k-f_{2}^{i,j,k}\cos\varphi_k) \\
& +\Big(f_{1}^{i,j,k}\sin 2\varphi_k-f_{2}^{i,j,k}\cos 2\varphi_k\Big)-(f_{1\varphi}^{i,j,k}\cos 2\varphi_k+f_{2\varphi}^{i,j,k}\sin 2\varphi_k)\Big],
\end{aligned}
$$

$$
b_7^{(i,j,k)}=\frac{1}{2\Delta x_1}\Big[f_{2}^{i,j,k}\cos 2\varphi_k-\ f_{1}^{i,j,k}\sin 2\varphi_k\ -(f_{1\varphi}^{i,j,k}\sin\varphi_k-f_{2\varphi}^{i,j,k}\cos\varphi_k)\sin\varphi_k\Big],
$$

$$
b_8^{(i,j,k)}=\frac{1}{2\Delta x_2}\Big[f_{1}^{i,j,k}\cos(2\varphi_k)+\ f_{2}^{i,j,k}\sin(2\varphi_k)+(f_{1\varphi}^{i,j,k}\sin\varphi_k-f_{2\varphi}^{i,j,k}\cos\varphi_k)\cos\varphi_k\Big],
$$

$$
\begin{aligned}
b_9^{(i,j,k)} \quad = \quad & \frac{1}{2\Delta\varphi}\Big[\frac{1}{2}\Big(\Big(f_{1\varphi}^{i,j,k}\Big)^2-\Big(f_{2\varphi}^{i,j,k}\Big)^2\Big)\sin 2\varphi_k-f_{1\varphi}^{i,j,k}f_{2\varphi}^{i,j,k}\cos 2\varphi_k \\
& -\Big(\Big(f_{1\varphi x_2}^{i,j,k}+f_{2x_2}^{i,j,k}\Big)\cos\varphi_k+\Big(f_{2\varphi x_2}^{i,j,k}-f_{1x_2}^{i,j,k}\Big)\sin\varphi_k\Big)\cos\varphi_k \\
& +\Big(\Big(f_{1\varphi x_1}^{i,j,k}+f_{2x_1}^{i,j,k}\Big)\cos\varphi_k+\Big(f_{2\varphi x_1}^{i,j,k}-f_{1x_1}^{i,j,k}\Big)\sin\varphi_k\Big)\sin\varphi_k \\
& +\Big(\Big(f_{1\varphi}^{i,j,k}+2f_{2}^{i,j,k}\Big)\cos\varphi_k+\Big(f_{2\varphi}^{i,j,k}-2f_{1}^{i,j,k}\Big)\sin\varphi_k\Big) \\
& \times\Big(f_{1}^{i,j,k}\cos\varphi_k+f_{2}^{i,j,k}\sin\varphi_k\Big)+\Big(f_{2}^{i,j,k}\cos\varphi_k-f_{1}^{i,j,k}\sin\varphi_k\Big) \\
& \times\Big(\Big(3f_{1\varphi}^{i,j,k}-f_{2\varphi\varphi}^{i,j,k}\Big)\sin\varphi_k-\Big(3f_{2\varphi}^{i,j,k}+f_{1\varphi\varphi}^{i,j,k}\Big)\cos\varphi_k\Big)\Big].
\end{aligned}
$$

From the first condition in (28), we have the following discrete boundary conditions:

$$\tilde{u}_{0,j}^{k} = u(a, x_{2j}, \varphi_k),\ \tilde{u}_{I+1,j}^{k} = u(b, x_{2j}, \varphi_k),$$
$$\tilde{u}_{i,0}^{k} = u(x_{1i}, c, \varphi_k),\ \tilde{u}_{i,J+1}^{k} = u(x_{1i}, d, \varphi_k),$$
$$(i = 0,1,...,I+1,\ j = 0,1,...,J+1,\ k = 1,2,...,K)$$

and also from the periodicity condition, we have $\tilde{u}_{i,j,0} = \tilde{u}_{i,j,K}$ and $\tilde{u}_{i,j,K+1} = \tilde{u}_{i,j,1}$. The approximate values $\tilde{u}_{i,j,k}$ are obtained at $I \times J \times K$ mesh points of Ω by solving system of linear algebraic equations (29).

To calculate λ numerically, the central-difference formulas are used in (24) and the difference equations:

$$\frac{\tilde{u}_{i+1,j}^{k} - \tilde{u}_{i-1,j}^{k}}{2\Delta x_1} \cos\varphi_k + \frac{\tilde{u}_{i,j+1}^{k} - \tilde{u}_{i,j-1}^{k}}{2\Delta x_2} \sin\varphi_k$$
$$+ \left(f_1^{i,j,k} \cos\varphi_k + f_2^{i,j,k} \sin\varphi_k \right) \frac{\tilde{u}_{i,j}^{k+1} - \tilde{u}_{i,j}^{k-1}}{2\Delta\varphi} = \tilde{\lambda}_{i,j}^{k}, \tag{30}$$
$$(i = 1,2,...,I\ ,\ \ j = 1,2,...,J\ ,\ \ k = 1,2,...,K)$$

are solved. Here $\tilde{\lambda}_{i,j,k}$ is the finite difference approximation to $\lambda(x_{1i}, x_{2j}, \varphi_k) = \lambda(a + i\Delta x_1, c + j\Delta x_2, k\Delta\varphi)$.

3.1 Numerical Experiments

The proposed method has been tested on many inverse problems. The computations are performed using MATLAB 7.0 program on a PC with Intel Core 2 T7200, 2 GHz. All the experiments have carried out using multiplicative random noise in the boundary data u_σ which is obtained by adding relative error to computed data u_{comp} according to the following expression:

$$u_\sigma(x_{1i}, x_{2j}, \varphi_k) = u_{comp}(x_{1i}, x_{2j}, \varphi_k) \left[1 + \frac{\alpha(u_{\max} - u_{\min})\sigma}{100} \right].$$

Here, $(x_{1i}, x_{2j}, \varphi_k)$ is a mesh point at the boundary $\partial\Omega$, α is a random number in the interval $[-1,1]$, $u_{\max}$ and $u_{\min}$ are maximal and minimal values of the computed data u_{comp}, respectively, and σ is the noise level in percents.

Example 1 *Let's consider the problem of determining* (u, λ) *in* $\Omega = (1,2) \times \left(0, \frac{1}{2}\right) \times (0, 2\pi)$ *that satisfies the equation*

$$u_{x_1} \cos\varphi + u_{x_2} \sin\varphi + \frac{1}{\sqrt{1 + (x_2 - x_1)^2}} u_\varphi (\cos\varphi + \sin\varphi) = \lambda$$

and the conditions

$$u(1,x_2,\varphi)=e^{\arcsin\left(\frac{x_2-1}{\sqrt{1+(x_2-1)^2}}\right)}(\cos\varphi-\sin\varphi),$$
$$u(2,x_2,\varphi)=e^{\arcsin\left(\frac{x_2-2}{\sqrt{1+(x_2-2)^2}}\right)}(\cos\varphi-\sin\varphi),$$
$$u(x_1,0,\varphi)=e^{\arcsin\left(\frac{-x_1}{\sqrt{1+x_1^2}}\right)}(\cos\varphi-\sin\varphi),$$
$$u(x_1,1/2,\varphi)=e^{\arcsin\left(\frac{1/2-x_1}{\sqrt{1+(1/2-x_1)^2}}\right)}(\cos\varphi-\sin\varphi),$$
$$u(x,\varphi)=u(x,\varphi+2\pi),\ \hat{L}\lambda=0.$$

The exact solution of the problem is

$$\begin{aligned} u(x_1,x_2,\varphi) &= e^{\arcsin\left(\frac{x_2-x_1}{\sqrt{1+(x_2-x_1)^2}}\right)}(\cos\varphi-\sin\varphi),\\ \lambda(x_1,x_2,\varphi) &= \frac{-2}{\sqrt{1+(x_2-x_1)^2}}e^{\arcsin\left(\frac{x_2-x_1}{\sqrt{1+(x_2-x_1)^2}}\right)}. \end{aligned}$$

Figure 1 presents a comparison between the exact solution and the computed numerical solution of the problem for $I=J=22$, $K=6$: (a) approximate u, (b) exact u. Approximate values of λ can be obtained using formula (30) according to the approximate values of u.

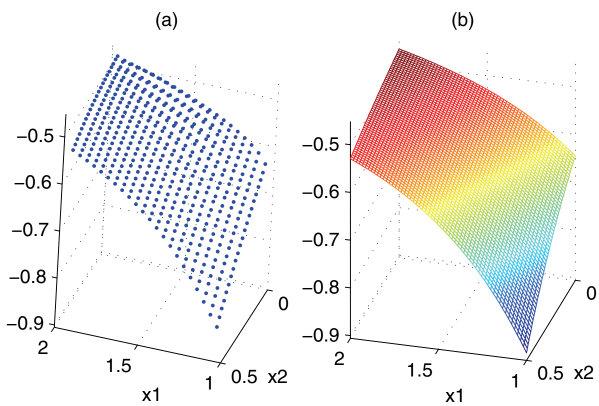

Figure 1: Exact and numerical solution of the problem.

Figure 2 displays the one dimensional cross sections $(x_2=0.45)$ of computed approximate solutions with different noise levels superimposed with the exact solution $(u(x_1,x_2,\varphi))$ of the inverse problem.

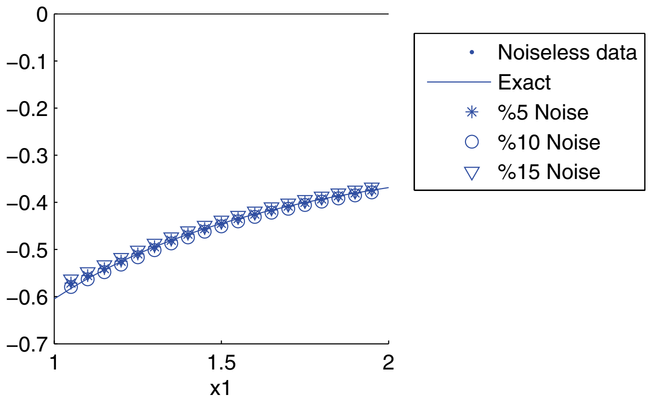

Figure 2: Approximate solutions for different noise levels at $x_2 = 0.45$.

Example 2 *Determine a pair of functions* (u,λ) *defined in* $\Omega = (0,1/2)\times(1/2,1)\times(0,2\pi)$ *that satisfies the equation*

$$u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + 3(x_2-x_1)^2 u_\varphi(\cos\varphi+\sin\varphi) = \lambda$$

and the following conditions

$$u(0,x_2,\varphi) = e^{x_2^3}(\cos\varphi-\sin\varphi), \quad u(1/2,x_2,\varphi) = e^{(x_2-1/2)^3}(\cos\varphi-\sin\varphi),$$

$$u(x_1,1/2,\varphi) = e^{(1/2-x_1)^3}(\cos\varphi-\sin\varphi), \quad u(x_1,1,\varphi) = e^{(1-x_1)^3}(\cos\varphi-\sin\varphi),$$

$$u(x,\varphi) = u(x,\varphi+2\pi),\ \hat{L}\lambda = 0.$$

Here the exact solution of the problem is

$$\begin{aligned} u(x_1,x_2,\varphi) &= e^{(x_2-x_1)^3}(\cos\varphi-\sin\varphi), \\ \lambda(x_1,x_2,\varphi) &= -6(x_2-x_1)^2 e^{(x_2-x_1)^3}. \end{aligned}$$

On Figure 3 below, numerical solution and exact solution $(u(x_1,x_2,\varphi))$ of the inverse problem are presented for $I = J = 24, K = 6$. On Figure 4, a comparison between the exact solution and the approximate solution of the problem for different noise levels is presented by one dimensional cross sections ($x_1 = 0.2$).

Consequently, numerical experiments show that proposed method is feasible for solving Problem 1. And it is robust to data noises. By using high performance computers, we can obtain high quality results for the greater values of I, J and K.

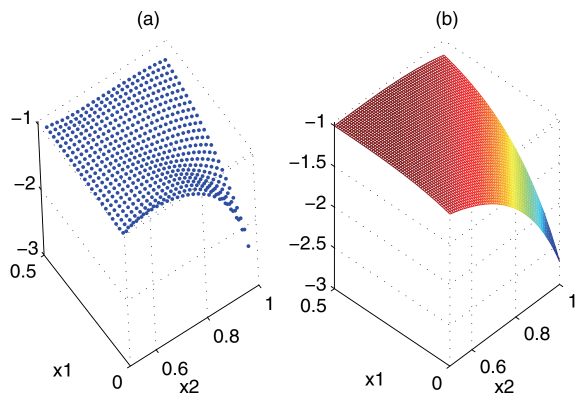

Figure 3: Exact and numerical solution of the problem.

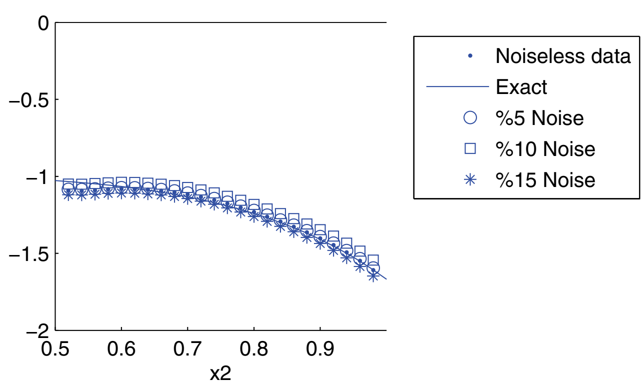

Figure 4: Approximate solutions for different noise levels at $x_1 = 0.2$.

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