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Interfacial Stresses Induced by a Point Heat Source in an Isotropic Plate with a Reinforced Elliptical Hole

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Abstract: A general analytical solution for a reinforced elliptical hole embedded in an infinite matrix subjected to a point heat source is provided in this paper. Based on the technique of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the general expressions of the temperature and stresses in the reinforcement layer and the matrix are derived explicitly in a series form. Some numerical results are provided to investigate the effects of the material combinations and geometric configurations on the interfacial stresses. The solution obtained can be treated as Green's functions which enable us to formulate an integral equation for a reinforced elliptical hole embedded in an infinite matrix with a crack.

Keywords: point heat source, reinforcement layer, conformal mapping

Nomenclature

a_1, a_2	semimajor of the two confocal ellipses
b_1, b_2	semiminor of the two confocal ellipses
$g(z)$	$\int \theta(z)dz$
k	heat conductivity
L_1, L_2	boundaries of the coated layer in the ζ-plane
l	$\sqrt{a_2^2 - b_2^2}$
q_0	the strength of the point heat source
q_x, q_y	components of heat flux in x and y direction
R	$\sqrt{\frac{a_2+b_2}{a_2-b_2}}$
S_1	the matrix in the ζ-plane

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S_2	the intermediate layer in the ζ-plane
T	temperature
U_{21}	$2k_1(k_1+k_2)^{-1}$
V_{21}	$(k_1-k_2)(k_1+k_2)^{-1}$
z	Cartesian coordinates
z_0	the location of the point heat source at z plane

Greek symbols

Ω_1	the matrix in the z-plane
Ω_2	the intermediate layer in the z-plane
Γ_1,Γ_2	boundaries of the coated layer in the z-plane
Π_{12}	$\frac{G_1-G_2}{\kappa_2 G_1+G_2}$
Λ_{12}	$\frac{G_1\kappa_2-G_2\kappa_1}{G_1+G_2\kappa_1}$
ζ	polar coordinates
ζ_0	the location of the point heat source at ζ plane
ρ_1,ρ_2	$\rho_i=\frac{a_i+b_i}{a_2+b_2} \quad i=1,2$
$\theta(\zeta)$	the temperature function
$\theta_0(\zeta)$	$-\frac{q_0}{2\pi k}\log(\zeta-\zeta_0)$

1 Introduction

Boundary value problems for an elliptical hole have received considerable attention from many researchers since those problems have applications to many different engineering structures. The stress field around the elliptical cavity under uniform loading for an isotropic and homogeneous material was first provided by [Muskhelishvili (1953)]. As to the thermoelastic problem, [Florence and Goodier (1960)] provided an exact solution for an isotropic medium containing a circular or ovaloid hole by the method of [England (1971)]. [Chen (1967)] solved the thermal stresses for orthotropic medium with an elliptic hole based on the complex variable technique developed by [Green and Zerna (1954)]. [Tarn and Wang (1993)] gave a closed form solution for anisotropic materials with a hole or a rigid inclusion based on [Lekhnitskii (1963)] complex potential approach. Based on [Stroh (1958)] formalism, [Hwu (1990)] found the thermal stresses for anisotropic body with an elliptical hole. The general solutions for an anisotropic solid with an elliptical inclusion under a remote uniform heat flow was given by [Chao and Shen

(1998)] using the method of analytical continuation and [Lekhnitskii (1963)] complex potential approach. Based on the complex potential approach, a thermoelastic solution was presented for a three phase elliptic inclusion problem subjected to a uniform temperature change [Ru (1998)]. Recently, an exact analytical solution of heat conduction problems for a three-phase elliptical composite was provided by [Chao, Chen and Chen (2009)].

In order to reduce the stress concentration, the reinforcement layer with appropriate geometry and material property is introduced to be bonded to a hole [Chen and Chao (2008); Chao, Lu, Chen and Chen (2009)]. This method has been widely applied to many practical problems where the elastic mismatch-induced stresses are of vital importance to mechanical integrity. For example, analytical solutions for a coated elliptical hole under a remote uniform heat flow were provided by [Chen and Chao (2008)]. One of the most difficult parts in solving the above mentioned problem with doubly connected regions is that the single-valued condition of the displacements and the stresses must be satisfied. The problem will become more complicated if singularities or point heat sources are considered. More specifically, the function forms of stress potentials must be properly chosen such that the single-valued condition of the displacements and the stresses can be automatically satisfied. When a reinforcement layer is introduced in the analysis, the interfacial stresses between a reinforcement layer and the matrix must be taken into account. In this work, we consider a reinforced elliptical hole in an infinite plate subject to a point heat source. The reinforcement layer is bounded by two confocal ellipses. The proposed method is based on the method of conformal mapping and the technique of analytical continuation that is alternately applied across two concentric circles. The plan of this paper is as follows. The general formulation for plane thermoelasticity and the method of conformal mapping are provided in Section 2. The series form solutions of the complex potentials of the temperature and the stresses are given in Section 3 and Section 4, respectively. Some numerical examples are solved in Section 5. Finally, Section 6 concludes the article.

2 Problem formulation

Consider a coated elliptical hole, which is assumed to be insulated from heat flow, in an unbounded matrix subjected to a point heat source (see Fig. 1). Let Ω_1 denote the matrix, Ω_2 denote the coated layer, respectively. The boundaries of the coated layer are two confocal ellipses Γ_1, Γ_2 with a_1, a_2 and b_1, b_2 being the semimajor and semiminor, respectively. For convenience of calculation, we introduce the

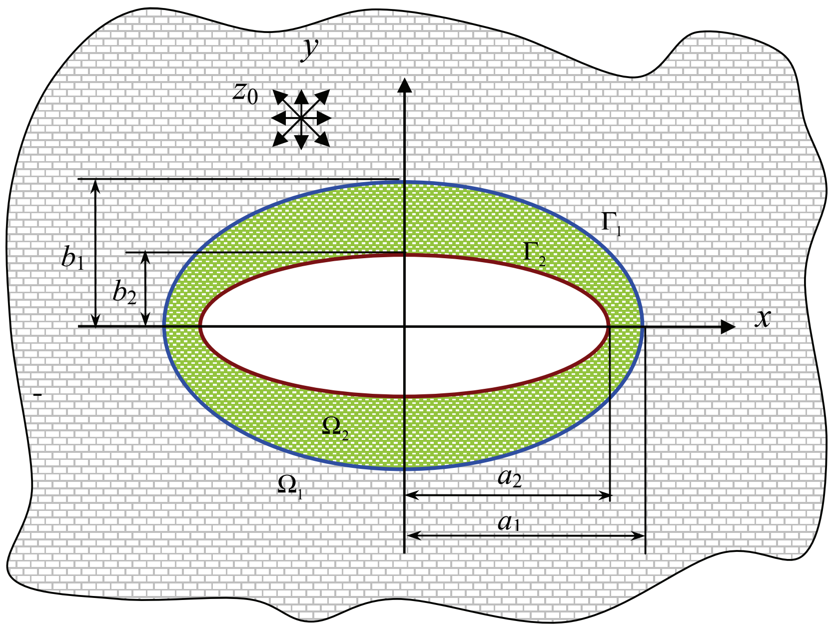

Figure 1: A coated elliptical hole in an infinite plane subject to a point heat source.

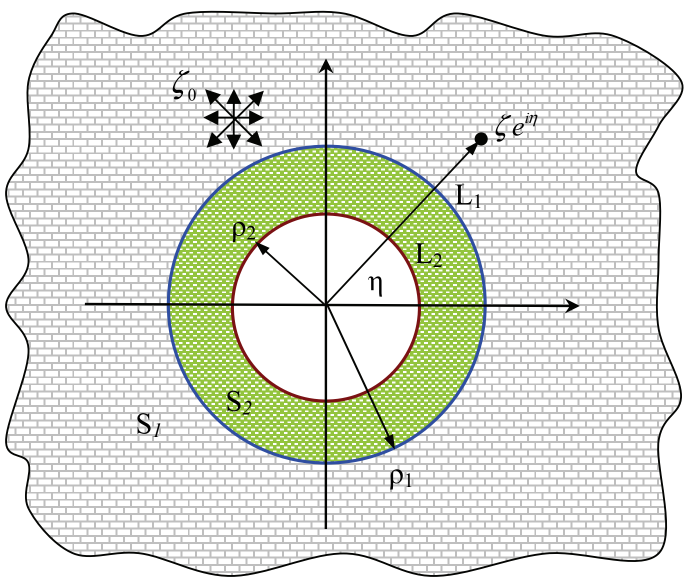

Figure 2: The problem in the ζ-plane.

following mapping function

$$z = m(\zeta) = \frac{l}{2}\left[R\zeta + \frac{1}{R\zeta}\right], \quad R\zeta = \frac{z}{l}\left\{1 + \left[1 - (\frac{l}{z})^2\right]^{1/2}\right\} \tag{1}$$

where

$$R = \sqrt{\frac{a_2 + b_2}{a_2 - b_2}}$$

$$l = \sqrt{a_2^2 - b_2^2}$$

This mapping function maps the confocal ellipses Γ_1, Γ_2 in the z-plane onto the concentric circles L_1, L_2 in the ζ-plane with radii ρ_1, ρ_2 (see Fig. 2) where

$$\rho_i = \frac{a_i + b_i}{a_2 + b_2} \quad i = 1,2$$

Note that the confocal ellipses Γ_1, Γ_2 in the z-plane are related by

$$a_1^2 - b_1^2 = a_2^2 - b_2^2$$

For a two-dimensional steady state heat conduction problem, the temperature function satisfies a harmonic equation. In the present study, the resultant heat flow Q and the temperature T are expressed in terms of a single complex potential $\theta(\zeta)$ as

$$Q = \int (q_x dy - q_y dx) = -k\mathrm{Im}[\theta(\zeta)] \tag{2}$$

$$T = \mathrm{Re}[\theta(\zeta)] \tag{3}$$

where Re and Im denote the real part and imaginary part of the bracketed expression, respectively. The quantities q_x, q_y in Eq. (2) are the components of heat flux in x and y direction, respectively, and k stands for the heat conductivity. Once the heat conduction problem is solved, the temperature function $\theta(\zeta)$ is determined. For a two-dimensional theory of thermoelasticity, the components of the displacement, stress and traction force can be expressed in terms of two stress functions and a temperature function as

$$2G(u_x + iu_y) = \kappa\phi(\zeta) - \frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\psi(\zeta)} + 2G\beta\int g'(\zeta)d\zeta \tag{4}$$

$$\sigma_{\xi\xi} + \sigma_{\eta\eta} = 2\left\{\frac{\phi'(\zeta)}{m'(\zeta)} + \frac{\overline{\phi'(\zeta)}}{\overline{m'(\zeta)}}\right\} \tag{5}$$

$$\sigma_{\xi\xi}+i\sigma_{\xi\eta}=\left\{\frac{\phi'(\zeta)}{m'(\zeta)}+\overline{\frac{\phi'(\zeta)}{m'(\zeta)}}\right\}-\left[\overline{\frac{m(\zeta)}{m'(\zeta)}}\overline{\frac{d}{d\zeta}\left\{\frac{\phi'(\zeta)}{m'(\zeta)}\right\}}+\overline{\frac{\psi'(\zeta)}{m'(\zeta)}}\right]\frac{\overline{\zeta m'(\zeta)}}{\zeta m'(\zeta)} \quad (6)$$

$$-Y+iX=\phi(\zeta)+\frac{m(\zeta)}{\overline{m'(\zeta)}}\overline{\phi'(\zeta)}+\overline{\psi(\zeta)} \quad (7)$$

where $\theta(z)=g'(z)$, and G is the shear modulus, $\kappa=3-4\nu$, $\beta=(1+\nu)\alpha$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$, $\beta=\alpha$ for plane stress with ν being the Poisson's ratio and α the thermal expansion coefficient. Here a superimposed bar represents the complex conjugate.

3 Temperature field

The complex function for a point heat source embedded in a homogeneous infinite plane can be trivially given as

$$\theta_0(\zeta)=-\frac{q_0}{2\pi k}\log(\zeta-\zeta_0) \quad (8)$$

where q_0 is the strength of the point heat source, and ζ_0 is the location of the point heat source.

The temperature function for a coated elliptic hole bonded in an infinite plane subject to a point heat source can be assumed as

$$\theta(\zeta)=\begin{cases}\sum\limits_{n=1}^{\infty}\theta_n(\zeta)+\theta_0(\zeta) & \zeta\in S_1\\ \sum\limits_{n=1}^{\infty}\theta_{an}(\zeta)+\sum\limits_{n=1}^{\infty}\theta_{bn}(\zeta) & \zeta\in S_2\end{cases} \quad (9)$$

where $\theta_n(\zeta)$ is holomorphic in region $|\zeta|\geq\rho_1$, $\theta_{an}(\zeta)$ and $\theta_{bn}(\zeta)$ are respectively holomorphic in regions $|\zeta|\leq\rho_1$ and $|\zeta|\geq\rho_2$, which can be expressed in terms of $\theta_0(\zeta)$ by the procedure as follows.

We first introduce two complex functions $\theta_1(\zeta)$ and $\theta_{a1}(\zeta)$ respectively holomorphic in $|\zeta|\geq\rho_1$ and $|\zeta|\leq\rho_1$ to satisfy the continuity conditions along the interface L_1 that

$$\theta_1(\sigma)+\theta_0(\sigma)+\overline{\theta_1(\sigma)}+\overline{\theta_0(\sigma)}=\theta_{a1}(\sigma)+\overline{\theta_{a1}(\sigma)} \quad \sigma\in L_1 \quad (10)$$

$$k_1\left[\theta_1(\sigma)+\theta_0(\sigma)-\overline{\theta_1(\sigma)}-\overline{\theta_0(\sigma)}\right]=k_2\left[\theta_{a1}(\sigma)-\overline{\theta_{a1}(\sigma)}\right] \quad \sigma\in L_1 \quad (11)$$

By analytical continuation method, we have

$$\theta_0(\zeta)+\overline{\theta_1}\left(\frac{\rho_1^2}{\zeta}\right)-\theta_{a1}(\zeta)=0 \quad |\zeta|\leq\rho_1 \quad (12)$$

$$-\overline{\theta_0}(\frac{\rho_1^2}{\zeta}) - \theta_1(\zeta) + \overline{\theta_{a1}}(\frac{\rho_1^2}{\zeta}) = 0 \quad |\zeta| \geq \rho_1 \tag{13}$$

and

$$k_1\theta_0(\zeta) - k_1\overline{\theta_1}(\frac{\rho_1^2}{\zeta}) - k_2\theta_{a1}(\zeta) = 0 \quad |\zeta| \leq \rho_1 \tag{14}$$

$$k_1\overline{\theta_0}(\frac{\rho_1^2}{\zeta}) - k_1\theta_1(\zeta) - k_2\overline{\theta_{a1}}(\frac{\rho_1^2}{\zeta}) = 0 \quad |\zeta| \geq \rho_1 \tag{15}$$

Solve Eqs. (14) and (15) to yield

$$\theta_{a1}(\zeta) = U_{21}\theta_0(\zeta) \tag{16}$$

$$\theta_1(\zeta) = V_{21}\overline{\theta_0}(\frac{\rho_1^2}{\zeta}) \tag{17}$$

where

$$U_{21} = 2k_1(k_1+k_2)^{-1}$$

$$V_{21} = (k_1-k_2)(k_1+k_2)^{-1}$$

Since $\theta_{a1}(\zeta)$ can not satisfy the boundary (adiabatic) condition at the interface L_2, the function $\theta_{b1}(\zeta)$ holomorphic in $|\zeta| \geq \rho_2$ is introduced to satisfy the boundary condition at L_2 such that

$$\theta_{a1}(\sigma) - \overline{\theta_{a1}(\sigma)} + \theta_{b1}(\sigma) - \overline{\theta_{b1}(\sigma)} = 0 \quad \sigma \in L_2 \tag{18}$$

By the same method, we have

$$\theta_{b1}(\zeta) = \overline{\theta_{a1}}(\frac{\rho_2^2}{\zeta}) \tag{19}$$

But the fields produced by $\theta_{b1}(\zeta)$ can not satisfy the continuity conditions of L_1. We again introduce another pair of functions $\theta_2(\zeta)$ and $\theta_{a2}(\zeta)$ respectively holomorphic in $|\zeta| \geq \rho_1$ and $|\zeta| \leq \rho_1$ to satisfy the continuity conditions along the interface L_1 that

$$\theta_2(\sigma) + \overline{\theta_2(\sigma)} = \theta_{b1}(\sigma) + \overline{\theta_{b1}(\sigma)} + \theta_{a2}(\sigma) + \overline{\theta_{a2}(\sigma)} \quad \sigma \in L_1 \tag{20}$$

$$k_1\left[\theta_2(\sigma) - \overline{\theta_2(\sigma)}\right] = k_2\left[\theta_{b1}(\sigma) - \overline{\theta_{b1}(\sigma)} + \theta_{a2}(\sigma) - \overline{\theta_{a2}(\sigma)}\right] \quad \sigma \in L_1 \tag{21}$$

By the same method, we have

$$\theta_{a2}(\zeta) = V_{12}\overline{\theta_{b1}}(\frac{\rho_1^2}{\zeta}) \tag{22}$$

$$\theta_2(\zeta) = U_{12}\theta_{b1}(\zeta) \tag{23}$$

Similarly, the fields produced by $\theta_{a2}(\zeta)$ can not satisfy the boundary condition of L_2. The term $\theta_{b2}(\zeta)$ holomorphic in $|\zeta| \geq \rho_2$ is introduced to satisfy the boundary condition at L_2. By the same procedure, we have

$$\theta_{b2}(\zeta) = \overline{\theta_{a2}}(\frac{\rho_2^2}{\zeta}) \tag{24}$$

Repetitions of the previous two steps are made until arriving at the results which have satisfied the continuity condition and the boundary condition. Finally, one can express the temperature function in terms of $\theta_0(\zeta)$ as follows

$$\theta(\zeta) = \begin{cases} \theta_0(\zeta) + V_{21}\overline{\theta_0}(\frac{\rho_1^2}{\zeta}) + U_{12}U_{21}\sum\limits_{n=1}^{\infty} V_{12}^{n-1}\overline{\theta_0}\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta}\right] & \zeta \in S_1 \\ U_{21}\sum\limits_{n=1}^{\infty} V_{12}^{n-1}\theta_0\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\zeta\right] + U_{21}\sum\limits_{n=1}^{\infty} V_{12}^{n-1}\overline{\theta_0}\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta}\right] & \zeta \in S_2 \end{cases} \tag{25}$$

Integration of Eq. (25) with z yields

$$g(\zeta)=\begin{cases}
\frac{-QlR}{4\pi k_1}(\zeta-\zeta_0)\left[\log(\zeta-\zeta_0)-1\right]+\frac{Ql}{4\pi k_1 R}\left[\left(\frac{1}{\zeta_0}-\frac{1}{\zeta}\right)\log(\zeta-\zeta_0)-\frac{\log\zeta}{\zeta_0}\right] \\
-\frac{U_{12}U_{21}QlR}{4\pi k_1}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\zeta-\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\overline{\zeta_0}}\right]\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right]+\frac{V_{21}QlR\rho_1^2\log\zeta}{4\pi k_1\overline{\zeta_0}} \\
+\frac{U_{12}U_{21}QlR\rho_2^2\log\zeta}{4\pi k_1\left(1-V_{12}\frac{\rho_2^2}{\rho_1^2}\right)\overline{\zeta_0}}+\frac{U_{12}U_{21}Ql}{4\pi k_1 R}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\frac{\overline{\zeta_0}}{\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\rho_2^2}-\frac{1}{\zeta}\right]\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right] \\
+\frac{V_{21}lQ}{4\pi k_1 R}\left[\frac{\overline{\zeta_0}}{\rho_1^2}-\frac{1}{\zeta}\right]\log\left[\frac{\rho_1^2}{\zeta}-\overline{\zeta_0}\right]-\frac{V_{21}QRl}{4\pi k_1}\left[\zeta-\frac{\rho_1^2}{\overline{\zeta_0}}\right]\log\left[\frac{\rho_1^2}{\zeta}-\overline{\zeta_0}\right]+\frac{lQ}{4\pi k_1 R}\frac{1}{\zeta} \\
\qquad \zeta\in S_1 \\
\\
-\frac{lRQU_{21}}{4\pi k_1}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\zeta-\frac{\zeta_0}{\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}}\right]\left\{\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\zeta-\zeta_0\right]-1\right\} \\
+\frac{U_{21}lQ}{4\pi k_1 R}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\frac{\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}}{\zeta_0}-\frac{1}{\zeta}\right]\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\zeta-\zeta_0\right]-\frac{U_{21}lQ}{4\pi k_1 R\zeta_0}\frac{\log\zeta}{1-V_{12}\frac{\rho_2^2}{\rho_1^2}} \\
+\frac{U_{21}QlR\rho_2^2\log\zeta}{4\pi k_1\left(1-V_{12}\frac{\rho_2^2}{\rho_1^2}\right)\overline{\zeta_0}}+\frac{U_{21}lQ}{4\pi k_1 R}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\frac{\overline{\zeta_0}}{\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\rho_2^2}-\frac{1}{\zeta}\right]\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right] \\
-\frac{U_{21}QlR}{4\pi k_1}\sum_{n=1}^{\infty}V_{12}^{n-1}\left[\zeta-\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\overline{\zeta_0}}\right]\log\left[\left(\frac{\rho_2^2}{\rho_1^2}\right)^{n-1}\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right]+\frac{lQ}{4\pi k_1 R}\frac{1}{\zeta} \\
\qquad \zeta\in S_2
\end{cases} \tag{26}$$

4 Stress field

For a region bounded by a circle, say $c=|\zeta|$, we introduce an auxiliary stress function $\omega(\zeta)$ such that [Chen and Chao (2008)]

$$\omega(\zeta)=\frac{\overline{m}\left(\frac{c^2}{\zeta}\right)}{m'(\zeta)}\phi'(\zeta)+\psi(\zeta) \tag{27}$$

It is worthy to note that unlike the standard Muskhelishvili complex functions $\phi(\zeta)$ and $\psi(\zeta)$, the function $\omega(\zeta)$ is dependent on the radius of the elliptical interface.

In view of Eqs. (4), (7) and (26) to satisfy the single-valued conditions of traction and displacement, the stress functions must have the form [Chao, Chen and Shen

(2007)]

$$\phi(\zeta)=\begin{cases}A\log\zeta+\phi_h(\zeta)+\sum\limits_{n=1}^{\infty}\phi_n(\zeta) & \zeta\in S_1\\ B\log\zeta+\sum\limits_{n=1}^{\infty}\phi_{an}(\zeta)+\sum\limits_{n=1}^{\infty}\phi_{bn}(\zeta) & \zeta\in S_2\end{cases}\tag{28}$$

$$\omega(\zeta)=\begin{cases}\overline{A}\log\frac{\zeta}{\rho_1^2}+\overline{B}\log\frac{\rho_1^2}{\rho_2^2}+\omega_h(\zeta)+\sum\limits_{n=1}^{\infty}\omega_n(\zeta) & \zeta\in S_1\\ \overline{B}\log\frac{\zeta}{\rho_2^2}+\sum\limits_{n=1}^{\infty}\omega_{an}(\zeta)+\sum\limits_{n=1}^{\infty}\omega_{bn}(\zeta) & \zeta\in S_2\end{cases}\tag{29}$$

where

$$A=\frac{G_1\beta_1Ql}{2(1+\kappa_1)\pi k_1}\left[\frac{1}{R\zeta_0}-\frac{V_{21}R\rho_1^2}{\overline{\zeta_0}}-\frac{U_{12}U_{21}R\rho_2^2}{(1-V_{12}\frac{\rho_2^2}{\rho_1^2})\overline{\zeta_0}}\right]$$

$$B=\frac{G_2\beta_2QlU_{21}}{2(1+\kappa_2)\pi k_1(1-V_{12}\frac{\rho_2^2}{\rho_1^2})}\left(\frac{1}{R\zeta_0}-\frac{R\rho_2^2}{\overline{\zeta_0}}\right)$$

$$\phi_h(\zeta)=\frac{G_1\beta_1Ql(R\zeta+\frac{1}{R\zeta}-R\zeta_0-\frac{1}{R\zeta_0})\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}-\frac{G_1\beta_1Ql\log(-\zeta_0)}{2\pi k_1(1+\kappa_1)R\zeta}$$

$$\omega_h(\zeta)=\frac{(\frac{R\rho_1^2}{\zeta}+\frac{\zeta}{R\rho_1^2})}{(R-\frac{1}{R\zeta^2})}\phi_h'(\zeta)-\frac{G_1\beta_1Ql(R\overline{\zeta_0}+\frac{1}{R\overline{\zeta_0}})\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}$$

The alternating technique and the analytical continuation method are applied to derive the unknown stress functions as follows.

Step 1: Analytical continuation acrossL_1

The stress functions $\phi_{a1}(\zeta)$, $\omega_{a1}(\zeta)$ holomorphic in $|\zeta|\le\rho_1$ and the stress functions $\phi_1(\zeta)$, $\omega_1(\zeta)$ holomorphic in $|\zeta|\ge\rho_1$ are introduced to satisfy the continuity conditions alongL_1 that

$$\varphi_1(\sigma)+\overline{\omega_1(\sigma)}+\varphi_h(\sigma)+\overline{\omega_h(\sigma)}=\varphi_{a1}(\sigma)+\overline{\omega_{a1}(\sigma)}\quad \sigma\in L_1\tag{30}$$

$$\begin{aligned}&\frac{\kappa_1\varphi_1(\sigma)}{2G_1}-\frac{\overline{\omega_1(\sigma)}}{2G_1}+\frac{\kappa_1\varphi_h(\sigma)}{2G_1}-\frac{\overline{\omega_h(\sigma)}}{2G_1}\\&+\frac{\beta_1lQ}{4\pi k_1R}\frac{1}{\sigma}+\beta_1g_{1a}(\sigma)-\frac{\beta_1Ql\log(-\zeta_0)}{4\pi k_1R\sigma}+\beta_1g_{1b}(\sigma)-\frac{\beta_1QlR\sigma}{4\pi k_1}\log(-\overline{\zeta_0})\\&=\frac{\kappa_2\varphi_{a1}(\sigma)}{2G_2}-\frac{\overline{\omega_{a1}(\sigma)}}{2G_2}+\frac{\beta_2lQ}{4\pi k_1R}\frac{1}{\sigma}+\beta_2g_{2a}(\sigma)-\frac{\beta_2lQ\log(-\zeta_0)}{4\pi k_1R\sigma}\end{aligned}$$

$$+\beta_2 g_{2b}(\sigma) - \frac{\beta_2 QlR\sigma}{4\pi k_1}\log(-\overline{\zeta_0}) \quad \sigma \in L_1 \tag{31}$$

Here

$$g_{1a}(\zeta) = -\frac{Ql}{4\pi k_1}(R\zeta + \frac{1}{R\zeta} - R\zeta_0 - \frac{1}{R\zeta_0})\log(\zeta - \zeta_0) + \frac{QlR(\zeta - \zeta_0)}{4\pi k_1} + \frac{Ql\log(-\zeta_0)}{4\pi k_1 R}\frac{1}{\zeta}$$

$$g_{1b}(\zeta) = \frac{U_{12}U_{21}Ql}{4\pi k_1 R}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\frac{\overline{\zeta_0}}{(\frac{\rho_2^2}{\rho_1^2})^{n-1}\rho_2^2} - \frac{1}{\zeta}\right]\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right]$$

$$+\frac{V_{21}lQ}{4\pi k_1 R}\left[\frac{\overline{\zeta_0}}{\rho_1^2} - \frac{1}{\zeta}\right]\log\left[\frac{\rho_1^2}{\zeta} - \overline{\zeta_0}\right] - \frac{V_{21}QRl}{4\pi k_1}\left[\zeta - \frac{\rho_1^2}{\overline{\zeta_0}}\right]\log\left[\frac{\rho_1^2}{\zeta} - \overline{\zeta_0}\right]$$

$$-\frac{U_{12}U_{21}QlR}{4\pi k_1}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\zeta - (\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\overline{\zeta_0}}\right]\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right] + \frac{QlR\zeta}{4\pi k_1}\log(-\overline{\zeta_0})$$

$$g_{2a}(\zeta) = -\frac{lRQU_{21}}{4\pi k_1}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\zeta - \frac{\zeta_0}{(\frac{\rho_2^2}{\rho_1^2})^{n-1}}\right]\left\{\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\zeta - \zeta_0\right] - 1\right\}$$

$$+\frac{U_{21}lQ}{4\pi k_1 R}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\frac{(\frac{\rho_2^2}{\rho_1^2})^{n-1}}{\zeta_0} - \frac{1}{\zeta}\right]\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\zeta - \zeta_0\right] + \frac{lQ\log(-\zeta_0)}{4\pi k_1 R\zeta}$$

$$g_{2b}(\zeta) = \frac{U_{21}lQ}{4\pi k_1 R}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\frac{\overline{\zeta_0}}{(\frac{\rho_2^2}{\rho_1^2})^{n-1}\rho_2^2} - \frac{1}{\zeta}\right]\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right]$$

$$-\frac{U_{21}QlR}{4\pi k_1}\sum_{n=1}^{\infty} V_{12}^{n-1}\left[\zeta - (\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\overline{\zeta_0}}\right]\log\left[(\frac{\rho_2^2}{\rho_1^2})^{n-1}\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right] + \frac{QlR\zeta}{4\pi k_1}\log(-\overline{\zeta_0})$$

By the standard analytical continuation arguments, it follows that

$$\varphi_1(\zeta) + \overline{\omega_h}(\frac{\rho_1^2}{\zeta}) - \overline{\omega_{a1}}(\frac{\rho_1^2}{\zeta}) + (a_{01} + a_{11})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R + \frac{\zeta}{\rho_1^2}} + (a_{02} + a_{12})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R - \frac{\zeta}{\rho_1^2}} = 0 \quad \zeta \in S_1 \tag{32}$$

$$\phi_{a1}(\zeta)-\phi_h(\zeta)-\overline{\omega_1}(\frac{\rho_1^2}{\zeta})$$
$$+(a_{01}+a_{11})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}+(a_{02}+a_{12})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_2 \quad (33)$$

$$\frac{\kappa_1}{2G_1}\phi_1(\zeta)-\frac{1}{2G_1}\overline{\omega_h}(\frac{\rho_1^2}{\zeta})+\frac{1}{2G_2}\overline{\omega_{a1}}(\frac{\rho_1^2}{\zeta})+\frac{(\beta_2-\beta_1)lQ\log(-\zeta_0)}{4\pi k_1R\zeta}+\frac{(\beta_1-\beta_2)lQ}{4\pi k_1R\zeta}$$
$$-\beta_2g_{2b}(\zeta)+\beta_1g_{1b}(\zeta)-\frac{(a_{01}+a_{11})}{2G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}$$
$$-\frac{(a_{02}+a_{12})}{2G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_1 \quad (34)$$

$$\frac{\kappa_2}{2G_2}\varphi_{a1}(\zeta)-\frac{\kappa_1}{2G_1}\varphi_h(\zeta)+\frac{1}{2G_1}\overline{\omega_1}(\frac{\rho_1^2}{\zeta})+\frac{(\beta_1-\beta_2)QlR\zeta}{4\pi k_1}\log(-\overline{\zeta_0})$$
$$-\beta_1g_{1a}(\zeta)+\beta_2g_{2a}(\zeta)-\frac{(a_{01}+a_{11})}{2G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}$$
$$-\frac{(a_{02}+a_{12})}{2G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_2 \quad (35)$$

where

$$a_{01}=\frac{1}{2}\overline{\phi_h'}(-\frac{1}{R}),\quad a_{02}=-\frac{1}{2}\overline{\phi_h'}(\frac{1}{R}),\quad a_{11}=-\frac{1}{2}\overline{\phi_{a1}'}(-\frac{1}{R}),\quad a_{12}=\frac{1}{2}\overline{\phi_{a1}'}(\frac{1}{R})$$

Solve Eqs. (32)-(34) to yield

$$\varphi_1(\zeta)=\Pi_{21}\overline{\omega_h}(\frac{\rho_1^2}{\zeta})+\frac{G_1G_2lQ(\beta_1-\beta_2)\left[\log(-\zeta_0)-1\right]}{2(\kappa_1G_2+G_1)\pi k_1R\zeta}$$
$$+\frac{2G_1G_2}{(\kappa_1G_2+G_1)}\left[\beta_2g_{2b}(\zeta)-\beta_1g_{1b}(\zeta)\right] \quad (36)$$

$$\phi_{a1}(\zeta) = (1+\Lambda_{21})\phi_h(\zeta) + \frac{(\beta_2-\beta_1)G_1G_2QlR\zeta}{2(\kappa_2G_1+G_2)\pi k_1}\log(-\overline{\zeta_0}) + \frac{2G_1G_2}{(\kappa_2G_1+G_2)}\left[\beta_1 g_{1a}(\zeta)-\beta_2 g_{2a}(\zeta)\right] + \Pi_{12}\left[(a_{01}+a_{11})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}} + (a_{02}+a_{12})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}\right] \quad (37)$$

$$\omega_1(\zeta) = \Lambda_{21}\overline{\phi_h}(\frac{\rho_1^2}{\zeta}) + \frac{2G_1G_2}{(\kappa_2G_1+G_2)}\left[\beta_1\overline{g_{1a}}(\frac{\rho_1^2}{\zeta})-\beta_2\overline{g_{2a}}(\frac{\rho_1^2}{\zeta})\right] + \frac{(\beta_2-\beta_1)G_1G_2QlR\rho_1^2}{2(\kappa_2G_1+G_2)\pi k_1\zeta}\log(-\zeta_0) + (1+\Pi_{12})\left[(\overline{a_{01}}+\overline{a_{11}})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}} + (\overline{a_{02}}+\overline{a_{12}})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}}\right] \quad (38)$$

$$\omega_{a1}(\zeta) = (1+\Pi_{21})\omega_h(\zeta) + (\overline{a_{01}}+\overline{a_{11}})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}} + (\overline{a_{02}}+\overline{a_{12}})\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}} + \frac{G_1G_2(\beta_1-\beta_2)lQ\left[\log(-\bar{\zeta}_0)-1\right]\zeta}{2\pi k_1(\kappa_1G_2+G_1)R\rho_1^2} + \frac{2G_1G_2}{(\kappa_1G_2+G_1)}\left[\beta_2\overline{g_{2b}}(\frac{\rho_1^2}{\zeta})-\beta_1\overline{g_{1b}}(\frac{\rho_1^2}{\zeta})\right] \quad (39)$$

where

$$\Pi_{12} = \frac{G_1-G_2}{\kappa_2G_1+G_2}, \quad \Lambda_{12} = \frac{G_1\kappa_2-G_2\kappa_1}{G_1+G_2\kappa_1}.$$

Step 2: Analytical continuation across L_2

Since the stress functions $\phi_{a1}(\zeta)$ and $\omega_{a1}(\zeta)$ can not satisfy the boundary condition at L_2, additional terms $\phi_{b1}(\zeta)$, $\omega_{b1}(\zeta)$ holomorphic in $|\zeta| \geq \rho_2$are introduced to satisfy the traction-free condition along L_2 that

$$\phi_{a1}(\sigma)+\overline{\omega_{a1}^{II}(\sigma)}+\phi_{b1}(\sigma)+\overline{\omega_{b1}(\sigma)} = 0 \qquad \sigma\in L_2 \quad (40)$$

where

$$\omega_{a1}^{II}(\zeta) = \omega_{a1}(\zeta) + \left[\frac{\frac{R(\rho_2^2-\rho_1^2)}{\zeta}+\frac{\zeta}{R}(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2})}{R-\frac{1}{R\zeta^2}}\right]\phi_{a1}'(\zeta)$$

By the analytical continuation method, we have

$$\phi_{a1}(\zeta)+a_{11}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R+\frac{\zeta}{\rho_2^2}}+a_{12}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R-\frac{\zeta}{\rho_2^2}}+\overline{\omega_{b1}}(\frac{\rho_2^2}{\zeta})=0 \tag{41}$$

$$\overline{\omega_{a1}^{II}}(\frac{\rho_2^2}{\zeta})-a_{11}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R+\frac{\zeta}{\rho_2^2}}-a_{12}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R-\frac{\zeta}{\rho_2^2}}+\phi_{b1}(\zeta)=0 \tag{42}$$

$$\phi_{b1}(\zeta)=-\overline{\omega_{a1}^{II}}(\frac{\rho_2^2}{\zeta})+a_{11}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R+\frac{\zeta}{\rho_2^2}}+a_{12}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R-\frac{\zeta}{\rho_2^2}} \tag{43}$$

$$\omega_{b1}(\zeta)=-\overline{\phi_{a1}}(\frac{\rho_2^2}{\zeta})-\overline{a_{11}}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R+\frac{1}{\zeta}}-\overline{a_{12}}\frac{R^2\rho_2^2+\frac{1}{R^2\rho_2^2}}{R-\frac{1}{\zeta}} \tag{44}$$

Step 3: Analytical continuation across L_1

Since the stress functions $\phi_{b1}(\zeta)$ and $\omega_{b1}(\zeta)$ can not satisfy the continuity conditions at L_1, additional terms $\phi_{a2}(\zeta)$, $\omega_{a2}(\zeta)$ and $\phi_2(\zeta)$,$\omega_2(\zeta)$ respectively holomorphic in $|\zeta|\leq\rho_1$ and $|\zeta|\geq\rho_1$ are introduced to satisfy the continuity conditions along L_1 that

$$\phi_2(\sigma)+\overline{\omega_2(\sigma)}=\phi_{b1}(\sigma)+\overline{\omega_{b1}^I(\sigma)}+\phi_{a2}(\sigma)+\overline{\omega_{a2}(\sigma)} \quad \sigma\in L_1 \tag{45}$$

$$\frac{\kappa_1}{G_1}\phi_2(\sigma)-\frac{1}{G_1}\overline{\omega_2(\sigma)}=\frac{\kappa_2}{G_2}\phi_{b1}(\sigma)-\frac{1}{G_2}\overline{\omega_{b1}^I(\sigma)}+\frac{\kappa_2}{G_2}\phi_{a2}(\sigma)-\frac{1}{G_2}\overline{\omega_{a2}(\sigma)} \quad \sigma\in L_1 \tag{46}$$

where

$$\omega_{b1}^I(\zeta)=\omega_{b1}(\zeta)+\frac{\frac{R(\rho_1^2-\rho_2^2)}{\zeta}+\frac{\zeta}{R}(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2})}{R-\frac{1}{R\zeta^2}}\phi_{b1}'(\zeta)$$

By the analytical continuation method, we have

$$\phi_2(\zeta)-\phi_{b1}(\zeta)-\overline{\omega_{a2}}(\frac{\rho_1^2}{\zeta})+a_{21}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}+a_{22}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_1 \tag{47}$$

$$\phi_{a2}(\zeta)+\overline{\omega_{b1}^{I}}(\frac{\rho_1^2}{\zeta})-\overline{\omega_2}(\frac{\rho_1^2}{\zeta})$$
$$+a_{21}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}+a_{22}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_2 \quad (48)$$

$$\frac{\kappa_1}{G_1}\phi_2(\zeta)-\frac{\kappa_2}{G_2}\phi_{b1}(\zeta)+\frac{1}{G_2}\overline{\omega_{a2}}(\frac{\rho_1^2}{\zeta})$$
$$-\frac{a_{11}}{G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}-\frac{a_{12}}{G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_1 \quad (49)$$

$$\frac{\kappa_2}{G_2}\phi_{a2}(\zeta)-\frac{1}{G_2}\overline{\omega_{b1}^{I}}(\frac{\rho_1^2}{\zeta})+\frac{1}{G_1}\overline{\omega_2}(\frac{\rho_1^2}{\zeta})$$
$$-\frac{a_{11}}{G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}-\frac{a_{12}}{G_2}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}=0 \quad \zeta\in S_2 \quad (50)$$

where

$$a_{21}=-\frac{1}{2}\overline{\phi_{a2}'}(-\frac{1}{R}), \quad a_{22}=\frac{1}{2}\overline{\phi_{a2}'}(\frac{1}{R}).$$

Solve Eqs. (47)-(50) to yield

$$\phi_2(\zeta)=(1+\Lambda_{12})\phi_{b1}(\zeta) \quad \zeta\in S_1 \quad (51)$$

$$\omega_2(\zeta)=(1+\Pi_{12})\omega_{b1}^{I}(\zeta)+(1+\Pi_{12})(\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}}\overline{a_{21}}+\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}}\overline{a_{22}}) \quad (52)$$

$$\phi_{a2}(\zeta)=\Pi_{12}\overline{\omega_{b1}^{I}}(\frac{\rho_1^2}{\zeta})+\Pi_{12}(a_{21}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}}+a_{22}\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}) \quad (53)$$

$$\omega_{a2}(\zeta)=\Lambda_{12}\overline{\phi_{b1}}(\frac{\rho_1^2}{\zeta})+\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}}\overline{a_{21}}+\frac{R^2\rho_1^2+\frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}}\overline{a_{22}} \quad (54)$$

Repetitions of the previous two steps are made until arriving at the results which have satisfied the continuity conditions and the boundary condition. Finally, one can express all the functions $\phi_n(\zeta)$, $\omega_n(\zeta)$, $\phi_{an}(\zeta)$, $\omega_{an}(\zeta)$, $\phi_{bn}(\zeta)$ and $\omega_{bn}(\zeta)(n = 2,3,4\cdots)$ in terms of $\phi_{a1}(\zeta)$ and $\phi_{b1}(\zeta)$ as follows

$$
\begin{aligned}
\phi_{an}(\zeta) =& -\Pi_{12}\phi_{a(n-1)}\left[(\frac{\rho_2^2}{\rho_1^2}\zeta)\right] + \Pi_{12}\overline{t_{21}}(\frac{\rho_1^2}{\zeta})\overline{\phi'_{b(n-1)}}(\frac{\rho_1^2}{\zeta}) \\
&+ \Pi_{12}\left[S^I(\zeta)a_{n1} - S^{II}(\frac{\rho_2^2}{\rho_1^2}\zeta)a_{(n-1)1} + R^I(\zeta)a_{n2} - R^{II}(\frac{\rho_2^2}{\rho_1^2}\zeta)a_{(n-1)2}\right] \\
\omega_{an}(\zeta) =& \Lambda_{12}\overline{\phi_{b(n-1)}}(\frac{\rho_1^2}{\zeta}) + \overline{S^I}(\frac{\rho_1^2}{\zeta})\overline{a_{n1}} + \overline{R^I}(\frac{\rho_1^2}{\zeta})\overline{a_{n2}} \\
\phi_n(\zeta) =& (1+\Lambda_{12})\phi_{b(n-1)}(\zeta) \\
\omega_n(\zeta) =& -(1+\Pi_{12})\overline{\phi_{a(n-1)}}(\frac{\rho_2^2}{\zeta}) + (1+\Pi_{12})t_{21}(\zeta)\phi'_{b(n-1)}(\zeta) \\
&+ (1+\Pi_{12})\left[\overline{S^I}(\frac{\rho_1^2}{\zeta})\overline{a_{n1}} - \overline{S^{II}}(\frac{\rho_2^2}{\zeta})\overline{a_{(n-1)1}} + \overline{R^I}(\frac{\rho_1^2}{\zeta})\overline{a_{n2}} - \overline{R^{II}}(\frac{\rho_2^2}{\zeta})\overline{a_{(n-1)2}}\right]
\end{aligned}
$$

for $n = 2,3,4\ldots$ (55)

$$
\begin{aligned}
\varphi_{bn}(\zeta) =& -\Lambda_{12}\varphi_{b(n-1)}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right] + \Pi_{12}(\frac{\rho_2}{\rho_1})^2\overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{\varphi'_{a(n-1)}}\left[(\frac{\rho_2}{\rho_1})^2\frac{\rho_2^2}{\zeta}\right] \\
&+ \Pi_{12}\frac{\rho_1^2\zeta^2}{\rho_2^4}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})t'_{21}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right]\varphi'_{b(n-1)}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right] \\
&+ \Pi_{12}\frac{\rho_1^2\zeta^2}{\rho_2^4}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})t_{21}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right]\varphi''_{b(n-1)}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right] \\
&+ a_{n1}\left[S^{II}(\zeta) - S^I(\frac{\rho_1^2}{\rho_2^2}\zeta)\right] + a_{n2}\left[R^{II}(\zeta) - R^I(\frac{\rho_1^2}{\rho_2^2}\zeta)\right] \\
&+ \Pi_{12}\overline{a_{(n-1)1}}\frac{\rho_2^2}{\rho_1^2}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{S^{II'}}(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta}) \\
&+ \Pi_{12}\overline{a_{(n-1)2}}\frac{\rho_2^2}{\rho_1^2}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{R^{II'}}(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta}) - \Pi_{12}\overline{a_{n1}}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{S^{I'}}(\frac{\rho_2^2}{\zeta}) \\
&- \Pi_{12}\overline{a_{n2}}\overline{t_{12}}(\frac{\rho_2^2}{\zeta})\overline{R^{I'}}(\frac{\rho_2^2}{\zeta}) \\
\omega_{bn}(\zeta) =& \Pi_{12}\overline{\varphi_{a(n-1)}}\left[(\frac{\rho_2}{\rho_1})^2\frac{\rho_2^2}{\zeta}\right] - \Pi_{12}t_{21}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right]\varphi'_{b(n-1)}\left[(\frac{\rho_1}{\rho_2})^2\zeta\right] \\
&- \overline{a_{n1}}\overline{S^{II}}(\frac{\rho_2^2}{\zeta}) - \overline{a_{n2}}\overline{R^{II}}(\frac{\rho_2^2}{\zeta}) \\
&- \Pi_{12}\left[\overline{S^I}(\frac{\rho_2^2}{\zeta})\overline{a_{n1}} - \overline{S^{II}}(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta})\overline{a_{(n-1)1}} + \overline{R^I}(\frac{\rho_2^2}{\zeta})\overline{a_{n2}} - \overline{R^{II}}(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta})\overline{a_{(n-1)2}}\right]
\end{aligned}
$$

for $n = 2, 3, 4\ldots$ (56)

where

$$S^I(\zeta) = \frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R + \frac{\zeta}{\rho_1^2}}, \quad R^I(\zeta) = \frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R - \frac{\zeta}{\rho_1^2}}$$

$$S^{II}(\zeta) = \frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R + \frac{\zeta}{\rho_2^2}}, \quad R^{II}(\zeta) = \frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R - \frac{\zeta}{\rho_2^2}}$$

$$t_{ij}(\zeta) = \frac{\frac{R(\rho_j^2-\rho_i^2)}{\zeta} + \frac{\zeta}{R}(\frac{1}{\rho_j^2} - \frac{1}{\rho_i^2})}{R - \frac{1}{R\zeta^2}} \quad i = 1,2; \quad j = 1,2$$

$$a_{n1} = -\frac{1}{2}\overline{\phi'_{an}}(-\frac{1}{R}), \quad a_{n2} = \frac{1}{2}\overline{\phi'_{an}}(\frac{1}{R})$$

For the special case when the regions S_1 and S_2 are made of the same material and $a_2 = b_2$ for the corresponding circle hole problem, Eqs. (28) and (29) can be simplified to an exact form as

$$\begin{aligned}\varphi(\zeta) =& \frac{G_1\beta_1 QRl(\zeta-\zeta_0)\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)} - \frac{\beta_1 G_1 QRl(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}\log(\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}) \\ &+ \frac{G_1\beta_1 QRl\zeta}{2\pi k_1(1+\kappa_1)}\log(-\overline{\zeta_0}) - \frac{G_1\beta_1 Ql}{2(1+\kappa_1)\pi k_1}\frac{R\rho_2^2}{\overline{\zeta_0}}\log\zeta\end{aligned} \tag{57}$$

$$\begin{aligned}\psi(\zeta) =& -\frac{G_1\beta_1 QRl}{2\pi k_1(1+\kappa_1)}\left\{\left(\frac{\rho_2^2}{\zeta_0}\log\zeta - \frac{1}{\overline{\zeta_0}}\frac{\rho_2^4}{\zeta^2}\right) + \overline{\zeta_0}\left[\log(\zeta-\zeta_0) - \log(\frac{\rho_2^2}{\zeta} - \overline{\zeta_0})\right]\right. \\ &\left. + \frac{\rho_2^2}{\zeta}\left[1 + \log(-\zeta_0) + \log(-\overline{\zeta_0})\right] + \frac{\rho_2^2}{\zeta}(\zeta-\zeta_0)\frac{\frac{\rho_2^2}{\zeta^2}}{\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}}\right\}\end{aligned} \tag{58}$$

which are found to be the same as the results provided by [Chao, Chen and Shen (2006)].

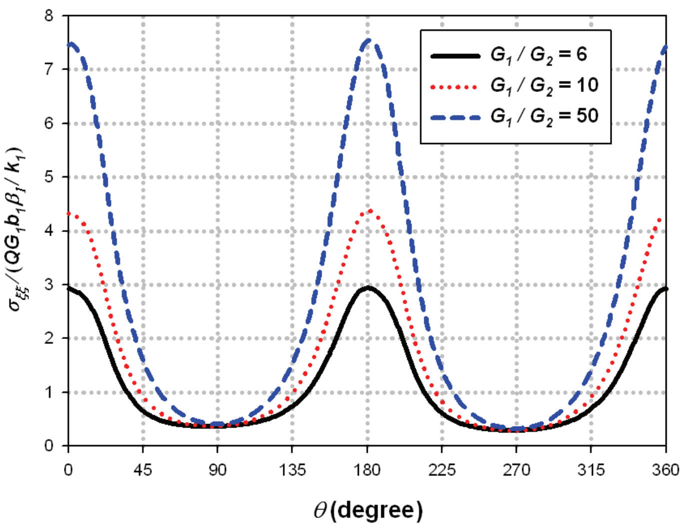

Figure 3: Angular variations of the interfacial normal stress for different shear modulus ratios $(a_1/b_1 = 1.5, a_2/a_1 = 0.9, \beta_1/\beta_2 = 2, k_1/k_2 = 2, h/a_2 = 3, \nu_1 = \nu_2 = 0.3)$.

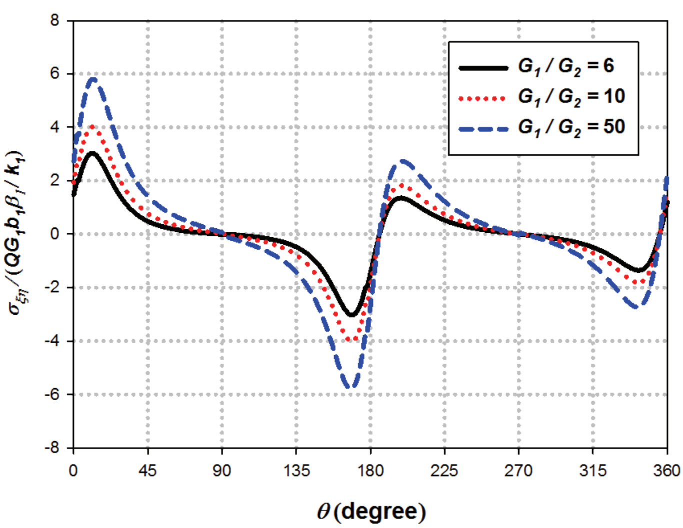

Figure 4: Angular variations of the interfacial shear stress for different shear modulus ratios $(a_1/b_1 = 1.5, a_2/a_1 = 0.9, \beta_1/\beta_2 = 2, k_1/k_2 = 2, h/a_2 = 3, \nu_1 = \nu_2 = 0.3)$.

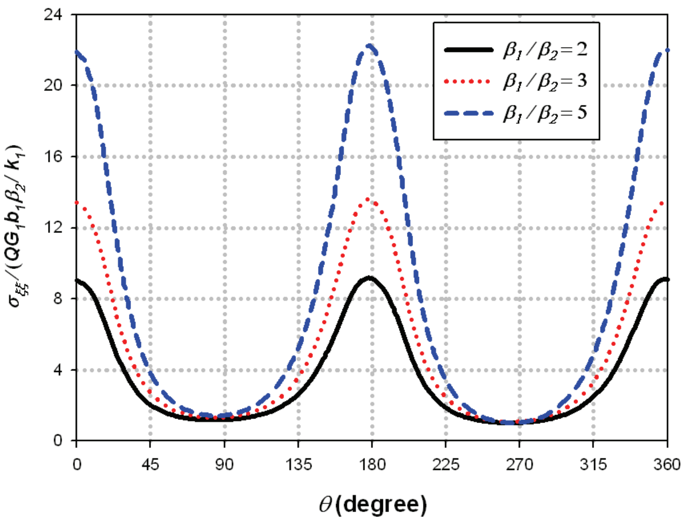

Figure 5: Angular variations of the interfacial normal stress for different thermal modulus ratios $(a_1/b_1 = 1.5, a_2/a_1 = 0.9, k_1/k_2 = 2, h/a_2 = 3, G_1/G_2 = 10, \nu_1 = \nu_2 = 0.3)$.

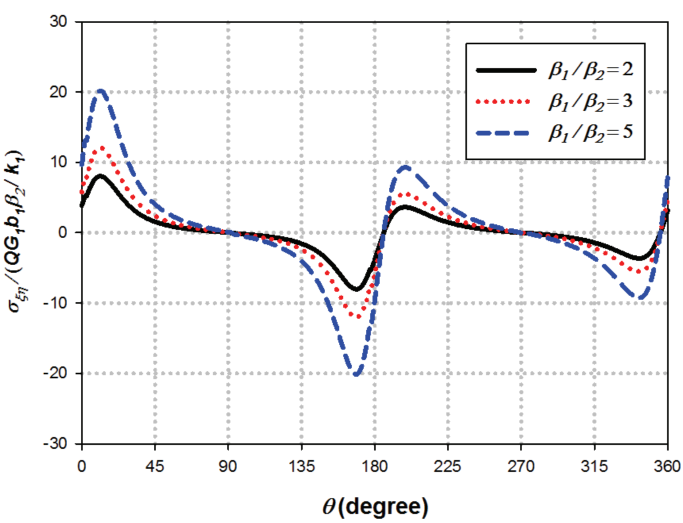

Figure 6: Angular variations of the interfacial shear stress for different thermal modulus ratios $(a_1/b_1 = 1.5, a_2/a_1 = 0.9, k_1/k_2 = 2, h/a_2 = 3, G_1/G_2 = 10, \nu_1 = \nu_2 = 0.3)$

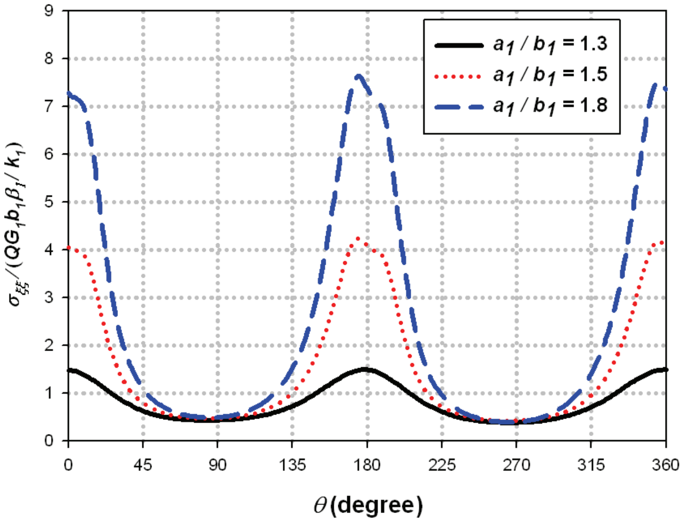

Figure 7: Angular variations of the interfacial normal stress for different aspect ratios $(a_2/a_1 = 0.9, k_1/k_2 = 2, h/a_2 = 3, G_1/G_2 = 10, \nu_1 = \nu_2 = 0.3)$

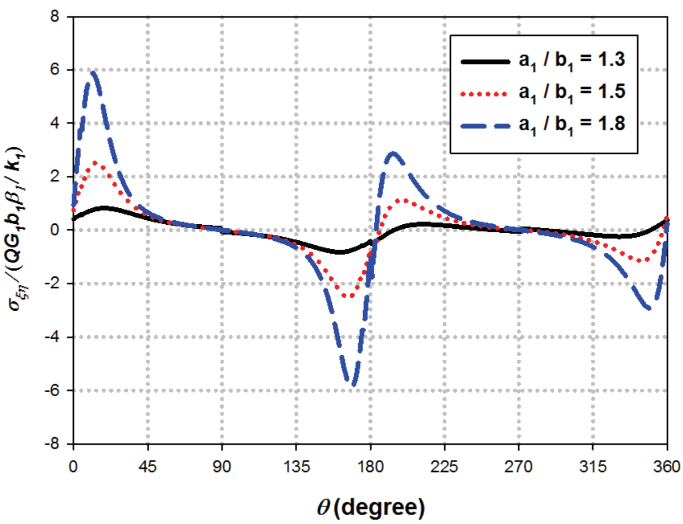

Figure 8: Angular variations of the interfacial shear stress for different aspect ratios $(a_2/a_1 = 0.9, k_1/k_2 = 2, h/a_2 = 3, G_1/G_2 = 10, \nu_1 = \nu_2 = 0.3)$

5 Results and discussion

The stress functions $\phi_{an}(z)$, $\phi_{bn}(z)$, $\varphi_n(z)$, $\omega_{an}(z)$, $\omega_{bn}(z)$ and $\omega_n(z)$ (n=2, 3, 4...) in Eqs. (28) and (29) can be calculated from $\phi_{a1}(z)$ and $\phi_{b1}(z)$ by the recurrence formulae (55) and (56). The rate of the convergence depends on two non-dimensional bimaterial constants Λ_{12} and Π_{12}. For most combinations of materials, Λ_{12} and Π_{12} are less than 1 and 0.5, respectively, which guarantees rapid convergence. In the following discussion, a point heat source is assumed to situate at the point along the positive y-axis ($z_0 = ih$). The angular variations of the interfacial normal stress and interfacial shear stress between the coated layer and the matrix for different shear modulus ratios are shown in Figures 3 and 4 respectively. As expected, the interfacial normal stress is symmetric about y-axis and is asymmetric about x-axis while the interfacial shear stress is symmetric about x-axis and is asymmetric about y-axis. Both the magnitude of the normal stress and shear stress increase with the difference of the shear moduli G_1 and G_2, but the trend is getting slighter for larger difference of ratio of G_1 and G_2. Figures 5 and 6 respectively show the angular variations of the interfacial normal stress and interfacial shear stress for different thermal modulus ratios. It is seen that both the interfacial normal and shear stresses are strongly dependent on the thermal moduli of the neighboring materials. One can find the interfacial stresses monotonously increase with larger difference of the thermal moduli of the neighboring materials. Figure 7 and Figure 8 show the angular variations of the interfacial normal and shear stresses, respectively between the coated layer and the matrix for different aspect ratios. What is of particular interest is the maximum interfacial shear stress, which is often responsible for debonding of the coated layer from the matrix. This figure clearly shows that the maximum interfacial shear stress increases with the aspect ratio. Note that all the calculated results are determined by summing up to $n = 10$ in Eq. (55) and Eq. (56), since they are checked to achieve a good accuracy for the current problem. A good accuracy for the current problem can be demonstrated by the contribution of the leading terms appearing in Eq. (28) and Eq. (29). The contribution of the stresses for the leading terms of a series solution is 35.03%, 15.14%, 5.35% and 0.13%, respectively (see Table 1). The contribution accounts for the ratio of each term to the summation of the first ten terms of a series solution. The leading ten terms have over 99% contribution, making the series solution rapidly convergent. This demonstrates the accuracy and the efficiency of our proposed method. Note that the convergence rate depends on the combinations of material properties and geometric configurations. In general, the convergence rate becomes more rapid if the differences of the elastic constants of the neighboring materials get smaller and the ratio a_1/b_1 (or a_2/b_2) approaches one.

Table 1: Contribution of the leading terms n =1∼10 for σ_{nn} and σ_{nt}

Term (n)	Error (%)
1	35.03
3	15.14
5	5.35
10	0.13

6 Conclusion

An analytical solution to the problem of a coated elliptical hole embedded in an unbounded matrix subjected to a point heat source is provided in this paper. Note that the conventional Laurent series expansion technique is unable to avoid solving a system of simultaneous equations for a large number of unknown constants for the current problem. Based on the method of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the temperature and elastic fields are obtained as a transformation on the solution to the corresponding homogeneous solution. It is found that the thermal modulus of the confocal ring has a strong effect on the interfacial thermal stresses of the current problem. The present proposed method can also be extended to solve the corresponding elliptical inclusion problem with any number of layered media. This will be left for our future study.

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Appendix

When S_1 and S_2 are the same, we have $U_{12} = 1,\ V_{12} = \Pi_{12} = \Lambda_{12} = 0$ and Eqs. (28) and (29) can be simplified as

$$\phi(\zeta) = \begin{cases} A\log\zeta + \phi_h(\zeta) + \sum\limits_{n=1}^{\infty}\phi_n(\zeta) & \zeta \in S_1 \\ B\log\zeta + \sum\limits_{n=1}^{\infty}\phi_{an}(\zeta) + \sum\limits_{n=1}^{\infty}\phi_{bn}(\zeta) & \zeta \in S_2 \end{cases}$$

$$\omega(\zeta)=\begin{cases}\overline{A}\log\frac{\zeta}{\rho_1^2}+\overline{B}\log\frac{\rho_1^2}{\rho_2^2}+\omega_h(\zeta)+\sum\limits_{n=1}^{\infty}\omega_n(\zeta) & \zeta\in S_1\\ \overline{B}\log\frac{\zeta}{\rho_2^2}+\sum\limits_{n=1}^{\infty}\omega_{an}(\zeta)+\sum\limits_{n=1}^{\infty}\omega_{bn}(\zeta) & \zeta\in S_2\end{cases}$$

where

$$\begin{aligned}A=&\frac{G_1\beta_1Ql}{2(1+\kappa_1)\pi k_1}\left[\frac{1}{R\zeta_0}-\frac{V_{21}R\rho_1^2}{\overline{\zeta_0}}-\frac{U_{12}U_{21}R\rho_2^2}{(1-V_{12}\frac{\rho_2^2}{\rho_1^2})\overline{\zeta_0}}\right]\\ =&\frac{G_1\beta_1Ql}{2(1+\kappa_1)\pi k_1}\left[\frac{1}{R\zeta_0}-\frac{R\rho_2^2}{\overline{\zeta_0}}\right]\end{aligned}$$

$$B=\frac{G_2\beta_2QlU_{21}}{2(1+\kappa_2)\pi k_1(1-V_{12}\frac{\rho_2^2}{\rho_1^2})}\left(\frac{1}{R\zeta_0}-\frac{R\rho_2^2}{\overline{\zeta_0}}\right)=\frac{G_1\beta_1Ql}{2(1+\kappa_1)\pi k_1}\left(\frac{1}{R\zeta_0}-\frac{R\rho_2^2}{\overline{\zeta_0}}\right)$$

$$\varphi_h(\zeta)=\frac{G_1\beta_1Ql(R\zeta+\frac{1}{R\zeta}-R\zeta_0-\frac{1}{R\zeta_0})\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}-\frac{G_1\beta_1Ql\log(-\zeta_0)}{2\pi k_1(1+\kappa_1)R\zeta}$$

$$\omega_h(\zeta)=\frac{(\frac{R\rho_1^2}{\zeta}+\frac{\zeta}{R\rho_1^2})}{(R-\frac{1}{R\zeta^2})}\phi_h'(\zeta)-\frac{G_1\beta_1Ql(R\overline{\zeta_0}+\frac{1}{R\overline{\zeta_0}})\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}$$

$$\begin{aligned}g_{1a}(\zeta)=&-\frac{Ql}{4\pi k_1}(R\zeta+\frac{1}{R\zeta}-R\zeta_0-\frac{1}{R\zeta_0})\log(\zeta-\zeta_0)\\ &+\frac{QlR(\zeta-\zeta_0)}{4\pi k_1}+\frac{Ql\log(-\zeta_0)}{4\pi k_1R}\frac{1}{\zeta}\end{aligned}$$

$$\begin{aligned}g_{1b}(\zeta)=&\frac{Ql}{4\pi k_1R}\left[\frac{\overline{\zeta_0}}{\rho_2^2}-\frac{1}{\zeta}\right]\log\left[\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right]\\ &-\frac{QlR}{4\pi k_1}\left[\zeta-\frac{\rho_2^2}{\overline{\zeta_0}}\right]\log\left[\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}\right]+\frac{QlR\zeta}{4\pi k_1}\log(-\overline{\zeta_0})\end{aligned}$$

$$\begin{aligned}g_{2a}(\zeta)=&-\frac{lRQ}{4\pi k_1}(\zeta-\zeta_0)\left[\log(\zeta-\zeta_0)-1\right]\\ &+\frac{lQ}{4\pi k_1R}\left[\frac{1}{\zeta_0}-\frac{1}{\zeta}\right]\log\left[\zeta-\zeta_0\right]+\frac{lQ\log(-\zeta_0)}{4\pi k_1R\zeta}\end{aligned}$$

$$g_{2b}(\zeta) = \frac{lQ}{4\pi k_1 R}\left(\frac{\overline{\zeta_0}}{\rho_2^2} - \frac{1}{\zeta}\right)\log\left(\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right) - \frac{QlR}{4\pi k_1}\left(\zeta - \frac{\rho_2^2}{\overline{\zeta_0}}\right)\log\left(\frac{\rho_2^2}{\zeta} - \overline{\zeta_0}\right) + \frac{QlR\zeta}{4\pi k_1}\log(-\overline{\zeta_0})$$

$$\varphi_1(\zeta) = \Pi_{21}\overline{\omega_h}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{G_1G_2lQ(\beta_1-\beta_2)\left[\log(-\zeta_0)-1\right]}{2(\kappa_1G_2+G_1)\pi k_1 R\zeta} + \frac{2G_1G_2}{(\kappa_1G_2+G_1)}\left[\beta_2 g_{2b}(\zeta) - \beta_1 g_{1b}(\zeta)\right] = 0$$

$$a_{01} = \frac{1}{2}\overline{\phi_h'}\left(-\frac{1}{R}\right), \quad a_{02} = -\frac{1}{2}\overline{\phi_h'}\left(\frac{1}{R}\right)$$

$$a_{11} = -\frac{1}{2}\overline{\phi_{a1}'}\left(-\frac{1}{R}\right) = -\frac{1}{2}\overline{\phi_h'}\left(-\frac{1}{R}\right) = -a_{01}$$

$$a_{12} = \frac{1}{2}\overline{\phi_{a1}'}\left(\frac{1}{R}\right) = \frac{1}{2}\overline{\phi_h'}\left(\frac{1}{R}\right) = -a_{02}$$

$$\phi_{a1}(\zeta) = (1+\Lambda_{21})\phi_h(\zeta) + \frac{(\beta_2-\beta_1)G_1G_2QlR\zeta}{2(\kappa_2G_1+G_2)\pi k_1}\log(-\overline{\zeta_0}) + \frac{2G_1G_2}{(\kappa_2G_1+G_2)}\left[\beta_1 g_{1a}(\zeta) - \beta_2 g_{2a}(\zeta)\right] + \Pi_{12}\left[(a_{01}+a_{11})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R+\frac{\zeta}{\rho_1^2}} + (a_{02}+a_{12})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R-\frac{\zeta}{\rho_1^2}}\right] = \varphi_h(\zeta)$$

$$\omega_1(\zeta) = \Lambda_{21}\overline{\phi_h}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{2G_1G_2}{(\kappa_2G_1+G_2)}\left[\beta_1\overline{g_{1a}}\left(\frac{\rho_1^2}{\zeta}\right) - \beta_2\overline{g_{2a}}\left(\frac{\rho_1^2}{\zeta}\right)\right] + \frac{(\beta_2-\beta_1)G_1G_2QlR\rho_1^2}{2(\kappa_2G_1+G_2)\pi k_1\zeta}\log(-\zeta_0) + (1+\Pi_{12})\left[(\overline{a_{01}}+\overline{a_{11}})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}} + (\overline{a_{02}}+\overline{a_{12}})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}}\right] = 0$$

$$\omega_{a1}(\zeta) = (1+\Pi_{21})\omega_h(\zeta) + (\overline{a_{01}} + \overline{a_{11}})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R+\frac{1}{\zeta}}$$

$$+(\overline{a_{02}} + \overline{a_{12}})\frac{R^2\rho_1^2 + \frac{1}{R^2\rho_1^2}}{R-\frac{1}{\zeta}} + \frac{G_1G_2(\beta_2-\beta_1)lQ\,[\log(-\zeta_0)-1]\,\zeta}{2\pi k_1(\kappa_1 G_2+G_1)R\rho_1^2}$$

$$+\frac{2G_1G_2}{(\kappa_1G_2+G_1)}\left[\beta_2\overline{g_{2b}}(\frac{\rho_1^2}{\zeta}) - \beta_1\overline{g_{1b}}(\frac{\rho_1^2}{\zeta})\right] = \omega_h(\zeta)$$

$$\omega_{a1}^{II}(\zeta) = \omega_{a1}(\zeta) + \left[\frac{\frac{R(\rho_2^2-\rho_1^2)}{\zeta} + \frac{\zeta}{R}(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2})}{R-\frac{1}{R\zeta^2}}\right]\phi'_{a1}(\zeta) = \psi_h(\zeta) + \frac{\frac{R\rho_2^2}{\zeta} + \frac{\zeta}{R\rho_2^2}}{R-\frac{1}{R\zeta^2}}\phi'_h(\zeta)$$

$$\phi_{b1}(\zeta) = -\overline{\omega_{a1}^{II}}(\frac{\rho_2^2}{\zeta}) + a_{11}\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R+\frac{\zeta}{\rho_2^2}} + a_{12}\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R-\frac{\zeta}{\rho_2^2}}$$

$$= -\overline{\omega_{a1}^{II}}(\frac{\rho_2^2}{\zeta}) + \left[a_{11}(R-\frac{\zeta}{\rho_2^2}) + a_{12}(R+\frac{\zeta}{\rho_2^2})\right]\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R^2-\frac{\zeta^2}{\rho_2^4}}$$

$$= -\overline{\psi_h}(\frac{\rho_2^2}{\zeta}) - \left[\frac{R\zeta + \frac{1}{R\zeta}}{R-\frac{\zeta^2}{R\rho_2^4}}\right]\overline{\varphi'_h}(\frac{\rho_2^2}{\zeta})$$

$$+\left[a_{11}(R-\frac{\zeta}{\rho_2^2}) + a_{12}(R+\frac{\zeta}{\rho_2^2})\right]\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R^2-\frac{\zeta^2}{\rho_2^4}}$$

$$\omega_{b1}(\zeta) = -\overline{\varphi_{a1}}(\frac{\rho_2^2}{\zeta}) - \overline{a_{11}}\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R+\frac{1}{\zeta}} - \overline{a_{12}}\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R-\frac{1}{\zeta}}$$

$$= -\overline{\varphi_h}(\frac{\rho_2^2}{\zeta}) - [\overline{a_{11}}(R-\frac{1}{\zeta}) - \overline{a_{11}}(R+\frac{1}{\zeta})]\frac{R^2\rho_2^2 + \frac{1}{R^2\rho_2^2}}{R^2-\frac{1}{\zeta^2}}$$

$$\phi_2(\zeta) = (1+\Lambda_{12})\phi_{b1}(\zeta) = \phi_{b1}(\zeta)$$

$$\omega_2(\zeta) = (1+\Pi_{12})\omega_{b1}^I(\zeta) = \omega_{b1}^I(\zeta)$$

$$\omega_{b1}^I(\zeta) = \omega_{b1}(\zeta) + \frac{\frac{R(\rho_1^2-\rho_2^2)}{\zeta} + \frac{\zeta}{R}(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2})}{R-\frac{1}{R\zeta^2}}\phi'_{b1}(\zeta)$$

$$\psi_h(\zeta) = -\frac{G_1\beta_1 Ql(R\overline{\zeta_0} + \frac{1}{R\overline{\zeta_0}})\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}$$

$$\psi_2(\zeta) = \omega_2(\zeta) - \frac{\frac{R\rho_1^2}{\zeta} + \frac{\zeta}{R\rho_1^2}}{R - \frac{1}{R\zeta^2}} \varphi_2'(\zeta)$$
$$= \omega_{b1}^I(\zeta) - \frac{\frac{R\rho_1^2}{\zeta} + \frac{\zeta}{R\rho_1^2}}{R - \frac{1}{R\zeta^2}} \varphi_{b1}'(\zeta)$$
$$= \omega_{b1}(\zeta) - \frac{\frac{R\rho_2^2}{\zeta} + \frac{\zeta}{R\rho_2^2}}{R - \frac{1}{R\zeta^2}} \varphi_{b1}'(\zeta)$$

For the corresponding hole problem, the aspect ratio a_2/b_2=1 and we have $1/R$=0 leading to

$$a_{11} = -\frac{1}{2}\overline{\phi_{a1}'}(0) = -\frac{G_1\beta_1 QRl\left[\log(-\overline{\zeta_0})+1\right]}{4\pi k_1(1+\kappa_1)},$$

$$a_{12} = \frac{1}{2}\overline{\phi_{a1}'}(0) = \frac{G_1\beta_1 QRl\left[\log(-\overline{\zeta_0})+1\right]}{4\pi k_1(1+\kappa_1)}$$

$$\phi_{b1}(\zeta) = -\overline{\omega_{a1}^{II}}(\frac{\rho_2^2}{\zeta}) + \frac{G_1\beta_1 QRl\zeta\left[\log(-\overline{\zeta_0})+1\right]}{2\pi k_1(1+\kappa_1)}$$

$$\varphi_{b1}(\zeta) = -\overline{\psi_h}(\frac{\rho_2^2}{\zeta}) - \zeta\overline{\varphi_h'}(\frac{\rho_2^2}{\zeta}) + \frac{G_1\beta_1 QRl\zeta\left[\log(-\overline{\zeta_0})+1\right]}{2\pi k_1(1+\kappa_1)}$$
$$= -\frac{G_1\beta_1 QRl(\zeta-\zeta_0)\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)} + \frac{G_1\beta_1 QRl\zeta\log(-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)}$$

$$\varphi(\zeta) = A\log\zeta + \varphi_h(\zeta) + \varphi_2(\zeta)$$
$$= \frac{-G_1\beta_1 QRl\rho_2^2}{2(1+\kappa_1)\pi k_1\overline{\zeta_0}}\log\zeta + \frac{G_1\beta_1 QRl(\zeta-\zeta_0)\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}$$
$$-\frac{G_1\beta_1 QRl(\zeta-\zeta_0)\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)(R)\rho_2^4} + \frac{G_1\beta_1 QRl\zeta\log(-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)}$$
$$= \frac{G_1\beta_1 QRl(\zeta-\zeta_0)\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)} - \frac{\beta_1 G_1 QRl(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)}\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})$$
$$+\frac{G_1\beta_1 QRl\zeta}{2\pi k_1(1+\kappa_1)}\log(-\overline{\zeta_0}) - \frac{G_1\beta_1 Ql}{2(1+\kappa_1)\pi k_1}\frac{R\rho_2^2}{\overline{\zeta_0}}\log\zeta$$

$$A = \frac{-G_1\beta_1 Ql}{2(1+\kappa_1)\pi k_1}\frac{R\rho_2^2}{\overline{\zeta_0}} = B$$

$$\begin{aligned}
\psi_2(\zeta) =& \omega_{b1}(\zeta) - \frac{\rho_2^2}{\zeta}\phi_{b1}'(\zeta) \\
=& -\frac{G_1\beta_1 QRl(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)} - \frac{G_1\beta_1 QRl\rho_2^2\left[\log(-\zeta_0)+1\right]}{2\pi k_1(1+\kappa_1)\zeta} \\
&+ \frac{G_1\beta_1 QRl\rho_2^2\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)\zeta} - \frac{G_1\beta_1 QRl(\zeta-\zeta_0)\rho_2^4}{2\pi k_1(1+\kappa_1)(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})\zeta^3} \\
&- \frac{G_1\beta_1 QRl\rho_2^2\log(-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)\zeta}
\end{aligned}$$

$$\begin{aligned}
\psi(\zeta) =& \overline{A}\log\frac{\zeta}{\rho_1^2} + \overline{B}\log\frac{\rho_1^2}{\rho_2^2} - \frac{\overline{m}(\frac{c^2}{\zeta})}{m'(\zeta)}\frac{A}{\zeta} + \psi_h(\zeta) + \psi_2(\zeta) \\
=& \overline{A}\log\frac{\zeta}{\rho_2^2} - \frac{\rho_2^2}{\zeta}\frac{A}{\zeta} - \frac{G_1\beta_1 QlR\overline{\zeta_0}\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)} + \psi_2(\zeta) \\
=& \overline{A}\log\frac{\zeta}{\rho_2^2} - \frac{\rho_2^2}{\zeta}\frac{A}{\zeta} - \frac{G_1\beta_1 QlR\overline{\zeta_0}\log(\zeta-\zeta_0)}{2\pi k_1(1+\kappa_1)} \\
&+ \frac{G_1\beta_1 QRl\overline{\zeta_0}\log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})}{2\pi k_1(1+\kappa_1)} - \frac{G_1\beta_1 QRl\rho_2^2\left[\log(-\zeta_0)+\log(-\overline{\zeta_0})+1\right]}{2\pi k_1(1+\kappa_1)\zeta} \\
&- \frac{G_1\beta_1 QRl(\zeta-\zeta_0)\rho_2^4}{2\pi k_1(1+\kappa_1)(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})\zeta^3} \\
=& -\frac{G_1\beta_1 QRl}{2\pi k_1(1+\kappa_1)}\left\{\left(\frac{\rho_2^2}{\zeta_0}\log\zeta - \frac{1}{\overline{\zeta_0}}\frac{\rho_2^4}{\zeta^2}\right) + \overline{\zeta_0}\left[\log(\zeta-\zeta_0) - \log(\frac{\rho_2^2}{\zeta}-\overline{\zeta_0})\right]\right. \\
&\left. + \frac{\rho_2^2}{\zeta}\left[1+\log(-\zeta_0)+\log(-\overline{\zeta_0})\right] + \frac{\rho_2^2}{\zeta}(\zeta-\zeta_0)\frac{\frac{\rho_2^2}{\zeta^2}}{\frac{\rho_2^2}{\zeta}-\overline{\zeta_0}}\right\}
\end{aligned}$$