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Reconstruction of Boundary Data in Two-Dimensional Isotropic Linear Elasticity from Cauchy Data Using an Iterative MFS Algorithm

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Abstract: We investigate the implementation of the method of fundamental solutions (MFS), in an iterative manner, for the algorithm of Kozlov, Maz′ya and Fomin (1991) in the case of the Cauchy problem in two-dimensional isotropic linear elasticity. At every iteration, two mixed well-posed and direct problems are solved using the Tikhonov regularization method, while the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. An efficient regularizing stopping criterion is also presented. The iterative MFS algorithm is tested for Cauchy problems for isotropic linear elastic materials to confirm the numerical convergence, stability and accuracy of the method.

Keywords: Inverse Problem; Cauchy Problem; Isotropic Linear Elasticity; Iterative Method of Fundamental Solutions (MFS); Regularization.

1 Introduction

In the case of inverse boundary value problems in solid mechanics, the lack of complete boundary conditions is usually overcome by supplying additional information in the form of either internal displacement, strain or stress measurements, or over-specified boundary conditions on the aforementioned boundary, the latter being referred to as the *Cauchy problem*. It is well known that such inverse problems are in general ill-posed, in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed, see Hadamard (1923). There are numerous important contributions in the literature (see e.g. Bonnet and Constantinescu (2005) for an extensive overview of inverse problems in solid mechanics) and various approaches devoted to the theoretical and numerical solutions of in-

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verse boundary value problems in elasticity. Generally speaking, two major classes of regularization methods are employed for the stable solution of inverse boundary value problems in elasticity, namely non-iterative (direct) and iterative methods.

The first class is usually based on either the minimization of a Tikhonov functional (or, equivalently, the resolution of the normal equation) (Tikhonov and Arsenin , 1986) or the decomposition of the matrix corresponding to the discretised system of equations, for example using the singular value decomposition (SVD) (Hansen , 1998), which is successively used to solve a sequence of well-conditioned problems depending on the regularization parameter. Finally, the optimal value of the regularization parameter and, consequently, the corresponding optimal solution, are selected using an appropriate criterion, such as the discrepancy principle of Morozov (Morozov , 1966), the generalized cross-validation (GCV) criterion (Wahba , 1977) or Hansen's L-curve criterion (Hansen , 1998).

Maniatty, Zabaras and Stelson (1989) employed the finite element method (FEM) and a first-order spatial regularization scheme using the measurements of internal strains and displacements to solve for the boundary traction reconstruction in terms of shape and magnitude. Schnur and Zabaras (1990) presented a boundary condition reconstruction and the so-called keynode method, which consists of specifying a polynomial to represent the missing boundary condition. Spatial regularization and the boundary element method (BEM) were also used by Zabaras, Morellas and Schnur (1989) for the resolution of the same problem. Later, Maniatty and Zabaras (1994) applied Bayesian statistical theory for general inverse problems to inverse elasticity problems and also compared it to the method proposed in Schnur and Zabaras (1990). Martin, Haldermann and Dulikravich (1995) combined the BEM and the SVD to determine the numerical solution of Cauchy problems in two-dimensional elasticity. Both Turco (1999) and Marin and Lesnic (2002a) used the BEM to discretise the problem and the Tikhonov regularization method completed by the GCV criterion and the L-curve method, respectively, to make the solution process entirely automatic. The BEM-based system of linear equations was successfully solved via the CGM and a stopping criterion based on a Monte-Carlo simulation of the GCV by Turco (2001). The SVD, in conjunction with the BEM, was employed by Marin and Lesnic (2002b) to determine the numerical solutions to Cauchy problems in linear elasticity. Bilotta and Turco (2009) solved the Cauchy problem in two-dimensional isotropic linear elasticity by using a standard FEM approach, the Tikhonov regularization method and the GCV criterion. Marin and Lesnic (2004) and Marin (2005) proposed the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method, for solving the Cauchy problem in two- and three-dimensional isotropic linear elasticity, respectively.

With respect to iterative methods, it should be mentioned that every iteration consists of the resolution of two or three well-posed direct problems and the iterative procedure has to be stopped according to a suitable regularizing stopping criterion. In this case, the role of the regularization parameter is played by the iteration number at which the iterative process is stopped. Both the non-iterative (direct) and the iterative methods work in a very similar manner as far as regularization is concerned and the choice of one method over another is usually related to the specific problem under investigation.

The Cauchy problem in elasticity was studied theoretically by Yeih, Koya and Mura (1993), who analysed its existence, uniqueness and continuous dependence on the data and proposed the fictitious boundary indirect method based on simple and double layer potential theory. The numerical implementation of the aforementioned method was undertaken by Koya, Yeih and Mura (1993), who employed the BEM and the Nyström method for discretising the integrals. The iterative algorithm of Kozlov, Maz′ya and Fomin (1991), which reduces the Cauchy problem to solving a sequence of well-posed boundary value problems, was implemented using the BEM for linear elastic materials by Marin, Elliott, Ingham and Lesnic (2001, 2002a) and Comino, Marin and Gallego (2007). Ellabib and Nachaoui (2008) investigated numerically the relaxation of the algorithm of Kozlov, Maz′ya and Fomin (1991). Further investigations were carried out by Marin and Johansson (2010) who also proposed alternative ways of relaxation of both the prescribed displacements and tractions on the over-specified boundary, proved the convergence of these schemes and introduced appropriate optimal stopping rules. Huang and Shih (1997) and Marin, Háo and Lesnic (2002) used the CGM, as a result of the variational approach, combined with the BEM in order to solve the two-dimensional Cauchy problem in linear elasticity. Four regularization methods for solving stably the Cauchy problem in linear elasticity, namely the Tikhonov regularization, the SVD, the CGM and the algorithm of Kozlov, Maz′ya and Fomin (1991), were compared in Marin, Elliott, Ingham and Lesnic (2002b). It was found that the truncated SVD outperforms the Tikhonov regularization method, whilst the latter outperforms the CGM. The Cauchy problem in elasticity with L^2-boundary data was approached by combining the BEM with the Landweber-Fridman method and the minimal error method by Marin and Lesnic (2005) and Marin (2009), respectively. Andrieux and Baranger (2008) reformulated the Cauchy problem for three-dimensional elastic media as an energy error minimization problem.

The MFS is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of solutions to linear partial differential equations when a fundamental solution of the governing equation is explicitly known. Since its introduction as a numerical method by Mathon and Johnston (1977), it has been successfully applied to a large variety of physical problems, an account of which may be found in the survey papers by Fairweather and Karageorghis (1998), Fairweather, Karageorghis and Martin (2003) and Cho, Golberg, Muleshkov and Li (2004). The MFS with fixed singularities has been applied to several direct problems in elasticity, such as two-dimensional [Redekop (1982); Burgess and Maharejin (1984, 1985); Mahajerin (1985)], axisymmetric [Redekop and Cheung (1987)] and three-dimensional problems [Redekop and Thompson (1983); Poullikkas, Karageorghis and Georgiou (2002)]. The ease of implementation of the MFS and its low computational cost make it an ideal candidate for inverse problems as well. For these reasons, the MFS, mostly in conjunction with the Tikhonov regularization method or the SVD, have been used increasingly over the last decade for the numerical solution of inverse problems.

In this paper, we investigate the numerical implementation of the algorithm of Kozlov, Maz′ya and Fomin (1991) for the Cauchy problem in two-dimensional isotropic linear elasticity using the MFS in an iterative manner. More precisely, at every iteration, two mixed well-posed and direct problems are solved using the MFS, in conjunction with the Tikhonov regularization method, while the optimal value of the regularization parameter is selected according to the GCV criterion. An efficient regularizing stopping criterion which terminates the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, is also presented. Finally, the iterative MFS algorithm is tested for Cauchy problems in isotropic linear elasticity in various geometries.

2 Mathematical formulation

Consider an open bounded domain $\Omega \subset \mathbb{R}^d$, where d is the dimension of the space where the problem is posed, usually $d \in \{1,2,3\}$, occupied by an isotropic medium and assume that Ω is bounded by a smooth or piecewise smooth curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. In the absence of body forces, the equilibrium equations are given by, see Aliabadi (2002),

$$\mathscr{L}\mathbf{u}(\mathbf{x}) \equiv -\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}(\mathbf{x})) = \mathbf{0}, \quad \mathbf{x} \in \Omega. \tag{1}$$

Here $\mathscr{L}$ is the Lamé (Navier) differential operator, $\boldsymbol{\sigma}(\mathbf{u}(\mathbf{x})) = [\sigma_{ij}(\mathbf{u}(\mathbf{x}))]_{1\leq i,j,\leq d}$ is the stress tensor associated with the displacement vector $\mathbf{u}(\mathbf{x}) = (\mathrm{u}_1(\mathbf{x}),\ldots,\mathrm{u}_d(\mathbf{x}))^{\mathsf{T}}$, whilst on assuming small deformations, the corresponding strain tensor $\boldsymbol{\epsilon}(\mathbf{u}(\mathbf{x})) = [\epsilon_{ij}(\mathbf{u}(\mathbf{x}))]_{1\leq i,j,\leq d}$ is given by the kinematic relations:

$$\boldsymbol{\epsilon}(\mathbf{u}(\mathbf{x})) = \frac{1}{2}\left(\nabla\mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})^{\mathsf{T}}\right), \quad \mathbf{x} \in \overline{\Omega} = \Omega \cup \partial\Omega. \tag{2}$$

These tensors are related by the constitutive law, namely

$$\sigma(\mathbf{u}(\mathbf{x})) = \mathbf{C}\epsilon(\mathbf{u}(\mathbf{x})), \quad \mathbf{x} \in \overline{\Omega}, \tag{3}$$

where $\mathbf{C} = \left[\mathrm{C}_{ijkl}\right]_{1\leq i,j,k,l\leq d}$ is the fourth-order elasticity tensor.

We now let $\mathbf{n}(\mathbf{x}) = (\mathrm{n}_1(\mathbf{x}),\ldots,\mathrm{n}_d(\mathbf{x}))^{\mathsf{T}}$ be the outward unit normal vector at $\mathbf{x} \in \partial\Omega$, and $\mathbf{t}(\mathbf{x}) = (\mathrm{t}_1(\mathbf{x}),\ldots,\mathrm{t}_d(\mathbf{x}))^{\mathsf{T}}$ be the traction vector at a point $\mathbf{x} \in \partial\Omega$, defined by

$$\mathbf{t}(\mathbf{x}) \equiv \sigma(\mathbf{u}(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{4}$$

If we assume that it is possible to measure both the displacement and traction vectors on a part of the boundary $\partial\Omega$, say Γ_1, then this leads to the mathematical formulation of the Cauchy problem consisting of the partial differential equations (1) and the boundary conditions

$$\mathbf{u}(\mathbf{x}) = \widetilde{\mathbf{u}}(\mathbf{x}), \qquad \mathbf{t}(\mathbf{x}) = \widetilde{\mathbf{t}}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_1, \tag{5}$$

where $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{t}}$ are prescribed vector valued functions on Γ_1. It can be seen from the boundary conditions (5) that the boundary Γ_1 is over-specified by prescribing both the displacement $\mathbf{u}\big|_{\Gamma_1} = \widetilde{\mathbf{u}}$ and the traction $\mathbf{t}\big|_{\Gamma_1} = \widetilde{\mathbf{t}}$ vectors, while the boundary Γ_2 is under-specified since both the displacement $\mathbf{u}\big|_{\Gamma_2}$ and the traction $\mathbf{t}\big|_{\Gamma_2}$ vectors are unknown and have to be determined. We also assume that data are chosen such that there exists a solution to this Cauchy problem. This solution is unique according to the so-called unique continuation properties for elliptic equations.

It should be mentioned that, if we denote by G, ν and δ_{ij} the shear modulus, the Poisson ratio and the Kronecker delta tensor, respectively, then the components of the fourth-order elasticity tensor for an isotropic linear elastic material are given by

$$\mathrm{C}_{ijkl} = G\left(\frac{2\nu}{1-\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{6}$$

3 Description of the algorithm

Let $\mathrm{H}^1(\Omega)^d$ be the Sobolev space and $\mathrm{H}^{1/2}(\partial\Omega)^d$ be the space of traces on $\partial\Omega$ corresponding to $\mathrm{H}^1(\Omega)^d$, see e.g. Lions and Magenes (1972). We denote by $\mathrm{H}^{1/2}(\Gamma_i)^d$ the space of functions from $\mathrm{H}^{1/2}(\partial\Omega)^d$ that are bounded on Γ_i and by $\left(\mathrm{H}^{1/2}(\Gamma_i)^d\right)^*$ the dual space of $\mathrm{H}^{1/2}(\Gamma_i)^d$, for $i = 1,2$.

Kozlov, Maz′ya and Fomin (1991) proposed the following iterative algorithm for the simultaneous reconstruction of the unknown displacement $\mathbf{u}\big|_{\Gamma_2}$ and traction $\mathbf{t}\big|_{\Gamma_2}$ vectors on the under-specified boundary:

Step 1. (i) If $k = 1$ then specify an initial guess for the boundary displacement vector on Γ_2, namely $\mathbf{u}^{(2k-1)} \in \mathrm{H}^{1/2}(\Gamma_2)^d$.

(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$\begin{cases} \mathscr{L}\mathbf{u}^{(2k-1)}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega, \\ \mathbf{u}^{(2k-1)}(\mathbf{x}) = \widetilde{\mathbf{u}}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \\ \mathbf{t}^{(2k-1)}(\mathbf{x}) = \mathbf{t}^{(2k-2)}(\mathbf{x}), & \mathbf{x} \in \Gamma_2, \end{cases} \tag{7}$$

to determine $\mathbf{u}^{(2k-1)}(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and $\mathbf{u}^{(2k-1)}(\mathbf{x})$ for $\mathbf{x} \in \Gamma_2$.

Step 2. Having constructed the approximation $\mathbf{u}^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\begin{cases} \mathscr{L}\mathbf{u}^{(2k)}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega, \\ \mathbf{t}^{(2k)}(\mathbf{x}) = \widetilde{\mathbf{t}}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \\ \mathbf{u}^{(2k)}(\mathbf{x}) = \mathbf{u}^{(2k-1)}(\mathbf{x}), & \mathbf{x} \in \Gamma_2, \end{cases} \tag{8}$$

is solved in order to determine $\mathbf{u}^{(2k)}(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and $\mathbf{t}^{(2k)}(\mathbf{x})$ for $\mathbf{x} \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 1. Kozlov, Maz′ya and Fomin (1991) showed that if $\partial\Omega$ is smooth, $\widetilde{\mathbf{u}} \in \mathrm{H}^{1/2}(\Gamma_1)^d$ and $\widetilde{\mathbf{t}} \in \left(\mathrm{H}^{1/2}(\Gamma_1)^d\right)^*$, then the alternating iterative algorithm based on steps $1-3$ produces two sequences of approximate solutions $\left\{\mathbf{u}^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{\mathbf{u}^{(2k)}\right\}_{k\geq 1}$ which both converge in $\mathrm{H}^1(\Omega)^d$ to the solution $\mathbf{u}$ of the Cauchy problem (1) and (5) for any initial guess $\mathbf{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)^d$, provided that a solution to this Cauchy problem exists. Furthermore, Kozlov, Maz′ya and Fomin (1991) proved that the alternating iterative algorithm has a regularizing character.

Remark 2. Also, the same conclusion holds if in step 1 one specifies an initial guess for the unknown traction vector on Γ_2, i.e. $\mathbf{t}^{(1)} \in \left(\mathrm{H}^{1/2}(\Gamma_2)^d\right)^*$, instead of an initial guess for the displacement vector, $\mathbf{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)^d$, and we modify steps 1 and 2 accordingly.

4 Method of fundamental solutions

The fundamental solution matrix $\mathbf{U} = [\mathrm{U}_{ij}]_{1 \le i,j \le 2}$ of the two-dimensional Lamé system, i.e. $d = 2$, of isotropic linear elasticity (1) for the displacement vector is given by, see Aliabadi (2002)

$$\begin{aligned} &\mathrm{U}_{ij}(\mathbf{x}, \boldsymbol{\xi}) = -\frac{1}{8\pi G(1-\overline{\nu})} \left[(3-4\overline{\nu}) \ln \|\mathbf{x} - \boldsymbol{\xi}\| \, \delta_{ij} - \frac{\mathrm{x}_i - \xi_i}{\|\mathbf{x} - \boldsymbol{\xi}\|} \frac{\mathrm{x}_j - \xi_j}{\|\mathbf{x} - \boldsymbol{\xi}\|} \right], \\ &\mathbf{x} \in \overline{\Omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}, \quad i, j = 1, 2, \end{aligned} \tag{9}$$

where $\boldsymbol{\xi}$ is a singularity or source point, and $\overline{\nu} = \nu$ in the plane strain state and $\overline{\nu} = \nu/(1+\nu)$ in the plane stress state.

The main idea of the MFS consists of the approximation of the displacement vector in the solution domain and on its boundary by a linear combination of fundamental solutions with respect to M singularities $\boldsymbol{\xi}^{(m)}, m = 1, \ldots, M$, in the form

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}^M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{m=1}^{M} \mathbf{U}(\mathbf{x}, \boldsymbol{\xi}^{(m)}) \, \mathbf{c}^{(m)}, \quad \mathbf{x} \in \overline{\Omega}, \tag{10}$$

where $\mathbf{c} \in \mathbb{R}^{2M}$ is a vector containing the components of the unknown two-dimensional vectors $\mathbf{c}^{(m)} = \left(\mathrm{c}_1^{(m)}, \mathrm{c}_2^{(m)}\right)^{\mathsf{T}}$, $m = 1, \ldots, M$, i.e. $\mathbf{c} = \left(\mathrm{c}_1^{(1)}, \mathrm{c}_2^{(1)}, \ldots, \mathrm{c}_1^{(M)}, \mathrm{c}_2^{(M)}\right)^{\mathsf{T}} \in \mathbb{R}^{2M}$, and $\boldsymbol{\xi} \in \mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\boldsymbol{\xi}^{(m)}$.

From Eqs. (2), (4) and (9), it follows that the traction vector at a point $\mathbf{x} \in \partial\Omega$ defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$ can be approximated by

$$\mathbf{t}(\mathbf{x}) \approx \mathbf{t}^M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{m=1}^{M} \mathbf{T}(\mathbf{x}, \boldsymbol{\xi}^{(m)}) \, \mathbf{c}^{(m)}, \quad \mathbf{x} \in \partial\Omega. \tag{11}$$

Here $\mathbf{T} = [\mathrm{T}_{ij}]_{1 \le i,j \le 2}$ is the fundamental solution matrix for the traction vector, whose components are given by

$$\begin{aligned} \mathrm{T}_{1j}(\mathbf{x}, \boldsymbol{\xi}) \; &= \; \frac{2G}{1-2\overline{\nu}} \left[(1-\overline{\nu}) \frac{\partial \mathrm{U}_{1j}(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathrm{x}_1} + \overline{\nu} \, \frac{\partial \mathrm{U}_{2j}(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathrm{x}_2} \right] \mathrm{n}_1(\mathbf{x}) \\ &+ \; G \left[\frac{\partial \mathrm{U}_{1j}(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathrm{x}_2} + \frac{\partial \mathrm{U}_{2j}(\mathbf{x}, \boldsymbol{\xi})}{\partial \mathrm{x}_1} \right] \mathrm{n}_2(\mathbf{x}), \end{aligned} \tag{12a}$$

and

$$\begin{aligned} \mathrm{T}_{2j}(\mathbf{x},\boldsymbol{\xi}) \;=\; & \frac{2G}{1-2\overline{\nu}}\left[\frac{\partial \mathrm{U}_{1j}(\mathbf{x},\boldsymbol{\xi})}{\partial \mathrm{x}_2}+\frac{\partial \mathrm{U}_{2j}(\mathbf{x},\boldsymbol{\xi})}{\partial \mathrm{x}_1}\right]\mathrm{n}_1(\mathbf{x}) \\ & +\; G\left[\overline{\nu}\,\frac{\partial \mathrm{U}_{1j}(\mathbf{x},\boldsymbol{\xi})}{\partial \mathrm{x}_1}+(1-\overline{\nu})\,\frac{\partial \mathrm{U}_{2j}(\mathbf{x},\boldsymbol{\xi})}{\partial \mathrm{x}_2}\right]\mathrm{n}_2(\mathbf{x}), \end{aligned} \tag{12b}$$

for $\mathbf{x}\in\partial\Omega$, $\boldsymbol{\xi}\in\mathbb{R}^2\setminus\overline{\Omega}$ and $j=1,2$.

Next, we select N_1 MFS collocation points $\{\mathbf{x}^{(n)}\}_{n=1}^{N_1}$ on the boundary Γ_1 and N_2 MFS collocation points $\{\mathbf{x}^{(n)}\}_{n=N_1+1}^{N_1+N_2}$ on the boundary Γ_2, such that the total number of MFS collocation points used to discretise the boundary $\partial\Omega$ of the solution domain Ω is given by $N=N_1+N_2$.

According to the MFS approximations (10) and (11), the discretised versions of the boundary value problems (7) and (8) may be recast as

$$\mathbf{A}\,\mathbf{c}=\mathbf{b}, \tag{13}$$

where $\mathbf{A}$ is the corresponding MFS matrix, the right-hand side vector $\mathbf{b}$ contains the boundary data associated with the boundary value problems (7) and (8) and the vector $\mathbf{c}$ contains the corresponding unknown boundary data. Eq. (13) represents a system of $2N$ linear algebraic equations with $2M$ unknowns, which can be uniquely determined if the number N of MFS boundary collocation points and the number M of singularities satisfy the inequality $M\le N$. However, Eq. (13) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution in the case of noisy Cauchy data on Γ_1.

In the case of the MFS, it is essential to determine the location of the singularities and this is usually achieved by considering either the static or the dynamic approach. In the first approach, the singularities are pre-assigned and kept fixed throughout the solution process, whilst in the latter, the singularities and the unknown coefficients are determined simultaneously during the solution process, see Fairweather and Karageorghis (1998). Thus the dynamic approach transforms the inverse problem into a more difficult nonlinear ill-posed problem which is also computationally much more expensive. On accounting for the findings of Gorzelańczyk and Kołodziej (2008), we decided to employ the static approach in our computations, with the shape of the pseudo-boundary on which the source points are located similar to that of the boundary of the solution domain.

5 The Tikhonov regularization method

Since the MFS matrix $\mathbf{A}$ is severely ill-conditioned, a suitable regularization method should be employed to obtain an accurate and stable solution of Eq. (13). Several regularization techniques used for the stable solution of systems of linear and non-linear algebraic equations are available in the literature, such as the SVD (Hansen , 1998), the Tikhonov regularization method (Tikhonov and Arsenin , 1986) and various iterative methods (Kunisch and Zou , 1998). Recently, Liu and Atluri (2008) proposed a new and robust numerical technique for the stable solution of ill-posed large systems of non-linear algebraic equations, namely the fictitious time integration method (FTIM). This method consists of introducing a fictitious time variable that plays the role of a regularization parameter, while its filtering effect is better than that of the Tikhonov and exponential filters. Liu and Atluri (2009) showed that, when applied to solving an ill-posed system of linear equations, the general FTIM may be viewed a special case of the Tiknonov regularization method.

Consider the system of linear algebraic equations given by Eq. (13), where $N \geq M$, $\mathbf{A} \in \mathbb{R}^{2N\times 2M}$, $\mathbf{c} \in \mathbb{R}^{2M}$ and $\mathbf{b} \in \mathbb{R}^{2N}$. The Tikhonov regularized solution to Eq. (13) is sought as, see Tikhonov and Arsenin (1986),

$$\mathbf{c}_\lambda : \quad \mathscr{F}_\lambda(\mathbf{c}_\lambda) = \min_{\mathbf{c} \in \mathbb{R}^{2M}} \mathscr{F}_\lambda(\mathbf{c}), \tag{14}$$

where $\mathscr{F}_\lambda$ represents the Tikhonov regularization functional given by

$$\mathscr{F}_\lambda(\cdot) : \mathbb{R}^{2M} \longrightarrow [0,\infty), \quad \mathscr{F}_\lambda(\mathbf{c}) = \left\|\mathbf{A}\,\mathbf{c} - \mathbf{b}\right\|^2 + \lambda^2 \left\|\mathbf{c}\right\|^2, \tag{15}$$

and $\lambda > 0$ is the regularization parameter to be prescribed. Formally, the Tikhonov regularized solution, $\mathbf{c}_\lambda$, of the problem (14) is given as the solution of the normal equation, i.e.

$$\mathbf{c}_\lambda = \mathbf{A}^\dagger\,\mathbf{b}, \quad \mathbf{A}^\dagger \equiv \left(\mathbf{A}^\mathsf{T}\mathbf{A} + \lambda^2 \mathbf{I}_{2M}\right)^{-1} \mathbf{A}^\mathsf{T}, \tag{16}$$

where $\mathbf{I}_{2M} \in \mathbb{R}^{2M\times 2M}$ is the identity matrix.

In this paper, we employ the GCV criterion (Wahba , 1977) to determine the optimal regularization parameter, λ_{opt}, for the Tikhonov regularization method, namely

$$\lambda_{\text{opt}} : \quad \mathscr{G}\left(\lambda_{\text{opt}}\right) = \min_{\lambda > 0} \mathscr{G}(\lambda). \tag{17}$$

Here

$$\mathscr{G}(\cdot) : (0,\infty) \longrightarrow [0,\infty), \quad \mathscr{G}(\lambda) = \frac{\left\|\mathbf{A}\,\mathbf{c}_\lambda - \mathbf{b}^\varepsilon\right\|^2}{\left[\text{trace}\left(\mathbf{I}_{2N} - \mathbf{A}\,\mathbf{A}^\dagger\right)\right]^2}, \tag{18}$$

where $\mathbf{c}_\lambda$ is obtained from Eq. (16) with $\mathbf{b} = \mathbf{b}^\varepsilon$ and $\left\|\mathbf{b}^\varepsilon - \mathbf{b}\right\| \leq \varepsilon$ is an estimate of the noisy Cauchy data, $\mathbf{b}^\varepsilon$, on the over-specified boundary Γ_1.

6 Numerical results and discussion

6.1 Examples

We consider an isotropic linear elastic medium characterised by the material constants $G = 3.35 \times 10^{10}\,\mathrm{N/m^2}$ and $\nu = 0.34$ corresponding to a copper alloy, and we solve the Cauchy problem given by Eqs. (1) and (5) for three typical examples in the following geometries:

Example 1. (Doubly connected domain with a smooth boundary) We consider the following analytical solution for the displacements:

$$\mathrm{u}_i^{(\mathrm{an})}(\mathbf{x}) = \frac{1}{2G(1+\nu)}\left[\mathrm{V}(1-\nu)\,\mathrm{x}_i - \mathrm{W}(1+\nu)\,\frac{\mathrm{x}_i}{\mathrm{x}_1^2+\mathrm{x}_2^2}\right], \quad \mathbf{x} \in \overline{\Omega}, \quad i = 1,2, \tag{19}$$

with

$$\mathrm{V} = -\frac{\sigma_o r_o^2 - \sigma_i r_i^2}{r_o^2 - r_i^2}, \quad \mathrm{W} = \frac{(\sigma_o - \sigma_i)\, r_o^2 r_i^2}{r_o^2 - r_i^2}, \quad \sigma_o = 2\sigma_i = 2.0 \times 10^{10}\ \mathrm{N/m^2}, \tag{20}$$

in the annulus $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \,|\, r_i < \rho(\mathbf{x}) < r_o\right\}$, where $\rho(\mathbf{x}) = \sqrt{\mathrm{x}_1^2 + \mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$, $r_i = 2$ and $r_o = 4$, which corresponds to constant internal and external pressures σ_i and σ_o, respectively, for which the stress tensor is given by

$$\sigma_{ij}^{(\mathrm{an})}(\mathbf{x}) = \left[\mathrm{V} + (-1)^{i+1}\,\mathrm{W}\,\frac{\mathrm{x}_1^2 - \mathrm{x}_2^2}{(\mathrm{x}_1^2+\mathrm{x}_2^2)^2}\right]\delta_{ij} + 2\mathrm{W}\,\frac{\mathrm{x}_1^2 - \mathrm{x}_2^2}{\mathrm{x}_1\,\mathrm{x}_2}\,(1-\delta_{ij}), \qquad \mathbf{x} \in \overline{\Omega}, \quad i,j = 1,2. \tag{21}$$

Here $\Gamma_1 = \Gamma_i = \left\{\mathbf{x} \in \partial\Omega \;\middle|\; \rho(\mathbf{x}) = r_i\right\}$ and $\Gamma_2 = \Gamma_o = \left\{\mathbf{x} \in \partial\Omega \;\middle|\; \rho(\mathbf{x}) = r_o\right\}$.

Example 2. (Simply connected domain with a smooth boundary) We consider the following analytical solution for the displacements:

$$\mathrm{u}_i^{(\mathrm{an})}(\mathbf{x}) = \frac{1-\nu}{2G(1+\nu)}\,\sigma_0\,\mathrm{x}_i, \quad \mathbf{x} \in \overline{\Omega}, \quad i = 1,2, \tag{22}$$

in the disk $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \,|\, \rho(\mathbf{x}) < r\right\}$, where $\sigma_0 = 1.5 \times 10^{10}\ \mathrm{N/m^2}$ and $r = 1$, which corresponds to the uniform hydrostatic stress

$$\sigma_{ij}^{(\mathrm{an})}(\mathbf{x}) = \sigma_0\,\delta_{ij}, \quad \mathbf{x} \in \overline{\Omega}, \quad i,j = 1,2. \tag{23}$$

Here $\Gamma_1 = \{\mathbf{x} \in \partial\Omega \mid 0 \le \theta(\mathbf{x}) < \pi/8\} \cup \{\mathbf{x} \in \partial\Omega \mid 3\pi/8 < \theta(\mathbf{x}) < 2\pi\}$ and $\Gamma_2 = \{\mathbf{x} \in \partial\Omega \mid \pi/8 \le \theta(\mathbf{x}) \le 3\pi/8\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.

Example 3. (Simply connected domain with a piecewise smooth boundary) We consider the following analytical solution for the displacements:

$$\mathrm{u}_i^{(\mathrm{an})}(\mathbf{x}) = \frac{1}{2G(1+\nu)}\,\sigma_0\,(\mathrm{x}_1\,\delta_{i1} - \nu\,\mathrm{x}_2\,\delta_{i2})\,, \quad \mathbf{x} \in \overline{\Omega}, \quad i = 1,2, \tag{24}$$

in the square $\Omega = (-1,1) \times (-1,1)$, where $\sigma_0 = 1.5 \times 10^{10}\ \mathrm{N/m^2}$, which corresponds to a uniform traction stress given by

$$\sigma_{ij}^{(\mathrm{an})}(\mathbf{x}) = \sigma_0\,\delta_{i1}\,\delta_{j1}, \quad \mathbf{x} \in \overline{\Omega}, \quad i,j = 1,2. \tag{25}$$

Here $\Gamma_1 = [-1,1] \times \{\pm 1\} \cup \{-1\} \times (-1,1)$ and $\Gamma_2 = \{1\} \times (-1,1)$.

The inverse problems investigated in this paper have been solved using a uniform distribution of both the MFS boundary collocation points $\mathbf{x}^{(n)}$, $n = 1,\ldots,N$, and the singularities $\boldsymbol{\xi}^{(m)}$, $m = 1,\ldots,M$. Furthermore, the numbers of MFS boundary collocation points N_1 and N_2 corresponding to the boundaries Γ_1 and Γ_2, respectively, and singularities M, as well as the distance d_S between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_\mathrm{S}$ on which the singularities are located, were set to:

(i) $N_1 \in \{40, 60, 80\}$, $N_2 = N_1/2$, $M = N_1 + N_2/2$, and $d_\mathrm{S} = r_i/2 = 1.0$ and $d_\mathrm{S} = r_o = 4.0$ for Γ_i and Γ_o, respectively, for Example 1;

(ii) $N_1 = 60$, $N_2 = 20$, $M = N/2 = 40$ and $d_\mathrm{S} = 2\,r = 2.0$ for Example 2;

(iii) $N_1 = 57$, $N_2 = 19$, $M = N/2 = 38$ and $d_\mathrm{S} = r = 1.0$ for Example 3.

6.2 Initial guess

An arbitrary vector valued function $\mathbf{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)^d$ or $\mathbf{t}^{(1)} \in \left(\mathrm{H}^{1/2}(\Gamma_2)^d\right)^*$ may be specified as an initial guess for the unknown displacement or traction vector on Γ_2. In order to improve the rate of convergence of the iterative algorithm, one may choose a vector valued function which ensures the continuity of the boundary displacement or traction vector at the common endpoints of the boundaries Γ_1 and Γ_2, respectively, and which is also linear with respect to either the angular polar coordinate θ for Example 2, or the Cartesian x_2-coordinate for Example 3, see Marin, Elliott, Ingham and Lesnic (2001) and Comino, Marin and Gallego (2007). However, in the general situation when the boundaries Γ_1 and Γ_2 have no common points, as is the case of Example 1, one cannot use the procedure described above.

Therefore, in this paper we use the following initial guesses for the unknown displacement and traction vectors on Γ_2

$$\mathbf{u}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2, \tag{26a}$$

and

$$\mathbf{t}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2, \tag{26b}$$

respectively. In this way, the most general situations regarding the geometry of the domain are accounted for and the robustness of the algorithm with respect to the initial guess for the unknown displacement or traction vector on Γ_2 is also tested.

6.3 Convergence of the algorithm

If N_i collocation points, $\{\mathbf{x}^{(n)}\}_{n=1}^{N_i}$, are considered on the boundary $\Gamma_i \subset \partial\Omega$ then the *root mean square error* (RMS error) associated with the vector valued function $\mathbf{v}(\cdot) = \big(\mathrm{v}_1(\cdot), \ldots, \mathrm{v}_d(\cdot)\big)^{\mathsf{T}} : \Gamma_i \longrightarrow \mathbb{R}^d$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_i}(\mathbf{v}) = \sqrt{\frac{1}{N_i}\sum_{n=1}^{N_i}\left[\frac{1}{d}\sum_{i=1}^{d}\mathrm{v}_i\big(\mathbf{x}^{(n)}\big)^2\right]}. \tag{27}$$

In order to investigate the convergence of the algorithm, at every iteration, $k \geq 1$, we evaluate the following accuracy errors corresponding to the displacement and traction vectors on Γ_2, which are defined as *relative RMS errors*, i.e.

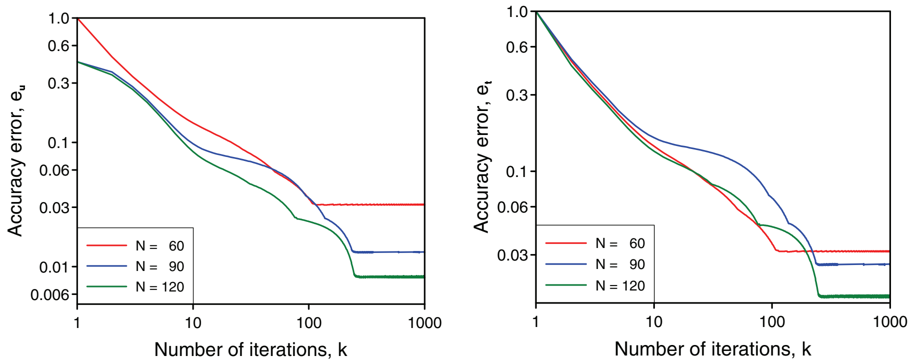
$$\mathrm{e}_{\mathbf{u}}(k) = \frac{\mathrm{RMS}_{\Gamma_2}\big(\mathbf{u}^{(2k-1)} - \mathbf{u}^{(\mathrm{an})}\big)}{\mathrm{RMS}_{\Gamma_2}\big(\mathbf{u}^{(\mathrm{an})}\big)} = \frac{\big\|\mathbf{u}^{(2k-1)}|_{\Gamma_2} - \mathbf{u}^{(\mathrm{an})}|_{\Gamma_2}\big\|_2}{\big\|\mathbf{u}^{(\mathrm{an})}|_{\Gamma_2}\big\|_2}, \quad k \geq 1, \tag{28a}$$

and

$$\mathrm{e}_{\mathbf{t}}(k) = \frac{\mathrm{RMS}_{\Gamma_2}\big(\mathbf{t}^{(2k)} - \mathbf{t}^{(\mathrm{an})}\big)}{\mathrm{RMS}_{\Gamma_2}\big(\mathbf{t}^{(\mathrm{an})}\big)} = \frac{\big\|\mathbf{t}^{(2k)}|_{\Gamma_2} - \mathbf{t}^{(\mathrm{an})}|_{\Gamma_2}\big\|_2}{\big\|\mathbf{t}^{(\mathrm{an})}|_{\Gamma_2}\big\|_2}, \quad k \geq 1, \tag{28b}$$

where $\mathbf{u}^{(2k-1)}$ and $\mathbf{t}^{(2k)}$ are the displacement and traction vectors on Γ_2 retrieved after k iterations by solving the boundary value problems (7) and (8), respectively.

Figs. 1(a) and (b) show the errors $\mathrm{e}_{\mathbf{u}}$ and $\mathrm{e}_{\mathbf{t}}$, respectively, as functions of the number of iterations, k, obtained using exact Cauchy data on Γ_1, $N \in \{60, 90, 120\}$ and



(a) Example 1: Accuracy error $\mathrm{e}_{\mathbf{u}}$ (b) Example 1: Accuracy error $\mathrm{e}_{\mathbf{t}}$

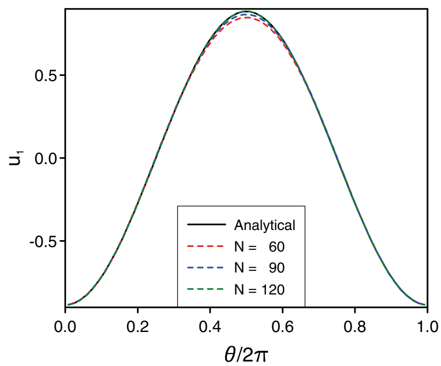
Figure 1: The accuracy errors (a) $\mathrm{e}_{\mathbf{u}}$ and (b) $\mathrm{e}_{\mathbf{t}}$, as functions of the number of iterations, k, obtained using the alternating iterative algorithm with initial guess $\mathbf{t}^{(1)}\big|_{\Gamma_2} = \mathbf{0}$, exact Cauchy data on Γ_1 and various numbers of MFS boundary collocation points and singularities, namely $N \in \{60, 90, 120\}$ and $M \in \{50, 75, 100\}$, respectively, for Example 1.

$M \in \{50, 75, 100\}$, for Example 1. It can be seen from these figures that both errors $\mathrm{e}_{\mathbf{u}}$ and $\mathrm{e}_{\mathbf{t}}$ decrease even after a large numbers of iterations, e.g. $k = 1000$, and as expected $\mathrm{e}_{\mathbf{u}} < \mathrm{e}_{\mathbf{t}}$ for all MFS discretisations employed, i.e. boundary tractions are more inaccurate than boundary displacements. Furthermore, as N increases, the errors $\mathrm{e}_{\mathbf{u}}$ and $\mathrm{e}_{\mathbf{t}}$ decrease showing that in the case of Example 1, $N \geq 90$ ensures a sufficient discretisation for the accuracy to be achieved.

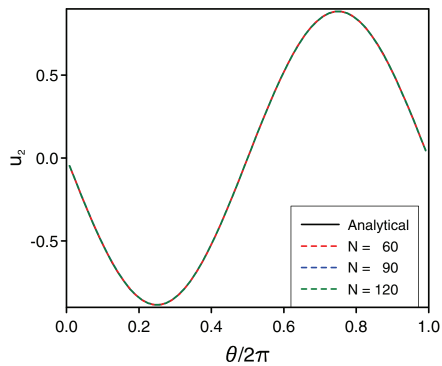
The numerical solutions for the displacement $\mathbf{u}\big|_{\Gamma_2}$ and traction $\mathbf{t}\big|_{\Gamma_2}$ vectors, obtained after $k = 1000$ iterations, using exact Cauchy data on Γ_1 and various numbers of MFS boundary collocation points and singularities, for Example 1, are presented in Figs. 2(a)–(d). From these figures, it can be seen that the accuracy in predicting both the boundary displacements and tractions on Γ_2 is very good. Similar results have also been obtained for the other examples investigated in this study and, therefore, these are not presented herein.

6.4 Regularizing stopping criterion

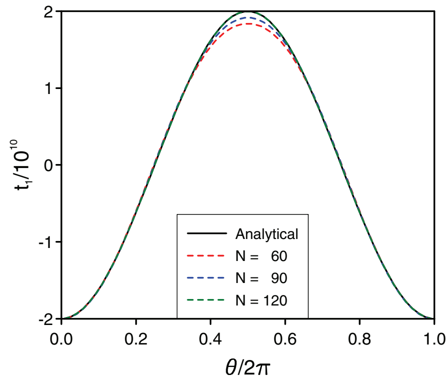
Once the convergence with respect to increasing N of the numerical solution to the exact solution has been established, we fix $N = 90$ and $M = 75$, and investigate the stability of the numerical solution corresponding to the alternating iterative algorithm described in Section 3 with the initial guess (26b), for Example 1. In what follows, the prescribed displacement, $\mathbf{u}\big|_{\Gamma_1} = \mathbf{u}^{(\mathrm{an})}\big|_{\Gamma_1}$, and/or the traction vectors,



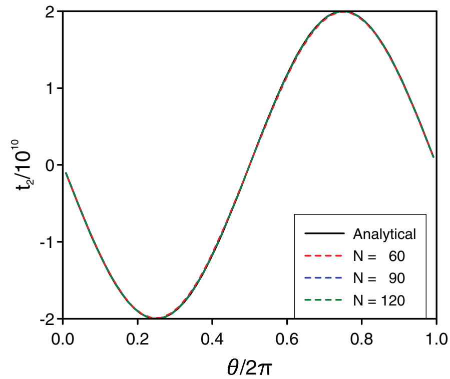
(a) Example 1: Displacement $\mathrm{u}_1\big|_{\Gamma_2}$



(b) Example 1: Displacement $\mathrm{u}_2\big|_{\Gamma_2}$



(c) Example 1: Traction $\mathrm{t}_1\big|_{\Gamma_2}$



(d) Example 1: Traction $\mathrm{t}_2\big|_{\Gamma_2}$

Figure 2: The analytical and numerical displacements (a) $\mathrm{u}_1\big|_{\Gamma_2}$ and (b) $\mathrm{u}_2\big|_{\Gamma_2}$, and tractions (c) $\mathrm{t}_1\big|_{\Gamma_2}$ and (d) $\mathrm{t}_2\big|_{\Gamma_2}$, obtained using the alternating iterative algorithm with initial guess $\mathbf{t}^{(1)}\big|_{\Gamma_2} = \mathbf{0}$, exact Cauchy data on Γ_1, $k = 1000$ iterations and various numbers of MFS boundary collocation points and singularities, namely $N \in \{60, 90, 120\}$ and $M \in \{50, 75, 100\}$, respectively, for Example 1.

$\mathbf{t}\big|_{\Gamma_1} = \mathbf{t}^{(\mathrm{an})}\big|_{\Gamma_1}$, have been perturbed as

$$\begin{aligned} \widetilde{\mathrm{u}}_i^{\varepsilon}\big|_{\Gamma_1} &= \mathrm{u}_i\big|_{\Gamma_1} + \delta\mathrm{u}_i, \quad \delta\mathrm{u}_i = \mathtt{G05DDF}(0, \sigma_{\mathrm{u}_i}), \quad \sigma_{\mathrm{u}_i} = \max_{\Gamma_1} |\mathrm{u}_i| \times (p_{\mathbf{u}}/100), \\ \widetilde{\mathrm{t}}_i^{\varepsilon}\big|_{\Gamma_1} &= \mathrm{t}_i\big|_{\Gamma_1} + \delta\mathrm{t}_i, \quad \delta\mathrm{t}_i = \mathtt{G05DDF}(0, \sigma_{\mathrm{t}_i}), \quad \sigma_{\mathrm{t}_i} = \max_{\Gamma_1} |\mathrm{t}_i| \times (p_{\mathbf{t}}/100), \end{aligned} \tag{29}$$

for $i = 1,2$, respectively. Here δu_i and δt_i are Gaussian random variables with mean zero and standard deviations σ_{u_i} and σ_{t_i}, respectively, generated by the NAG subroutine `G05DDF` (NAG Library Mark 21 , 2007), while $p_{\mathbf{u}}\%$ and $p_{\mathbf{t}}\%$ are the percentages of additive noise included into $\mathbf{u}\big|_{\Gamma_1}$ and $\mathbf{t}\big|_{\Gamma_1}$, respectively.

Figs. $3(a)$ and $3(b)$ present the accuracy errors $e_{\mathbf{u}}$ and $e_{\mathbf{t}}$, respectively, for various levels of Gaussian random noise $p_{\mathbf{t}} \in \{1\%, 3\%, 5\%\}$. From these figures it can be seen that as $p_{\mathbf{t}}$ decreases then $e_{\mathbf{u}}$ and $e_{\mathbf{t}}$ decrease. However, the errors in predicting the displacement and traction vectors on Γ_2 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to terminate the iterative process at the point where the errors in the numerical solutions start increasing.

After each iteration, k, we evaluate the following convergence error which is associated with the displacement vectors on the over-specified boundary, Γ_1, namely

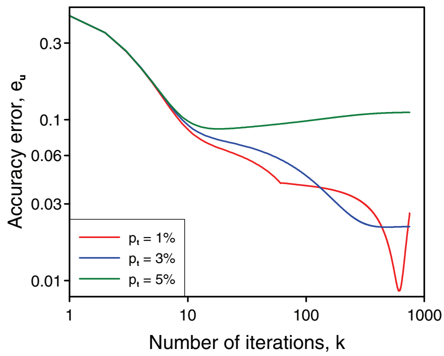
$$\mathrm{E}(k) = \frac{\mathrm{RMS}_{\Gamma_1}\left(\mathbf{u}^{(2k)} - \widetilde{\mathbf{u}}^{\varepsilon}\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{\mathbf{u}}^{\varepsilon}\right)} = \frac{\|\mathbf{u}^{(2k)}|_{\Gamma_1} - \widetilde{\mathbf{u}}^{\varepsilon}|_{\Gamma_1}\|_2}{\|\widetilde{\mathbf{u}}^{\varepsilon}|_{\Gamma_1}\|_2}, \quad k \geq 1, \tag{30}$$

where $u^{(2k)}$ is the displacement vector on Γ_1 retrieved numerically after k iterations by solving the boundary value problem (8). This error E should tend to zero as the sequences $\left\{\mathbf{u}^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{\mathbf{u}^{(2k)}\right\}_{k\geq 1}$ tend to the analytical solution, $\mathbf{u}^{(\mathrm{an})}$, in the space $\mathrm{H}^1(\Omega)^d$ and hence they are expected to provide an appropriate stopping criterion. Indeed, if we investigate the error E obtained at every iteration for Example 1 for various levels of Gaussian random noise added into the input displacement data $\mathbf{u}\big|_{\Gamma_1}$, we obtain the curves graphically represented in Fig. $3(c)$. By comparing Figs. 3(a)–(c), it can be noticed that the convergence error E, as well as the accuracy errors $e_{\mathbf{u}}$ and $e_{\mathbf{t}}$, attain their corresponding minimum at around the same number iterations. Therefore, a natural stopping criterion terminates the MFS iterative algorithm at the optimal number of iterations, k_{opt}, given by:

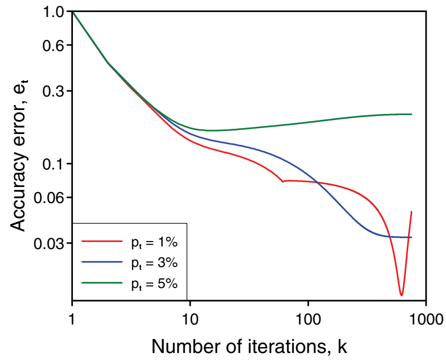
$$k_{\mathrm{opt}}: \quad \mathrm{E}(k_{\mathrm{opt}}) = \min_{k\geq 1} \mathrm{E}(k). \tag{31}$$

6.5 Stability of the algorithm

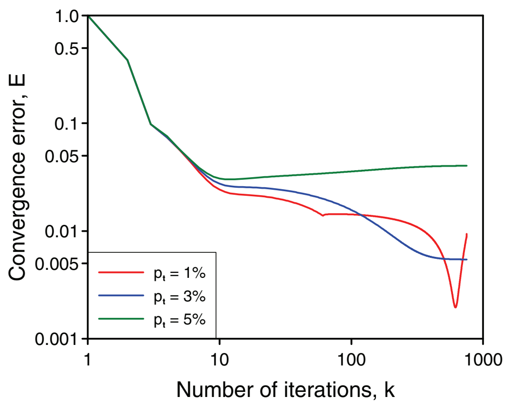
Based on the stopping criterion (31), the analytical and numerical values for the displacement, **u**, and traction vectors, **t**, on Γ_2, obtained using the initial guess (26b) and various levels of noise added into the Dirichlet data on Γ_1 for Example 1, are illustrated in Figs. 4(a)–(d). From these figures it can be seen that the numerical



(a) Example 1: Accuracy error $e_{\mathbf{u}}$



(b) Example 1: Accuracy error $e_{\mathbf{t}}$

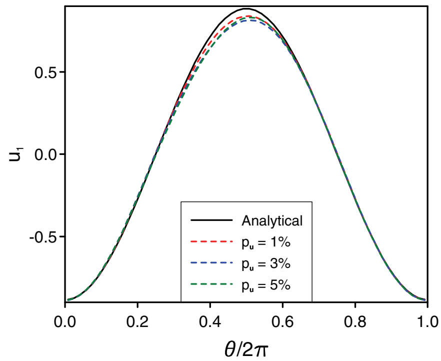


(c) Example 1: Convergence error E

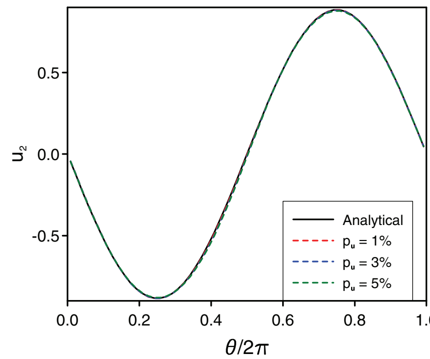
Figure 3: The accuracy errors (a) $e_{\mathbf{u}}$ and (b) $e_{\mathbf{t}}$, and (c) the convergence error E, as functions of the number of iterations, k, obtained using the alternating iterative algorithm with initial guess $\mathbf{t}^{(1)}\big|_{\Gamma_2} = \mathbf{0}$, and various amounts of noise added into the traction vector $\mathbf{t}\big|_{\Gamma_1}$, i.e. $p_{\mathbf{t}} \in \{1\%, 3\%, 5\%\}$, for Example 1.

solution is a stable approximation for the exact solution, free of unbounded and rapid oscillations, and it also converges to the exact solution as $p_{\mathbf{u}}$ decreases.

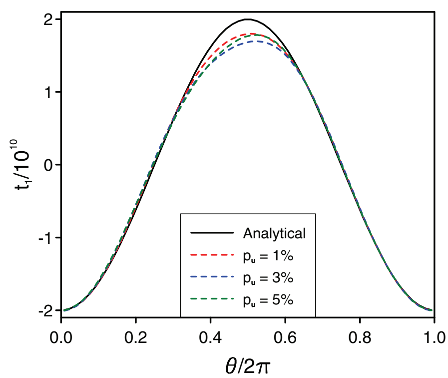
For Example 1, very satisfactory results have also been retrieved for both the unknown displacement, $\mathbf{u}\big|_{\Gamma_2}$, and traction vectors, $\mathbf{t}\big|_{\Gamma_2}$, when using the stopping criterion (31), the initial guess (26b) and various levels of noise added into the Neumann data on Γ_1, namely $p_{\mathbf{t}} \in \{1\%, 3\%, 5\%\}$, and these are presented in Figs. 5(a)–(d). By comparing Figs. 4 and 5 we can conclude that the numerical results



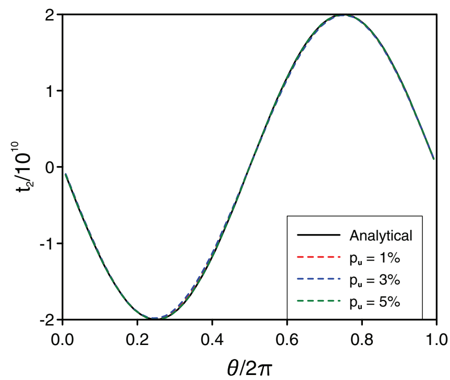
(a) Example 1: Displacement $u_1|_{\Gamma_2}$



(b) Example 1: Displacement $u_2|_{\Gamma_2}$



(c) Example 1: Traction $t_1|_{\Gamma_2}$

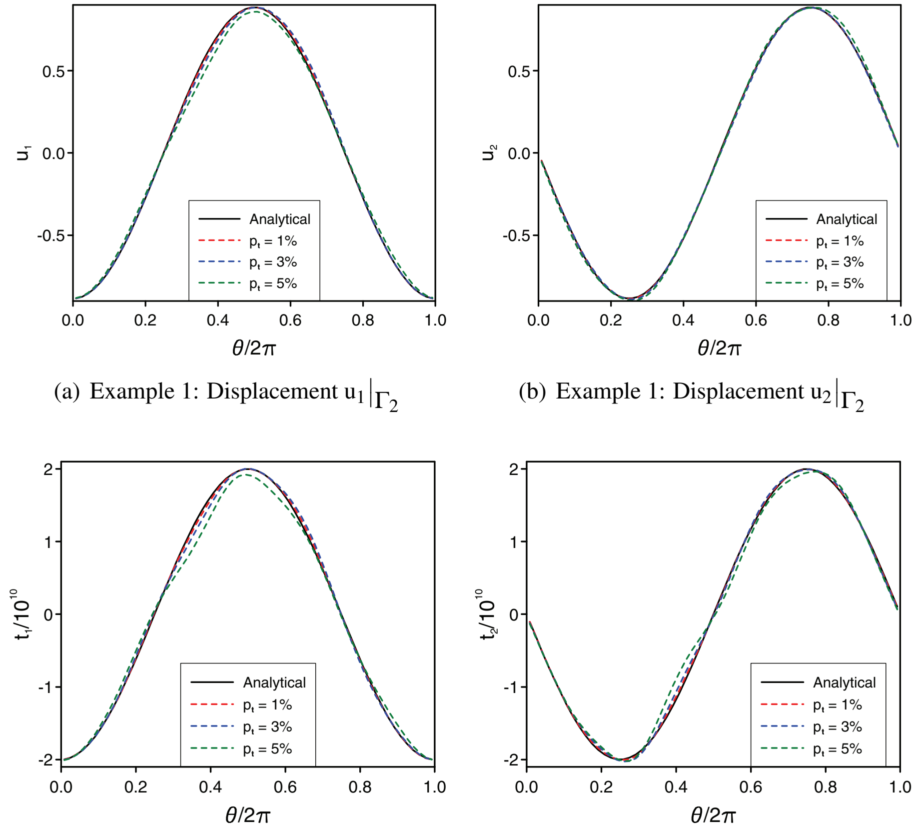


(d) Example 1: Traction $t_2|_{\Gamma_2}$

Figure 4: The analytical and numerical displacements (a) $u_1|_{\Gamma_2}$ and (b) $u_2|_{\Gamma_2}$, and tractions (c) $t_1|_{\Gamma_2}$ and (d) $t_2|_{\Gamma_2}$, obtained using the alternating iterative algorithm with initial guess $\mathbf{t}^{(1)}|_{\Gamma_2} = \mathbf{0}$, and various amounts of noise added into the displacement vector $\mathbf{u}|_{\Gamma_1}$, i.e. $p_{\mathbf{u}} \in \{1\%, 3\%, 5\%\}$, for Example 1.

obtained using the proposed MFS iterative algorithm, in conjunction with the stopping criterion (31), are more sensitive to perturbations in the displacements on the over-specified boundary than to noisy boundary tractions on Γ_1.

Similar stable numerical results for both the unknown displacement, $\mathbf{u}|_{\Gamma_2}$, and traction vectors, $\mathbf{t}|_{\Gamma_2}$, which are at the same time free of unbounded and rapid oscillations, have been obtained for the Cauchy problem (1) and (5) corresponding to an isotropic linear elastic solid occupying a simply connected domain with a smooth



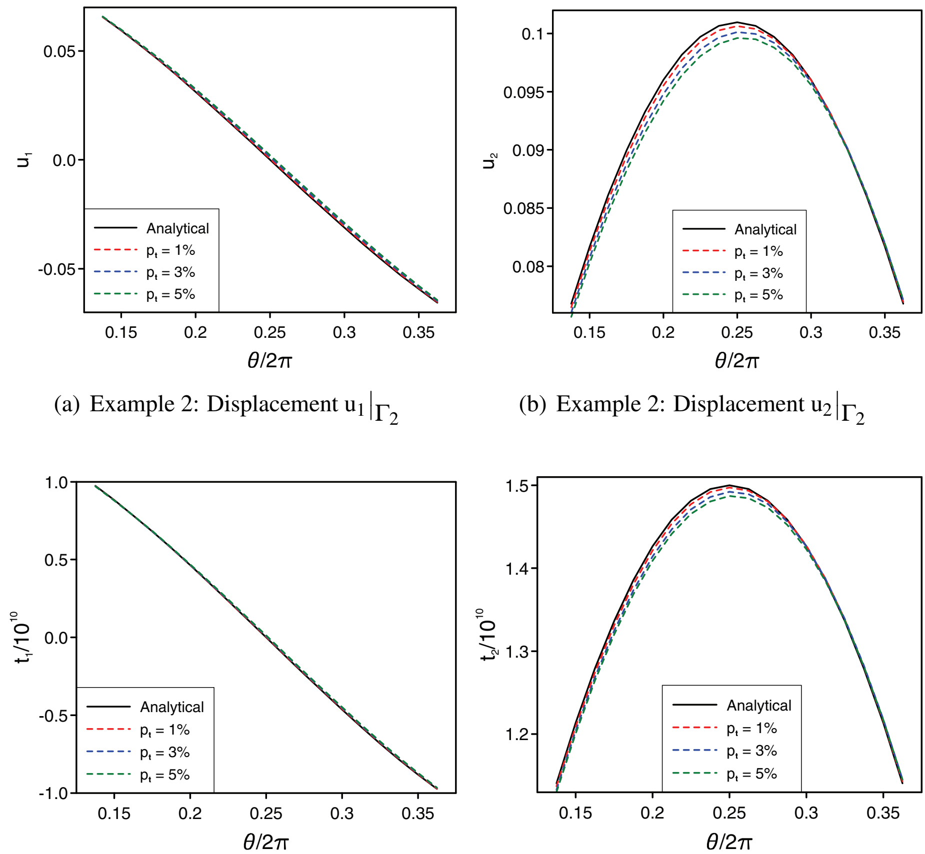
(a) Example 1: Displacement $u_1|_{\Gamma_2}$ (b) Example 1: Displacement $u_2|_{\Gamma_2}$

(c) Example 1: Traction $t_1|_{\Gamma_2}$ (d) Example 1: Traction $t_2|_{\Gamma_2}$

Figure 5: The analytical and numerical displacements (a) $u_1|_{\Gamma_2}$ and (b) $u_2|_{\Gamma_2}$, and tractions (c) $t_1|_{\Gamma_2}$ and (d) $t_2|_{\Gamma_2}$, obtained using the alternating iterative algorithm with initial guess $\mathbf{t}^{(1)}|_{\Gamma_2} = \mathbf{0}$, and various amounts of noise added into $\mathbf{t}|_{\Gamma_1}$, i.e. $p_{\mathbf{t}} \in \{1\%, 3\%, 5\%\}$, for Example 1.

boundary, namely the disk considered in Example 2. Figs. 6(a)–(d) illustrate the numerical results for displacements and tractions on the boundary Γ_2, obtained using the stopping criterion (31), the initial guess (26a) and various amounts of noise added into the traction data on Γ_1, namely $p_{\mathbf{t}} \in \{1\%, 3\%, 5\%\}$, in comparison with their corresponding analytical values, in the case of Example 2.

The proposed MFS-alternating iterative algorithm, in conjunction with the stopping criterion (31), also works reasonably for the Cauchy problem (1) and (5) associated



(a) Example 2: Displacement $u_1|_{\Gamma_2}$

(b) Example 2: Displacement $u_2|_{\Gamma_2}$

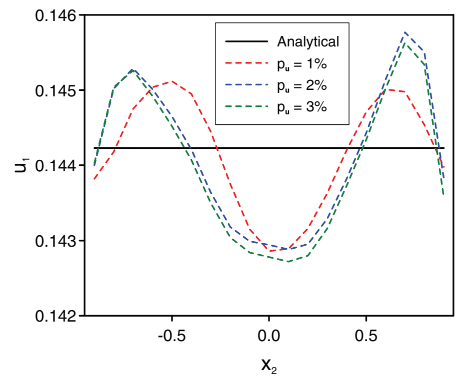
(c) Example 2: Traction $t_1|_{\Gamma_2}$

(d) Example 2: Traction $t_2|_{\Gamma_2}$

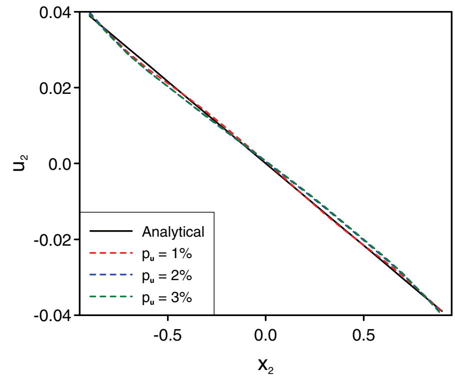
Figure 6: The analytical and numerical displacements (a) $u_1|_{\Gamma_2}$ and (b) $u_2|_{\Gamma_2}$, and tractions (c) $t_1|_{\Gamma_2}$ and (d) $t_2|_{\Gamma_2}$, obtained using the alternating iterative algorithm with initial guess $\mathbf{u}^{(1)}|_{\Gamma_2} = \mathbf{0}$, and various amounts of noise added into $\mathbf{t}|_{\Gamma_1}$, i.e. $p_\mathbf{t} \in \{1\%, 3\%, 5\%\}$, for Example 2.

with an isotropic linear elastic material occupying a simply connected domain with a piecewise smooth boundary, such as the rectangle investigated in Example 3. Figs. 7(a)–(d) show the analytical and numerical values for displacements and tractions on the boundary Γ_2, retrieved using the stopping criterion (31), the initial guess (26a) and various amounts of noise added into the displacement vector on Γ_1, namely $p_\mathbf{u} \in \{1\%, 2\%, 3\%\}$, for Example 3.

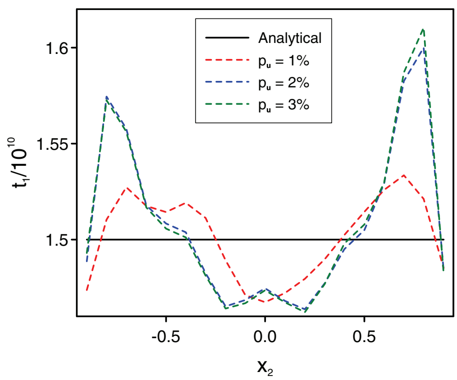
The numerical results obtained using the MFS-based iterative algorithm of



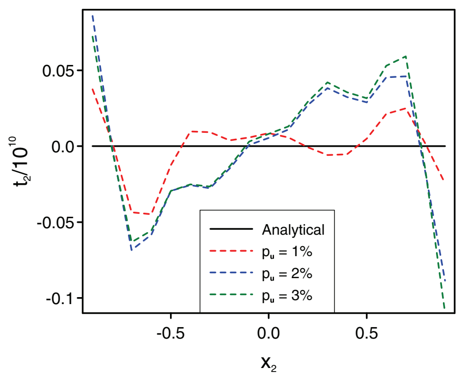
(a) Example 3: Displacement $u_1|_{\Gamma_2}$



(b) Example 3: Displacement $u_2|_{\Gamma_2}$



(c) Example 3: Traction $t_1|_{\Gamma_2}$



(d) Example 3: Traction $t_2|_{\Gamma_2}$

Figure 7: The analytical and numerical displacements (a) $u_1|_{\Gamma_2}$ and (b) $u_2|_{\Gamma_2}$, and tractions (c) $t_1|_{\Gamma_2}$ and (d) $t_2|_{\Gamma_2}$, obtained using the alternating iterative algorithm with initial guess $\mathbf{u}^{(1)}|_{\Gamma_2} = \mathbf{0}$, and various amounts of noise added into $\mathbf{t}|_{\Gamma_1}$, i.e. $p_{\mathbf{u}} \in \{1\%, 2\%, 3\%\}$, for Example 3.

Kozlov, Maz′ya and Fomin (1991) for Cauchy problems in isotropic linear elasticity in simply connected domains with a smooth or piecewise smooth boundary, such as those given by Examples 2 and 3, respectively, are remarkable. More specifically, both the reconstructed displacement and traction vectors using the MFS iterative algorithm described in Sections $3-5$ are more accurate than their counterparts retrieved by employing a similar but BEM-based iterative algorithm, see e.g. Marin, Elliott, Ingham and Lesnic (2001).

7 Conclusions

In this paper, the iterative algorithm of Kozlov, Maz′ya and Fomin (1991) was implemented, for the Cauchy problem in two-dimensional isotropic linear elasticity, using a meshless method. The two mixed, well-posed and direct problems corresponding to every iteration of the numerical procedure were solved using the MFS, in conjunction with the Tikhonov regularization method, while the optimal value of the regularization parameter was selected according to the GCV criterion. An efficient regularizing stopping criterion which terminates the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, was also presented. The MFS-based iterative algorithm was tested for Cauchy problems associated with isotropic linear elastic materials occupying simply and doubly connected two-dimensional domains, with smooth or piecewise smooth boundaries.

From the numerical results presented in this study, it can be concluded that the proposed method is consistent, accurate, convergent with respect to increasing the number of MFS boundary collocation points and stable with respect to decreasing the amount of noise added into the Cauchy data. One possible disadvantage of the MFS-based iterative algorithm is related to the optimal choice of the regularization parameter associated with the Tikhonov regularization method which requires, at each step of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), additional iterations with respect to the regularization parameter. However, this inconvenience can be overcome by introducing relaxation procedures in the MFS iterative algorithm and this is currently under investigation.

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