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An Enhanced Fictitious Time Integration Method for Non-Linear Algebraic Equations With Multiple Solutions: Boundary Layer, Boundary Value and Eigenvalue Problems

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Abstract: When problems in engineering and science are discretized, algebraic equations appear naturally. In a recent paper by Liu and Atluri, non-linear algebraic equations (NAEs) were transformed into a nonlinear system of ODEs, which were then integrated by a method labelled as the Fictitious Time Integration Method (FTIM). In this paper, the FTIM is enhanced, by using the concept of a *repellor* in the theory of *nonlinear dynamical systems*, to situations where multiple-solutions exist. We label this enhanced method as MSFTIM. MSFTIM is applied and illustrated in this paper through solving boundary-layer problems, boundary-value problems, and eigenvalue problems with multiple solutions.

Keywords: Non-linear algebraic equations, Ordinary differential equations, Multiple-Solution Fictitious Time Integration Method (MSFTIM), Repellor, Attracting set

1 Introduction

Many problems in engineering and science require the solutions of non-linear equations. Systems of finitely many Non-Linear Algebraic Equations (NAEs) in several real variables occur in many fields of applications and, correspondingly, they differ widely in form and properties.

For solving the engineering problems, numerical methods used in computational mechanics, such as those demonstrated by Atluri and Zhu (1998), Zhu, Zhang and Atluri (1999), Atluri (2002), Atluri and Shen (2002), and Atluri, Liu and Han (2003) lead to the solution of a system of linear algebraic equations for a linear

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problem, and of a NAEs system for a non-linear problem. The numerical solutions of infinite-dimensional operator equations generally involve the construction of finite-dimensional analogues, and usually these approximations inherit non-linearities and multiplicity of solutions existent in the original operators. Besides, to deal with many practical non-linear engineering problems, the methods such as the finite element method, the boundary element method, the distinct element method, and the meshless method, etc., also lead to a system of non-linear algebraic equations.

Over the past several decades, many contributions have been made towards the numerical solutions of NAEs. Among these methods, Newton's method is perhaps the best known method for finding approximations to the roots of a real-valued non-linear system. Since it converges quadratically, Newton's method can often converge remarkably quickly if the initial guess is sufficiently close to the root. However, a Newton-like algorithm is sensitive to the initial guess of solution, which is known to be locally convergent, and is expensive in the computations of the Jacobian matrix and its inverse at each iterative step. In addition, Newton's method may stagnate at a point in which the solution search may fail when a singular Jacobian matrix is encountered. For solving a large system of non-linear algebraic equations, a novel method, namely the fictitious time integration method (FTIM), has been proposed by Liu and Atluri . Based on a novel continuation method, the FTIM embeds the linear or non-linear algebraic equations into a system of nonautonomous first order ordinary differential equations (ODEs) in a time-like or fictitious variable.

The fixed point of these ODEs, which is the root for the original algebraic equation, is obtained by applying numerical integrations to the resultant nonlinear ODEs, which do not require the information of the Jacobian and its inverse. Based on a time marching algorithm, Liu introduced the use of the FTIM to solve the non-linear obstacle problems. The FTIM has also been adopted to tackle two-dimensional quasilinear elliptic boundary value problems. It is interesting that the FTIM can easily deal with the non-linear boundary value problems and has high efficiency as well as high accuracy. Liu and Atluri proposed the idea of using the FTIM for solving a non-linear optimization problem (NOP) under multiple equality and inequality constraints. The Kuhn-Tucker optimality conditions are used to transform the NOP into a mixed complementarity problem. With the aid of NCP-functions a set of non-linear algebraic equations are obtained; then the FTIM is used to solve these non-linear equations. Furthermore, Liu and Atluri proposed the use of the FTIM for solving the discretized inverse Sturm-Liouville problems, and Liu solved the m-point boundary value problems (BVPs). The FTIM was proved also effective for solving the Poisson-type non-linear partial differential equations by Tsai, Liu and Yeih (2010). The FTIM has recently been demonstrated to be an important numerical tool in its ability to solve a certain class of problems more effectively than the Newton-like methods [Dennis and More (1974,1977); Spedicato and Huang (1997)], in that the FTIM does not need to calculate the Jacobian matrix and its inverse and is thus very efficient, and that the FTIM is insensitive to the guessing of initial conditions, and it is thus easy to find the solutions. It is believed that this new method may bring a major revolution to the computational fields. With the aim of improving the convergence and ease of implementation of the FTIM, a new time-like function with the incorporation of the FTIM is proposed by Ku, Yeih, Liu and Chi. By adding a control parameter, the FTIM with the new time-like function can improve the performance of FTIM for solving highly non-linear BVPs and gives an important control to assure the convergence of the solution during the time integration process.

The Sturm-Liouville problem has been of considerable physical interest and is rather important in many fields, including partial differential equations, vibrations of continua, and quantum mechanics. In most cases, it is not possible to obtain the eigenvalues of Sturm-Liouville problem analytically. However, there are various numerical methods to approximate them. Pryce (1993) has provided a comprehensive review of the mathematical background of Sturm-Liouville problems, and their numerical solutions, as well as a detailed discussion of applications.

There is a continued interest in the numerical solution of Sturm-Liouville problems and associated Schrödinger equations with the aim to improve convergence rates, and ease of implementation of different algorithms. In order to obtain more efficient numerical results, several numerical methods have been developed in the past several years, e.g., Andrew (1994, 2000a), Andrew and Paine (1985, 1986), Celik (2005a, 2005b), Celik and Gokmen (2005), Condon (1999), Ghelardoni (1997), Ghelardoni, Gheri and Marletta (2001, 2006), Vanden Berghe and De Meyer (1991, 1995, 2007), and Yücel (2006). Among them, the most influential one is the algebraically asymptotic correction method, which is reviewed by Andrew (2000b). Ghelardoni and Gheri (2001) have discussed a shooting technique for computing eigenvalues, and furthermore, Liu has proposed a Lie-group shooting method to solve the eigenvalue problems of the Sturm-Liouville type.

Usually, the following Sturm-Liouville problem:

$$-\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right]+q(x)y(x)=\lambda s(x)y(x),\ \ x_0<x<x_f, \tag{1}$$

$$y(x_0)=0,\ \ y(x_f)=0 \tag{2}$$

is discretized into a matrix eigenvalue problem:

$$\mathbf{A}\mathbf{x}=\lambda\mathbf{x}. \tag{3}$$

However, even by means of the most advanced numerical techniques, it is difficult to find all multiple solutions of the above eigenvalue problem.

Another famous problem which has multiple solutions is the boundary layer equation . When an incompressible flow passes in the vicinity of solid boundaries, the Navier-Stokes equations can be simplified drastically. The boundary layer theory was first proposed by Prandtl in 1904. It asserts that the viscous effect would be confined to a thin shear layer adjacent to the solid boundary in the case of motion of a fluid with very little viscosity. Hence, the fluid motion is split into two parts: near the boundary the viscosity effect is important and the fluid is said to be viscous, and far away from the boundary the fluid viscous effect is unimportant and the fluid can be treated as being inviscid.

As a computational method for finding all the solutions of NAEs, interval analysis based techniques are well known, and various algorithms have been developed, for example, Nakaya and Oishi (1998), Yamamura, Kawata and Tokue (1998), Yamamura (2000), Yamamura and Tanaka (2002), and Yamamura and Fujioka (2003).

In this paper we will study the boundary-layer equations by utilizing the method of Differential Quadrature (DQ), and the FTIM is used for finding the multiple solutions. This paper is arranged as follows. In Section 2 we introduce some continuous methods to solve the NAEs. Section 3 is devoted to the development of an improved FTIM to deal with situations of multiple solutions to the nonlinear algebraic wquations (MSFTIM), based on the concept of "decomposition of factors" and "repellor". Then we apply the MSFTIM to solve some NAEs with multiple solutions in Section 4. In Section 5 we introduce the methods of differential quadrature and integral quadrature, which are then used to derive the NAEs for non-linear boundary value problems in Section 6, and the MSFTIM is used to search for multiple solutions. Finally, we draw some conclusions in Section 7.

2 From discrete to continuous methods for solving the NAEs

For the following algebraic equations:

$$F_i(x_1,\dots,x_n)=0,\ \ i=1,\dots,n, (\text{or } \ \mathbf{F}(\mathbf{x})=\mathbf{0}), \tag{4}$$

the Newton method is given by

$$\mathbf{x}_{k+1}=\mathbf{x}_k-[\mathbf{B}(\mathbf{x}_k)]^{-1}\mathbf{F}(\mathbf{x}_k), \tag{5}$$

where we use $\mathbf{x}:=(x_1,\dots,x_n)^{\mathrm{T}}$ and $\mathbf{F}:=(F_1,\dots,F_n)^{\mathrm{T}}$ to represent the vectors, and $\mathbf{B}$ is an $n\times n$ Jacobian matrix with its ij-th component given by $B_{ij}=\partial F_i/\partial x_j$. Starting from an initial guess of solution by $\mathbf{x}_0$, Eq. (5) can be used to generate a sequence of $\mathbf{x}_k$, $k = 1, 2, \ldots$. When $\mathbf{x}_k$ are convergent under a specified convergence criterion, the solutions of Eq. (4) are obtained.

The Newton method has a great advantage that it is quadratically convergent. However, it still has some drawbacks of not being easy to guess the initial solution, and the computational burden of finding $[\mathbf{B}(\mathbf{x}_k)]^{-1}$.

Hirsch and Smale (1979) and many others have derived a continuous Newton method governed by the following differential equation:

$$\dot{\mathbf{x}}(t) = -\mathbf{B}^{-1}(\mathbf{x})\mathbf{F}(\mathbf{x}), \tag{6}$$

$$\mathbf{x}(0) = \mathbf{a}, \tag{7}$$

where t is an artificial time, and $\mathbf{a}$ is an initial guess of $\mathbf{x}$. It can be seen that the ODEs in Eq. (6) are difficult to integrate, because they involve a matrix inversion.

The corresponding dynamics of Eq. (6) has been studied by several researchers, such as, Alber (1971), Boggs and Dennis (1976), Smale (1976), Chu (1988), Maruster (2001), and Ascher, Huang and van den Doel (2007). Presently, this artificial time embedding technique does not bring out any practically useful result pertaining to the Newton's algorithm.

More simpler is the following system of ODEs:

$$\dot{\mathbf{x}}(t) = -\mathbf{F}(\mathbf{x}), \tag{8}$$

$$\mathbf{x}(0) = \mathbf{a}, \tag{9}$$

which are proposed by Ramm (2007). However, the iteration procedure generated from Eq. (8) is very sensitive to the initial guess and may have a very slow convergence.

Iterative processes for the solution of finite-dimensional non-linear equations vary almost as widely in form and properties as do the equations themselves. The major advantages in recasting the NAEs into an ODE form are that we can use the ideas from the theory of finite-dimensional non-linear dynamical systems, to guide us in developing more effective iteration methods.

The *repellor* defind in the theory of non-linear dynamical systems is a fixed point, which repels nearby trajectories. Here, we will develop a new system of ODEs, which is equivalent to Eq. (4), where the concept of *repellor* will be slightly relaxed to the singular point, but with the same *repelling property*. Then, a natural technique of recasting the NAEs into a system of ODEs, as developed by Liu and Atluri, will be combined with a new technique for finding the multiple solutions of NAEs. Corresponding to the artificial embedding technique, which is not yet proven to be useful, our embedding technique of transforming Eq. (4) into a continuous form in a space, which is one-dimension higher, may be found to be very useful. Here we develop an additional novel technique to find the all multiple solutions.

3 The FTIM in the presence of multiple solutions

Liu and Atluri have embedded the n-dimensional NAEs, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, into a system of nonlinear ODEs:

$$\dot{\mathbf{x}} = -\frac{\nu}{q(t)}\mathbf{F}(\mathbf{x}). \tag{10}$$

In the above, $\dot{\mathbf{x}} = d\mathbf{x}/dt$, t being a fictitious time-like parameter. We should stress that the factor $-\nu/q(t)$ in front of $\mathbf{F}(\mathbf{x})$ is important to ensure the convergence of solution. As a special case, Liu and Atluri have taken $q(t) = 1+t$.

To motivate the present approach for solving the NAEs, in the presence of multiple solutions, by the FTIM, we consider a simple NAE in one variable with three solutions:

$$F(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

It is easy to check that $x = 1$ is a solution. Then we divide $F(x)$ by $x-1$ and search the second solution by

$$F_1(x) = \frac{F(x)}{x-1} = x^2 - 5x + 6 = 0.$$

When $x = 2$ is a solution, we solve

$$F_2(x) = \frac{F_1(x)}{x-2} = \frac{F(x)}{(x-1)(x-2)} = x - 3 = 0.$$

Thus, $x = 3$ is the third solution. The above process is usually known as the method of decomposition of factors.

Suppose that Eq. (4) has m solutions. In order to obtain the m multiple solutions of Eq. (4) we use the following technique:

$$\begin{cases} \dot{\mathbf{x}} = -\frac{\nu_1}{q(t)}\mathbf{F}(\mathbf{x}), & \mathbf{x}(0) = \mathbf{x}_{10}, \\ \dot{\mathbf{x}} = -\frac{\nu_2}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_1\|}, & \mathbf{x}(0) = \mathbf{x}_{20} = \mathbf{x}_1 + \mathbf{T}_1, \\ \dot{\mathbf{x}} = -\frac{\nu_3}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_1\|\|\mathbf{x}-\mathbf{x}_2\|}, & \mathbf{x}(0) = \mathbf{x}_{30} = \mathbf{x}_2 + \mathbf{T}_2, \\ \vdots & \vdots \\ \dot{\mathbf{x}} = -\frac{\nu_m}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_1\|\|\mathbf{x}-\mathbf{x}_2\|\ldots\|\mathbf{x}-\mathbf{x}_{m-1}\|}, & \mathbf{x}(0) = \mathbf{x}_{m0} = \mathbf{x}_{m-1} + \mathbf{T}_{m-1}. \end{cases} \tag{11}$$

For simplicity, we only take $q(t) = 1+t$, even though other monotonically increasing functions of t can be chosen for $q(t)$. Here, $\mathbf{x}_1$ is a steady-state solution of the first equation with a guessed initial value $\mathbf{x}_{10}$. When $\mathbf{x}_1$ satisfies a certain convergence criterion, say $\|\mathbf{F}(\mathbf{x}_1)\| < \varepsilon$, we obtain the first solution of the NAEs in Eq. (4). $\mathbf{T}_1$ is a translation vector, in order to push out the initial condition of the second equation from the attracting set of the first set of roots, and so on, $\mathbf{x}_2$ is a steady-state solution of the second equation with an initial value $\mathbf{x}_1 + \mathbf{T}_1$, and $\mathbf{T}_2$ is a translation vector, in order to push out the initial condition of the third equation from the attracting set of the second set of roots.

In the above, we divide each previous equation by a factor $\|\mathbf{x}-\mathbf{x}_i\|$, which is similar to the above introduced *decomposition of factors* for the scalar equation. Moreover, at each point of $\mathbf{x}_i$, the ODEs are singular, which results in a predominant effect that $\mathbf{x}_i$ is a *repellor*, repelling all the nearby trajectories of $\mathbf{x}(t)$ in the evolution process. Therefore, the path of $\mathbf{x}$ does not tend to these points again when we integrate the new ODEs, and it must tend to other solutions.

Sometimes, the singularity produced by $\|\mathbf{x}-\mathbf{x}_i\|$ in the denominator is not strong enough to avoid the path of $\mathbf{x}$ tending to that point $\mathbf{x}_i$ again, for example, $\mathbf{x}_i$ a double-root, and we may consider the following equations with exponential singularities:

$$\begin{cases}
\dot{\mathbf{x}} = -\frac{\nu_1}{q(t)}\mathbf{F}(\mathbf{x}), & \mathbf{x}(0) = \mathbf{x}_{10}, \\
\dot{\mathbf{x}} = -\frac{\nu_2}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\exp(-1/\|\mathbf{x}-\mathbf{x}_1\|)}, & \mathbf{x}(0) = \mathbf{x}_{20} = \mathbf{x}_1 + \mathbf{T}_1, \\
\dot{\mathbf{x}} = -\frac{\nu_3}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\exp(-1/\|\mathbf{x}-\mathbf{x}_1\|)\exp(-1/\|\mathbf{x}-\mathbf{x}_2\|)}, & \mathbf{x}(0) = \mathbf{x}_{30} = \mathbf{x}_2 + \mathbf{T}_2, \\
\vdots & \vdots \\
\dot{\mathbf{x}} = -\frac{\nu_m}{q(t)}\frac{\mathbf{F}(\mathbf{x})}{\exp(-1/\|\mathbf{x}-\mathbf{x}_1\|)\exp(-1/\|\mathbf{x}-\mathbf{x}_2\|)\ldots\exp(-1/\|\mathbf{x}-\mathbf{x}_{m-1}\|)}, & \mathbf{x}(0) = \mathbf{x}_{m0} = \mathbf{x}_{m-1} + \mathbf{T}_{m-1}.
\end{cases} \tag{12}$$

Again, a simple choice for $q(t)$ can be $q(t) = 1+t$, even though $q(t)$ can be any monotonically increasing functions of t.

In Eqs. (11) and (12) we have introduced two different types of singularity in the denominators, such that when the path of $\mathbf{x}$ approaches to the previous solution, the singularity works and gives a strong force to push out $\mathbf{x}$ from the previous solution. For convenience we may call the singularities in Eqs. (11) and (12) as the algebraic and exponential singularities, respectively.

4 The NAEs with multiple solutions

4.1 Example 1

We apply the MSFTIM technique in Eq. (11) to the following set of NAEs:

$$F_1(x,y) = x^2 - y - 1 = 0, \tag{13}$$

$$F_2(x,y) = y^2 - x - 1 = 0. \tag{14}$$

There are four solutions: the first root is $(x,y) = (-1,0)$, the second root is $(x,y) = (0,-1)$, the third root is $(x,y) = ((1-\sqrt{5})/2, (1-\sqrt{5})/2)$, and the fourth root is $(x,y) = ((1+\sqrt{5})/2, (1+\sqrt{5})/2)$.

Liu, Yeih, Kuo and Atluri solved this problem by using the scalar homotopy method, and they pointed out that the third root and the fourth root are hardly solved by using the FTIM.

First we solve the following ODEs:

$$\frac{dx}{dt} = -\frac{\nu}{1+t}[x^2 - y - 1], \tag{15}$$

$$\frac{dy}{dt} = -\frac{\nu}{1+t}[y^2 - x - 1] \tag{16}$$

by the Euler forward scheme. It is interesting to investigate the attracting set of each fixed point in the plane of initial conditions of $(x(0), y(0))$. Starting from any initial condition in the domain of $-2 < x(0) < 2$, $-2 < y(0) < 2$ we apply the FTIM under a converegence criterion of $\varepsilon = 10^{-4}$, and with $\nu = 0.05$ and $\Delta t = 0.01$ to find its terminal location, and determine which attracting set it belongs by a small disk with a ceneter on each fixed point. We limit the number of solution steps to be smaller than 10000. Consequently, as shown in Fig. 1(a) most points do not converge within 10000 steps. The attracting sets are marked in Fig. 1(a) for each solution.

Next, we demonstrate that the technique of MSFTIM can improve the above situation. For this purpose we consider the following ODEs:

$$\frac{dx}{dt} = -\frac{\nu}{1+t}\frac{x^2 - y - 1}{\sqrt{(x+1)^2 + y^2}\sqrt{x^2 + (y+1)^2}}, \tag{17}$$

$$\frac{dy}{dt} = -\frac{\nu}{1+t}\frac{y^2 - x - 1}{\sqrt{(x+1)^2 + y^2}\sqrt{x^2 + (y+1)^2}}, \tag{18}$$

which are the two-dimensional special cases of the third equation in Eq. (11), and similarly we integrate them by using the Euler forward scheme with the same

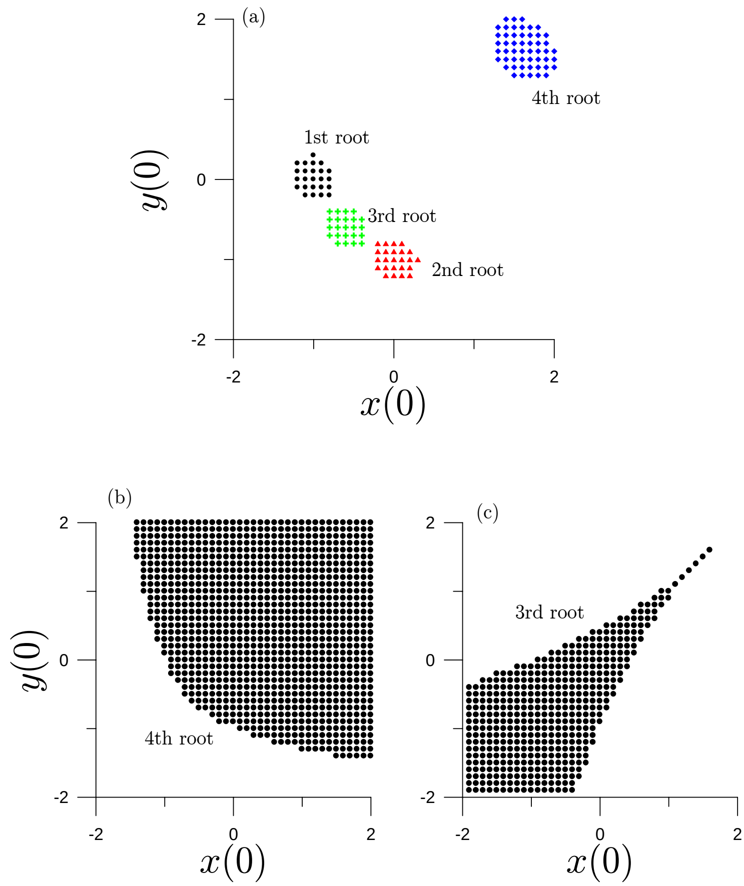

Figure 1: Comparing the attracting sets (a) obtained by the FTIM, (b) obtained by the MSFTIM for the fourth root, and (c) obtained by the MSFTIM third root.

$\Delta t = 0.01$. For $\nu = 5$ (this value is much large than the above one with $\nu = 0.05$, because when we use $\nu = 5$, the numerical integration is still stable), the attracting set for the fourth root is shown in Fig. 1(b), whose size is much large than that shown in Fig. 1(a). For the ODEs in Eqs. (17) and (18) the two points $(-1,0)$ and $(0,-1)$ (the first root and the second root) become repellors, and thus in the plane of $-2 < x(0) < 2,\ -2 < y(0) < 2$ there does not exist an attracting set for these two solutions. Similarly, for $\nu = -5$ the attracting set for the third root is shown in Fig. 1(c), whose size is also much larger than that shown in Fig. 1(a). Therefore, we can use the above MSFTIM technique to easily find the third and the fourth solutions. Originally, by using the FTIM it was hard to find the third and the fourth solutions, because a suitable initial condition is very hard to find.

4.2 Example 2

As a test we apply the MSFTIM technique in Eq. (11) to the following set of NAEs:

$$F_1(x,y) = x^3 - 3xy^2 + a_1(2x^2 + xy) + b_1y^2 + c_1x + a_2y = 0, \tag{19}$$

$$F_2(x,y) = 3x^2y - y^3 - a_1(4xy - y^2) + b_2x^2 + c_2 = 0, \tag{20}$$

with $a_1 = 25$, $b_1 = 1$, $c_1 = 2$, $a_2 = 3$, $b_2 = 4$ and $c_2 = 5$.

Liu and Atluri have solved the above problem by using the FTIM. Liu, Yeih, Kuo and Atluri also solved this problem by using the scalar homotopy method, while Atluri, Liu and Kuo (2009) used the modified Newton method. They found three solutions by guessing three different initial conditions.

In this calculation by using the MSFTIM, we apply the group-preserving scheme [Liu (2001)] to integrate the resulting ODEs with a fixed step size $\Delta t = 0.01$, and with a convergence criterion $\varepsilon = 10^{-8}$. Other parameters used in the calculation are $\nu_1 = 0.1$, $\nu_2 = -1.5$, $\nu_3 = 50$, $\nu_4 = 5000$, $(x_0, y_0) = (5,5)$, $\mathbf{T}_1 = (15,15)$, $\mathbf{T}_2 = (25,25)$, and $\mathbf{T}_3 = (40,-40)$. We found that there exist four roots as shown in Fig. 2. The first root we obtain is $(-50.3970755, -0.80424262)$ through 805 iteration steps with a residual error 7.88×10^{-9}; the second root is $(0.62774247, 22.2444123)$ through 1486 iteration steps with a residual error 9.95×10^{-9}; the third root is $(36.045401914, 36.80750808)$ through 1879 iteration steps with a residual error 9.86×10^{-9}; and the fourth root is $(50.465039997, -37.2634179)$ through 1754 iteration steps with a residual error 9.91×10^{-9}.

Indeed, the second root was not found previously by Liu and Atluri. As shown in Fig. 3(a) the path generated from the second equation in Eq. (11) tends to a point between the attracting sets of other solutions. This is the reason why it is hard to find the second root by using the original differential equtions derived from the FTIM. However, as shown in Fig. 3(b) the attracting set of the second equation of the MSFTIM for the second solution is quite large in the whole domain of $-60 < x(0) < 0$, and $0 < y(0) < 40$, and thus we are able to find the second solution. It is interesting that the above domain is originally an attracting set for the first root as shown in Fig. 3(a), but now it becomes the attracting set for the second root as shown in Fig. 3(b). From the above discussions, it can be seen that the performance of the MSFTIM in finding all the solutions is very high, and a good accuracy can be achieved.

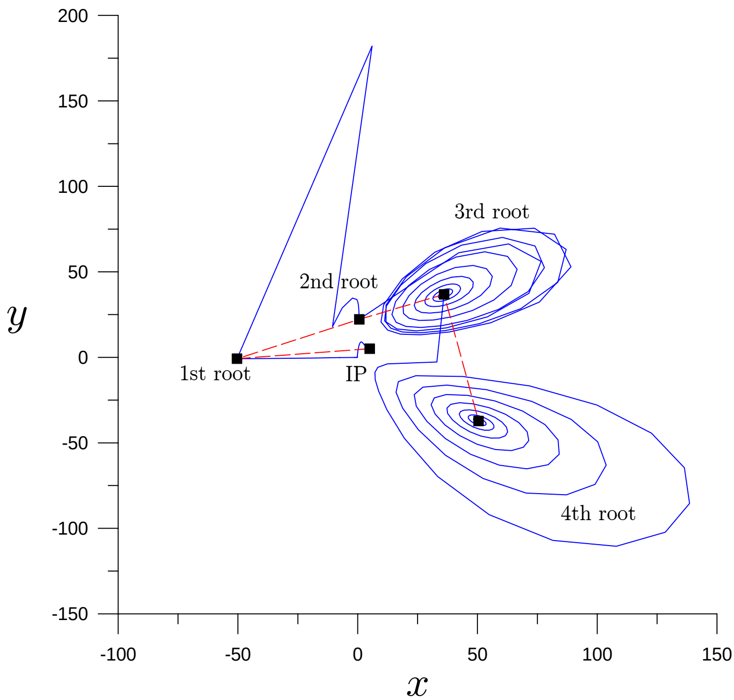

Figure 2: The path for four solutions obtained by the MSFTIM.

4.3 A matrix eigenvalue problem

Taking the inner product of Eq. (3) with $\mathbf{x}$, leads to

$$\lambda = \frac{\mathbf{x} \cdot (\mathbf{A}\mathbf{x})}{\|\mathbf{x}\|^2}. \tag{21}$$

Upon inserting it into Eq. (3) we can obatin a system of NAEs:

$$\mathbf{A}\mathbf{x} - \frac{\mathbf{x} \cdot (\mathbf{A}\mathbf{x})}{\|\mathbf{x}\|^2}\mathbf{x} = \mathbf{0}. \tag{22}$$

Introducing

$$\mathbf{n} := \frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{23}$$

as a unit eigenvector, we can write

$$\mathbf{A}\mathbf{n} - \mathbf{n} \cdot (\mathbf{A}\mathbf{n})\mathbf{n} = \mathbf{0}. \tag{24}$$

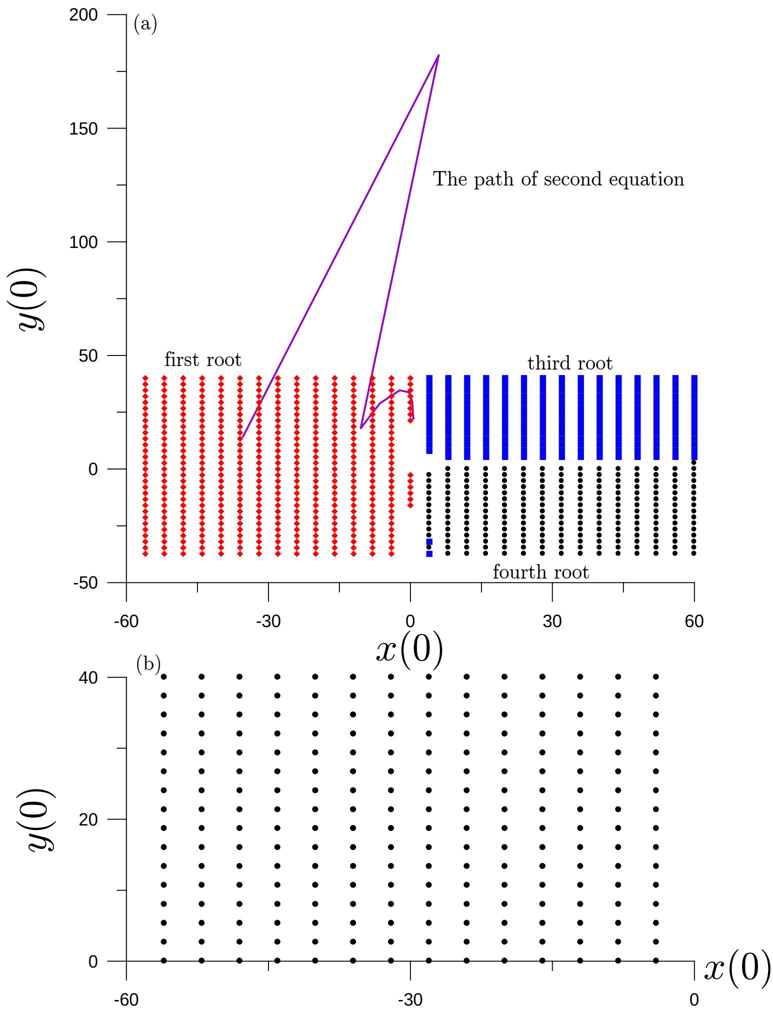

Figure 3: Comparing the attracting sets and the path of second equation in (a), and (b) displaying the attracting set for the second root obtained by the MSFTIM.

However, for the solution of $\mathbf{n}$ to be unique we can further impose the condition of $n_1 \geq 0$, where n_1 is the first component of $\mathbf{n}$.

We apply the MSFTIM technique (12) to the matrix eigenvalue problem (3) with:

$$\mathbf{A} = \begin{bmatrix} -1.6407 & 1.0814 & 1.2014 & 1.1539 \\ 1.0814 & 4.1573 & 7.4035 & -1.0463 \\ 1.2014 & 7.4035 & 2.7890 & -1.5737 \\ 1.1539 & -1.0463 & -1.5737 & 8.6944 \end{bmatrix}, \tag{25}$$

which has the following exact eigenvalues: $\lambda(\mathbf{A}) = \{12, 8, -4, -2\}$.

We simply use the Euler forward scheme with $\Delta t = 0.001$ to integrate the resulting ODEs, with a convergence criterion $\varepsilon = 10^{-5}$. Other parameters used in the calculation are $v_1 = 10$, $v_2 = -15$, $v_3 = -5$, $v_4 = 5$, $\mathbf{x}_{10} = \mathbf{1}$, $\mathbf{T}_1 = 5\mathbf{1}$, $\mathbf{T}_2 = -0.5\mathbf{1}$, and $\mathbf{T}_3 = -10\mathbf{1}$, where $\mathbf{1} = (1,1,1,1)^{\mathrm{T}}$. The first eigenvalue we obtain is -4.000072 through 654 iteration steps, the second eigenvalue is 12.00005 through 83 iteration steps, the third eigenvalue is 7.99996 through 25220 iteration steps, and the fourth eigenvalue is -1.99994 through 9868 iteration steps. It can be seen that we can obtain all the eigenvalues with an accuracy in the order of 10^{-5}. The patterns of the corresponding unit eigenvectors are plotted in Fig. 4, where n_k denotes the k-th component of $\mathbf{n}$.

5 Differential quadrature and integral quadrature

Since the pioneering work of Bellman and Casti (1971), and Bellman, Kashef and Casti (1972), Differential Quadrature (DQ) has been developed and employed by many researchers. However, due to its ill-conditioned property, this method is limited to the small scale engineering problems. Shu (2000) has developed a systematic method to compute the weighting coefficients, under the analysis of a high-order polynomial approximation and the analysis of a linear vector space.

Bellman and Casti (1971), and Bellman, Kashef and Casti (1972) first proposed the Differential Quadrature (DQ) approximation of derivatives to mimic the integral quadrature. Here, we consider a scalar function $f(x)$ defined in a closed interval $x \in [a,b]$. It is supposed that there are n grid points with coordinates $x_1 = a, x_2, \ldots, x_n = b$. The function $f(x)$ is assumed to be differentiable at any grid point, so that its first-order derivative $f'(x)$ at any grid point x_i can be approximated by

$$f'(x_i) = \sum_{j=1}^{n} a_{ij} f(x_j). \tag{26}$$

In the first approach of Bellman, Kashef and Casti (1972), the test functions are chosen as

$$g_k(x) = x^k, \;\; k = 0, 1, \ldots, n-1, \tag{27}$$

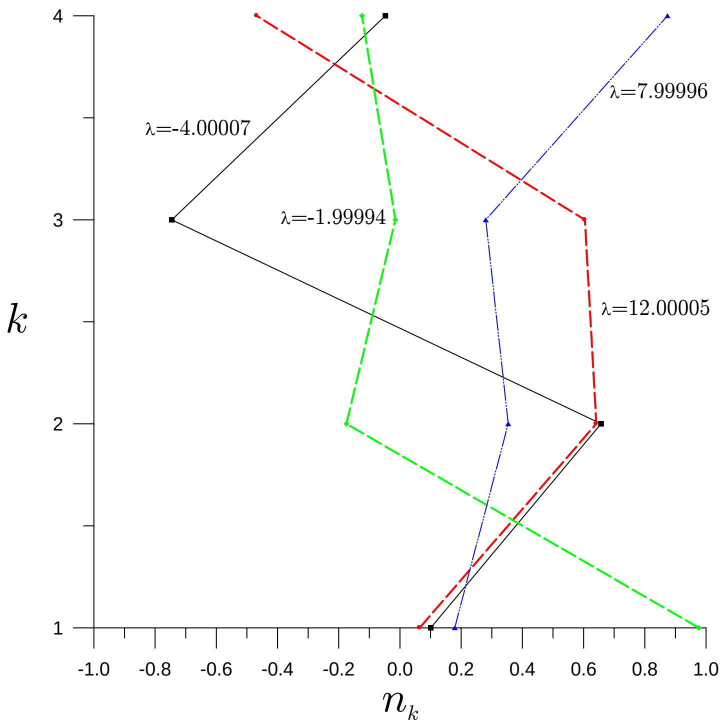

Figure 4: Applying the MSFTIM to a matrix eigenvalue problem with eigenvectors being shown.

such that we have the following algebraic equations to determine the weighting coefficients a_{ij}:

$$\begin{cases} \sum_{j=1}^{n} a_{ij} = 0, \\ \sum_{j=1}^{n} a_{ij} x_j = 1, \\ \sum_{j=1}^{n} a_{ij} x_j^k = k x_i^{k-1}, \quad k = 2, \dots, n-1. \end{cases} \tag{28}$$

Similarly, for the integral quadrature:

$$\int_a^b f(x)dx = \sum_{i=1}^{n} b_i f(x_i), \tag{29}$$

we can derive

$$\begin{cases} \sum_{i=1}^{n} b_i = b - a, \\ \sum_{i=1}^{n} b_i x_i = \frac{b^2 - a^2}{2}, \\ \sum_{i=1}^{n} b_i x_i^k = \frac{b^{k+1} - a^{k+1}}{k+1}, \quad k = 2, \dots, n-1. \end{cases} \tag{30}$$

By inspection, we can see that the above systems are with the Vandermonde matrix as the coefficient matrix. Therefore, we can apply the technique described by Liu and Atluri to solve the above system, i.e., we solve

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \frac{x_1}{R_0} & \frac{x_2}{R_0} & \cdots & \frac{x_{n-1}}{R_0} & \frac{x_n}{R_0} \\ \left(\frac{x_1}{R_0}\right)^2 & \left(\frac{x_2}{R_0}\right)^2 & \cdots & \left(\frac{x_{n-1}}{R_0}\right)^2 & \left(\frac{x_n}{R_0}\right)^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \left(\frac{x_1}{R_0}\right)^{n-2} & \left(\frac{x_2}{R_0}\right)^{n-2} & \cdots & \left(\frac{x_{n-1}}{R_0}\right)^{n-2} & \left(\frac{x_n}{R_0}\right)^{n-2} \\ \left(\frac{x_1}{R_0}\right)^{n-1} & \left(\frac{x_2}{R_0}\right)^{n-1} & \cdots & \left(\frac{x_{n-1}}{R_0}\right)^{n-1} & \left(\frac{x_n}{R_0}\right)^{n-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b-a \\ \frac{b^2-a^2}{2R_0} \\ \vdots \\ \frac{b^{k+1}-a^{k+1}}{(k+1)R_0^k} \\ \vdots \\ \frac{b^n-a^n}{nR_0^{n-1}} \end{bmatrix}, \tag{31}$$

where R_0 is a scaling factor.

6 Non-linear boundary value problems

6.1 Power law fluids

The boundary layer equations are encountered in many engineering applications, such as airfoil, liquid transport by belt conveyor, and many others. We can obtain [Hussaini and Lakin (1986); Soewono, Vajravelu and Mohapatra (1991)] the following ODE:

$$(|f''(\eta)|^{N-1}f''(\eta))' + f(\eta)f''(\eta) = 0, \tag{32}$$

subject to the following boundary conditions:

$$f(0) = -C, \;\; f'(0) = \xi, \;\; f'(+\infty) = 1. \tag{33}$$

In above, $\xi = U_w/U_\infty$ is the velocity ratio. When $0 < \xi < 1$, the speed of the oncoming fluid is larger than that of the plate. When $\xi > 1$, the speed of the moving plate is faster than the speed of the oncoming fluid. When $\xi = 0$ for a resting plate, and $N = 1$ further, the Blasius equation is recovered. The term $C = (N+1)BV_w/U_\infty$ is a constant related to suction if it is negative or injection if it is positive. When $\xi < 0$, there is a reverse flow near the boundary.

Since the 1960s the researchers working on this problem have been using the Crocco transformation in which the tangential velocity f' becomes the new independent variable, by setting

$$z = f'(\eta), \tag{34}$$

while the new dependent variable is the shear force

$$g(z) = [f''(\eta)]^N. \tag{35}$$

Here we propose new method for the computation of the following second-order boundary layer equation:

$$|g|^{1/N} g'' = -z, \;\; \xi < z < 1, \tag{36}$$

$$g'(\xi) = C, \;\; g(1) = 0, \tag{37}$$

which are obtained by the above transformations in Eqs. (34) and (35) applied to the boundary layer equations (32) and (33).

We apply the technique of DQ to Eq. (36) for obtaining a system of NAEs, and then we apply the new MSFTIM in Eq. (11) to obtain the multiple solutions. A numerical example is shown in Fig. 5. Here we simply use the Euler forward scheme with $\Delta t = 0.0001$ to integrate the resulting ODEs, with a convergence criterion $\varepsilon = 10^{-5}$. Other parameters used in the calculation are $N = 0.8$, $\xi = -0.2$, $C = 0.2$, $\nu_1 = -0.5$, $\nu_2 = -1.5$, $\nu_3 = -1.9$, $\mathbf{x}_{10} = 0.08\mathbf{1}$, $\mathbf{T}_1 = 0.8\mathbf{1}$, and $\mathbf{x}_{30} = -0.02\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$.

6.2 A non-linear BVP

Let us consider the following BVP [Kubicek and Hlavacek (1983)]:

$$u'' = a_0^2 u \exp\left[\frac{a_1(1-u)}{1+a_2(1-u)}\right], \tag{38}$$

$$u'(0) = 0, \;\; u(1) = 1. \tag{39}$$

This problem is of the mixed type boundary conditions and has three solutions under $a_0 = 0.16$, $a_1 = 14$ and $a_2 = 0.7$. This problem has been solved by Liu (2006) using the Lie-group shooting method (LGSM). We apply the technique of DQ to Eq. (38) obtaining a system of NAEs, and then we apply the new MSFTIM in Eq. (11) to obtain multiple solutions. The parameters used in this problem are $n = 40$, $\nu_1 = -0.5$, $\nu_2 = -8.5$, $\nu_3 = -36$, $\mathbf{x}_{10} = \mathbf{0}$, $\mathbf{x}_{20} = 0.63\mathbf{1}$, and $\mathbf{x}_{30} = 0.982\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$. In Fig. 6 we can see that the present solutions by using the MSFTIM are rather close to those obtained by the LGSM.

6.3 Sturm-Liouville problem

For this example we consider a Sturm-Liouville problem with [Ghelardoni, Gheri and Marletta (2001); Yücel (2006)]:

$$-y''(x) + e^x y(x) = \lambda y(x), \tag{40}$$

$$y(0) = y(\pi) = 0. \tag{41}$$

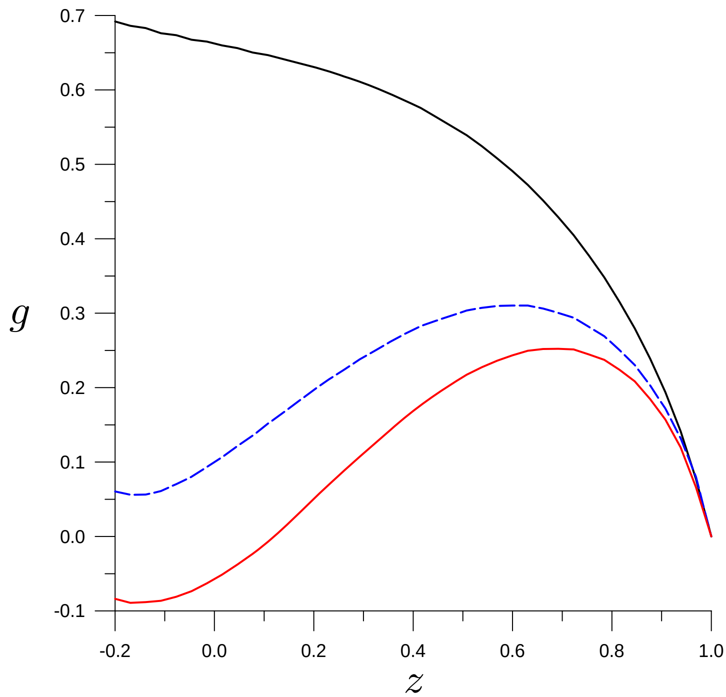

Figure 5: Applying the MSFTIM to power-law boundary layer problem with three solutions.

The eigenvalue did not have a closed-form solution, and Liu first employed the Lie-group shooting method to solve it.

Now, we rearrange it to be a non-linear equation. Multiplying Eq. (40) by y and integrating it in the whole interval with the aid of boundary conditions in Eq. (41), we can obtain:

$$\lambda = \frac{\int_0^\pi [y'(x)^2 + e^x y^2(x)]dx}{\int_0^\pi y^2(x)dx}. \tag{42}$$

Here we can normalize $y(x)$ by $\int_0^\pi y^2(x)dx = 1$, such that

$$\lambda = \int_0^\pi [y'(x)^2 + e^x y^2(x)]dx. \tag{43}$$

When we insert Eq. (43) into Eq. (40), and apply the DQ for Eq. (40) and the IQ for Eq. (43), we can obtain a system of NAEs to solve $y(x)$ at the node points x_i. Then the new MSFTIM can help us to find the multiple solutions as shown in

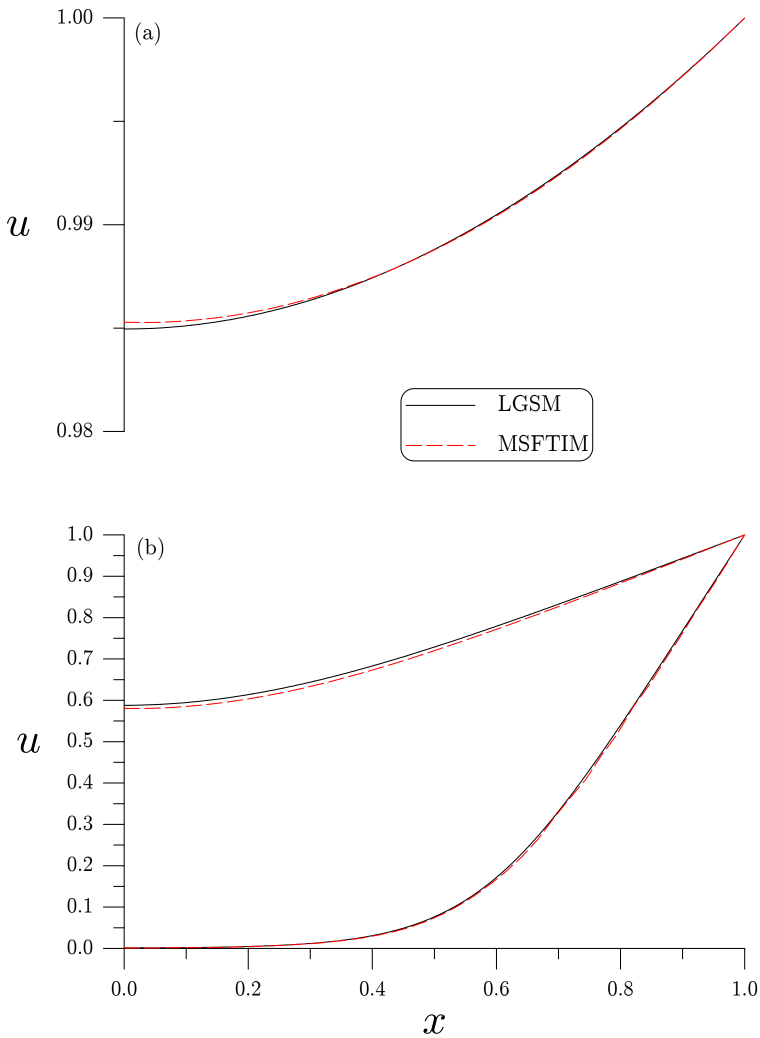

Figure 6: Applying the MSFTIM to a nonlinear boundary value problem with three solutions being shown.

Fig. 7 for the first five eigenfunctions. The parameters used are: $n = 60$, $\nu_1 = -1$, $\nu_2 = -0.01$, $\nu_3 = -0.03$, $\nu_4 = -0.5$, $\nu_5 = -4.25$, $\mathbf{x}_{10} = 0.01\mathbf{1}$, $\mathbf{x}_{20} = \{\sin 2x_i,\ i = 1,\ldots,n\}$, $\mathbf{x}_{30} = \{\sin 3x_i,\ i = 1,\ldots,n\}$, $\mathbf{x}_{40} = \{\sin 4x_i,\ i = 1,\ldots,n\}$, and $\mathbf{x}_{50} = \{\sin 5x_i,\ i = 1,\ldots,n\}$. The eigenvalues computed are, respectively, $\lambda_1 = 4.89665$, $\lambda_2 = 10.32542$, $\lambda_3 = 15.54427$, $\lambda_4 = 22.45851$, and $\lambda_5 = 32.26437$, which are very close to those calculated by Liu.

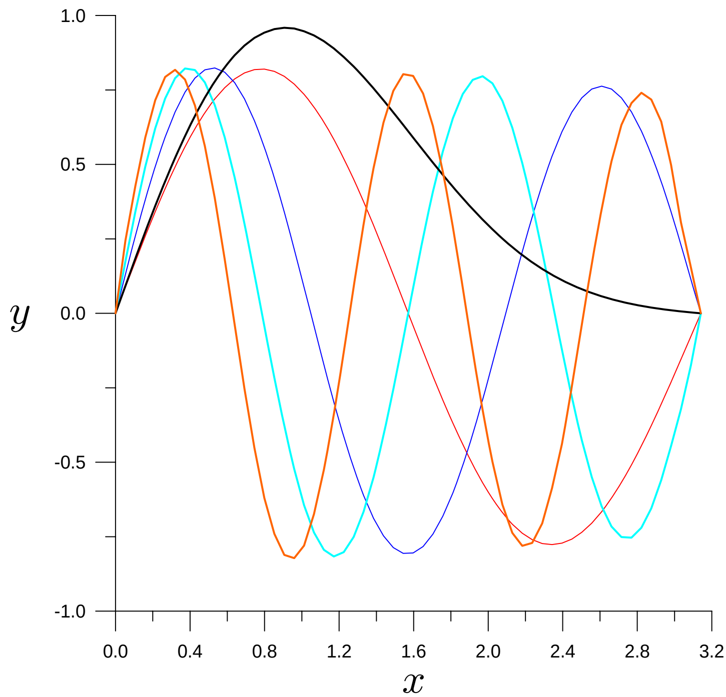

Figure 7: Applying the MSFTIM to a Sturm-Liouville eigenvalue problem with the first five modes being shown.

7 Conclusions

The solutions of NAEs involved at least two overlapping areas, namely, (1) the analysis of the solvability properties of NAEs, and (2) the development and study of suitable numerical methods. The present paper partially addresses the second area, and developed a novel numerical method based on the idea of FTIM, which is very suitable to find all the multiple-solutions of the NAEs. We modified the vector fields of ODEs according to the concepts of *decomposition of factors* and *repellor*. Then the previous solutions became repellors of the new ODEs derived from the multiple-solution FTIM (MSFTIM). We have used two examples to demonstrate that the MSFTIM can change the attracting set, such that, when it is difficult to choose a suitable initial condition by using the FTIM, the MSFTIM makes it easy to choose initial conditions and enables one to find all the multiple solutions. The MSFTIM was employed in this paper to solve the multiple solutions of boundary value problems, boundary layer problems, as well as the eigenvalue problems. The new methods have very unified and simple structures, such that engineers can easily learn and apply them to solve different and difficult engineering problems with multiple solutions.

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