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A Unified Computational Approach to Instability of Periodic Laminated Materials

M.V. Menshykova[1], I.A. Guz and O.V. Menshykov

Abstract: The present work is devoted to the investigation of the internal instability in laminated materials. The paper is concerned with the development of a unified computational procedure for numerical realisation of the method as applied to various constitutive equations of the layers, different loading schemes (uniaxial or biaxial loading) and different precritical conditions (large or small precritical deformations). It contains many examples of critical stresses/strains calculations for particular composites as well as analysis of different buckling modes.

Keywords: Layered materials; Microstructure; Microbuckling.

1 Introduction

The compressive strength of currently used advanced layered materials – e.g., cross-ply and multi-directional carbon fibre reinforced plastics, glass fibre reinforced plastics, aramid fibre-aluminium laminates, glass fibre-aluminium laminates, carbon fibre-aluminium laminates, hybrid titanium composite laminates etc. – is generally 30–40% lower than the tensile strength, see Budiansky and Fleck (1994), Soutis and Turkmen (1995), Schultheisz and Waas (1996), Niu and Talreja (2000). Thus it is recognised that the compressive strength is often a design-limiting factor.

A better understanding of the compressive strength and failure mechanisms is therefore crucial to the development of improved modern materials [Zhang, Huang and Atluri (2008), Fitzgerald, Goldbeck-Wood, Kung, Petersen, Subramanian and Wescott (2008)]. The task of deriving three-dimensional (3-D) analytical solutions to describe the response of laminated materials is considered as one of great importance [Kouri and Atluri (1993), Patrício, Mattheij and With (2008, 2009), Solano, Costales, Francisco, Martín Pendás, Miguel Blanco, Lau, He and Ravindra Pandey (2008), Kashtalyan and Menshykova (2007, 2009), Menshykova, Menshykov and Guz (2009)]. Analytical solutions, if obtained, enable us to analyse the behaviour

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of a structure on the wide range of material properties, and loading schemes, without the restrictions imposed by simplified approximate methods.

The work of Dow and Grunfest (1960) was the first to consider the microbuckling of fibres as a form of fracture of a unidirectional composite under compression. Since then, the beginning of fracture process under compression is usually associated with the buckling of the microstructure of the material when the critical load is determined by parameters characterising the microstructure of the composite rather than by the dimensions and shape of the specimen or structural member, i.e. with the internal instability phenomena according to Biot (1965). In this paper we adopt the same assumption of linking the onset of fracture and the loss of stability in the internal structure of the material.

This paper concerns with the most accurate approach to study this phenomenon. It is based on the model of a piecewise-homogeneous medium, when the behaviour of each component of the material is described by the 3-D equations of solid mechanics, without any simplifications for the modelling of material behaviour in the vicinity of the cracks. The corresponding equations for the plane problem are obtained in the usual way from the set of equations for three-dimensional problems. The results of this approach give the most accurate information about the considered phenomenon and can be used as a benchmark for simplified models.

Probably the first solutions to the problem of internal instability for a layered material obtained within the most accurate (exact) approach were reported by Guz (1969), Babich and Guz (1969, 1972), where the problem for linear-elastic layers under uniaxial compression was solved. This solution was included in numerous books, for example, Guz, (1990), and comprehensive reviews on the topic, e.g., by Guz (1992, 2009). This problem seems to remain topical for more than thirty years and is still being "re-examined". A paper by Parnes and Chiskis (2002) reports the solution (by a very approximate method, based on the modelling rigid layers as 2-D beams embedded in the matrix) of precisely the same problem that was solved more than thirty(!) years ago in Guz (1969), Babich and Guz (1972) within the exact approach.

Later the exact solutions were derived also for more complex problems: for orthotropic, non-linear elastic and elastic-plastic, compressible and incompressible layers including the case of large (finite) deformations – see, for example, Guz (1989a,b, 1998), Guz and Dekret (2006, 2009a,b), Guz and Guz (2000a,b,c), Guz and Soutis (2001a,b), Guz and Herrmann (2003), Dekret (2008a,b) and the reviews by Guz (1990, 1992, 2005, 2009). Recently, the application of the developed methods to the materials with pronounced heterogeneity at nano-scale was also examined [Guz and Rodger, Guz and Rushchitsky (2007), Guz, Rushchitsky and Guz, (2007), Guz and Rushchitsky (2004, 2007)].

The importance and the complexity of the considered phenomena caused a large number of publications which put forward various approximate methods aimed at tackling the problems with different levels of accuracy – see, for example, Rosen (1965), Schuerch (1966), Sadovsky, Pu and Hussain (1967) and the reviews by Budiansky and Fleck (1994), Soutis and Turkmen (1995), Schultheisz and Waas (1996), Niu and Talreja (2000). It was concluded after the detailed analyses [Guz (1990, 1992), Soutis and Turkmen (1995), Niu and Talreja (2000)], that the approximate methods are not very accurate when compared to experimental measurements and observations.

For instance, the model suggested by Rosen (1965) involves considerable simplifications, modelling the reinforcement layers by the thin beam theory and the matrix as an elastic material using one-dimensional stress analysis. It makes the results of this method inaccurate even for simple cases. It was shown by Guz (1990, 1992), Guz and Soutis (2000), Soutis and Guz (2001) that the approximate model can give a significant discrepancy in comparison with the exact approach and with experimental data even for the simplest case of a composite with linear elastic compressible layers undergoing small pre-critical deformations and considered within the scope of geometrically linear theory. For small fibre volume fractions the approximate approach gives physically unrealistic critical strains. It does not describe the phenomenon under consideration even on the qualitative level, since it predicts a different mode of stability loss from that obtained by the 3-D exact analysis. For more complex models, which take into account large deformations and geometrical and physical non-linearity (e.g. those considered in this chapter), the approximate theories are definitely inapplicable and one can expect even a bigger difference between the exact and approximate approaches. The exact approach utilised throughout this paper allows us to take into account large deformations, geometrical and physical non-linearities and load biaxiality that the simplified methods cannot consider.

Another approach, which is commonly used, is based on the investigation of fibre kinking. From the early literature on compressive fracture it was easy to get the impression that fibre instability (microbuckling) and kinking are competing mechanisms. However, it is now accepted that a kink band is an outcome of the microbuckling failure of actual fibres, as observed experimentally by Guynn, Bradley and Ochoa (1992) and Moran, Liu and Shih (1995). Fibre microbuckling occurs first, followed by propagation of this local damage to form a kink band. A comprehensive comparative analysis of the Rosen model, Argon-Budiansky (kinking) model, and Batdorf-Ko model was presented by Soutis and Turkmen (1995). Studies of the kinking phenomenon were also reviewed by Budiansky and Fleck (1994). It was shown by Soutis and Turkmen (1995) that the existing kinking analyses are able to account for some, but not all, of the experimental observations. They correctly predict that shear strength and fibre imperfections are important parameters affecting the compressive strength of the composite. However, within this model it is not possible to say exactly how the strength will vary with fibre content; and the value of misalignment is chosen arbitrarily. This model requires knowledge of the shear strength properties, the initial fibre misalignment and, the most importantly, the kink-band orientation angle which is a post-failure geometric parameter.

The simplified models are not analysed in this paper. This paper is concerned with the development of a unified computational procedure for numerical realisation of the most accurate ("exact") method as applied to various constitutive equations of the layers, different loading schemes (uniaxial or biaxial loading) and different precritical conditions (large or small precritical deformations). It contains many examples of calculation of critical stresses/strains for particular composites as well as analysis of different buckling modes. Some comparisons with available experimental data were discussed earlier by Guz (1990), Berbinau, Soutis and Guz (1999), Winiarski and Guz (2008).

2 Formulation of the problem

Let us briefly consider the statement of the problem of internal instability (microbuckling) for layered composites. The detailed formulations for particular types of layered materials were given, for example, in Guz and Soutis (2000), Guz and Herrmann (2003).

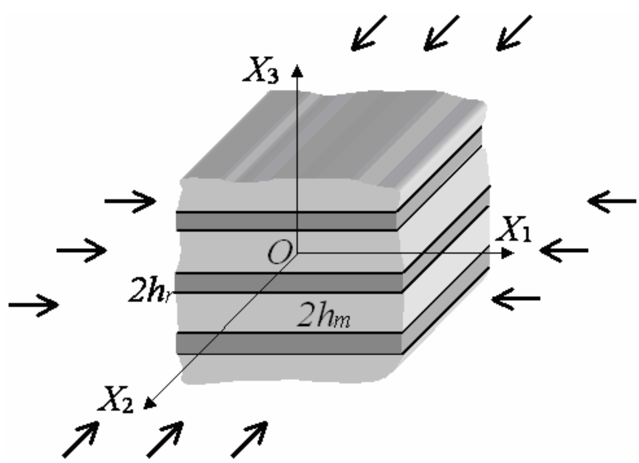

Figure 1: Cartesian co-ordinate system and loads for a layered composite system.

The thicknesses of the layers of a composite are $2h_r$ and $2h_m$, see Fig. 1. Two different loading schemes are studied: the uniaxial compression and the biaxial compression. The solution of the problem is obtained for four modes of stability loss see Guz and Herrmann (2003).

Using the piecewise-homogeneous medium model and the equations of the 3-D stability theory Guz (1999) the following eigen-value problem must be solved.

The stability equations for each layer are, see Guz (1999),

$$\frac{\partial}{\partial x_i} t_{ij}^r = 0, \quad \frac{\partial}{\partial x_i} t_{ij}^m = 0; \quad i, j = 1,\, 2,\, 3, \tag{1}$$

where t_{ij} is the non-symmetrical Piola-Kirchhoff stress tensor (nominal stress tensor). Tensor t_{ij} has the following form:

$$t_{ij} = \kappa_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \delta_{ij}\lambda_j^{-1}p, \quad t_{ij} = \omega_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \tag{2}$$

for incompressible solids ($\lambda_1\lambda_2\lambda_3 = 1$ is the incompressibility condition, λ_j is the elongation/shortening factor in the direction of the OX_j axis) and

$$t_{ij} = \omega_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \tag{3}$$

for compressible solids. The components of the tensors $\kappa_{ij\alpha\beta}$ and $\omega_{ij\alpha\beta}$ depend on the properties of the layers and the loads. The most general expressions for $\kappa_{ij\alpha\beta}$ and $\omega_{ij\alpha\beta}$ could be found in Guz (1999):

$$\kappa_{ij\alpha\beta} = \lambda_j\lambda_\alpha[\delta_{ij}\delta_{\alpha\beta}A_{\beta\, i} + (1-\delta_{ij})(\delta_{i\alpha}\delta_{j\beta}\mu_{ij} + \delta_{i\beta}\delta_{j\alpha}\mu_{ji})] + \delta_{i\beta}\delta_{j\alpha}S^0_{\beta\beta}, \tag{4}$$

$$\omega_{ij\alpha\beta} = \lambda_j\lambda_\alpha[\delta_{ij}\delta_{\alpha\beta}A'_{\beta\, i} + (1-\delta_{ij})(\delta_{i\alpha}\delta_{j\beta}\mu'_{ij} + \delta_{i\beta}\delta_{j\alpha}\mu'_{ji})] + \delta_{i\beta}\delta_{j\alpha}S^0_{\beta\beta},$$

where $A_{ij}(A'_{ij})$ and $\mu_{ij}(\mu'_{ij})$ are the quantities which characterise the axial and shear stiffnesses. The quantity characterising the precritical state (the stress component S^0_{11} or the strain component ε^0_{11}) is the parameter in respect to which the eigen-value problem should be solved.

To complete the problem statement, the boundary conditions for each interface should be written. The layer interfaces could consist of zones of perfectly connected (bonded) layers and defects such as cracks or delaminations. In this study we consider the composites with perfectly bonded layers or "perfectly lubricated" (sliding without friction) interfaces. For the perfectly bonded layers we have the continuity conditions for the stresses and displacements

$$t^r_{3i} = t^m_{3i},\; u^r_i = u^m_i, \quad i = \overline{1,3}, \tag{5}$$

and for “the perfectly lubricated layers”, see Aboudi (1987), Librescu and Schmidt (2001), only the continuity of the normal components is retained at the surface, with boundary conditions for perturbations of stresses and displacements in the form of

$$t_{31}^{r}=t_{31}^{m}=t_{32}^{r}=t_{32}^{m}=0,\ t_{33}^{r}=t_{33}^{m},\ u_{3}^{r}=u_{3}^{m}. \tag{6}$$

Note that in practical cases the assumption of perfect bonding between neighbouring layers in composites does not correspond to reality due to different imperfections always present in real laminated materials. Unfortunately considering the composite with such defects it is sometimes difficult to identify a set of the defects and its influence on the stability of the composite material. Hence, we suggest the following estimation. It is obvious that the critical strain ε_{cr} for a real composite with imperfections of interfacial adhesion must be larger than the critical strain ε_{cr}^{pl} for the same structure with perfectly lubricated layers, but smaller than the critical strain ε_{cr}^{pb} for the structure with perfectly bonded layers. Thus, we obtain the following bounds for the critical strain:

$$\varepsilon_{cr}^{pl}\le\varepsilon_{cr}\le\varepsilon_{cr}^{pb}. \tag{7}$$

3 Analytical solution

Solutions of equation (1) (i.e. perturbations of stresses and displacements) for each of the layers can be expressed through the functions X and Ψ, which are the solutions of the following equations, see Guz (1999)

$$\left(\Delta_1+\xi_1^2\frac{\partial^2}{\partial x_3^2}\right)\Psi=0, \tag{8}$$

$$\left(\Delta_1+\xi_2^2\frac{\partial^2}{\partial x_3^2}\right)\left(\Delta_1+\xi_3^2\frac{\partial^2}{\partial x_3^2}\right)\mathrm{X}=0,$$

where $\Delta_1=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$.

The parameter ξ_j depends on the components of the tensor $\kappa_{ij\alpha\beta}$ (or $\omega_{ij\alpha\beta}$) and, therefore, on the properties of the layers and on the loads. It was proved in Guz (1989) that for elastic compressible and elastic incompressible layers $\xi_j^2>0$, Im $\xi_j^2=0$ and for elastic-plastic incompressible layers $\mathrm{Im}\xi_{2,3}^2\neq 0$, $\xi_3^2=\overline{\xi_2^2}$.

The characteristic determinants associated with the four modes of stability loss were derived earlier in Guz (1989a,b, 1990, 1992) for various constitutive equations of the layers, different loading schemes (uniaxial or biaxial loading) and different precritical conditions (large or small precritical deformations). Similarly, the characteristic equations can be derived for other modes of stability loss. However, the modes with the larger periods in transverse direction are usually not of practical interest. In this paper, the characteristic determinants are presented in the unified form in order to facilitate a uniform computational procedure for solving them:

- for perfectly bonded layers

$$\det \begin{Vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{Vmatrix} = 0, \tag{9}$$

- for perfectly lubricated layers

$$\det \begin{Vmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ 0 & 0 & \beta_{13} & \beta_{14} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{Vmatrix} = 0. \tag{10}$$

The elements of the determinant for different types of materials and for different loading schemes are given in Guz (1990, 1992), Guz and Soutis (2000, 2001a,b), Guz and Herrmann (2003), Soutis and Guz (2001, 2006).

4 Computational procedure

To facilitate the analysis of characteristic determinants, the software package with the graphical user-friendly interface was developed using MATLAB 7.6.0 (R2008a). The software contains the database of material properties for typical layered composites and the library of components of the tensors $\kappa_{ij\alpha\beta}$ and $\omega_{ij\alpha\beta}$, Eq. (4). The fully automated numerical procedure consists of the following steps. First, the characteristic determinants, Eqs. (9) and (10), are computed depending on the user's choice of loading schemes (uniaxial or biaxial loading), initial conditions (large or small precritical deformations), and interfacial properties (perfectly bonded and perfectly lubricated layers). Then the results are analysed, and the critical controlled parameters of the internal instability (including the critical wavelength) are searched for. This analysis is conducted for all four considered modes of stability loss. At the final stage the modes are compared and the critical mode is found.

Some of the results for the cases of perfectly bonded and perfectly lubricated layers are presented in the next Section of this paper.

5 Results for different layered materials

5.1 Hyperelastic incompressible layered materials

Let the composite consists of alternating non-linear hyperelastic layers – many new materials fall into this category, see Ling and Atluri (2007).

Suppose that the materials of these layers are incompressible and a simplified version of the Mooney's potential, namely the so-called neo-Hookean potential, may be chosen to describe them in the following form

$$\Phi^r = 2C_{10}^r I_1^r(\varepsilon_{ij}^0), \quad \Phi^m = 2C_{10}^m I_1^m(\varepsilon_{ij}^0), \tag{11}$$

where Φ is the strain energy density function (elastic potential), C_{10} is a material constant, and $I_1(\varepsilon)$ is the first algebraic invariant of the Cauchy-Green strain tensor. This potential is also called the Treloar's potential, after the author who obtained it from an analysis of a model for rubber regarded as a macromolecular network structure made of very long and flexible interlinking chains, see Treloar (1975).

Then the characteristic equations (9) and (10) can be specified for particular modes of stability loss following Guz (1989a) and Guz and Herrmann (2003). The resulting transcendental equations in terms of λ_1 (shortening factor) and α_r (normalised wavelength) will be different for each of the modes. In the case of biaxial loading Guz and Herrmann (2003):

1) for perfectly bonded layers

- for the first (shear) mode

$$\begin{aligned}
&-\lambda^{-3}(1+\lambda_1^6)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-3} \tanh\alpha_m\lambda_1^{-3}\\
&-4\lambda_1^3[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \tanh\alpha_m\\
&+[2-(1+\lambda_1^6)C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-3} \tanh\alpha_m\\
&+[1+\lambda_1^6-2C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \tanh\alpha_m\lambda_1^{-3}\\
&+(1-\lambda_1^6)^2 C_{10}^r(C_{10}^m)^{-1}(\tanh\alpha_r \tanh\alpha_r\lambda_1^{-3}+\tanh\alpha_m \tanh\alpha_m\lambda_1^{-3}) = 0.
\end{aligned} \tag{12}$$

- for the second (extension) mode

$$\begin{aligned}
&-\lambda^{-3}(1+\lambda_1^6)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-3} \coth\alpha_m\lambda_1^{-3}\\
&-4\lambda_1^3[1-C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \coth\alpha_m\\
&+[2-(1+\lambda_1^6)C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r\lambda_1^{-3} \coth\alpha_m\\
&+[1+\lambda_1^6-2C_{10}^r(C_{10}^m)^{-1}]^2 \tanh\alpha_r \coth\alpha_m\lambda_1^{-3}
\end{aligned}$$

$$+(1-\lambda_1^6)^2 C_{10}^r (C_{10}^m)^{-1}(\tanh\alpha_r \tanh\alpha_r\lambda_1^{-3} + \coth\alpha_m \coth\alpha_m\lambda_1^{-3}) = 0. \tag{13}$$

- for the third mode

$$\begin{aligned}
&-\lambda^{-3}(1+\lambda_1^6)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \coth\alpha_r\lambda_1^{-3}\coth\alpha_m\lambda_1^{-3}\\
&-4\lambda_1^3[1-C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\coth\alpha_m\\
&+[2-(1+\lambda_1^6)C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\lambda_1^{-3}\coth\alpha_m\\
&+[1+\lambda_1^6-2C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\coth\alpha_m\lambda_1^{-3}\\
&+(1-\lambda_1^6)^2C_{10}^r(C_{10}^m)^{-1}(\coth\alpha_r\coth\alpha_r\lambda_1^{-3}+\coth\alpha_m\coth\alpha_m\lambda_1^{-3}) = 0.
\end{aligned} \tag{14}$$

- for the fourth mode

$$\begin{aligned}
&-\lambda^{-3}(1+\lambda_1^6)^2[1-C_{10}^r(C_{10}^m)^{-1}]^2 \coth\alpha_r\lambda_1^{-3}\tanh\alpha_m\lambda_1^{-3}\\
&-4\lambda_1^3[1-C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\tanh\alpha_m\\
&+[2-(1+\lambda_1^6)C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\lambda_1^{-3}\tanh\alpha_m\\
&+[1+\lambda_1^6-2C_{10}^r(C_{10}^m)^{-1}]^2\coth\alpha_r\tanh\alpha_m\lambda_1^{-3}\\
&+(1-\lambda_1^6)^2C_{10}^r(C_{10}^m)^{-1}(\coth\alpha_r\coth\alpha_r\lambda_1^{-3}\\
&+\tanh\alpha_m\tanh\alpha_m\lambda_1^{-3}) = 0.
\end{aligned} \tag{15}$$

2) for perfectly lubricated layers

- for the first (shear) mode

$$4\lambda_1^3\left(\frac{C_{10}^r}{C_{10}^m}\tanh\alpha_r+\tanh\alpha_m\right)-\left(1+\lambda_1^6\right)^2\left(\frac{C_{10}^r}{C_{10}^m}\tanh\frac{\alpha_r}{\lambda_1^3}+\tanh\frac{\alpha_m}{\lambda_1^3}\right)=0. \tag{16}$$

- for the second (extension) mode

$$4\lambda_1^3\left(\frac{C_{10}^r}{C_{10}^m}\tanh\alpha_r+\coth\alpha_m\right)-\left(1+\lambda_1^6\right)^2\left(\frac{C_{10}^r}{C_{10}^m}\tanh\frac{\alpha_r}{\lambda_1^3}+\coth\frac{\alpha_m}{\lambda_1^3}\right)=0. \tag{17}$$

- for the third mode

$$4\lambda_1^3\left(\frac{C_{10}^r}{C_{10}^m}\coth\alpha_r+\coth\alpha_m\right)-\left(1+\lambda_1^6\right)^2\left(\frac{C_{10}^r}{C_{10}^m}\coth\frac{\alpha_r}{\lambda_1^3}+\coth\frac{\alpha_m}{\lambda_1^3}\right)=0. \tag{18}$$

- for the fourth mode

$$4\lambda_1^3\left(\frac{C_{10}^r}{C_{10}^m}\coth\alpha_r+\tanh\alpha_m\right)-\left(1+\lambda_1^6\right)^2\left(\frac{C_{10}^r}{C_{10}^m}\coth\frac{\alpha_r}{\lambda_1^3}+\tanh\frac{\alpha_m}{\lambda_1^3}\right)=0. \tag{19}$$

The shortening factor λ_1 is related to the value of strain ε_{11}^0 by the following equation

$$u_i^0 = (\lambda_i - 1)x_i, \quad \lambda_i = const, \quad \varepsilon_{ij}^0 = (\lambda_i - 1)\delta_{ij}, \tag{20}$$

where u_i^0 is the axial displacement and ε_{ij}^0 is the strain (in terms of the elongation/shortening factor λ_j in the direction of the OX_j axis). The values of displacement and strain corresponding to the precritical state are marked by the superscript '0' to distinguish them from perturbations of the same values (u_i^0 and u_i, ε_{ij}^0 and ε_{ij} respectively).

In order to obtain the characteristic equations for the uniaxial loading, λ_1^{-3}, λ_1^3, and λ_1^6 should be replaced respectively with λ_1^{-2}, λ_1^2, and λ_1^4 in Eqs. (12–19).

The critical value for the particular mode ($\lambda_{cr}^{(N)}$, N is the number of the mode) can be found as a maximum of the corresponding curve. The maximum of these values will be the critical shortening factor of the internal instability for the considered layered material

$$\lambda_{cr}^{pl} = \max_N \{\lambda_{cr}^{(N)}\} = \max_N \{\max_{\alpha_r} \lambda_1^{(N)}\}, \tag{21}$$

$$\lambda_{cr}^{pb} = \max_N \{\lambda_{cr}^{(N)}\} = \max_N \{\max_{\alpha_r} \lambda_1^{(N)}\}.$$

Note that maximum shortening factors correspond to minimal strains and, therefore, to minimal loads according to Eq. (20). The curves corresponding to the 3^{rd} and the 4^{th} modes lie beneath the curves corresponding to the 1^{st} and the 2^{nd} modes, see Guz and Herrmann (2003). Therefore, the 1^{st} and 2^{nd} modes appear to be the most common modes of practical interest.

The computed critical values of shortening factors for hyperelastic composites with perfectly bonded layers under uniaxial loading are presented in Figs. 2–4. The comparison of the results for the fist and second modes of stability loss are presented in Fig. 2 and Fig. 3. The shortening factor tends to zero with the decrease of the material constants ratio and the difference between results for the first and the second modes of stability loss becomes smaller (Fig. 2). In Fig. 3 one can see how the ratio of the layer thicknesses influences the value of shortening factor for first two modes. The shortening factors for the first and the second modes coincide while the reinforcement layer is thin comparing to the matrix layer. In the considered case the difference between the results for the fist and the second modes becomes noticeable when the ratio of the layer thicknesses reaches a certain value (0.09 for the case of Fig. 3). It increases with the increase of the ratio of the layer thicknesses.

The 3-D plots in Fig. 4 show the whole picture of the dependences between the shortening factor, the ratio of layer thicknesses and the ratio of material constants for the first and the second modes of the stability loss.

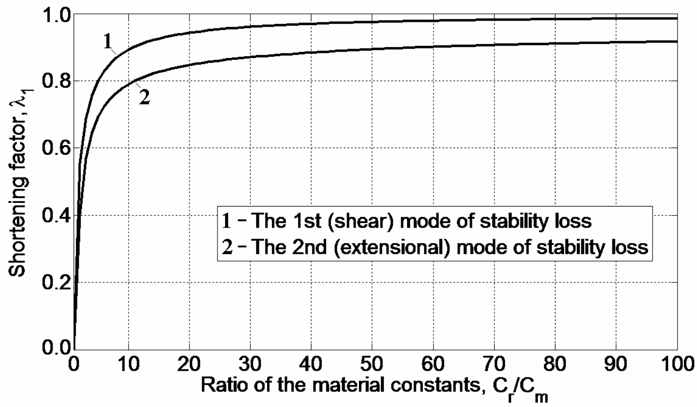

Figure 2: Shortening factor against the ratio of material constants; $h_r/h_m = 0.5$

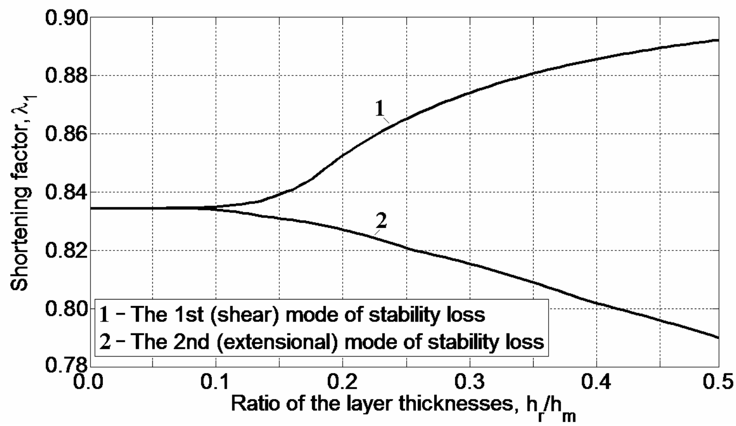

Figure 3: Shortening factor against the ratio of the layer thicknesses; $C_r/C_m = 0.2$

5.2 Compressible linear elastic layered materials

Let us consider a composite consisting of alternating linear-elastic isotropic compressible layers with different elastic properties (the Young's moduli E and the

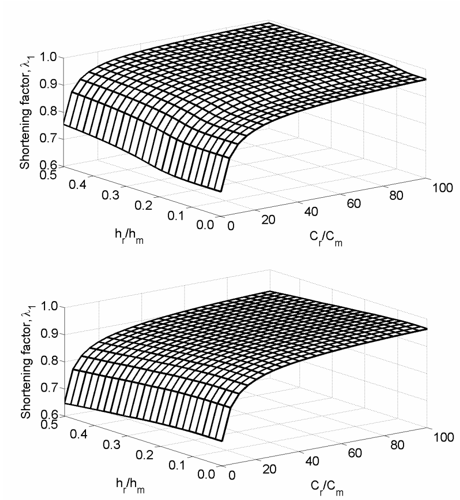

Figure 4: The 1[st] and the 2[nd] modes of stability loss

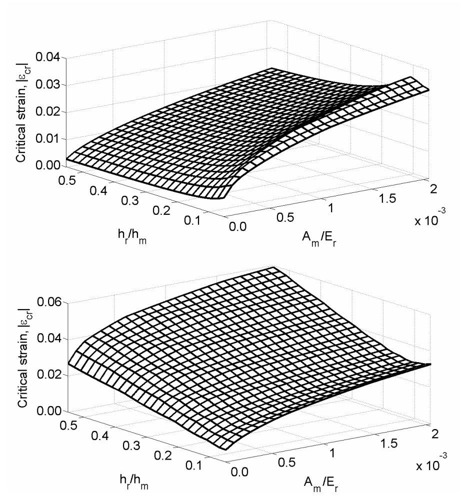

Figure 5: The 1[st] and the 2[nd] modes, $k_m = 0.25$, $\nu_r = 0.21$

Poisson's ratios ν). Then for the reinforcement layer we have

$$(\sigma_{ij}^0)^r = \delta_{ij}\frac{E_r\nu_r}{(1+\nu_r)(1-2\nu_r)}\varepsilon_{nn}^0 + \frac{E_r}{1+\nu_r}\varepsilon_{ij}^0, \tag{22}$$

and for the matrix

$$(\sigma_{ij}^0)^m = \delta_{ij}\frac{E_m\nu_m}{(1+\nu_m)(1-2\nu_m)}\varepsilon_{nn}^0 + \frac{E_m}{1+\nu_m}\varepsilon_{ij}^0. \tag{23}$$

The components of tensor $\omega_{ij\alpha\beta}$ for such materials are given by Guz (1999) and Guz and Soutis (2000) for different types of loading. Following the procedure described in the previous Subsection, i.e. substituting the expressions for $\omega_{ij\alpha\beta}$ into the characteristic equations (9) and (10), the characteristic equation can be specified for the considered type of composite material, see Guz (1990, 1992), Guz and Soutis (2000, 2001a) for more details.

For all modes we have transcendental equations in terms of two variables, ε_{11}^0 (applied strain) and α_r (normalised half-wavelength). Solving the characteristic equations for different modes of stability loss, the dependences $\varepsilon_{11}^{(N)}(\alpha_r)$ are obtained ($N = 1, 2, 3, 4$ is the number of the mode). A minimum of the corresponding dependence is the critical value for the particular mode – $\varepsilon_{cr}^{(N)}$. The critical strain of internal instability for the considered layered material is the minimal of these four values (ε_{cr}^{pl} in the case of perfectly lubricated layers, and ε_{cr}^{pb} in the case of perfectly bonded layers):

$$\varepsilon_{cr}^{pl} = \min_{N} \varepsilon_{cr}^{(N)} = \min_{N} \left(\min_{\alpha_r} \varepsilon_{11}^{(N)} \right), \tag{24}$$

$$\varepsilon_{cr}^{pb} = \min_{N} \varepsilon_{cr}^{(N)} = \min_{N} \left(\min_{\alpha_r} \varepsilon_{11}^{(N)} \right).$$

5.3 Materials containing elastic-plastic layers

Now, let us consider the following layered composite: the reinforcement behaves as a linear-elastic isotropic compressible material, Eq. (22), and the matrix response is elastic-plastic incompressible described by the following relationship for equivalent stress (σ_I^0) and strain (ε_I^0):

$$\sigma_I^0 = A_m (\varepsilon_I^0)^{k_m}, \tag{25}$$

where k_m and A_m are material constants for elastic-plastic matrix. The constitutive equation (25) is typical for metal matrix composites, see Honeycombe (1968), Pinnel and Lawley (1970), Guz (1989b, 1998). Again, using the expressions for $\omega_{ij\alpha\beta}$ and $\kappa_{ij\alpha\beta}$, Guz (1990), Guz and Herrmann (2003), one can deduce the transcendental equations for each of the considered modes of stability loss, see Guz (1989b, 1998).

The computed results for metal matrix elastic-plastic composites under biaxial loading are presented in Figs. 5–8. The results show how the bonds between the layers affect the solution for the first two modes of stability loss. The 3-D plots in Figs. 5–8 give the critical strain versus different properties of the material. In Fig. 5 and Fig. 6 the results for perfectly bonded layers are shown and the results for perfectly lubricated layers are presented in Fig. 7 and Fig. 8.

5.4 Bounds for the critical controlled parameters

In this subsection, the critical values of controlled parameters for perfectly bonded and perfectly lubricated layers are compared for hyperelastic and metal matrix composites under different types of loading.

According to Eq. (7), these values form the bounds for the critical controlled parameters (i.e. either for critical strains or for critical shortening factors) for practical composites with imperfections of interfacial adhesion. If for critical strain the bounds have the form of Eq. (7), for critical shortening factors taking into account Eq. (20) they are

$$\lambda_{cr}^{pb} \le \lambda_{cr} \le \lambda_{cr}^{pl}. \tag{26}$$

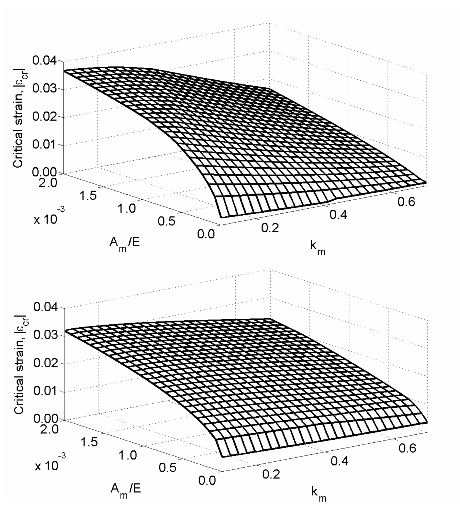

Figure 6: The 1^{st} and the 2^{nd} modes, $\nu_r = 0.21$, $h_r/h_m = 0.1$

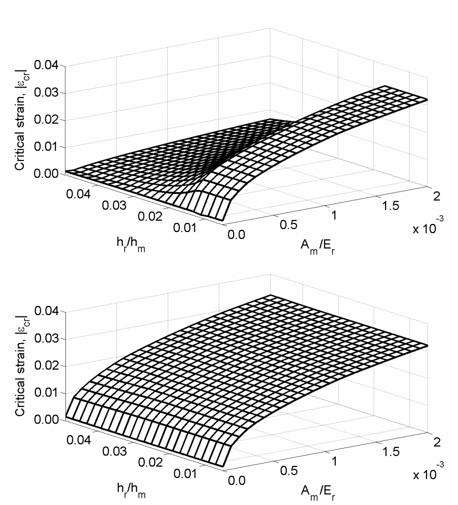

Figure 7: The 1^{st} and the 2^{nd} modes, $k_m = 0.25$, $\nu_r = 0.21$

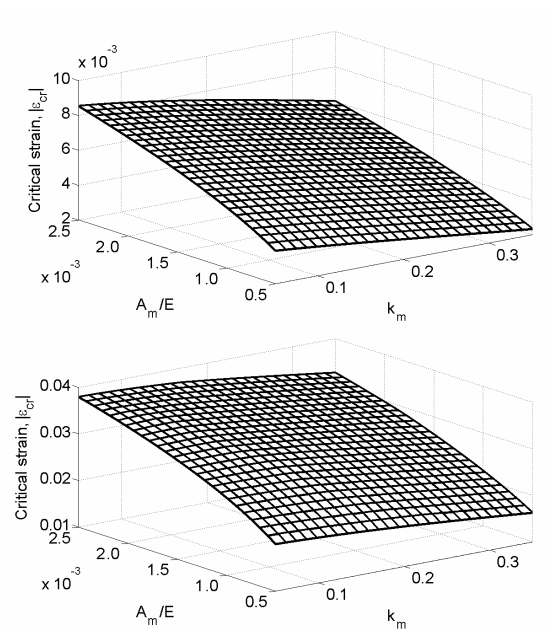

Figure 8: The 1^{st} and the 2^{nd} modes, $\nu_r = 0.21$,$h_r/h_m = 0.05$

It should be underlined that practical composites contain not only interlaminar, but also various sorts of intralaminar defects. The effect of intralaminar damage can be accounted for by considering layers with reduced stiffness properties – see, for example, Kashtalyan and Soutis (2001, 2006, 2007).

The computed results for four modes of stability loss of the hyperelastic incompressible layered material are shown in Figs. 9 and 10. One can see that the bounds for shortening factor are wider when the ratio of material constants is lower. For the second mode the results for perfectly bonded and perfectly lubricated layers start to coincide when the ratio of material constants reaches a certain value (40 for the case of Fig. 9). The results of computation for layered composites with elastic-plastic matrix are shown in Figs. 11 and 12.

For the first and second modes of stability loss the critical strain remains constant and the difference between the results for perfectly bonded and perfectly lubricated layers does not change while the ratio of the layer thicknesses is lower than a certain value (0.027 for the case of Fig. 11). Then with the increase of the difference between layer thicknesses the bounds for critical strain become narrower.

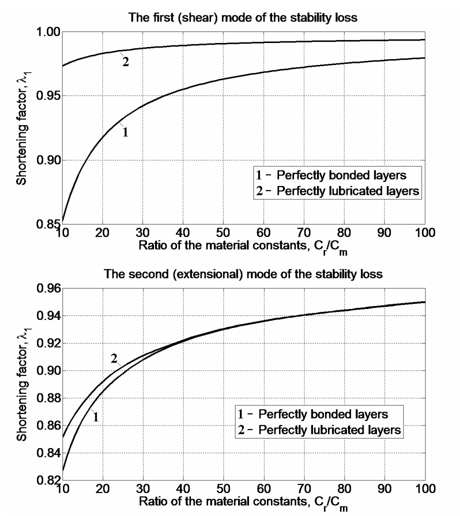

Figure 9: The bounds for the 1^{st} and the 2^{nd} modes of the hyperelastic composite under uniaxial loading; $h_r/h_m = 0.2$

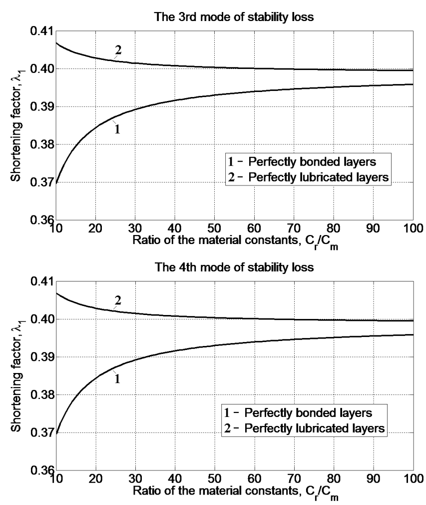

Figure 10: The bounds for the 3^{rd} and the 4^{th} modes of the hyperelastic composite under uniaxial loading; $h_r/h_m = 0.2$

The bounds for critical strain are shown in Fig. 12 as a function of k_m. With the increase of the coefficient k_m, the distance between the upper and the lower curves

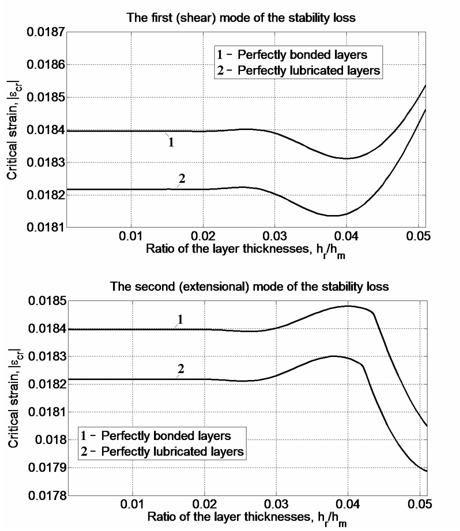

Figure 11: The bounds for the 1^{st} and the 2^{nd} modes of the metal matrix composite under biaxial loading; $A_m/E = 0.0005$, $\nu_r = 0.21$, $k_m = 0.25$.

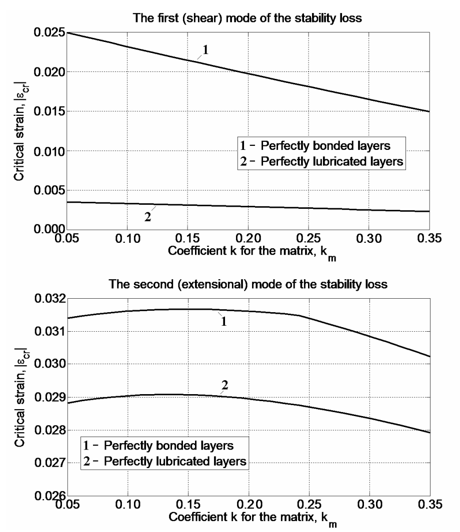

Figure 12: The bounds for the 1^{st} and the 2^{nd} modes of the metal matrix composite under biaxial loading; $A_m/E = 0.0001$, $h_r/h_m = 0.25$, $\nu_r = 0.21$.

significantly decreases for the first mode of stability loss and remains almost the same for the second mode.

The computed bounds appear to give a reasonable estimation for the critical controlled parameters and may be considered as the first approximation on the way to the exact solution of the problem of stability in compression along interfacial defects. Further work is required to compare the results with experimental observations and measurements.

6 Conclusions

In the paper the investigation of the internal instability for different types of layered materials, namely hyperelastic incompressible, compressible linear elastic and materials with elastic-plastic layers was held. The analysis of different loading schemes and precritical conditions was carried out using developed software package with fully automated numerical procedure. MATLAB was used to create the software which has graphical user friendly interface and the database of material properties.

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