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A New Time Domain Boundary Integral Equation and Efficient Time Domain Boundary Element Scheme of Elastodynamics

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Abstract: The traditional time domain boundary integral equation (TDBIE) of elastodynamics is formulated based on the time dependent fundamental solution and the reciprocal theorem of elastodynamics. The time dependent fundamental solution of the elastodynamics is the response of the infinite elastic medium under a unit concentrate impulsive force subjected at a point and at an instant, including not only the pressure wave and shear wave, but also the Laplace wave with speed between that of P and S waves. In this paper, a new TDBIE is derived directly from the initial boundary value problem of the partial differential equation of elastodynamics, and using the integral equation in weighted residual format. In the new TDBIE the D'Alembert solution of the elastodynamics, namely the spherical convergent pressure wave and shear wave are applied as the kernel functions respectively. In this way, the system of TDBIE obtained is much simpler than the traditional one.

In the traditional time domain boundary element method (TDBEM) of elastodynamics, the boundary solutions can be obtained in time step by step. At the first steps, the matrix of the algebraic equation system is quite sparse, because the elements which the wave front has not reached need not be computed. But the wave front reaches more and more elements as the computation continues step by step. To further enhance the efficiency, the impulsive waves of spherical convergent pressure and shear waves are applied as the kernel functions. It is not difficult in the new TDBIE of elastodynamics, which can be realized simply by the superposition of two successive and opposite spherical convergent wave components. To guarantee the equivalence of the TDBIE with the corresponding partial differential equation of elastodynamics, the width of the impulse should be greater than the maximum length of the lines in the elastic domain connecting the convergent boundary point with all other boundary points. The width of the impulse can be optimized in future work.

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1 Introduction

Because the elastodynamic problems exist widely in different engineering fields, the elastodynamics is an important research field of the boundary element methods in solid mechanics. In the literature on the boundary integral equation method for elastodynamics, the first paper (Cruse and Rizzo, 1968; Cruse, 1968) combined the boundary integral equation with the Laplace transform and the elliptical equations in the transformed domain are solved. Manolis and Beskos made some improvement on it (Manolis and Beskos, 1988). Time domain approach of the boundary element formulation was firstly presented in 1978 for antiplanar problems (Cole, Kosloff and Minster, 1978), Niwa, Kobayashi and Kitahara presented its general formulation (Niwa, Kobayashi and Kitahara, 1980). Some further improvements can be found in literature (Mansur, 1983; Karabalis and Beskos, 1984; Antes, 1985 and so on).

The problems governed by the equations of elastodynamics include not only the elastic wave problems, but also the vibration problems of elastic solids. For the later, Nardini and Brebbia derived the mass matrix and stiffness matrix based on the elastostatic formulation (Nardini and Brebbia, 1982), which has been developed as dual reciprocal approach to transform the domain integral of inertial forces into the boundary integrals. Some authors have applied it to the dynamic analysis of anisotropic elastic solids (Kogl and Gaul, 2000). The boundary element methods for elastodynamics have been widely applied in engineering design and analysis, which attracted much attention of the researchers and engineers. The corresponding methods applied in the fields of soil-structure interaction and dynamic fracture mechanics have been developed in recent decades. In recent years, some authors have presented the investigation on elastodynamics using local boundary integral equation method, local Petrov-Galerkin method (Sladek et al, 2004; Sladek et al, 2009), or BEM/FEM coupling method (Soares and Mansur, 2005). In the monographs of boundary element methods (such as Aliabadi, 2002), there is at least a separated chapter for the topics of elastodynamics.

The traditional TDBIE of elastodynamics is briefly introduced in this paper, which is based on the reciprocal theorem of elastodynamics and using the time dependent fundamental solution of the elastodynamics equations. This fundamental solution is the response of the infinite elastic medium under a unit concentrated impulsive force subjected at a point and at an instant, which includes not only the pressure wave and shear wave, but also the Laplace wave with speed between that of P and S waves. And then a new TDBIE is derived directly from the initial boundary value problem of the partial differential equation of elastodynamics, and using the integral equation in weighted residual format. In the new TDBIE the D'Alembert solution of the elastodynamics, namely the spherical convergent pressure wave and shear wave are applied as the kernel functions respectively. In this way, the system of TDBIE obtained is much simpler than the traditional one.

The traditional TDBEM of elastodynamics is directly applied to the new TDBIE, the boundary solutions can be obtained in time step by step. At the first several steps, the matrix of the algebraic equation system is quite sparse, because the elements which the wave front has not reached need not be computed. But the wave front reaches more and more elements as the computation continues step by step. To further enhance the efficiency, the impulsive waves of spherical convergent pressure and shear waves are applied as the kernel functions. It is not difficult in the new TDBIE of elastodynamics, which can be realized simply by the superposition of two successive and opposite spherical convergent wave components. To guarantee the equivalence of the TDBIE with the corresponding initial and boundary value problem of the partial differential equation of elastodynamics, the width of the impulse should be greater than the maximum length of the lines in the elastic domain connecting the convergent boundary point with all other boundary points.

2 Traditional TDBIE of elastodynamics

The formulation of the traditional TDBIE is based on the reciprocal theorem of elastodynamics, and using the time dependent fundamental solution.

2.1 The time dependent fundamental solution of elastodynamics

The time dependent fundamental solution of elastodynamics satisfies the governing equation

$$\rho\left(c_1^2-c_2^2\right)u^s_{kj},_{ij}+\rho c_2^2 u^s_{ki},_{jj}-\rho\ddot{u}^s_{ki}=-\delta_{ki}\Delta(\mathrm{P},\mathrm{Q})\Delta(\tau,t) \tag{1}$$

where the physical meaning of $u^s_{kj}(\mathrm{P},\tau;\,\mathrm{Q},t)$is the displacement component in x_j direction of a field point Q of the infinite elastic medium at instant t resulted by the unit concentrate force in the direction of x_k subjected at the source point P and instant τ; c_1, c_2 are the wave speed of the pressure wave and shear wave respectively, ρ the mass density of the elastic medium, δ_{ki} the Kronecker δ, and $\Delta(\tau,\,t)$, $\Delta(\mathrm{P},\mathrm{Q})$ the Dirac Delta function,

$$c_1=\sqrt{\lambda+2G/\rho},\; c_2=\sqrt{G/\rho},$$

$$\Delta(\tau,\,t)=0\;\forall\tau\neq t,\;\int_{-\infty}^{\infty}\Delta(\tau,\,t)\,\mathrm{d}\tau=\int_{-\infty}^{\infty}\Delta(\tau,\,t)\,\mathrm{d}t=1,$$

$$\Delta(\mathrm{P},\;\mathrm{Q})=0\;\forall\mathrm{P}\neq\mathrm{Q},\;\int_V\Delta(\mathrm{P},\;\mathrm{Q})\,\mathrm{d}V(\mathrm{Q})=1.$$

The displacement fundamental solution can be written as

$$u^s_{ki}(\mathrm{P},\,\tau;\,\mathrm{Q},t) = \frac{1}{4\pi\rho r}\left\{\frac{t'}{r^2}(3r_{,k}r_{,i}-\delta_{ki})\left[\mathrm{H}\left(t'-\frac{r}{c_1}\right)-\mathrm{H}\left(t'-\frac{r}{c_2}\right)\right]\right.$$
$$\left.+r_{,k}r_{,i}\left[\frac{1}{c_1^2}\Delta\left(t',\,\frac{r}{c_1}\right)-\frac{1}{c_2^2}\Delta\left(t',\,\frac{r}{c_2}\right)\right]+\frac{\delta_{ki}}{c_2^2}\Delta\left(t',\,\frac{r}{c_2}\right)\right\} \quad (2)$$

where $t' = t-\tau$, H is the Heaviside function,

$$\mathrm{H}\left(t',\,a\right) = \begin{cases}1 & \forall t' > a\\ 0 & \forall t' < a\end{cases} \qquad \mathrm{H}\left(t'-a\right) = \int_{-\infty}^{t'}\Delta(t,\,a)\,\mathrm{d}t$$

The corresponding traction fundamental solution is

$$t^s_{ki}(\mathrm{P},\,\tau;\,\mathrm{q},t) = \frac{1}{4\pi}\left[\left(\frac{\partial\psi}{\partial r}-\frac{\chi}{r}\right)\left(\frac{\partial r}{\partial n}\delta_{ki}+r_{,i}n_k\right)-2\frac{\chi}{r}\left(r_{,k}n_i-2r_{,k}r_{,i}\frac{\partial r}{\partial n}\right)\right.$$
$$\left.-2\frac{\partial\chi}{\partial r}r_{,k}r_{,i}\frac{\partial r}{\partial n}+\left(\frac{c_1^2}{c_2^2}-2\right)\left(\frac{\partial\psi}{\partial r}-\frac{\partial\chi}{\partial r}-2\frac{\chi}{r}\right)r_{,k}n_i\right] \quad (3)$$

where

$$\psi = \frac{c_2^2}{r^3}t'\left[\mathrm{H}\left(t'-\frac{r}{c_2}\right)-\mathrm{H}\left(t'-\frac{r}{c_1}\right)\right]+\frac{1}{r}\Delta\left(t',\,\frac{r}{c_2}\right)$$

$$\chi = 3\psi-\frac{2}{r}\Delta\left(t',\,\frac{r}{c_2}\right)-\frac{c_2^2}{c_1^2}\frac{1}{r}\Delta\left(t',\,\frac{r}{c_1}\right)$$

$$\frac{\partial\psi}{\partial r} = -\frac{\chi}{r}-\frac{1}{r^2}\left[\Delta\left(t',\,\frac{r}{c_2}\right)+\frac{r}{c_2}\dot{\Delta}\left(t',\,\frac{r}{c_2}\right)\right]$$

$$\frac{\partial\chi}{\partial r} = -\frac{3\chi}{r}-\frac{1}{r^2}\left[\Delta\left(t',\,\frac{r}{c_2}\right)+\frac{r}{c_2}\dot{\Delta}\left(t',\,\frac{r}{c_2}\right)\right]$$
$$+\frac{c_2^2}{c_1^2}\frac{1}{r^2}\left[\Delta\left(t',\,\frac{r}{c_1}\right)+\frac{r}{c_1}\dot{\Delta}\left(t',\,\frac{r}{c_1}\right)\right]$$

In Eq. (2) Q denotes an arbitrary field point of an elastic solid, as the source point P of the fundamental solution, namely the subjected point of the concentrated impulsive force, approaches the boundary, it will be denoted by p in lower case.

This time dependent fundamental solution includes not only pressure wave and shear wave in the infinite elastic medium, but also the Laplace wave, which has a speed between that of pressure and shear waves.

2.2 The traditional TDBIE of elastodynamics

The Betti reciprocal theorem in elasticity has been generalized to the elastodynamic case by Graf (Graff, 1975). For the two independent dynamic state of the same elastic solid in same time space domain, $(u_i^{(1)},\ t_i^{(1)},\ f_i^{(1)},\ \ddot{u}_i^{0(1)})$ and $\left(u_i^{(2)},\ t_i^{(2)},\ f_i^{(2)},\ \ddot{u}_i^{0(2)}\right)$, the reciprocal theorem can be written as

$$\begin{aligned}&\int_S t_i^{(1)}(\mathrm{q},t)*u_i^{(2)}(\mathrm{q},t)\,\mathrm{d}S(\mathrm{q})+\int_V f_i^{(1)}(\mathrm{Q},t)*u_i^{(2)}(\mathrm{Q},t)\,\mathrm{d}V(\mathrm{Q})\\&-\int_V\rho\ddot{u}_i^{(1)}(\mathrm{Q},t)*u_i^{(2)}(\mathrm{Q},t)\,\mathrm{d}V(\mathrm{Q})=\int_S t_i^{(2)}(\mathrm{q},t)*u_i^{(1)}(\mathrm{q},t)\,\mathrm{d}S(\mathrm{q})\\&\qquad+\int_V f_i^{(2)}(\mathrm{Q},t)*u_i^{(1)}(\mathrm{Q},t)\,\mathrm{d}V(\mathrm{Q})-\int_V\rho\ddot{u}_i^{(2)}(\mathrm{Q},t)*u_i^{(1)}(\mathrm{Q},t)\,\mathrm{d}V(\mathrm{Q})\qquad(4)\end{aligned}$$

where * denotes the convolution integral.

If the dynamic state to be solved is taken as the state (1), and the state corresponding to the fundamental solution is taken as state (2), then

$$\begin{aligned}&u_i^{(2)}(\mathrm{q},t)=u_{ki}^s(\mathrm{p},\tau;\ \mathrm{q},t)\equiv u_{ki}^s(\mathrm{p},\mathrm{q};\ t-\tau)\\&u_i^{(2)}(\mathrm{Q},t)=u_{ki}^s(\mathrm{p},\tau;\ \mathrm{Q},t)\equiv u_{ki}^s(\mathrm{p},\mathrm{Q};\ t-\tau)\\&t_i^{(2)}(\mathrm{q},t)=t_{ki}^s(\mathrm{p},\tau;\ \mathrm{q},t)\equiv t_{ki}^s(\mathrm{p},\mathrm{q};\ t-\tau)\\&f_i^{(2)}(\mathrm{Q},t)=\delta_{ki}\Delta(\mathrm{p},\mathrm{Q})\,\Delta(\tau,\ t)\\&\dot{u}_i^{(2)}(\mathrm{Q},t)=\dot{u}_{ki}^s(\mathrm{p},\tau;\ \mathrm{Q},t)\equiv\dot{u}_{ki}^s(\mathrm{p},\mathrm{Q};\ t-\tau)\\&\ddot{u}_i^{(2)}(\mathrm{Q},t)=\ddot{u}_{ki}^s(\mathrm{p},\tau;\ \mathrm{Q},t)\equiv\ddot{u}_{ki}^s(\mathrm{p},\mathrm{Q};\ t-\tau)\end{aligned}$$

The convolution integral in Eq. (4) should be expressed as

$$t_i^{(1)}(\mathrm{q},t)*u_i^{(2)}(\mathrm{q},t)\equiv\int_{t_0}^{t}u_{ki}^s(\mathrm{p},\mathrm{q};t-\tau)\,t_i(\mathrm{q},\tau)\,\mathrm{d}\tau,\ \ldots\ldots$$

Therefore, the Eq. (4) can be rewritten as

$$
\begin{aligned}
&C_{ki}(\mathrm{p})\,u_i(\mathrm{p},t) + \int_S \int_{t_0}^{t} t_{ki}^{s}(\mathrm{p},\mathrm{q};\,t-\tau)\,u_i(\mathrm{q},\tau)\,\mathrm{d}\tau\mathrm{d}S(\mathrm{q}) \\
&-\int_V \int_{t_0}^{t} \rho\ddot{u}_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-\tau)\,u_i(\mathrm{Q},\tau)\,\mathrm{d}\tau\mathrm{d}V(\mathrm{Q}) \\
&=\int_V \int_{t_0}^{t} u_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-\tau)\,f_i(\mathrm{Q},\tau)\,\mathrm{d}\tau\mathrm{d}V(\mathrm{Q}) + \int_S \int_{t_0}^{t} u_{ki}^{s}(\mathrm{p},\mathrm{q};\,t-\tau)\,t_i(\mathrm{q},\tau)\,\mathrm{d}\tau\mathrm{d}S(\mathrm{q}) \\
&\qquad -\int_V \int_{t_0}^{t} \rho u_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-\tau)\,\ddot{u}_i(\mathrm{Q},\tau)\,\mathrm{d}\tau\mathrm{d}V(\mathrm{Q}) \qquad (5)
\end{aligned}
$$

Integrating the last integrals in both sides by parts, the time domain displacement boundary integral equation can be finally obtained,

$$
\begin{aligned}
&C_{ki}(\mathrm{p})\,u_i(\mathrm{p},t) + \int_S \int_{t_0}^{t} t_{ki}^{s}(\mathrm{p},\mathrm{q};\,t-\tau)\,u_i(\mathrm{q},\tau)\,\mathrm{d}\tau\mathrm{d}S(\mathrm{q}) \\
&+\rho \int_V \dot{u}_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-t_0)\,u_i(\mathrm{Q},t_0)\,\mathrm{d}V(\mathrm{Q}) \\
&=\int_V \int_{t_0}^{t} u_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-\tau)\,f_i(\mathrm{Q},\tau)\,\mathrm{d}\tau\mathrm{d}V(\mathrm{Q}) + \int_S \int_{t_0}^{t} u_{ki}^{s}(\mathrm{p},\mathrm{q};\,t-\tau)\,t_i(\mathrm{q},\tau)\,\mathrm{d}\tau\mathrm{d}S(\mathrm{q}) \\
&\qquad +\rho \int_V u_{ki}^{s}(\mathrm{p},\mathrm{Q};\,t-t_0)\,\dot{u}_i(\mathrm{Q},t_0)\,\mathrm{d}V(\mathrm{Q}) \qquad (6)
\end{aligned}
$$

This boundary integral equation can be solved by TDBEM.

3 Derivation of a new TDBIE of elastodynamics

To derive a new TDBIE, it is directly started from the governing partial differential equation of elastodynamics, and using the weighted residual integration form. This method is a basic method for the derivation of the boundary integral equation from the corresponding partial differential equations, which has been applied in author's early work on BIE-BEM started 30 years ago (Du and Yao, 1982, in Chinese). In literature, similar approach can be found in many papers (Grannell and Atluri, 1978; Atluri, 1984, and so on); and some papers developed this approach further to formulate more efficient boundary integral equations, for example: which has been applied to derive a novel displacement gradient BEM for elastic stress analysis with high accuracy (Okada, Rajiyah and Atluri, 1988), the non-hyper-singular integral-representations for velocity (displacement) gradients in elastic/plastic solids (Okada, Rajiyah and Atluri, 1989). It was shown that, by using certain linearly independent combination of the first and higher-order derivatives of the fundamental solutions as the test function (and their various physical properties), it is possible to derive more desirable and only weakly-singular forms of integral equations (and hence BIE) for the first and higher-order derivatives of the primary-variables (such as displacements) in the problem of elasticity and elasto-plasticity (Chien, Rajiyah and Atluri, 1990A; Okada, Rajiyah and Atluri, 1990B; Han and Atluri, 2003; Atluri, 2005; Han and Atluri, 2007).

3.1 The weighted residual integration form of the elastodynamics

The governing equation of the elastodynamics is wellknown,

$$\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)=0 \tag{7a}$$

which is a vectorial field equation; therefore the divergence and the curl of this equation should also be zero, namely

$$\begin{cases}\left[\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)\right]_{,i}=0\\ e_{kmi}\left[\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)\right]_{,m}=0\end{cases} \tag{7}$$

This equation system (7) is nearly identical with the original governing equation (7a). The possible difference is only the solution of the following equation

$$\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)=\text{const}$$

which is corresponding to a virtual static deformation state resulted by uniformly distributed constant body forces. As the initial displacement condition is prescribed, the above mentioned possible difference will be eliminated automatically.

The weighted residual integral equations corresponding to Eq. (7) can be written as

$$\begin{cases}\int_V\int_{t_0}^{t_1}\left[\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)\right]_{,i}\\ \qquad w^{(1)}(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V=0\\ \int_V\int_{t_0}^{t_1}e_{kmi}\left[\rho\left(c_1^2-c_2^2\right)u_{j,ij}(\mathrm{Q},t)+\rho c_2^2 u_{i,jj}(\mathrm{Q},t)+f_i(\mathrm{Q},t)-\rho\ddot{u}_i(\mathrm{Q},t)\right]_{,m}\\ \qquad w_k^{(2)}(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V=0\end{cases} \tag{8}$$

As $w^{(1)}(\mathrm{Q},t)$, $w_k^{(2)}(\mathrm{Q},t)$ are arbitrary weighted functions, these equations are also identical with Eq. (7).

These integral equations can be transformed using the generalized Gauss identities. For example, the first equation of Eq. (8) can be finally converted to

$$\begin{aligned}
&\int_V \int_{t_0}^{t_1} -\Big[\rho\left(c_1^2-2c_2^2\right) w^{(1)}{}_{,jji}(\mathrm{Q},t)+\rho c_2^2\left(w^{(1)}{}_{,jij}(\mathrm{Q},t)+w^{(1)}{}_{,ijj}(\mathrm{Q},t)\right)\\
&-\rho \ddot{w}^{(1)}{}_{,i}(\mathrm{Q},t)\Big]\, u_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V\\
&+\int_S \int_{t_0}^{t_1} \left[\rho\left(c_1^2-2c_2^2\right) u_{j,ji}(\mathrm{q},t)+\rho c_2^2\left(u_{j,ij}(\mathrm{q},t)+u_{i,jj}(\mathrm{q},t)\right)+f_i(\mathrm{q},t)\right.\\
&\left.-\rho \ddot{u}_i(\mathrm{q},t)\right] n_i w^{(1)}(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S\\
&-\int_S \int_{t_0}^{t_1} \left[\rho\left(c_1^2-2c_2^2\right) u_{k,k}(\mathrm{q},t)\,\delta_{ij}+\rho c_2^2\left(u_{j,i}(\mathrm{q},t)+u_{i,j}(\mathrm{q},t)\right)\right] w^{(1)}{}_{,i}(\mathrm{q},t)\, n_j \mathrm{d}t\mathrm{d}S\\
&+\int_S \int_{t_0}^{t_1} \left[\rho\left(c_1^2-2c_2^2\right) w^{(1)}{}_{,kk}(\mathrm{q},t)\,\delta_{ij}+\rho c_2^2\left(w^{(1)}{}_{,ji}(\mathrm{q},t)+w^{(1)}{}_{,ij}(\mathrm{q},t)\right)\right]\\
&u_i(\mathrm{q},t)\, n_j \mathrm{d}t\mathrm{d}S\\
&-\int_V \int_{t_0}^{t_1} f_i(\mathrm{Q},t)\, w^{(1)}{}_{,i}(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V+\int_V \left[\rho \dot{u}_i(\mathrm{Q},t)\, w^{(1)}{}_{,i}(\mathrm{Q},t)\right]_{t=t_0}^{t=t_1}\mathrm{d}V\\
&\qquad\qquad -\int_V \left[\rho u_i(\mathrm{Q},t)\,\dot{w}^{(1)}{}_{,i}(\mathrm{Q},t)\right]_{t=t_0}^{t=t_1}\mathrm{d}V=0 \quad (9)
\end{aligned}$$

In the first integral term, the integrand related to the weighted function is

$$\begin{aligned}
&\rho\left(c_1^2-2c_2^2\right) w^{(1)}{}_{,jji}(\mathrm{Q},t)+\rho c_2^2\left(w^{(1)}{}_{,jij}(\mathrm{Q},t)+w^{(1)}{}_{,ijj}(\mathrm{Q},t)\right)-\rho \ddot{w}^{(1)}{}_{,i}(\mathrm{Q},t)\\
&\qquad =\rho c_1^2\left(w^{(1)}{}_{,i}\right){}_{,jj}(\mathrm{P},\tau;\,\mathrm{Q},t)-\rho \ddot{w}^{(1)}{}_{,i}(\mathrm{P},\tau;\,\mathrm{Q},t) \quad (10)
\end{aligned}$$

It can be seen that $w^{(1)}$ is just the scalar potential function of the elastic pressure wave, and $w^{(1)}{}_{,i}$ is the displacement component of the pressure wave. In order to eliminate the time end terms related to instant $t=t_1$ in the last two integration terms of Eq. (9), it should take the spherical convergent pressure wave as the weighted function, which is convergent at the source point p at the instant τ.

Because the response of the concentrated impulsive force subjected at a point P of an infinite elastic medium includes not only the spherical convergent pressure wave, the spherical convergent pressure wave convergent to the point P in an infinite elastic medium cannot result in a concentrated impulsive force subjected at that point. Therefore, Eq. (9) cannot be deduced to an integral equation similar to the Somigliana identity in elastostatic case. Perhaps this is the reason to explain, why the more complex traditional fundamental solution of elastodynamics has been widely applied.

3.2 Derivation of a TDBIE related to spherical convergent pressure wave

To derive the boundary integral equation directly, let the spherical convergent pressure wave is convergent to a boundary point p at the instant τ, and $w^{(1)}{}_{,i}$ is rewritten as u^s_{1i}. For convenience, the direction of x_1 of the Cartesian coordinate system is taken the outward normal direction at boundary point p. To extract the singular point from the integral domain, a small spherical surface with the center at point p and a radius of δ is applied (Fig. 1). In this way, Eq. (9) should be rewritten as

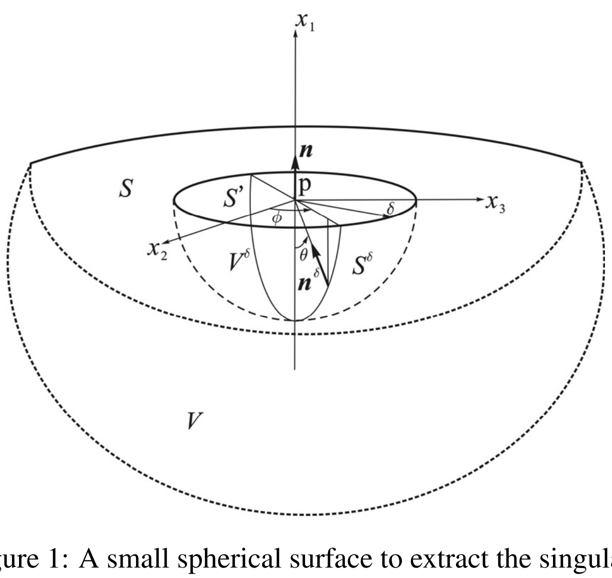


Figure 1: A small spherical surface to extract the singular point p

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}}\left[\rho\left(c_1^2-2c_2^2\right)u_{1k},_k^s\,\delta_{ij}+\rho c_2^2\left(u_{1j},_i^s+u_{1i},_j^s\right)\right]\left(\mathrm{p},\,\tau;\,\mathrm{q}^\delta,t\right)\\
&n_j(\mathrm{q})\,u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S\\
&-\lim_{\delta\to 0}\int_{S-S'+S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}}u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{q},t)\left[\rho\left(c_1^2-2c_2^2\right)u_{k},_k\,\delta_{ij}+\rho c_2^2\left(u_j,_i+u_i,_j\right)\right](\mathrm{q},t)\\
&n_j(\mathrm{q})\,\mathrm{d}t\mathrm{d}S\\
&+\lim_{\delta\to 0}\int_{S-S'}\int_{t_0}^{\tau-\frac{\delta}{c_1}}\left[\rho\left(c_1^2-2c_2^2\right)u_{1k},_k^s\,\delta_{ij}+\rho c_2^2\left(u_{1j},_i^s+u_{1i},_j^s\right)\right](\mathrm{p},\,\tau;\,\mathrm{q},t)\,n_j(\mathrm{q})\\
&u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S\\
&-\lim_{\delta\to 0}\int_{V-V^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}}u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t)\,f_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V\\
&-\lim_{\delta\to 0}\int_{V-V^\delta}u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,\rho\dot{u}_i(\mathrm{Q},t_0)\,\mathrm{d}V\\
&\qquad\qquad+\lim_{\delta\to 0}\int_{V-V^\delta}\rho\dot{u}_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,u_i(\mathrm{Q},t_0)\mathrm{d}V=0 \qquad (11)
\end{aligned}
$$

It should be noted here that in this equation instant t_1 is taken as $\tau-\delta/c_1$ to avoid the singularity, the first integral is removed because the kernel function of spherical pressure wave satisfies the homogeneous wave equation of pressure wave, and the second integral is removed because the dynamic response to be solved satisfies the equations of motion. In this formulation, the equation of motion has been once more differentiated in Eq. (7); therefore the requirement of its differentiability is one order higher than conventional case, both for the domain points and boundary points.

The limit of the integrals in Eq. (11) except the first one is quite simple; therefore Eq. (11) can be rewritten as

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}}t_{1i}^s\left(\mathrm{p},\,\tau;\,\mathrm{q}^\delta,t\right)u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S\\
&-\int_S\int_{t_0}^{\tau}u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{q},t)\,t_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S+\fint_S\int_{t_0}^{\tau}t_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{q},t)\,u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S\\
&-\int_V\int_{t_0}^{\tau}u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t)\,f_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V-\int_V u_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,\rho\dot{u}_i(\mathrm{Q},t_0)\mathrm{d}V\\
&\qquad\qquad+\int_V\rho\dot{u}_{1i}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,u_i(\mathrm{Q},t_0)\mathrm{d}V=0 \qquad (12)
\end{aligned}
$$

where the third integral term is a Cauchy principal value integral term.

In the spherical coordinate system shown in Fig. 1, where the point p is taken as the origin, the scalar potential of the spherical convergent wave can be expressed as

$$w^{(1)} = \frac{a}{r}\mathrm{H}\left(r - c_1\left(\tau - t\right)\right) = \frac{a}{r}\mathrm{H}\left(r - c_1 t'\right) \tag{13}$$

which satisfies the wave equation

$$\rho c_1^2 w^{(1)}{}_{,jj}\left(\mathrm{p},\, \tau;\ \mathrm{Q}, t\right) - \rho \ddot{w}^{(1)}\left(\mathrm{p},\, \tau;\ \mathrm{Q}, t\right) = 0 \tag{14}$$

The corresponding non-zero components of displacement, strain and stress can be expressed in the spherical coordinate system as follows:

$$u_r = \frac{a}{r}\left(\Delta\left(r,\ c_1 t'\right) - \frac{\mathrm{H}\left(r - c_1 t'\right)}{r}\right) \tag{15}$$

$$\begin{aligned} \varepsilon_{rr} &= \frac{a}{r}\left(\Delta'\left(r,\ c_1 t'\right) - \frac{2}{r}\Delta\left(r,\ c_1 t'\right) + \frac{2}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right) \\ \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} &= \frac{a}{r}\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right) \end{aligned} \tag{16}$$

$$\begin{aligned} \sigma_{rr} &= \frac{a}{r}\left[\rho c_1^2 \Delta'\left(r,\ c_1 t'\right) - 4\rho c_2^2\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right] \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \frac{a}{r}\left[\rho\left(c_1^2 - 2c_2^2\right)\Delta'\left(r,\ c_1 t'\right) + 2\rho c_2^2\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right] \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta\left(r,\ c_1 t'\right) &= \frac{\partial}{\partial r}\mathrm{H}\left(r - c_1 t'\right) \\ \Delta'\left(r,\ c_1 t'\right) &= \frac{\partial}{\partial r}\Delta\left(r,\ c_1 t'\right) \end{aligned} \tag{18}$$

In the Cartesian coordinate system, the displacement can be written as

$$u_{1i}^{s} = \frac{a r_{,i}}{r}\left(\Delta\left(r,\ c_1 t'\right) - \frac{\mathrm{H}\left(r - c_1 t'\right)}{r}\right) \tag{19}$$

and the corresponding traction is

$$t^s_{1i} = \frac{an_i}{r}\left[\rho\left(c_1^2 - 2c_2^2\right)\Delta'\left(r,\ c_1 t'\right) + 2\rho c_2^2\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right]$$
$$+ n_j r_{,j} r_{,i}\frac{a}{r}\left[2\rho c_2^2\Delta'\left(r,\ c_1 t'\right) - 6\rho c_2^2\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right] \quad (20)$$

On the small spherical surface S^δ, there is only uniformly distributed normal traction applied. For a boundary point p on the smooth part of boundary, the resultant of the traction should be in the normal direction,

$$\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} t^s_{1i}\left(\mathrm{p},\ \tau;\ \mathrm{q}^\delta, t\right) u_i\left(\mathrm{q}^\delta, t\right)\mathrm{d}t\mathrm{d}S$$
$$= \lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} -\frac{ar_{,i}}{\delta}\left[\rho c_1^2\Delta'\left(\delta,\ c_1(\tau - t)\right)\right.$$
$$\left. -4\rho c_2^2\left(\frac{1}{\delta}\Delta\left(\delta,\ c_1(\tau - t)\right) - \frac{1}{\delta^2}\mathrm{H}\left(\delta - c_1(\tau - t)\right)\right)\right] u_i\left(\mathrm{q}^\delta, t\right)\mathrm{d}t\mathrm{d}S \quad (21)$$

where q^δ denotes the boundary point on the small hemisphere surface S^δ, and

$$\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} -\frac{ar_{,i}}{\delta}\rho c_1^2\Delta'\left(\delta,\ c_1(\tau - t)\right) u_i\left(\mathrm{q}^\delta, t\right)\mathrm{d}t\mathrm{d}S$$
$$= \lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} -\frac{ar_{,i}}{\delta}\rho c_1^2\frac{1}{c_1}\dot{\Delta}\left(\delta,\ c_1(\tau - t)\right) u_i\left(\mathrm{q}^\delta, t\right)\mathrm{d}t\mathrm{d}S$$
$$= \lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} \frac{ar_{,i}}{\delta}\rho c_1^2\frac{1}{c_1}\Delta\left(\delta,\ c_1(\tau - t)\right) \dot{u}_i\left(\mathrm{q}^\delta, t\right)\mathrm{d}t\mathrm{d}S$$
$$= \lim_{\delta\to 0}\int_{S^\delta}\frac{ar_{,i}}{\delta}\rho c_1^2\frac{1}{c_1^2}\dot{u}_i\left(\mathrm{q}^\delta,\ \tau - \frac{\delta}{c_1}\right)\mathrm{d}S = \lim_{\delta\to 0}\left[-\pi\delta^2\frac{an_i}{\delta}\rho\dot{u}_i\left(\mathrm{q}^\delta,\ \tau - \frac{\delta}{c_1}\right)\right] = 0 \quad (22)$$

In the derivation, it should be noted that

$$\dot{\Delta}\left(r,\ c_1(\tau - t)\right) = \frac{\partial}{\partial t}\Delta\left(r,\ c_1(\tau - t)\right)$$

$$\dot{\Delta}\left(r,\ c_1(\tau - t)\right) = \dot{\Delta}\left(r - c_1(\tau - t)\right) = \dot{\Delta}(y) = \frac{\partial}{\partial t}\Delta(y) = \frac{\partial}{\partial y}\Delta(y)\frac{\partial y}{\partial t} = c_1\frac{\partial}{\partial y}\Delta(y)$$
$$= c_1\Delta'\left(r,\ c_1(\tau - t)\right)$$

$$\int_{t_0}^{\tau-\frac{\delta}{c_1}} \dot{\Delta}(r,\ c_1(\tau-t))\,u_i(\mathrm{q},t)\,\mathrm{d}t$$
$$= \int_{t_0}^{\tau-\frac{\delta}{c_1}} \left[\frac{\partial}{\partial t}\left(\Delta(r,\ c_1(\tau-t))\,u_i(\mathrm{q},t)\right) - \Delta(r,\ c_1(\tau-t))\,\dot{u}_i(\mathrm{q},t)\right]\mathrm{d}t$$
$$= -\int_{t_0}^{\tau-\frac{\delta}{c_1}} \Delta(r,\ c_1(\tau-t))\,\dot{u}_i(\mathrm{q},t)\,\mathrm{d}t$$
$$= -\int_{t_0}^{c_1\tau-\delta} \Delta(r,\ c_1(\tau-t))\,\dot{u}_i(\mathrm{q},t)\,\frac{1}{c_1}\mathrm{d}(c_1 t) = -\frac{1}{c_1}\dot{u}_i\left(\mathrm{q},\ \tau-\frac{\delta}{c_1}\right) \quad (23)$$

For the second term in Eq. (21),

$$\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} \frac{ar_{,i}}{\delta}4\rho c_2^2\frac{1}{\delta}\Delta(\delta,\ c_1(\tau-t))\,u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S$$
$$= \lim_{\delta\to 0}\int_{S^\delta}\frac{ar_{,i}}{\delta}4\rho\frac{c_2^2}{c_1}\frac{1}{\delta}u_i\left(\mathrm{q}^\delta,\ \tau-\frac{\delta}{c_1}\right)\mathrm{d}S = \lim_{\delta\to 0}\left[-\pi\delta^2\frac{an_i}{\delta}4\rho\frac{c_2^2}{c_1}\frac{1}{\delta}u_i\left(\mathrm{q}^\delta,\ \tau-\frac{\delta}{c_1}\right)\right]$$
$$= -4\pi a\rho\frac{c_2^2}{c_1}n_i u_i(\mathrm{p},\tau) = -4\pi a\rho\frac{c_2^2}{c_1}u_1(\mathrm{p},\tau) \quad (24)$$

and for the last term

$$\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} -\frac{ar_{,i}}{\delta}4\rho c_2^2\frac{1}{\delta^2}\mathrm{H}(\delta-c_1(\tau-t))\,u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S = 0 \quad (25)$$

It should be noted that here the time integration is equal to zero before $\delta\to 0$, because only at one instant the integrand takes a non-zero and finite value.
Finally it is obtained

$$\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_1}} t_{1i}^s\left(\mathrm{p},\ \tau;\ \mathrm{q}^\delta,t\right)u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S = -4\pi a\rho\frac{c_2^2}{c_1}u_1(\mathrm{p},\tau) \quad (26)$$

If we take the constant $a = \frac{c_1}{4\pi\rho c_2^2}$, Eq. (12) can be rewritten as

$$u_1(\mathrm{p},\tau) = -\int_S\int_{t_0}^{\tau} u_{1i}^s(\mathrm{p},\ \tau;\ \mathrm{q},t)\,t_i(\mathrm{q},t)\,n_j(\mathrm{q})\,\mathrm{d}t\mathrm{d}S$$
$$+\int_S\int_{t_0}^{\tau} t_{1i}^s(\mathrm{p},\ \tau;\ \mathrm{q},t)\,n_j(\mathrm{q})\,u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S$$
$$-\int_V\int_{t_0}^{\tau} u_{1i}^s(\mathrm{p},\ \tau;\ \mathrm{Q},t)\,f_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V - \int_V u_{1i}^s(\mathrm{p},\ \tau;\ \mathrm{Q},t_0)\,\rho\dot{u}_i(\mathrm{Q},t_0)\mathrm{d}V$$
$$+\int_V \rho\dot{u}_{1i}^s(\mathrm{p},\ \tau;\ \mathrm{Q},t_0)\,u_i(\mathrm{Q},t_0)\mathrm{d}V \quad (27)$$

This is the first one of the new TDBIE, where the spherical convergent pressure wave is applied as the kernel function,

$$u_{1i}^{s} = \frac{c_1 r_{,i}}{4\pi\rho c_2^2 r}\left(\Delta\left(r,\ c_1 t'\right) - \frac{\mathrm{H}\left(r - c_1 t'\right)}{r}\right) \tag{28}$$

$$\begin{aligned} t_{1i}^{s} = \frac{c_1 n_i}{4\pi r}&\left[\left(\frac{c_1^2}{c_2^2} - 2\right)\Delta'\left(r,\ c_1 t'\right) + 2\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right] \\ &+ \frac{c_1 n_j r_{,j} r_{,i}}{2\pi r}\left[\Delta'\left(r,\ c_1 t'\right) - 3\left(\frac{1}{r}\Delta\left(r,\ c_1 t'\right) - \frac{1}{r^2}\mathrm{H}\left(r - c_1 t'\right)\right)\right] \end{aligned} \tag{29}$$

For the simpler cases with zero initial conditions and without body force, Eq. (27) can be simplified as

$$\begin{aligned} u_1\left(\mathrm{p},\ \tau\right) = &-\int_S \int_{t_0}^{\tau} u_{1i}^{s}\left(\mathrm{p},\ \tau;\ \mathrm{q},\ t\right) t_i\left(\mathrm{q},\ t\right) \mathrm{d}t\mathrm{d}S \\ &+ \int_S \int_{t_0}^{\tau} t_{1i}^{s}\left(\mathrm{p},\ \tau;\ \mathrm{q},\ t\right) u_i\left(\mathrm{q},\ t\right) \mathrm{d}t\mathrm{d}S \end{aligned} \tag{30}$$

3.3 TDBIE related to spherical convergent shear waves

The TDBIE related to the spherical convergent shear waves can be derived from the second equation of Eq. (8).

The vectorial potential of the spherical convergent shear waves can be expressed as

$$w_2^{(2)} = \frac{b}{r}\mathrm{H}\left(r - c_2 t'\right),\ w_3^{(2)} = -\frac{b}{r}\mathrm{H}\left(r - c_2 t'\right) \tag{31}$$

The corresponding displacements can be written as

$$\begin{aligned} u_{2j}^{s} &= -e_{ji3}\frac{b r_{,i}}{r}\left(\Delta\left(r,\ c_2 t'\right) - \frac{\mathrm{H}\left(r - c_2 t'\right)}{r}\right) \\ u_{3j}^{s} &= e_{ji2}\frac{b r_{,i}}{r}\left(\Delta\left(r,\ c_2 t'\right) - \frac{\mathrm{H}\left(r - c_2 t'\right)}{r}\right) \end{aligned} \tag{32}$$

For simplicity, it is denoted that

$$\Psi\left(r,\ c_2 t'\right) \triangleq \Delta'\left(r,\ c_2 t'\right) - \frac{2\Delta\left(r - c_2 t'\right)}{r} + \frac{2\mathrm{H}\left(r - c_2 t'\right)}{r^2} \tag{33}$$

in the following formulae for the corresponding strain and stress components:

$$
\begin{aligned}
&\varepsilon_{11}^{s(2)} = -\frac{br_{,2}\,r_{,1}}{r}\Psi\left(r,\ c_2t'\right),\ \varepsilon_{22}^{s(2)} = \frac{br_{,1}\,r_{,2}}{r}\Psi\left(r,\ c_2t'\right)\\
&\varepsilon_{12}^{s(2)} = \varepsilon_{21}^{s(2)} = -\frac{b\left(r_{,2}\,r_{,2} - r_{,1}\,r_{,1}\right)}{2r}\Psi\left(r,\ c_2t'\right)\\
&\varepsilon_{13}^{s(2)} = \varepsilon_{31}^{s(2)} = -\frac{br_{,2}\,r_{,3}}{2r}\Psi\left(r,\ c_2t'\right),\ \varepsilon_{23}^{s(2)} = \varepsilon_{32}^{s(2)} = \frac{br_{,1}\,r_{,3}}{2r}\Psi\left(r,\ c_2t'\right)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&\varepsilon_{11}^{s(3)} = -\frac{br_{,3}\,r_{,1}}{r}\Psi\left(r,\ c_2t'\right)\ \varepsilon_{33}^{s(3)} = \frac{br_{,1}\,r_{,3}}{r}\Psi\left(r,\ c_2t'\right)\\
&\varepsilon_{13}^{s(3)} = \varepsilon_{31}^{s(3)} = -\frac{b\left(r_{,3}\,r_{,3} - r_{,1}\,r_{,1}\right)}{2r}\Psi\left(r,\ c_2t'\right)\\
&\varepsilon_{12}^{s(3)} = \varepsilon_{21}^{s(3)} = -\frac{br_{,3}\,r_{,2}}{2r}\Psi\left(r,\ c_2t'\right)\ \varepsilon_{23}^{s(3)} = \varepsilon_{32}^{s(3)} = \frac{br_{,1}\,r_{,2}}{2r}\Psi\left(r,\ c_2t'\right)
\end{aligned}
\tag{35}
$$

$$
\sigma_{ij}^{s(2)} = 2\rho c_2^2\varepsilon_{ij}^{s(2)},\ \sigma_{ij}^{s(3)} = 2\rho c_2^2\varepsilon_{ij}^{s(3)}
\tag{36}
$$

The corresponding traction components can be expressed as

$$
\begin{aligned}
&t_{21}^{s} = n_i\sigma_{i1}^{s(2)} = -\rho c_2^2\frac{b}{r}\left(2n_1r_{,2}\,r_{,1} + n_2\left(r_{,2}\,r_{,2} - r_{,1}\,r_{,1}\right) + n_3r_{,2}\,r_{,3}\right)\Psi\left(r,\ c_2t'\right)\\
&t_{22}^{s} = n_i\sigma_{i2}^{s(2)} = -\rho c_2^2\frac{b}{r}\left(n_1\left(r_{,2}\,r_{,2} - r_{,1}\,r_{,1}\right) - 2n_2r_{,1}\,r_{,2} - n_3r_{,1}\,r_{,3}\right)\Psi\left(r,\ c_2t'\right)\\
&t_{23}^{s} = n_i\sigma_{i3}^{s(2)} = -\rho c_2^2\frac{b}{r}\left(n_1r_{,2}\,r_{,3} - n_2r_{,1}\,r_{,3}\right)\Psi\left(r,\ c_2t'\right)
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&t_{31}^{s} = n_i\sigma_{i1}^{s(3)} = \rho c_2^2\frac{b}{r}\left(-2n_1r_{,3}\,r_{,1} - n_2r_{,3}\,r_{,2} - n_3\left(r_{,3}\,r_{,3} - r_{,1}\,r_{,1}\right)\right)\Psi\left(r,\ c_2t'\right)\\
&t_{32}^{s} = n_i\sigma_{i2}^{s(3)} = \rho c_2^2\frac{b}{r}\left(-n_1r_{,3}\,r_{,2} + n_3r_{,1}\,r_{,2}\right)\Psi\left(r,\ c_2t'\right)\\
&t_{33}^{s} = n_i\sigma_{i3}^{s(3)} = \rho c_2^2\frac{b}{r}\left(n_1\left(r_{,1}\,r_{,1} - r_{,3}\,r_{,3}\right) + n_2r_{,1}\,r_{,2} + 2n_3r_{,1}\,r_{,3}\right)\Psi\left(r,\ c_2t'\right)
\end{aligned}
\tag{38}
$$

The equations derived from the second equation of Eq. (8) are similar to Eq. (12), namely

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{S^\delta}\int_{t_0}^{\tau-\frac{\delta}{c_2}} t_{ki}^s\left(\mathrm{p},\,\tau;\,\mathrm{q}^\delta,t\right)u_i\left(\mathrm{q}^\delta,t\right)\mathrm{d}t\mathrm{d}S\\
&-\int_S\int_{t_0}^{\tau} u_{ki}^s(\mathrm{p},\,\tau;\,\mathrm{q},t)\,t_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S+\fint_S\int_{t_0}^{\tau} t_{ki}^s(\mathrm{p},\,\tau;\,\mathrm{q},t)\,u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S\\
&-\int_V\int_{t_0}^{\tau} u_{ki}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t)\,f_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V-\int_V u_{ki}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,\rho\dot{u}_i(\mathrm{Q},t_0)\mathrm{d}V\\
&+\int_V \rho\dot{u}_{ki}^s(\mathrm{p},\,\tau;\,\mathrm{Q},t_0)\,u_i(\mathrm{Q},t_0)\mathrm{d}V=0
\end{aligned}$$

$$k=2,\ 3 \quad (39)$$

On the semispherical surface S^δ, in the spherical coordinate system, it can be noted that

$$\begin{aligned}
&r_{,1}=-\cos\theta,\ n_1=\cos\theta\\
&r_{,2}=\sin\theta\cos\phi,\ n_2=-\sin\theta\cos\phi\\
&r_{,3}=\sin\theta\sin\phi,\ n_3=-\sin\theta\sin\phi
\end{aligned}\tag{40}$$

Therefore only t_{22}^s and t_{33}^s have nonzero resultant forces. The first integral in Eq. (39) for the case of $k=2$ can be expressed in the spherical coordinate system as

$$\begin{aligned}
&\lim_{\delta\to 0}\int_0^{\pi}\int_0^{2\pi}\int_{t_0}^{\tau-\frac{\delta}{c_2}} -\rho c_2^2\frac{b}{r}\left(n_1\left(r_{,2}r_{,2}-r_{,1}r_{,1}\right)-2n_2r_{,1}r_{,2}-n_3r_{,1}r_{,3}\right)\Psi\left(r,\ c_2t'\right)\\
&u_2\left(\mathrm{q}^\delta,\ t\right)\mathrm{d}t\delta^2\sin\theta\mathrm{d}\phi\mathrm{d}\theta\\
&=\lim_{\delta\to 0}\int_0^{\pi}\int_0^{2\pi} -\rho c_2^2\frac{b}{\delta}\left(n_1\left(r_{,2}r_{,2}-r_{,1}r_{,1}\right)-2n_2r_{,1}r_{,2}-n_3r_{,1}r_{,3}\right)\\
&\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\Delta'\left(r,\ c_2t'\right)-\frac{2\Delta\left(r,\ c_2t'\right)}{\delta}+\frac{2\mathrm{H}\left(r,\ c_2t'\right)}{\delta^2}\right)u_2\left(\mathrm{q}^\delta,\ t\right)\mathrm{d}t\delta^2\sin\theta\mathrm{d}\phi\mathrm{d}\theta\\
&=\int_0^{\pi}\int_0^{2\pi} -\rho c_2^2 b\left(n_1\left(r_{,2}r_{,2}-r_{,1}r_{,1}\right)-2n_2r_{,1}r_{,2}-n_3r_{,1}r_{,3}\right)\\
&\left[\lim_{\delta\to 0}\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\Delta'\left(r,\ c_2t'\right)\delta-2\Delta\left(r,\ c_2t'\right)+\frac{2\mathrm{H}\left(r,\ c_2t'\right)}{\delta}\right)u_2\left(\mathrm{q}^\delta,\ t\right)\mathrm{d}t\right]\sin\theta\mathrm{d}\phi\mathrm{d}\theta
\end{aligned}$$

where

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\Delta'\left(\delta,\ c_2t'\right)\delta-2\Delta\left(\delta,\ c_2t'\right)+\frac{2\mathrm{H}\left(\delta,\ c_2t'\right)}{\delta}\right)u_2\left(\mathrm{q}^{\delta},\ t\right)\mathrm{d}t\\
&=\lim_{\delta\to 0}\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\frac{1}{c_2}\dot{\Delta}\left(\delta,\ c_2t'\right)\delta-2\Delta\left(\delta,\ c_2t'\right)+\frac{2\mathrm{H}\left(\delta,\ c_2t'\right)}{\delta}\right)u_2\left(\mathrm{q}^{\delta},\ t\right)\mathrm{d}t\\
&=\lim_{\delta\to 0}\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\frac{1}{c_2}\Delta\left(\delta,\ c_2t'\right)\dot{u}_2\left(\mathrm{q}^{\delta},\ t\right)\delta-2\Delta\left(\delta,\ c_2t'\right)u_2\left(\mathrm{q}^{\delta},\ t\right)\right.\\
&\qquad\left.+\frac{2\mathrm{H}\left(\delta,\ c_2t'\right)}{\delta}u_2\left(\mathrm{q}^{\delta},\ t\right)\right)\mathrm{d}t=-\frac{2}{c_2}u_2\left(\mathrm{p},\ \tau\right)
\end{aligned}\tag{41}$$

and then

$$\begin{aligned}
&\int_0^{\pi}\int_0^{2\pi}-\rho c_2^2 b\left(n_1\left(r_{,2}\,r_{,2}-r_{,1}\,r_{,1}\right)-2n_2r_{,1}\,r_{,2}-n_3r_{,1}\,r_{,3}\right)\\
&\left[\lim_{\delta\to 0}\int_{t_0}^{\tau-\frac{\delta}{c_2}}\left(\Delta'\left(r,\ c_2t'\right)\delta-2\Delta\left(r,\ c_2t'\right)+\frac{2\mathrm{H}\left(r,\ c_2t'\right)}{\delta}\right)u_2\left(\mathrm{q}^{\delta},\ t\right)\mathrm{d}t\right]\sin\theta\mathrm{d}\phi\mathrm{d}\theta\\
&=\frac{2}{c_2}u_2\left(\mathrm{p},\ \tau\right)\int_0^{\pi}\int_0^{2\pi}\rho c_2^2 b\left(n_1\left(r_{,2}\,r_{,2}-r_{,1}\,r_{,1}\right)-2n_2r_{,1}\,r_{,2}-n_3r_{,1}\,r_{,3}\right)\sin\theta\mathrm{d}\phi\mathrm{d}\theta\\
&=\frac{2}{c_2}u_2\left(\mathrm{p},\ \tau\right)\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho c_2^2 b\left(\cos\theta\left(\sin^2\theta\cos^2\phi-\cos^2\theta\right)-2\sin^2\theta\cos^2\phi\cos\theta\right.\\
&\left.-\sin^2\theta\sin^2\phi\cos\theta\right)\sin\theta\mathrm{d}\phi\mathrm{d}\theta\\
&\qquad=-\frac{\rho c_2^2 b}{c_2}u_2\left(\mathrm{p},\ \tau\right)\int_0^{\frac{\pi}{2}}\sin 2\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi=-2\pi\rho c_2 b u_2\left(\mathrm{p},\ \tau\right)
\end{aligned}\tag{42}$$

If we take the constant $b=\frac{1}{2\pi\rho c_2}$, Eq. (39) for the case of $k=2$ can be finally rewritten as

$$\begin{aligned}
&u_2\left(\mathrm{p},\ \tau\right)=-\int_S\int_{t_0}^{\tau}u_{2i}^s\left(\mathrm{p},\ \tau;\ \mathrm{q},\ t\right)t_i\left(\mathrm{q},\ t\right)\mathrm{d}t\mathrm{d}S\\
&+\fint_S\int_{t_0}^{\tau}t_{2i}^s\left(\mathrm{p},\ \tau;\ \mathrm{q},\ t\right)u_i\left(\mathrm{q},\ t\right)\mathrm{d}t\mathrm{d}S\\
&-\int_V\int_{t_0}^{\tau}u_{2i}^s\left(\mathrm{p},\ \tau;\ \mathrm{Q},\ t\right)f_i\left(\mathrm{Q},\ t\right)\mathrm{d}t\mathrm{d}V-\int_V u_{2i}^s\left(\mathrm{p},\ \tau;\ \mathrm{Q},\ t_0\right)\rho\dot{u}_i\left(\mathrm{Q},\ t_0\right)\mathrm{d}V\\
&\qquad+\int_V\rho\dot{u}_{2i}^s\left(\mathrm{p},\ \tau;\ \mathrm{Q},\ t_0\right)u_i\left(\mathrm{Q},\ t_0\right)\mathrm{d}V
\end{aligned}\tag{43}$$

The corresponding equation for the case of $k=3$ can be derived similarly, and finally it can be summarized as

$$u_k(\mathrm{p},\tau) = -\int_S\int_{t_0}^{\tau} u_{ki}^s(\mathrm{p},\tau;\,\mathrm{q},t)\,t_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S$$
$$+\int_S\int_{t_0}^{\tau} t_{ki}^s(\mathrm{p},\tau;\,\mathrm{q},t)\,u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S$$
$$-\int_V\int_{t_0}^{\tau} u_{ki}^s(\mathrm{p},\tau;\,\mathrm{Q},t)\,f_i(\mathrm{Q},t)\,\mathrm{d}t\mathrm{d}V - \int_V u_{ki}^s(\mathrm{p},\tau;\,\mathrm{Q},t_0)\,\rho\dot{u}_i(\mathrm{Q},t_0)\mathrm{d}V$$
$$+\int_V \rho\dot{u}_{ki}^s(\mathrm{p},\tau;\,\mathrm{Q},t_0)\,u_i(\mathrm{Q},t_0)\mathrm{d}V$$
$$k=1,\,2,\,3 \quad (44)$$

and for the simpler cases with zero initial conditions and without body force, this equation can be simplified as

$$u_k(\mathrm{p},\tau) = -\int_S\int_{t_0}^{\tau} u_{ki}^s(\mathrm{p},\tau;\,\mathrm{q},t)\,t_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S + \int_S\int_{t_0}^{\tau} t_{ki}^s(\mathrm{p},\tau;\,\mathrm{q},t)\,u_i(\mathrm{q},t)\,\mathrm{d}t\mathrm{d}S \quad k=1,\,2,\,3 \quad (45)$$

4 An efficient scheme of TDBEM

For the solution of the new derived TDBIE, the traditional TDBEM can be applied. Considered the characteristics of the new TDBIE, an efficient scheme of TDBEM is then suggested.

4.1 Traditional TDBEM applied to the new derived equations

The whole boundary is divided into N_{e} boundary elements, and the time interval from t_0 to t_1 is divided into M time steps. For the time discretization, the traction and displacement can apply different shape functions as follows:

$$\begin{aligned} t_i(\mathrm{q},t) &= \sum_{m=1}^{M} \phi_m(t)\,t_i^m(\mathrm{q}) \\ u_i(\mathrm{q},t) &= \sum_{m=1}^{M} \left[\xi_{1m}(t)\,u_i^m(\mathrm{q}) + \xi_{2m}(t)\,u_i^{m-1}(\mathrm{q})\right] \end{aligned} \quad (46)$$

where

$$\phi_m(t) = H(t-(m-1)\Delta t) - H(t-m\Delta t)$$
$$\xi_{1m}(t) = \frac{t-(m-1)\Delta t}{\Delta t}\phi_m(t),\ \xi_{2m}(t) = \frac{m\Delta t - t}{\Delta t}\phi_m(t)$$

That is, the constant approximation is applied for the time interpolation of the traction, and for the time interpolation of displacement, linear approximation is applied. Introduced the above interpolation into the new derived TDBIE, Eq. (45) for the simpler cases with zero initial conditions and without body force, it can be rewritten as

$$u_j^\mu(\mathrm{p}) = -\sum_{m=1}^{\mu}\int_S\int_{(m-1)\Delta t}^{m\Delta t} u_{ji}^s(\mathrm{p},\mathrm{q};\ \mu\Delta t - t)\, t_i^m(\mathrm{q})\,\mathrm{d}t\mathrm{d}S(\mathrm{q}) + \sum_{m=1}^{\mu}\int_S\int_{(m-1)\Delta t}^{m\Delta t} t_{ji}^s(\mathrm{p},\mathrm{q};\ \mu\Delta t - t)\left[\xi_{1m}(t)\,u_i^m(\mathrm{q}) + \xi_{2m}(t)\,u_i^{m-1}(\mathrm{q})\right]\mathrm{d}t\mathrm{d}S(\mathrm{q}) \tag{47}$$

where $\mu = \tau/\Delta t$.

For the interpolation of boundary displacement, traction, the same shape functions can be applied, if the displacement and traction components of the α node of n element at instant $m\Delta t$ are denoted as $u_i^{nm\alpha}$, $t_i^{nm\alpha}$, and the corresponding geometrical coordinates are denoted as $x_i^{n\alpha}$, the TDBIE can finally be discretized as the following system of linear algebraic equations:

$$u_j^\mu(\mathrm{p}) = -\sum_{m=1}^{\mu}\sum_{n=1}^{N_e}\sum_{\alpha} t_i^{nm\alpha}\int_{-1}^{1}\int_{-1}^{1} U_{ji}^{\mu-m+1} N_\alpha J^n \mathrm{d}\eta_1\mathrm{d}\eta_2 + \sum_{m=1}^{\mu}\sum_{n=1}^{N_e}\sum_{\alpha} u_i^{nm\alpha}\int_{-1}^{1}\int_{-1}^{1}\left(T_{ji1}^{\mu-m+1} + T_{ji2}^{\mu-m}\right) N_\alpha J^n \mathrm{d}\eta_1\mathrm{d}\eta_2 \tag{48}$$

where $U_{ji}^{\mu-m+1}$, $T_{ji1}^{\mu-m+1}$, $T_{ji2}^{\mu-m}$ denote the time integration of kernel functions,

$$\begin{aligned} U_{ji}^{\mu-m+1}(\mathrm{p};\mathrm{q}) &= \int_{(m-1)\Delta t}^{m\Delta t} u_{ji}^s(\mathrm{p},\mathrm{q};\ \mu\Delta t - t)\,\mathrm{d}t \\ T_{ji1}^{\mu-m+1}(\mathrm{p};\mathrm{q}) &= \int_{(m-1)\Delta t}^{m\Delta t} t_{ji}^s(\mathrm{p},\mathrm{q};\ \mu\Delta t - t)\,\xi_{1m}(t)\,\mathrm{d}t \\ T_{ji2}^{\mu-m+1}(\mathrm{p};\mathrm{q}) &= \int_{(m-1)\Delta t}^{m\Delta t} t_{ji}^s(\mathrm{p},\mathrm{q};\ \mu\Delta t - t)\,\xi_{2m}(t)\,\mathrm{d}t \end{aligned} \tag{49}$$

The detailed formulae can be derived easily.

In the matrix form, Eq. (48) can be written as

$$\tilde{\mathbf{H}}^{\mu\mu}\mathbf{u}^\mu = \tilde{\mathbf{G}}^\mu\mathbf{t}^\mu + \sum_{m=1}^{\mu-1}\left(\tilde{\mathbf{G}}^{\mu m}\mathbf{t}^m - \tilde{\mathbf{H}}^{\mu m}\mathbf{u}^m\right) \tag{50}$$

where $\mathbf{u}^m$ stands for the displacement array of instant $m\Delta t$, $\mathbf{t}^m$ stands for the traction array of the time step m, and the matrices $\tilde{\mathbf{H}}^{\mu m}$, $\tilde{\mathbf{G}}^{\mu m}$ are formed by integrating the product of kernel function and shape function on boundary elements for each time step.

Considered the boundary conditions, and moved all unknowns to the left side, this equation can be rewritten as

$$\tilde{\mathbf{A}}^{\mu\mu}\mathbf{x}^{\mu} = \tilde{\mathbf{B}}^{\mu\mu}\mathbf{y}^{\mu} + \sum_{m=1}^{\mu-1}\left(\tilde{\mathbf{G}}^{\mu m}\mathbf{t}^{m} - \tilde{\mathbf{H}}^{\mu m}\mathbf{u}^{m}\right) \quad \mu = 1, 2, ..., M \tag{51}$$

In the practical computation starting from the time step $\mu = m = 1$, the matrices $\tilde{\mathbf{A}}^{\mu\mu}$, $\tilde{\mathbf{B}}^{\mu\mu}$ need to be computed only once. Actually all the matrices $\tilde{\mathbf{A}}^{\mu\mu}$, $\tilde{\mathbf{B}}^{\mu\mu}$, $\tilde{\mathbf{H}}^{\mu m}$, $\tilde{\mathbf{G}}^{\mu m}$ only depend on the difference of their two superscripts. During the computation of a new time step, only the matrices with maximum difference of their two superscripts, namely, $\tilde{\mathbf{H}}^{\mu 1}$ and $\tilde{\mathbf{G}}^{\mu 1}$ are new and need to be computed.

Finally, the equations can be rewritten as

$$\begin{aligned} &\tilde{\mathbf{A}}\mathbf{x}^{\mu} = \mathbf{f}^{\mu} \\ &\mathbf{f}^{\mu} = \tilde{\mathbf{B}}\mathbf{y}^{\mu} + \sum_{m=1}^{\mu-1}\left(\tilde{\mathbf{G}}^{\mu m}\mathbf{t}^{m} - \tilde{\mathbf{H}}^{\mu m}\mathbf{u}^{m}\right) \\ &\mu = 1,\ 2, ..., M \end{aligned} \tag{52}$$

This equation system can be solved time step by step. During the first several steps, the difference of two superscripts in $\tilde{\mathbf{H}}^{\mu m}$, $\tilde{\mathbf{G}}^{\mu m}$ are quite small, and the matrices are quite sparse. As the time step increases, the non-zero components in the matrices, and the computing cost per time step increases gradually and continuously.

4.2 An efficient scheme of TDBEM

For some large scale problems, not only the number of boundary nodes but also the time steps should be quite large. To make the computation more efficient, it should be improved the kernel functions in the TDBIE. Because the kernel functions applied are spherical convergent waves, they are D'Alembert solutions, and the wave form of such solution could be arbitrary. Above mentioned only the simplest one, we can adopt an efficient one to enhance the efficiency of the computation, the corresponding potential function is the solid line shown in Fig. 2, while the dashed line is the above mentioned simplest one.

The potential of the spherical convergent impulsive waves can be expressed as

$$\begin{aligned}
\phi^{(1)} &= \frac{a}{r}\mathrm{H}\left(r-c_1t'\right)-\frac{a}{r}\mathrm{H}\left(r-c_1T-c_1t'\right)\\
\psi_2^{(2)} &= \frac{a}{r}\mathrm{H}\left(r-c_2t'\right)-\frac{a}{r}\mathrm{H}\left(r-c_2T-c_2t'\right)\\
\psi_3^{(2)} &= -\frac{a}{r}\mathrm{H}\left(r-c_2t'\right)+\frac{a}{r}\mathrm{H}\left(r-c_2T-c_2t'\right)
\end{aligned} \tag{53}$$

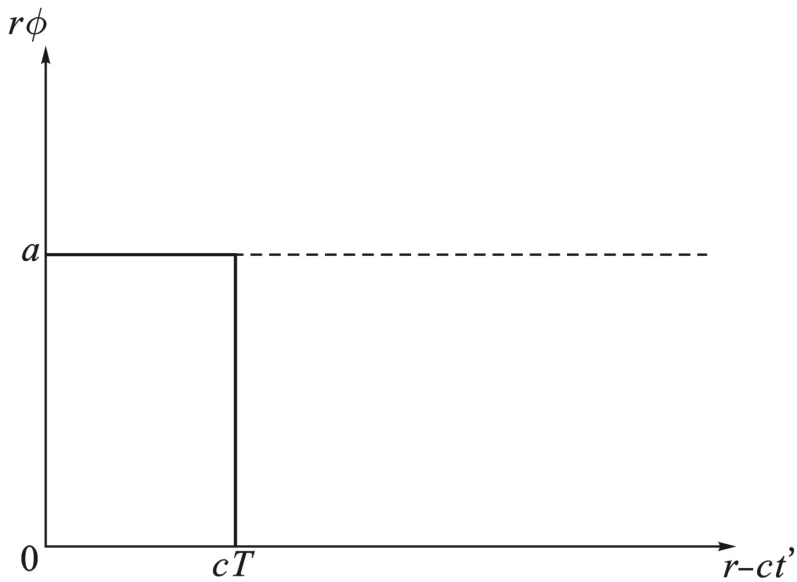


Figure 2: The potential of spherical convergent impulsive waves

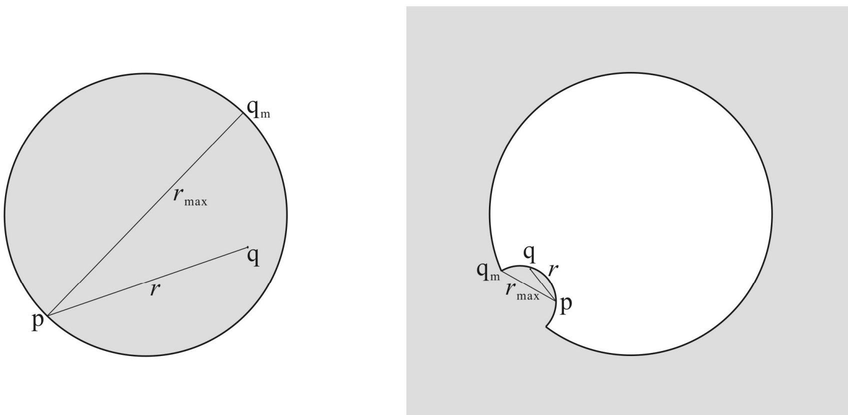
The corresponding displacements are

$$\begin{aligned}
u_{1i} &= \frac{ar_{,i}}{r}\left(\Delta\left(r,\ c_1t'\right)-\Delta\left(r,\ c_1t'+c_1T\right)-\frac{\mathrm{H}\left(r-c_1t'\right)}{r}+\frac{\mathrm{H}\left(r-c_1t'-c_1T\right)}{r}\right)\\
u_{2i} &= -e_{ij3}\frac{br_{,j}}{r}\left(\Delta\left(r,\ c_2t'\right)-\Delta\left(r,\ c_2t'+c_2T\right)-\frac{\mathrm{H}\left(r-c_2t'\right)}{r}+\frac{\mathrm{H}\left(r-c_2t'-c_2T\right)}{r}\right)\\
u_{3i} &= e_{ij2}\frac{br_{,j}}{r}\left(\Delta\left(r,\ c_2t'\right)-\Delta\left(r,\ c_2t'+c_2T\right)-\frac{\mathrm{H}\left(r-c_2t'\right)}{r}+\frac{\mathrm{H}\left(r-c_2t'-c_2T\right)}{r}\right)
\end{aligned} \tag{54}$$

As the spherical convergent impulsive waves are convergent to the point p, the nonzero displacements are localized in a sphere with a radius $r = c_1T$ for pressure wave, and $r = c_2T$ for shear wave.

In this way, Eq. (52) can be reduced to

$$\begin{aligned}
&\tilde{\mathbf{A}}\mathbf{x}^{\mu} = \mathbf{f}^{\mu} \\
&\mathbf{f}^{\mu} = \tilde{\mathbf{B}}\mathbf{y}^{\mu} + \sum_{m=m_1}^{\mu-1} \left(\tilde{\mathbf{G}}^{\mu m}\mathbf{t}^{m} - \tilde{\mathbf{H}}^{\mu m}\mathbf{u}^{m}\right) \\
&m_1 = \max\left(1,\ \mu - T/\Delta t\right) \\
&\mu = 1,\ 2,\ ...,\ M
\end{aligned} \tag{55}$$



(a) (b)

Figure 3: The definition of the maximum length $r_{\max}$ connecting p and q

It should be mentioned here, to guarantee the equivalence of the TDBIE with the corresponding initial and boundary value problem of the partial differential equation of elastodynamics, the width of the impulse $c_1T,\ c_2T$ should be greater than the maximum length $r_{\max}$ of the lines in the elastic domain connecting the convergent boundary point p with all other boundary points q as shown in Fig. 3. This is come from the completeness condition for the arbitrary weighted function in weighted residual integration form. Fig. 3(a) is an inner domain problem of an elastic sphere, $r_{\max}$ is just the diameter of the sphere. Fig. 3(b) is infinite elastic medium with a cavity. If the cavity has convex surface, such as spherical or ellipsoidal cavity, $r_{\max}$ is approaches zero, theoretically very short convergent impulsive wave can be applied in the above mentioned computation.

Before the discretization, the spherical convergent pressure wave and spherical convergent shear wave convergent to any boundary point p at any instant construct a complete function system, which can formulate arbitrary elastic waves in the elastic domain without body forces. This completeness guarantees the equivalence of the TDBIE with the partial differential equations of elastodynamics.

In some cases of the large scale problems, the computation efficiency could be enhanced dramatically by using the impulsive waves as the kernel functions. The width of the impulse could be optimized in detail in the future work under the consideration of both efficiency and accuracy.

5 Concluding Remarks

Based on the general method for the derivation of boundary integral equation, a system of new TDBIE has been derived directly from the partial differential equations of elastodynamics. The spherical convergent pressure and shear waves are taken as the kernel functions in the derived TDBIE respectively. In comparison with the traditional TDBIE, the new derived TDBIE is not only much simpler, but also with clear physical meaning.

For the solution of the new TDBIE, the traditional TDBEM can be applied easily, and the computational efficiency could be enhanced, because the kernel functions are much simpler than traditional one. The resulted linear algebraic equation system can be solved time step by step. During the first several steps, the matrices are quite sparse. As the time step increases, the non-zero components in the matrices, and the computing cost per time step increases gradually and continuously.

For some large scale problems, not only the number of boundary nodes but also the time steps should be quite large. To further enhance the computational efficiency, as the kernel functions in TDBIE is adopted the spherical convergent impulsive pressure and shear waves respectively. As the spherical convergent impulsive waves are convergent to the boundary point p, the nonzero displacements are localized in a sphere with a prescribed radius. But it should be mentioned that to guarantee the equivalence of the TDBIE with the corresponding partial differential equation of elastodynamics, the width of the impulse should be greater than the maximum length $r_{\max}$ of the lines in the elastic domain connecting the point p with all other boundary points q. In some cases of the large scale problems, the computation efficiency could expect to be enhanced dramatically by using the impulsive waves as the kernel functions. The width of the impulse could be optimized in future work.

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