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Dynamical Response of Two Axially Pre-Strained System Comprising of a Covering Layer and a Half Space to Rectangular Time-Harmonic Forces

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Abstract: The time-harmonic dynamical stress field in the system comprising two axially pre-stressed covering layer and two axially pre-stressed half space was studied under the action of uniformly distributed forces on free face plane of the covering layer. It is assumed that the forces are distributed within the rectangular area. The study was conducted within the scope of the piecewise homogeneous body model with the use of three-dimensional theory of elastic waves in an initially stressed bodies. The materials of the layer and half-space were assumed to be isotropic and homogeneous. The corresponding three-dimensional boundary-value-contact problem was solved by applying double Fourier exponential integral transformation. The numerical results were presented and discussed for the case where the material of the layer and half space were aluminum and steel, respectively. In this case, the main focus was the dependencies between the interfacial normal stress and frequency of the external forces. It was established that under the action of the statically equivalent forces in rectangular area the influence of the size of the area on the prescribed dependencies increased with increasing frequency of the forces. At the same time, it was shown that the influence of the pre-stretching of the covering layer on the stress was dependent on the frequency of the external forces. In particular, it was found that the influence became more significant as the frequency of the external forces approached the "resonance" frequency.

Keywords: Initial stress, time-harmonic forces, dynamical response, "resonance" frequency, stress distribution.

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1 Introduction

Studies of the dynamical response of the composite deformable systems with initial stresses to the acting of the external time-harmonic forces have a wide range of applications in various areas of natural sciences and engineering including mechanics of composites, seismology, biomechanics. The intensive investigations in this field was initiated in the second half of the 20^{th} century. Numerous theoretical and experimental investigations have been conducted. The systematic consideration and analyses of the results of these investigations were presented in monographs by Biot (1965), Eringen (1975) and Guz (2004). The review of the later researches has been presented in the papers by Guz (2002), Akbarov and Zhuk (2007). Almost all these investigations were carried out utilizing the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB). A considerable part of those referred to the study of the influence of the initial stresses of the wave propagation (dispersion) in various types of members of constructions and in layered materials. It should also be noted that the investigations regarding the study of the influence of the initial strains (or stresses) on the time-harmonic dynamical stress field in the homogeneous and layered materials had great significance in theoretical and practical sense. Therefore in the papers by Akbarov (2006a, 2006b, 2006c), Akbarov (2007), Akbarov and Guler (2007), Akbarov and Guler (2005), Akbarov, Zamanov and Suleimanov (2005), Emiroglu, Tasci and Akbarov (2004), Akbarov, Emiroglu and Tasci (2005), Yahnioglu (2007) and the other aforementioned studies were made for the system consisting of a pre-stressed covering layer and a pre-stressed half-space. However, almost all these investigations (except the papers by Emiroglu, Tasci and Akbarov (2004); Akbarov, Emiroglu and Tasci (2005) were related to the plane-strain state in the stratified half-plane under the action of the line located time-harmonic forces. However, in the papers by Emiroglu, Tasci and Akbarov (2004); Akbarov, Emiroglu and Tasci (2005) the three-dimensional Lamb's problem was studied for the system consisting of a bi-axially pre-stressed covering layer and a bi-axially pre-stressed half-space under the action of a normal point-located time-harmonic force on the free-face plane of the covering layer. It is known that "a point-located time-harmonic force" modeling of the external loading is suitable for the cases where the stress distribution is considered for the areas far from the loading region. Consequently, the study of stress distribution in the areas near the loading region, the mentioned model is not suitable (especially for the dynamical problems). Usually, in such cases "a continuous distributed time-harmonic force" modeling of the external loading is used. In connection with this, in the present paper, the investigations carried out in the papers by Emiroglu, Tasci and Akbarov (2004); Akbarov, Emiroglu and Tasci (2005) was developed for the case where the normal time-harmonic uniformly distributed force acts on the rectangular area lying on the free face plane of the covering layer. Moreover, in this paper the main attention is given to the dependence between the normal stress acting on the interfacial plane and the frequency of the external forces. The numerical results were presented for the case where the material of the covering layer and half space are aluminum (shortly Al) and steel (shortly St), respectively. Note that the corresponding two-dimensional problem for plane-strain state was studied in the paper by Akbarov and Guler (2005).

2 Formulation of the problem

Consider the half-space covered by a bi-axially pre- stretched layer. For generality, we assume that the half-space is also bi-axially pre-stressed. With the covering layer we associate a Lagrangian coordinate system, $Ox_1x_2x_3$, which in the undeformed state, would coincide with a Cartesian coordinate system. Note that the covering layer and the half-space occupy the regions,

$$\{-\infty < x_1 < +\infty,\ -h \leq x_2 \leq 0,\ -\infty < x_3 < +\infty\}$$

and

$$\{-\infty < x_1 < +\infty,\ -\infty \leq x_2 \leq -h,\ -\infty < x_3 < +\infty\}$$

respectively (Figure 1). We assume that, before contact, the layer and the half-space are stressed separately in the directions of Ox_1 and Ox_3 axes and homogeneous initial stress states appear in both materials.

The values related to the layer and the half-space are denoted by upper indices (1) and (2), respectively. The values related to the initial stresses are denoted by upper indices (m), 0 where $m = 1, 2$.

The linearly elastic material of the layer and the half-space are to be taken homogeneous and isotropic. The initial stresses in the layer and the half-space are determined within the framework of the classical linear theory of elasticity as follows

$$\sigma_{11}^{(m),0} = const_{1m}, \quad \sigma_{33}^{(m),0} = const_{3m}, \sigma_{ij}^{(m),0} = 0 \text{ for } ij \neq 11;33 \tag{1}$$

According to Guz (2004), the equations for the small initial deformation state version of the TLTEWISB are

$$\frac{\partial \sigma_{ij}^{(m)}}{\partial x_j} + \sigma_{11}^{(m),0} \frac{\partial^2 u_i^{(m)}}{\partial x_1^2} + \sigma_{33}^{(m),0} \frac{\partial^2 u_i^{(m)}}{\partial x_3^2} = \rho_0^{(m)} \frac{\partial^2 u_i^{(m)}}{\partial t^2}, i; j = 1, 2, 3, \quad m = 1, 2. \tag{2}$$

In Eq. (2), $\rho_0^{(m)}$ denotes the density of the m-th material in the natural state.

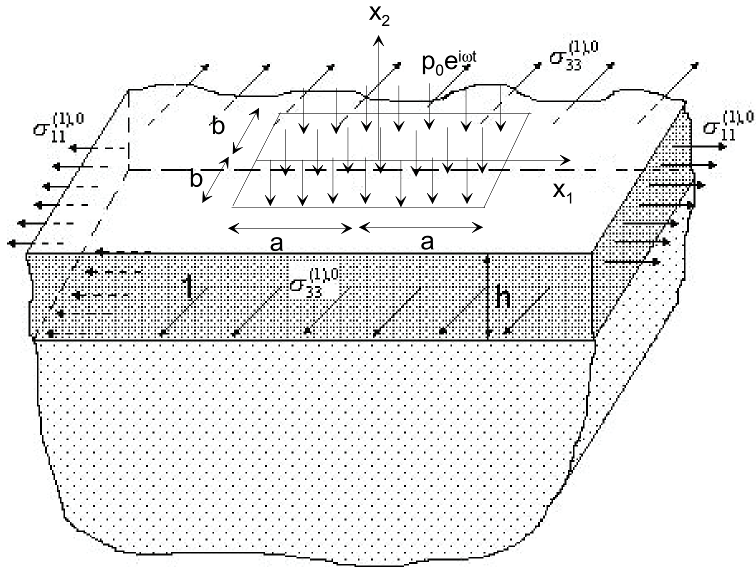

Figure 1: The geometry of the system considered

For an isotropic compressible material one can write the following mechanical relations.

$$\sigma_{ij}^{(m)} = \lambda^{(m)}\theta^{(m)}\delta_{ij} + 2\mu^{(m)}\varepsilon_{ij}^{(m)}, \theta^{(m)} = \varepsilon_{11}^{(m)} + \varepsilon_{22}^{(m)} + \varepsilon_{33}^{(m)} \tag{3}$$

where $\lambda^{(m)}$and $\mu^{(m)}$are Lamé's constants and

$$\varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right) \tag{4}$$

We assume that the following complete contact conditions exist between the layer and the half-space.

$$u_i^{(1)}\Big|_{x_2=-h} = u_i^{(2)}\Big|_{x_2=-h}, \quad \sigma_{i2}^{(1)}\Big|_{x_2=-h} = \sigma_{i2}^{(2)}\Big|_{x_2=-h}, \quad i = 1,2,3 \tag{5}$$

In the free face plane of the covering layer, the following conditions are satisfied

$$\sigma_{32}^{(1)}\Big|_{x_2=0} = \sigma_{12}^{(1)}\Big|_{x_2=0} = 0,$$

$$\sigma_{22}^{(1)}\Big|_{x_2=0}=\begin{cases}-p_0e^{i\omega t} & if(x_1,x_3)\in[-a,+a]\times[-b,b]\\ 0 & if(x_1,x_3)\notin[-a,+a]\times[-b,b]\end{cases}\tag{6}$$

By $[-a,a]\times[-b,b]$ we mean a rectangle with length (width) $2a\,(2b)$on the plane Ox_1x_3 (Figure 1). Moreover, we introduce a notation

$$P_0=p_04ab.\tag{7}$$

In addition to these, we also assume that as $x_2\to-\infty$ there is no reflection, which means all waves travel in the negative x_2 direction. In other words, we assume that

$$\left|u_i^{(2)}\right|,\quad \left|\sigma_{ij}^{(2)}\right|<M=\text{ constant, as } x_2\to-\infty.\tag{8}$$

This completes the formulation of the problem. Thus, within the framework of the equations (1)-(4), the contact conditions (5), the boundary conditions (6), (7) and the assumption (8), we investigate the forced vibration of the system caused by the foregoing time-harmonic forces acting within the rectangle $[-a,a]\times[-b,b]$.

It should be noted that in the case where $\sigma_{11}^{(m),0}=0$, $\sigma_{33}^{(m),0}=0\;(m=1,2)$, the problem transforms to the corresponding formulation of the classical linear theory.

3 Method of solution

Up to now, various types of numerical and semi-analytical methods have been developed for computer modeling to solve the various types of problems concerning deformable solid body mechanics. The present level of such methods is described, for example, in the papers by Lu and Zhu (2007), Chen, Fu and Zhang (2007), Gato and Shie (2008), Liu, Chen, Li and Cen (2008), Lin, Lee, Tsai, Chen, Wang and Lee (2008), Wang and Wang (2008), Guz and Dekret (2008), Dekret (2008a, 2008b) and in many others. Note that all these methods are based on the discretization or semi-discretization of the domain occupied by the body considered and field equations which are satisfied in this domain. It is known that as a result of the mentioned discretization the solution to the problem considered is reduced to the solution of the system of algebraic equations from which the numerical results required are attained. At the same time, there also other methods, so called analytical +numerical methods, according to which the mentioned numerical results are obtained from the complicate analytical solution through the employment modern PC algorithm (see: Akbarov and Guliyev (2009), Babich, Glukhov and Guz (2008a, 2008b), Akbarov, Emiroglu and Tasci (2005) and others). In the present paper, the latest version of computer modeling is employed.

As in Emiroglu, Tasci and Akbarov (2004); Akbarov, Emiroglu and Tasci (2005), we apply the double integral (Fourier) transformation method for the solution to the problem and first we derive from the equations (3), (4) and (2) the following linearized equations of motion in terms of the displacement perturbations:

$$\nabla^2 u_i^{(m)} + \left(1 + \frac{\lambda^{(m)}}{\mu^{(m)}}\right) \frac{\partial^2 u_j^{(m)}}{\partial x_j \partial x_i} + \frac{\sigma_{11}^{(m),0}}{\mu^{(m)}} \frac{\partial^2 u_i^{(m)}}{\partial x_1^2} + \frac{\sigma_{33}^{(m),0}}{\mu^{(m)}} \frac{\partial^2 u_i^{(m)}}{\partial x_3^2}$$

$$= \frac{1}{\left(c_2^{(m)}\right)^2} \frac{\partial^2 u_i^{(m)}}{\partial t^2}, \tag{9}$$

where

$$\nabla = \frac{\partial}{\partial x_k} \mathbf{e}_k, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad c_2^{(m)} = \sqrt{\frac{\mu^{(m)}}{\rho_0^{(m)}}} \tag{10}$$

In Eq. (10) $c_2^{(m)}$ is the speed of the distortion wave.
We attempt to use the Lamé representations for displacements [2]:

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}, \quad \nabla . \boldsymbol{\psi} = 0 \tag{11}$$

where

$$\mathbf{u} = (u_1, u_2, u_3), \quad \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \tag{12}$$

From Eqs. (11), (12) we can write

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3}, \quad u_2 = \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1}, \quad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}. \tag{13}$$

By substituting equations (13) in equation (9), we obtain, after some manipulation, the following equations for the functions ϕ, ψ_1, ψ_2 and ψ_3:

$$\nabla^2 \phi^{(\mathrm{m})} + \frac{\sigma_{11}^{(\mathrm{m}),0}}{\lambda^{(\mathrm{m})} + 2\mu^{(\mathrm{m})}} \frac{\partial^2 \phi^{(\mathrm{m})}}{\partial x_1^2} + \frac{\sigma_{33}^{(\mathrm{m}),0}}{\lambda^{(\mathrm{m})} + 2\mu^{(\mathrm{m})}} \frac{\partial^2 \phi^{(\mathrm{m})}}{\partial x_3^2}$$

$$= \frac{1}{\left(c_1^{(\mathrm{m})}\right)^2} \frac{\partial^2 \phi^{(\mathrm{m})}}{\partial t^2},$$

$$\nabla^2 \psi_i^{(\mathrm{m})} + \frac{\sigma_{11}^{(\mathrm{m}),0}}{\mu^{(\mathrm{m})}} \frac{\partial^2 \psi_i^{(\mathrm{m})}}{\partial x_1^2} + \frac{\sigma_{33}^{(\mathrm{m}),0}}{\mu^{(\underline{\mathrm{m}})}} \frac{\partial^2 \psi_i^{(\mathrm{m})}}{\partial x_3^2} = \frac{1}{\left(c_2^{(\mathrm{m})}\right)^2} \frac{\partial^2 \psi_i^{(\mathrm{m})}}{\partial t^2},$$

$$\frac{\partial \psi_1^{(m)}}{\partial x_1}+\frac{\partial \psi_2^{(m)}}{\partial x_2}+\frac{\partial \psi_3^{(m)}}{\partial x_3}=0, \tag{14}$$

where $c_1^{(m)}=\sqrt{\left(\lambda^{(\mathrm{m})}+2\mu^{(\mathrm{m})}\right)/\rho_0^{(\mathrm{m})}}$ is the speed of the dilatation wave.

Under the conditions $\sigma_{11}^{(m),0}=0$ and $\sigma_{33}^{(m),0}=0$, the equations (14) coincide with the corresponding ones derived in the classical linear theory of elastodynamics, see Eringen and Suhubi (1975).

Now we consider the solution to equations (14). Because, the external load is harmonic in time, only the stationary case will be considered; all dependent variables become harmonic and can be represented as:

$$\left\{u_n^{(m)},\sigma_{nj}^{(m)},\varepsilon_{nj}^{(m)},\phi^{(m)},\psi_n^{(m)}\right\}=\left\{\overline{u}_n^{(m)},\overline{\sigma}_{nj}^{(m)},\overline{\varepsilon}_{nj}^{(m)},\overline{\phi}^{(m)},\overline{\psi}_n^{(m)}\right\}e^{i\omega t} \tag{15}$$

where a superimposed dash denotes the amplitude of the relevant quantity. From here on, we will omit this superimposed dash.

If (15) is employed in (9)-(14), by replacing $\frac{\partial^2 u_i^{(m)}}{\partial t^2}$, $\frac{\partial^2 \phi^{(m)}}{\partial t^2}$, $\frac{\partial^2 \psi_n^{(m)}}{\partial t^2}$ with $-\omega^2 u_i^{(m)}$, $-\omega^2\phi^{(m)}$ and $-\omega^2\psi_n^{(m)}$, respectively, we obtain the same equations and conditions for the amplitude of the sought quantities. Consequently, introducing the dimensionless coordinates $x_i \rightarrow x_i/h$ and dimensionless frequency

$$\Omega=\frac{\omega h}{c_2^{(1)}}, \tag{16}$$

we obtain the following equations for the potentials $\phi^{(m)}$, $\psi_n^{(m)}$.

$$\nabla^2\phi^{(m)}+\frac{\sigma_{11}^{(m),0}}{\lambda^{(m)}+2\mu^{(m)}}\frac{\partial^2\phi^{(m)}}{\partial x_1^2}+\frac{\sigma_{33}^{(m),0}}{\lambda^{(m)}+2\mu^{(m)}}\frac{\partial^2\phi^{(m)}}{\partial x_3^2}+$$

$$\frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2\phi^{(m)}=0$$

$$\nabla^2\psi_n^{(m)}+\frac{\sigma_{11}^{(m),0}}{\mu^{(m)}}\frac{\partial^2\psi_n^{(m)}}{\partial x_1^2}+\frac{\sigma_{33}^{(m),0}}{\mu^{(m)}}\frac{\partial^2\psi_n^{(m)}}{\partial x_3^2}+\frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2\psi_n^{(m)}=0$$

$$\frac{\partial \psi_1^{(m)}}{\partial x_1}+\frac{\partial \psi_2^{(m)}}{\partial x_2}+\frac{\partial \psi_3^{(m)}}{\partial x_3}=0 \tag{17}$$

From Eqs. (17), we employ the double Fourier transformation with respect to the coordinates x_1 and x_3:

$$f_{13F}(s_1,x_2,s_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2,x_3)e^{-i(s_1x_1+s_3x_3)}dx_1dx_3 \tag{18}$$

It should be noted that the integral transformation methods are powerful mathematical tools and at present are employed intensively for the solution to the various problems of modern engineering and sciences (see: Sladek, Sladek and Solek (2009), Liu, Yeih and Atluri (2009), Duddeck (2006), Davies, Grann, Crann and Lai (2007), Sladek, Sladek and Atluri (2004), Sladek, Sladek, Tan and Atluri (2008), Dai and Wang (2007), Lu and Dai (2008), Rienstra and Tester (2008) and many others). These methods simplify significantly the solution procedure and give very high accurate analytical and numerical results. Taking this statement into account, in the present investigation we employ the integral transformation method (the double Fourier integral transformation method with respect to the space coordinates) for solution to the problem of interest.

After applying the transformation (18) to equation (17), the functions and are determined as follows:

$$\begin{aligned}
&\phi_{13F}^{(2)} = A_1^{(2)}(s_1,s_3)e^{\gamma_1^{(2)}(s_1,s_3)x_2},\\
&\psi_{n13F}^{(1)} = B_{1n}^{(1)}(s_1,s_3)e^{\gamma_2^{(1)}(s_1,s_3)x_2} + B_{2n}^{(1)}(s_1,s_3)e^{-\gamma_2^{(1)}(s_1,s_3)x_2},\\
&\psi_{n13F}^{(2)} = B_{1n}^{(2)}(s_1,s_3)e^{\gamma_2^{(2)}(s_1,s_3)x_2},
\end{aligned} \tag{19}$$

where

$$\left(\gamma_1^{(m)}\right)^2 = s_1^2\left(1+\frac{\sigma_{11}^{(m),0}}{\lambda^{(\underline{m})}+2\mu^{(m)}}\right) + s_3^2\left(1+\frac{\sigma_{33}^{(m),0}}{\lambda^{(m)}+2\mu^{(m)}}\right) - \frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2$$

$$\left(\gamma_2^{(m)}\right)^2 = s_1^2\left(1+\frac{\sigma_{11}^{(m),0}}{\mu^{(m)}}\right) + s_3^2\left(1+\frac{\sigma_{33}^{(m),0}}{\mu^{(m)}}\right) - \frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2, \tag{20}$$

From the last equation in (17) and (19) we obtain the following relations between $B_{1n}^{(m)}$ and $B_{2n}^{(m)}$

$$B_{12}^{(m)} = \frac{i}{\gamma_2^{(m)}}\left(s_1B_{11}^{(m)}+s_3B_{13}^{(m)}\right), \quad B_{22}^{(m)} = \frac{-i}{\gamma_2^{(m)}}\left(s_1B_{21}^{(m)}+s_3B_{23}^{(m)}\right), \quad i=\sqrt{-1}$$

(21)

Note that in writing expressions (19), the conditions (7) are taken into account.

Thus, we obtain from Eqs. (5) and (6) the algebraic system of equations for the unknowns

$$A_1^{(1)},\ A_2^{(1)},\ A_1^{(2)},\ B_{11}^{(1)},\ B_{11}^{(2)},\ B_{13}^{(1)},\ B_{13}^{(2)},\ B_{21}^{(1)} \text{ and } B_{23}^{(1)} \tag{22}$$

In this case we obtain the following expression for the double Fourier transformation (18) of the right hand side of the boundary condition for $\sigma_{22}^{(1)}$ at $x_2 = 0$:

$$-4p_0ab\frac{\sin(s_1a)}{s_1a}\frac{\sin(s_3b)}{s_3b} \tag{23}$$

Using the notation (7) we obtain from (23) the case considered Emiroglu, Tasci and Akbarov (2004), Akbarov, Emiroglu and Tasci (2005) as $|s_1a|, |s_3b| \to 0$. Thus, after determining the unknowns in (22) within the formula (23), we get the corresponding expressions for $u_{n13F}^{(m)}$, $\sigma_{nj13F}^{(m)}$ and $\varepsilon_{nj13F}^{(m)}$ from the relations (14), (3) and (4). The expressions of $u_{n13F}^{(m)}$, $\sigma_{nj13F}^{(m)}$ and $\varepsilon_{nj13F}^{(m)}$ are cumbersome, therefore we do not present them here.

The original unknowns that were sought can now be represented as

$$\left\{u_{\underline{n}}^{(m)}, \sigma_{\underline{nj}}^{(m)}, \varepsilon_{\underline{nj}}^{(m)}\right\} = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{u_{\underline{n}13F}^{(m)}, \sigma_{\underline{nj}13F}^{(m)}, \varepsilon_{\underline{nj}13F}^{(m)}\right\} x e^{i(s_1x_1+s_3x_3)} ds_1 ds_3. \tag{24}$$

The integral (24) is calculated with the algorithm developed and employed in the papers Emiroglu, Tasci and Akbarov (2004), Akbarov, Emiroglu and Tasci (2005).

4 The algorithm for calculation of the integrals (24)

As an example, we consider the calculation of the integral for $\sigma_{22}^{(m)}$, that is the integral

$$\sigma_{22}^{(m)} = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma_{2213F}^{(m)}(s_1,x_2,s_3)e^{i(s_1x_1+s_3x_3)}ds_1ds_3 \tag{25}$$

Introduce the following notation

$$\varphi(x_1,x_2,x_3) = \sigma_{22}^{(m)}(x_1,x_2,x_3), \quad \varphi_{13F}(s_1,x_2,s_3) = \sigma_{2213F}^{(m)}(s_1,x_2,s_3),$$

$$\varphi_{3F}(x_1,x_2,s_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_{13F}(s_1,x_2,s_3)e^{is_1x_1}ds_1. \tag{26}$$

Using the problem symmetry and Eq. (26), integral (25) can be represented as follows:

$$\varphi(x_1,x_2,x_3) = \frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty} \varphi_{13F}(s_1,x_2,s_3)\cos(s_1x_1)\cos(s_3x_3)ds_1ds_3 \tag{27}$$

The following explain how to reduced integral (25) is calculated. First, the integral (27) is replaced by a corresponding definite integral, by using the following approximation:

$$\int_0^{+\infty}\int_0^{+\infty} \varphi_{13F}(s_1,x_2,s_3)\cos(s_1x_1)\cos(s_3x_3)ds_1ds_3 \approx$$

$$\int_0^{s_{3*}}\int_0^{s_{1*}} \varphi_{13F}(s_1,x_2,s_3)\cos(s_1x_1)\cos(s_3x_3)ds_1ds_3 \tag{28}$$

The values of S_{1*} and S_{3*} in Eq. (28) are defined from the convergence requirement. For calculation of the definite integral in Eq. (28), first, the interval $[0, S_{3*}]$ is divided into shorter intervals $[S_{3i}, S_{3i+1}]$, i=0,1,2,...,N, $S_{30} = 0$, $S_{3N} = S_{3*}$, where $\bigcup_{i=0}^{N} [S_{3i}, S_{3i+1}] = [0, S_{3*}]$ and $\bigcap_{i=0}^{N} (S_{3i}, S_{3i+1}) = \emptyset$. Then, the definite integral becomes

$$\int_0^{S_{3*}}\int_0^{S_{1*}} (.)ds_1ds_3 = \sum_{i=0}^{N}\int_{S_{3i}}^{S_{3i+1}}\left(\int_0^{S_{1*}}(.)ds_1\right)ds_3, \tag{29}$$

where $(.)$ denotes the integrand.

Consequently, we obtain from Eqs. (29) and (26) that

$$\int_0^{S_{3*}}\int_0^{S_{1*}} (.)ds_1ds_3 = \sum_{i=0}^{N}\int_{S_{3i}}^{S_{3i+1}} \varphi_{3F}(x_1,x_2,s_3)ds_3. \tag{30}$$

For calculations of the integrals (30) in the intervals $[S_{3i}, S_{3i+1}]$, we use the Gauss integration algorithm, where it is necessary to know the values of $\varphi_{3F}(x_1,x_2,s_3)$ at certain nodal points $s_3 = s_{3k}^{/}$. Thus, the calculation of the integral (29) is reduced to the calculation of the integral

$$\varphi_{3F}(x_1, x_2, s_{3k}^{/}) = \int_0^{S_{1*}} \varphi_{13F}(s_1, x_2, s_{3k}^{/}) \cos(s_1 x_1) ds_1 \tag{31}$$

Now we consider the calculation of the integral (31). Numerical investigations show that the function $\varphi_{13F}(s_1, x_2, s_{3k}^{/})$ has a singular point for each selected $s_{3k}^{/}$ and the location of these singular points in the intervals $[0, S_{1*}]$ depends on $s_{3k}^{/}$.
It should be noted that, since the unknowns (22) can be expressed as

$$(A_1^{(1)}; A_2^{(1)}; ...; B_{13}^{(2)}; B_{21}^{(1)}; B_{23}^{(1)})(s_1, s_{3k}^{/}) =$$

$$\frac{1}{\det \left\| \alpha_{ij}(s_1, s_{3k}^{/}) \right\|} \left(\det \left\| \beta_{ij}^{A_1^{(1)}}(s_1, s_{3k}^{/}) \right\| ; \right.$$

$$\left. \det \left\| \beta_{ij}^{A_1^{(1)}}(s_1, s_{3k}^{/}) \right\| ; \ldots ; \det \left\| \beta_{ij}^{B_{23}^{(1)}}(s_1, s_{3k}^{/}) \right\| \right) \tag{32}$$

the aforementioned singular points coincide with the roots of the equation

$$\det \left\| \alpha_{ij}(s_1, s_{3k}^{/}) \right\| = 0, \quad i; j = 1, 2, 3, ..., 9 \tag{33}$$

in s_1, where $\alpha_{ij}(s_1, s_{3k}^{/})$ are the coefficients of the unknowns in the algebraic equation system obtained from the contact (5) and the boundary conditions (6). Note that the expressions for $\left\| \beta_{ij}^{A_1^{(1)}} \right\|, \ldots, \left\| \beta_{ij}^{B_{23}^{(1)}} \right\|$ are obtained from $\left\| \alpha_{ij}(s_1, s_{3k}^{/}) \right\|$ by replacing the corresponding column of $\left\| \alpha_{ij}(s_1, s_{3k}^{/}) \right\|$ with the right side of the algebraic equation system.
A numerical analysis shows that the order of the roots of the equation (33) is one. Therefore, the order of all singular points is also one. Taking this situation into account in the solution of equation (33), we employ the well-known bisection method.
Let us denote the roots of the equation (31) as

$$s_{11}(s_{3k}^{/}) < s_{12}(s_{3k}^{/}) < \ldots < s_{1k}(s_{3k}^{/}) < \ldots < s_{1M}(s_{3k}^{/}). \tag{34}$$

The number M in equation (34) depends mainly on the values of $s_{3k}^{/}$, the dimensionless frequency Ω(16) and the mechanical and geometrical parameters of the layer and half-space. After determining the roots (34), the interval of integration $[0, S_{1*}]$ in equation (31) is partitioned as follows

$$\int\limits_{0}^{S_{1*}} (.)ds_1 = \int\limits_{0}^{S_{11}(S_{3k}^{/})-\varepsilon} (.)ds_1 + \int\limits_{S_{11}(S_{3k}^{/})+\varepsilon}^{S_{12}(S_{3k}^{/})-\varepsilon} (.)ds_1 + \cdots + \int\limits_{S_{1M}(S_{3k}^{/})+\varepsilon}^{S_{1*}} (.)ds_1 \tag{35}$$

Then, the calculation of the integral (31) is performed in the Cauchy's principal value sense. Here ε is a very small value determined numerically from the convergence requirement of the integral (31). Each interval $\left[s_{1n}(s_{3k}^{/}) + \varepsilon,\ s_{1n+1}(s_{3k}^{/}) - \varepsilon\right]$ is further divided into a certain number of shorter intervals, which are used in Gauss integration algorithm. All these procedures are performed using the programmes written in C++. Note that in the above procedure, the values of x_1,x_2 and x_3 are fixed.

5 Numerical results and discussions

As it has been noted above, we assume that the covering layer material is aluminum (Al) with mechanical properties, $\rho = 2700 kg/m^3$, $\nu = 0.35$, $c_1 = 6420 m/s$, $c_2 = 3110 m/s$ the half-space material is steel with mechanical properties $\rho = 7680 kg/m^3$, $\nu = 0.29$, $c_1 = 5890 m/s$, $c_2 = 3210 m/s$, where ρ, ν, c_1 and c_2 are material density, Poisson coefficient, dilatation and distortion waves speed, respectively. Moreover, we introduce the dimensionless parameters

$$a_1 = \frac{a}{h}, \quad b_1 = \frac{b}{h}, \tag{36}$$

through which we will investigate the influence of the size of the rectangle in which the uniformly distributed forces with intensity p_0 act on the stress distribution. Moreover we introduce the notation

$$\chi_{22} = \left\{ \frac{10 \times \left(\sigma_{22}^{(1)}\Big|_{P.L.} - \sigma_{22}^{(1)}\Big|_{a_1=b_1>0} \right) h}{P_0} \right\} \Bigg|_{x_2/h=-1.0; x_1/h=0.0} \tag{37}$$

through which the influence of the size of the rectangle on the difference between the values of $\sigma_{22}^{(1)}$ caused by the point located force (in expression (26) and further the "P.L." means "point located") and the values of $\sigma_{22}^{(1)}$ caused by the foregoing uniformly distributed forces. Note that, according to the expression (7) the mentioned uniformly distributed forces and point located force are statically equivalent.

At the same time, we introduce the notation

$$\psi_{22}(x_1,x_2) = \frac{10^2 \times \left(\sigma_{22}^{(1)}(x_1,x_2)\Big|_{\eta_1=0.0} - \sigma_{22}^{(1)}(x_1,x_2)\Big|_{\eta_1\neq 0.0} \right) h}{P_0} \tag{38}$$

where $\eta_1 = \sigma_{11}^{(1)0}/\mu^{(1)}$; throughout the numerical investigation we will assume that $\eta_2 = \sigma_{11}^{(2)0}/\mu^{(2)} = 0.0$, here $\mu^{(1)}$ $\left(\mu^{(2)}\right)$ is a shear modulus of elasticity of the covering layer (half-space) material. Note that the notation (27) will characterize the influence of the initial stress in the covering layer on the stress distribution considered. The investigations carried out in the papers by Akbarov and Guler (2005), Akbarov (2006), Akbarov (2007), Akbarov and Guler (2007) and Emiroglu, Tasci and Akbarov (2004) and the others show that the mechanical behaviour of the forced vibration of the half-space covered with the layer is similar to that of the system comprising a mass, a spring and a dashpot. Therefore there exist such values of the frequency Ω (denoted by Ω_*) called the "resonance" frequency of the external force under which the stresses and displacements have their absolute maximum values. Taking this statement into account in the present investigation the main attention is focused on the dependencies between

$$\sigma_{22} = \left(\frac{\sigma_{22}^{(1)} h}{P_0} \right)\Bigg|_{x_2/h=-1.0;x_1/h=0.0} \tag{39}$$

and Ω. In the mean time, the influence of the problem parameters a_1, b_1 and η_1 on these dependencies is also studied. Throughout this study it is assumed that $a_1 = b_1$.

Thus, after the foregoing preparation discussion we consider the graphs of the dependencies between σ_{22} (28) and Ω. These graphs are given in Figure 2 for various values of a_1 under $\eta_1 = 0.0$(Figure 2a), 0.004(Figure 2b), 0.008(Figure 2c) and 0.01(Figure 2d). It follows from these graphs, as it can be predicted, that the values of σ_{22} decrease with a_1, i.e. the graphs considered move up wholly with a_1. In this case the values of the "resonance" frequency, i.e. the values of Ω_*, as well as the "resonance" values of the stress σ_{22} depend on the initial stretching of the covering layer, i.e. on the parameter η_1. For a clearer illustration of these influences in Figure 3 (Figure 4) the graphs of the dependencies between parameter χ_{22}(26) ($\psi_{22}(0,-h)$(27)) and Ω are given for the cases where $a_1 = 0.1$ (Figure 3a), $a_1 = 0.4$ (Figure 3b), $a_1 = 0.7$ (Figure 3c) ($\eta_1 = 0.0$ (Figure 4a), $\eta_1 = 0.004$ (Figure 4b) and $\eta_1 = 0.008$ (Figure 4c)). It follows from Figure 3 that the influence of the parameter a_1 on the values of σ_{22} increase with Ω.

Figure 4 shows that the character of the influence of the parameter $\eta_1$on the values of σ_{22} depends on Ω. So that, there exists a certain value of Ω (denoted by Ω')

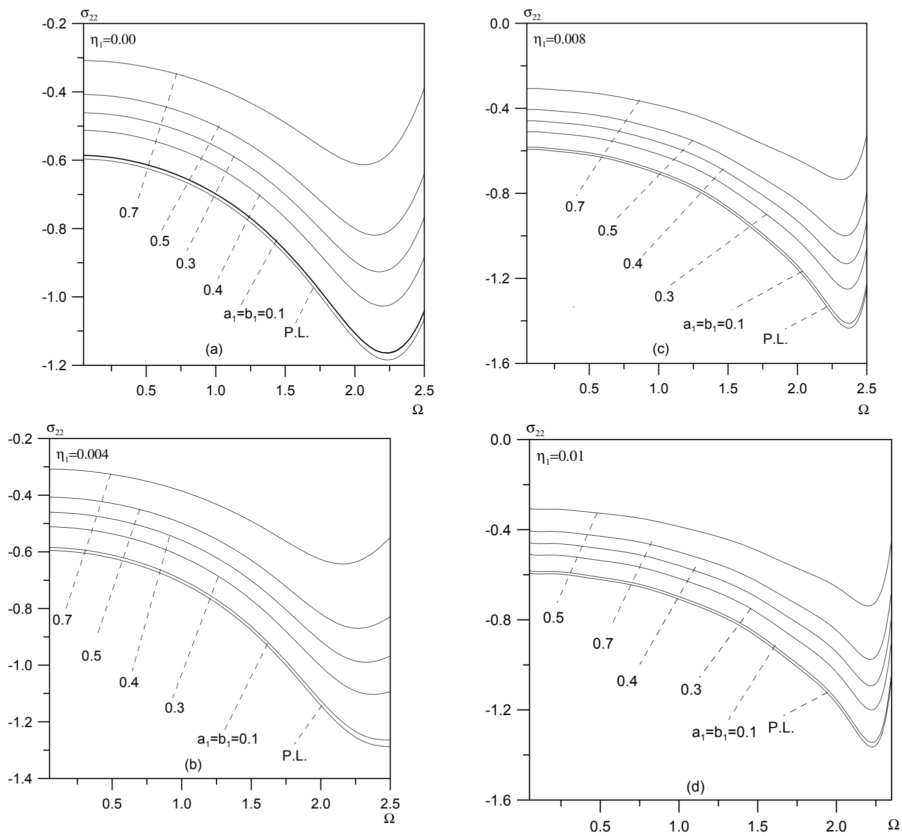

Figure 2: The graphs of the dependence between σ_{22} (28)and Ω for various values of a_1 under $\eta_1 = 0.0\,(a)$, $0.004\,(b)$, $0.008\,(c)$and $0.01\,(d)$.

before which, i.e. for $\Omega < \Omega^{/}$ stress σ_{22}decreases as a result of the initial stretching of the covering layer. But this decrease decays as $\Omega \to \Omega'$ and $\psi_{22}(0,-h) = 0.0$ for $\Omega = \Omega'$. However, for the case where $\Omega > \Omega'$ the existence of the initial stretching in the covering layer causes to increase in the absolute values of the stress σ_{22}, i.e. the values of $\psi_{22}(0,-h)$ are greater than zero and increase monotonically with Ω. Note that this increase becomes more significant as $\Omega \to \Omega_*$.

The graphs illustrated in Figure 4 show that the values of Ω' depend on the parameter a_1. For the cases considered we obtain that $\Omega' \approx 1.5$ for $a_1 = 0.1$, $\Omega' \approx 1.4$ for $a_1 = 0.4$ and $\Omega' \approx 1.3$ for $a_1 = 0.7$. Consequently the values of Ω' decrease with a_1.

Now we consider the distribution of the stress $\sigma_{22}(x_1)\left(= \sigma_{22}^{(1)}(x_1,-h)h/P_0\right)$ with

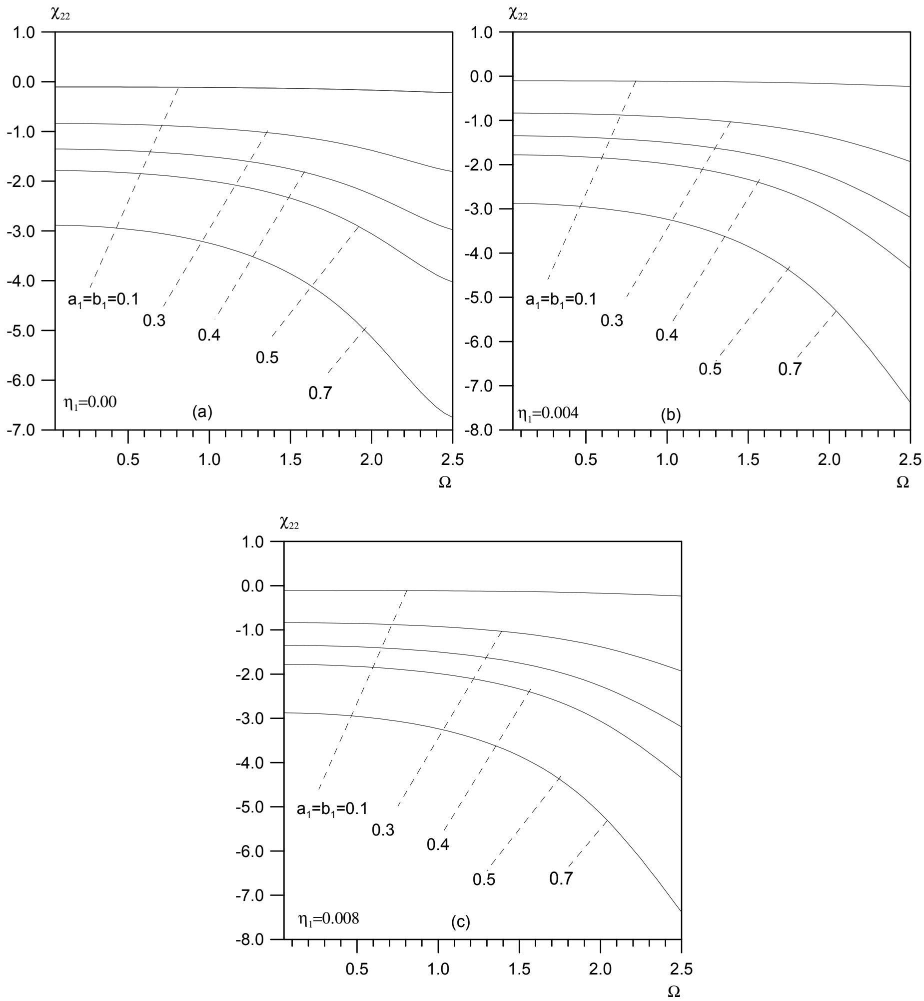

Figure 3: The graphs of the dependence between χ_{22} (26)and Ω for the cases where $a_1 = 0.1\,(a)$, $0.4\,(b)$ and $0.7\,(c)$.

respect to $x_1/h$and analyze the influence of the parameters a_1 and $\eta_1$on this distribution. For this purpose we consider the graphs given in Figure 5 and the graphs given in Figure 6. Figure 5 show the dependencies between $\sigma_{22}(x_1)$and $x_1/h$for various values of a_1 in the case where $\eta_1 = 0.0$, but Figure 6 shows the dependencies between $\psi_{22}(x_1,-h)$ and x_1/h for various values of a_1 in the cases where $\eta_1 = 0.004$(Figure 6a) and $\eta_1 = 0.008$(Figure 6b).

It follows from Figure 6 that the influence of the initial stretching of the covering

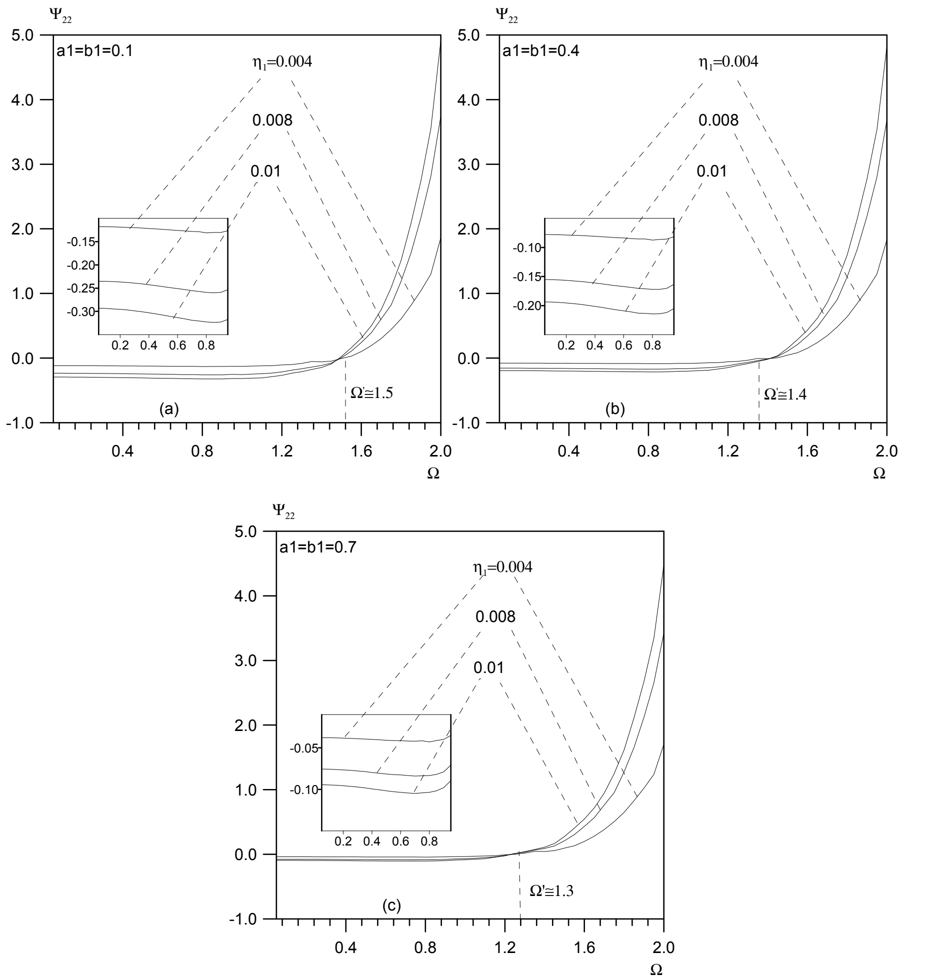

Figure 4: The graphs of the dependence between $\psi_{22}(0,-h)$ (27) and Ω for the cases $\eta_1 = 0.0\,(a)$, $0.004\,(b)$, $0.008\,(c)$.

layer on the values of the $\sigma_{22}^{(1)}(0,-h)h/P_0$ increase with decreasing a_1.

6 Conclusions

In this paper, the time-harmonic dynamical stress field in the system comprising two axially pre-stressed covering layer and a two axially pre-stressed half space is studied under the action of uniformly distributed normal forces on the free face plane of the covering layer. It is assumed that the forces are distributed within the

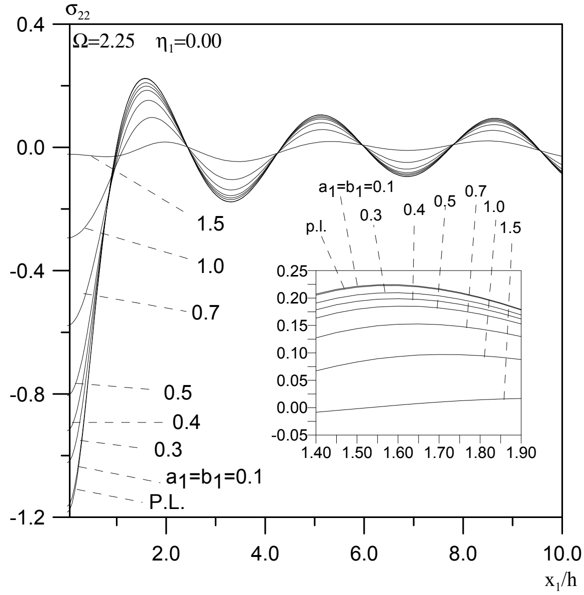

Figure 5: Distribution of the stress $\sigma_{22}(\chi_1)$ (28) with respect to χ_1/h for the case where $\Omega = 2.25$.

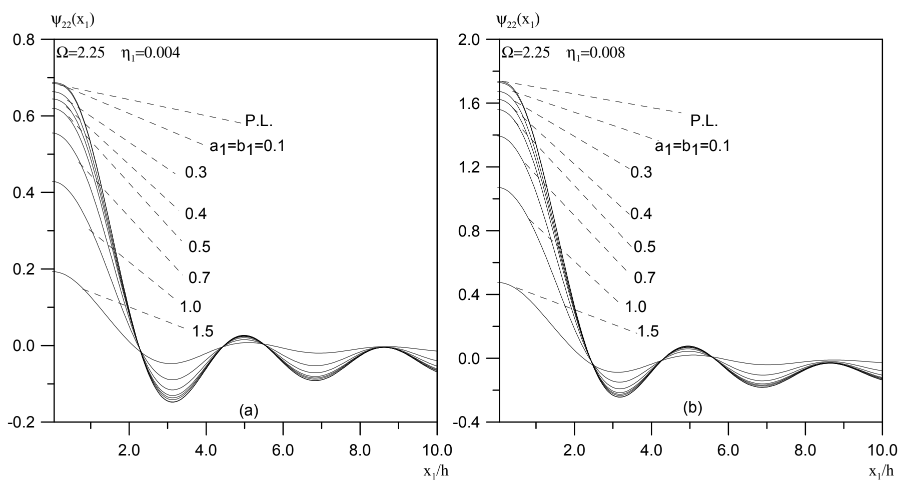

Figure 6: Distribution of the $\psi_{22}(\chi_1, -h)$with respect to χ_1/h for various values of a_1 in the cases where $\eta_1 = 0.004\,(a)$ and $0.008\,(b)$.

rectangular area. The study is made within the scope of the piecewise homogeneous body model with the use of the TLTEWISB. The corresponding three-dimensional boundary-value-contact problem is solved by applying double Fourier exponential integral transformation. The numerical results are presented and discussed for the case where the material of the layer and the half space are aluminum and steel, respectively. But the material of the half-space is steel. At the same time, the numerical results are obtained for the case where the mentioned rectangle is a square, i.e. for the case where $a_1 = b_1$ (the values of a_1 and $b_1$are determined through the expression (25)). The main attention is focused on the dependencies between the interface normal stress and a frequency of the external forces.

The numerical results indicate the following conclusions:

1. Under the action of the statically equivalent forces in the rectangular area, the influence of the size of this area on the dependencies between the stress σ_{22} (28) and Ω(16) increases monotonically with Ω;

2. The character of the influence of the initial stretching, i.e., of the parameter η_1(where$\eta_1 = \sigma_{11}^{(1)0} \big/ \mu^{(1)}$, $\sigma_{11}^{(1)0}$ is an initial stress acting in the covering layer, $\mu_1$is a shear modulus of the covering layer material) on the values of σ_{22} depends on Ω. So that, there exists a certain value of Ω (denoted by Ω') before which, i.e., for stress decreases as a result of the initial stretching of the covering layer. But this decreasing decays as $\Omega \to \Omega'$ and $\psi_{22}(0,-h) = 0.0$ for $\Omega = \Omega'$, where ψ_{22} is determined by the expression (27);

3. For the case where $\Omega > \Omega'$ the existence of the initial stretching in the covering layer causes to increase in the absolute values of the stress σ_{22}, i.e., the values of $\psi_{22}(0,-h)$ are greater than zero and increase monotonically with Ω. That this increasing becomes more significant as $\Omega \to \Omega_*$, where Ω_* is a "resonance" value of the frequency of the external forces;

4. The values of Ω' depend on the parameter a_1(25), i.e., the values of Ω' decrease with a_1;

5. The influence of the initial stretching of the covering layer on the values of the $\sigma_{22}^{(1)}(0,-h)h/P_0$ increases with decreasing a_1.

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