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A New Method for Fredholm Integral Equations of 1D Backward Heat Conduction Problems

Chein-Shan Liu[1]

Abstract: In this paper an analytical method for approximating the solution of backward heat conduction problem is presented. The Fourier series expansion technique is used to formulate a first-kind Fredholm integral equation for the temperature field $u(x,t)$ at any time $t < T$, when the data are specified at a final time T. Then we consider a direct regularization, instead of the Tikhonov regularization, by adding the term $\alpha u(x,t)$ to obtain a second-kind Fredholm integral equation. The termwise separable property of kernel function allows us by transforming it to a two-point boundary value problem, and thus a closed-form solution is derived. The uniform convergence and error estimate of the regularized solution $u^\alpha(x,t)$ are proved and a strategy to select the regularization parameter is provided. When numerical examples were tested, we find that the new method can retrieve the initial data very excellently, even the final data are seriously noised.

Keywords: Backward heat conduction problem, Ill-posed problem, Two-point boundary value problem, Fredholm integral equation, Fourier series, Regularized solution

1 Backward heat conduction problems

We consider a homogeneous rod of length ℓ, which is sufficiently slender so that the temperature is uniformly distributed over the cross section of the rod at any time t. The surface of the rod is insulated, and therefore there is no heat loss through the lateral boundary. In many practical engineering application areas we may want to recover all the past temperature distribution $u(x,t)$, where $t < T$, of which the temperature is assumed to be known at a given final time T. The problem we

[1] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan. E-mail: liucs@ntu.edu.tw

consider is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ \ 0 < x < \ell, \ \ 0 < t < T, \tag{1}$$

$$u(0,t) = 0, \ \ u(\ell,t) = 0, \ \ 0 \le t \le T, \tag{2}$$

$$u(x,T) = f(x), \ \ 0 \le x \le \ell. \tag{3}$$

This is the so-called a backward heat conduction problem (BHCP), which is known to be highly ill-posed, namely, the solution does not depend continuously on the input data $u(x,T)$. Really, the rapid decay of temperature with time results in a fast fading memory of initial conditions. Therefore, the numerical recovery of initial temperature from the data measured at a final time $T > 0$ is a rather difficult issue due to the influence of noise and computational error.

It is well known that the approach of ill-posed problems by numerical method is rather difficult [Han, Ingham and Yuan (1995); Mera, Elliott, Ingham and Lesnic (2001); Jourhmane and Mera (2002); Mera, Elliott and Ingham (2002); Kirkup and Wadsworth (2002); Chiwiacowsky and de Campos Velho (2003); Iijima (2004); Liu (2004); Mera (2005); Liu, Chang and Chang (2006); Liu, Yeih and Atluri (2009)]. Mera (2005) has mentioned that the backward problem is impossible to solve using classical numerical methods and requires special techniques to be employed.

One way to solve the ill-posed problem is to perturb it into a well-posed one. A number of perturbing techniques have been proposed, including a biharmonic regularization developed by Lattés and Lions (1969), a pseudo-parabolic regularization proposed by Showalter and Ting (1970), and a hyperbolic regularization proposed by Ames and Cobb (1997). It seems that Showalter (1983) first regularized the above problem by considering a quasi-boundary-value approximation to the final value problem, that is, to replace Eq. (3) by

$$\alpha u(x,0) + u(x,T) = f(x). \tag{4}$$

The problems (1), (2) and (4) have been shown to be well-posed for each $\alpha > 0$ by Clark and Oppenheimer (1994). Ames and Payne (1999) have discussed those regularizations from the continuous dependence of solution on the regularization parameter.

In this paper, after casting the backward heat conduction problem into a first-kind Fredholm integral equation we propose a direct regularization technique to transform it into a second-kind Fredholm integral equation, and using the kernel function separating characteristic and eigenfunctions expansion technique we can derive a closed-form solution of the second-kind Fredholm integral equation.

The new method would provide us an analytical solution, and renders a more compendious numerical implementation than other schemes to solve backward problems. The main motivation is placed on an effective solution of the BHCP, which is one of the inverse problem, and is different from the sideways heat conduction problem recently reviewed and calculated by Chang, Liu and Chang (2005). The degree of the ill-posedness of BHCP is over other inverse heat conduction problems including the sideways heat conduction problem, which is dealed with the reconstruction of unknown boundary conditions.

A similar second-kind Fredholm integral equation regularization method was first used by Liu (2007a) to solve a direct problem of elastic torsion in an arbitrary plane domain, where it was called a meshless regularized integral equation method. Liu (2007b, 2007c) extended it to solve the Laplace direct problem in arbitrary plane domains. Based on those good results and experiences, Liu (2009) used this new method to treat the inverse Robin coefficient problem of Laplace equation. At the same time, Liu, Chang and Chang (2009) used it to calculate the backward in time advection-dispersion equation. On the other hand, basing on the Lie-group shooting method [Liu (2006)], Chang, Liu and Chang (2007) solved the above one-dimensional quasi-boundary value problem very well. Then Chang, Liu and Chang (2009) extended their results to the multi-dimensional backward heat conduction problems by using the Lie-group shooting method.

The remaining sections of this paper are arranged as follows. In Section 2 we derive the second-kind Fredholm intergral equation by a direct regularization of the first-kind Fredholm intergral equation. We point out its difference with the Tikhonov regularization and the quasi-boundary regularization. In Section 3 we derive a two-point boundary value problem, which can help us to derive a closed-form solution of the second-kind Fredholm intergral equation in Section 4. In Section 5 we prove the uniform convergence of the obtained regularized solution, as well as give an error estimate. In Section 6 we demonstrate a selection principle of the regularization parameter, and use some numerical examples to test the new method. Then, we give some conclusions in Section 7.

2 The Fredholm integral equations

By utilizing the technique of separation of variables we are easy to write a formal series expansion of $u(x,t)$ satisfying Eqs. (1) and (2):

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-(k\pi/\ell)^2 t} \sin \frac{k\pi x}{\ell}, \tag{5}$$

where a_k are coefficients to be determined.

By imposing the final time condition (3) on the above equation we can obtain

$$u(x,T) = \sum_{k=1}^{\infty} a_k e^{-(k\pi/\ell)^2 T} \sin\frac{k\pi x}{\ell} = f(x). \tag{6}$$

Fixing any a $t < T$ and applying the Fourier sine series on Eq. (5) one has

$$a_k = \frac{2e^{(k\pi/\ell)^2 t}}{\ell} \int_0^\ell \sin\frac{k\pi\xi}{\ell} u(\xi,t)d\xi. \tag{7}$$

When substituting Eq. (7) for a_k into Eq. (6) and assuming that the order of summation and integral can be interchanged, the following integral equation can be derived:

$$(\mathbf{K}_x^{T-t} u(\cdot,t))(x) := \int_0^\ell K(x,\xi;T-t)u(\xi,t)d\xi = f(x), \tag{8}$$

where

$$K(x,\xi;t) = \frac{2}{\ell}\sum_{k=1}^{\infty} e^{-(k\pi/\ell)^2 t} \sin\frac{k\pi x}{\ell} \sin\frac{k\pi\xi}{\ell} \tag{9}$$

is a kernel function, and $\mathbf{K}_x^{T-t}$ is an integral operator generated from $K(x,\xi;T-t)$. Corresponding to the kernel $K(x,\xi;t)$, the operator is denoted by $\mathbf{K}_x^t$.

In order to recover the temperature $u(x,t)$ at any time $t < T$ from a given data $f(x)$ at a final time T we have to solve the first-kind Fredholm integral equation (8). This however is very difficult, since this integral equation is highly ill-posed [Liu and Atluri (2009)].

Ames and Epperson (1997) have used a Tikhonov regularization technique to treat Eq. (8) by the so-called normal equation:

$$(\mathbf{K}^\star\mathbf{K} + \alpha\mathbf{I})u(x,\cdot) = \mathbf{K}^\star f(x). \tag{10}$$

Refer Eq. (2.23) of the above cited paper. Below we propose a direct regularization of Eq. (8).

We assume that there exists a regularization parameter α, such that Eq. (8) can be regularized by

$$\alpha u(x,t) + \int_0^\ell K(x,\xi;T-t)u(\xi,t)d\xi = f(x), \tag{11}$$

which is a second-kind Fredholm integral equation.

Let t in Eq. (11) be zero,

$$\alpha u(x,0)+\int_0^\ell K(x,\xi;T)u(\xi,0)d\xi = f(x), \tag{12}$$

and note that

$$u(x,T)=\int_0^\ell K(x,\xi;T)u(\xi,0). \tag{13}$$

From these two equations we obtain the quasi-boundary condition (4). Conversely, Eq. (4) does not necessarily imply Eq. (11).

Taking $x=\eta$ in Eq. (12) we have

$$\alpha u(\eta,0)+\int_0^\ell K(\eta,\xi;T)u(\xi,0)d\xi = f(\eta), \tag{14}$$

and applying the operator $\mathbf{K}_x^t$ on the above equation and noting that

$$(\mathbf{K}_x^t u(\cdot,0))(x)=\int_0^\ell K(x,\eta;t)u(\eta,0)d\eta = u(x,t),$$
$$(\mathbf{K}_x^t \mathbf{K}_\eta^T u(\cdot,0))(x)=(\mathbf{K}_x^T \mathbf{K}_\eta^t u(\cdot,0))(x),$$

we obtain

$$\alpha u(x,t)+\int_0^\ell K(x,\xi;T)u(\xi,t)d\xi = F(x,t)=\int_0^\ell K(x,\xi;t)f(\xi). \tag{15}$$

This equation has been derived by Ames, Clark, Epperson and Oppenheimer (1998), and the numerical implementation has been carried out. It is also a second-kind Fredholm integral equation but is different from Eq. (11). It can be seen that our regularized equation (11) is simpler than the above equation and also than Eq. (10). Eq. (11) is our starting point.

3 Two-point boundary value problem

We assume that the kernel function in Eq. (11) can be approximated by m terms with

$$K(x,\xi;T-t)=\frac{2}{\ell}\sum_{k=1}^m e^{-(k\pi/\ell)^2(T-t)}\sin\frac{k\pi x}{\ell}\sin\frac{k\pi\xi}{\ell}, \tag{16}$$

because of $T-t>0$, and the higher terms decaying very fast with k increasing. The above kernel is termwise separable, which is also called the degenerate kernel or the Pincherle-Goursat kernel [Tricomi (1985)].

By the inspection of Eq. (16) we have

$$K(x,\xi;T-t) = \mathbf{P}(x;T-t)\cdot\mathbf{Q}(\xi), \tag{17}$$

where **P** and **Q** are m-vectors given by

$$\mathbf{P} := \frac{2}{\ell}\begin{bmatrix} e^{-(\pi/\ell)^2(T-t)}\sin\frac{\pi x}{\ell} \\ e^{-(2\pi/\ell)^2(T-t)}\sin\frac{2\pi x}{\ell} \\ \vdots \\ e^{-(m\pi/\ell)^2(T-t)}\sin\frac{m\pi x}{\ell} \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} \sin\frac{\pi\xi}{\ell} \\ \sin\frac{2\pi\xi}{\ell} \\ \vdots \\ \sin\frac{m\pi\xi}{\ell} \end{bmatrix}, \tag{18}$$

and the dot between **P** and **Q** denotes the inner product, which is sometimes written as $\mathbf{P}^{\mathrm{T}}\mathbf{Q}$, where the superscript T signifies the transpose.
With the aid of Eq. (17), Eq. (11) can be decomposed as

$$\alpha u(x,t) + \int_0^x \mathbf{P}^{\mathrm{T}}(x)\mathbf{Q}(\xi)u(\xi,t)d\xi + \int_x^\ell \mathbf{P}^{\mathrm{T}}(x)\mathbf{Q}(\xi)u(\xi,t)d\xi = f(x), \tag{19}$$

where we omit the parameter $T-t$ in **P** for clearity.
Let us define

$$\mathbf{u}_1(x) := \int_0^x \mathbf{Q}(\xi)u(\xi,t)d\xi, \tag{20}$$

$$\mathbf{u}_2(x) := \int_\ell^x \mathbf{Q}(\xi)u(\xi,t)d\xi, \tag{21}$$

and Eq. (19) can be expressed as

$$\alpha u(x,t) + \mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_1(x) - \mathbf{u}_2(x)] = f(x). \tag{22}$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ can be solved we can calculate $u(x,t)$.
Taking the differential of Eqs. (20) and (21) with respect to x we obtain

$$\mathbf{u}_1'(x) = \mathbf{Q}(x)u(x,t), \tag{23}$$

$$\mathbf{u}_2'(x) = \mathbf{Q}(x)u(x,t). \tag{24}$$

Inserting Eq. (22) for $u(x,t)$ into the above two equations we obtain

$$\alpha\mathbf{u}_1'(x) = \mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + f(x)\mathbf{Q}(x), \quad \mathbf{u}_1(0) = \mathbf{0}, \tag{25}$$

$$\alpha\mathbf{u}_2'(x) = \mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + f(x)\mathbf{Q}(x), \quad \mathbf{u}_2(\ell) = \mathbf{0}, \tag{26}$$

where the last two boundary conditions follow from Eqs. (20) and (21) readily. The above two equations constitute a two-point boundary value problem.

4 A closed-form solution

In this section we will find a closed-form solution of $u(x,t)$. From Eqs. (23) and (24) it can be seen that $\mathbf{u}_1' = \mathbf{u}_2'$, which means that

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{c}, \tag{27}$$

where $\mathbf{c}$ is a constant vector to be determined. By using the final condition in Eq. (26) we find that

$$\mathbf{u}_1(\ell) = \mathbf{u}_2(\ell) + \mathbf{c} = \mathbf{c}. \tag{28}$$

Substituting Eq. (27) into Eq. (25) we have

$$\alpha \mathbf{u}_1'(x) = -\mathbf{Q}(x)\mathbf{P}^{\mathrm{T}}(x)\mathbf{c} + f(x)\mathbf{Q}(x), \ \ \mathbf{u}_1(0) = \mathbf{0}. \tag{29}$$

Integrating and using the initial condition leads to

$$\mathbf{u}_1(x) = \frac{-1}{\alpha}\int_0^x \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\,\mathbf{c} + \frac{1}{\alpha}\int_0^x f(\xi)\mathbf{Q}(\xi)d\xi. \tag{30}$$

Taking $x = \ell$ in the above equation and imposing the condition (28) one obtains a governing equation for $\mathbf{c}$:

$$\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)\mathbf{c} = \int_0^\ell f(\xi)\mathbf{Q}(\xi)d\xi. \tag{31}$$

It is straightforward to write

$$\mathbf{c} = \left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1}\int_0^\ell f(\xi)\mathbf{Q}(\xi)d\xi. \tag{32}$$

On the other hand, from Eqs. (22) and (27) we have

$$\alpha u(x,t) = f(x) - \mathbf{P}(x)\cdot\mathbf{c}. \tag{33}$$

Inserting Eq. (32) into the above equation we obtain

$$\alpha u(x,t) = f(x) - \mathbf{P}(x)\cdot\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\right)^{-1}\int_0^\ell f(\xi)\mathbf{Q}(\xi)d\xi. \tag{34}$$

Due to the orthogonality of

$$\int_0^\ell \sin\frac{j\pi\xi}{\ell}\sin\frac{k\pi\xi}{\ell}d\xi = \frac{\ell}{2}\delta_{jk}, \tag{35}$$

where δ_{jk} is the Kronecker delta, the $m \times m$ matrix can be written as

$$\int_0^\ell \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi = \mathrm{diag}\left[e^{-(\pi/\ell)^2(T-t)}, e^{-(2\pi/\ell)^2(T-t)}, \ldots, e^{-(m\pi/\ell)^2(T-t)}\right], \tag{36}$$

where diag means that the $m \times m$ matrix is a diagonal matrix.

Inserting Eq. (36) into Eq. (34) we thus obtain

$$\begin{aligned} u(x,t) = & \frac{1}{\alpha} f(x) - \frac{1}{\alpha} \mathbf{P}^{\mathrm{T}}(x) \\ & \cdot \mathrm{diag}\left[\frac{1}{\alpha + e^{-(\pi/\ell)^2(T-t)}}, \frac{1}{\alpha + e^{-(2\pi/\ell)^2(T-t)}}, \ldots, \frac{1}{\alpha + e^{-(m\pi/\ell)^2(T-t)}}\right] \\ & \cdot \int_0^\ell f(\xi)\mathbf{Q}(\xi)d\xi. \end{aligned} \tag{37}$$

While we use Eq. (18) for $\mathbf{P}$ and $\mathbf{Q}$, we can get

$$u(x,t) = \frac{1}{\alpha} f(x) - \frac{2}{\alpha\ell} \sum_{k=1}^{\infty} \frac{e^{-(k\pi/\ell)^2(T-t)}}{\alpha + e^{-(k\pi/\ell)^2(T-t)}} \int_0^\ell \sin\frac{k\pi x}{\ell} \sin\frac{k\pi\xi}{\ell} f(\xi)d\xi, \tag{38}$$

where the summation upper bound m is now replaced by ∞, because our argument is independent of m.

At this moment it is impossible to take the limit of $\alpha = 0$ in Eq. (38). In order to get a formula where the limit of $\alpha \to 0$ can be carried out, we let $t = 0$ in Eq. (38) to obtain an initial condition:

$$u(x,0) = \frac{1}{\alpha} f(x) - \frac{2}{\alpha\ell} \sum_{k=1}^{\infty} \frac{e^{-(k\pi/\ell)^2 T}}{\alpha + e^{-(k\pi/\ell)^2 T}} \int_0^\ell \sin\frac{k\pi x}{\ell} \sin\frac{k\pi\xi}{\ell} f(\xi)d\xi. \tag{39}$$

For a given $f(x)$, through some integrals one may employ the above equation to calculate $u(x,0)$.

If $u(x,0)$ is available we can calculate $u(x,t)$ at any time $t < T$ by

$$u^\alpha(x,t) = \sum_{k=1}^{\infty} a_k e^{-(k\pi/\ell)^2 t} \sin\frac{k\pi x}{\ell}, \tag{40}$$

where

$$a_k = \frac{2}{\ell} \int_0^\ell \sin\frac{k\pi\xi}{\ell} u(\xi,0)d\xi. \tag{41}$$

Inserting Eq. (39) into the above equation and utilizing the orthogonality equation (35) it is verified that

$$a_k = \frac{2}{\ell[\alpha + e^{-(k\pi/\ell)^2 T}]} \int_0^\ell \sin\frac{k\pi\xi}{\ell} f(\xi)d\xi. \tag{42}$$

Eqs. (40) and (42) constitute an analytical solution of the BHCP. In order to distinct it from the exact solution $u(x,t)$ we use the symbol $u^{\alpha}(x,t)$ to denote that the solution is obtained from a regularization method, which is named a regularized solution.

5 Two main results

In the previous section we have derived a regularized solution $u^{\alpha}(x,t)$ of Eqs. (1)-(3) under the regularized format (11) with a regularization parameter $\alpha > 0$.

We can prove the following main result.

Theorem 1: If the final data $f(x)$ is bounded in the interval $x \in [0,\ell]$, then for any $\alpha > 0$ and $T > t_0 > 0$ the regularized solution $u^{\alpha}(x,t)$ converges uniformly for all $t \geq t_0$ and $x \in [0,\ell]$.

Proof: Since $\alpha > 0$, and $f(x)$ is bounded, say $|f(x)| \leq C^{\star}$, $x \in [0,\ell]$, for some $C^{\star} > 0$, from Eq. (42) we have

$$|a_k| = \frac{2}{\ell[\alpha + e^{-(k\pi/\ell)^2 T}]} \left| \int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right| \leq \frac{2}{\ell\alpha} \int_0^{\ell} |f(\xi)| d\xi \leq \frac{2C^{\star}}{\alpha} =: C, \quad (43)$$

where C is a positive constant. Thus, for any $t \geq t_0 > 0$ we have

$$\left| a_k e^{-(k\pi/\ell)^2 t} \sin \frac{k\pi x}{\ell} \right| \leq C e^{-(k\pi/\ell)^2 t_0}. \quad (44)$$

Through the ratio test it is obvious that the series $e^{-(k\pi/\ell)^2 t_0}$ converges. Hence, by the Weierstrass M-test, the series in Eq. (40) converges uniformly with respect to x and t whenever $t \geq t_0$ and $x \in [0,\ell]$. This ends the proof. □

Taking $\alpha = 0$ in Eq. (42) and inserting it into Eq. (40) we have a formal exact solution of Eqs. (1)-(3):

$$u(x,t) = \sum_{k=1}^{\infty} a_k^{\star} e^{-(k\pi/\ell)^2 (t-T)} \sin \frac{k\pi x}{\ell}, \quad (45)$$

where

$$a_k^{\star} = \frac{2}{\ell} \int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi. \quad (46)$$

The above solution might be divergent, unless the final data $f(x)$ satisfies the following condition:

$$\sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2 T} \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 < \infty. \quad (47)$$

The above inequality is available by applying the Parseval equality on the Fourier sine series of $u(x,0) \in L^2(0,\ell)$:

$$u(x,0) = \sum_{k=1}^{\infty} a_k^{\star} e^{(k\pi/\ell)^2 T} \sin \frac{k\pi x}{\ell}, \tag{48}$$

which is obtained from Eq. (45) by inserting $t = 0$.

About $u(x,t)$ and $u^{\alpha}(x,t)$ we can prove the following result.

Theorem 2: If the final data $f(x)$ satisfies condition (47) and there exists an $\varepsilon \in (0,1)$, such that moreover,

$$4 \sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2(1+\varepsilon)T} \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 := M^2(\varepsilon) < \infty, \tag{49}$$

then for any $\alpha > 0$ and $t \geq 0$ the regularized solution $u^{\alpha}(x,t)$ satisfies the following error estimation:

$$\| u^{\alpha}(x,t) - u(x,t) \|_{L^2(0,\ell)} \leq \alpha^{\varepsilon} M(\varepsilon). \tag{50}$$

Proof: From Eqs. (40), (42), (45) and (46) it follows that

$$u(x,t) - u^{\alpha}(x,t) = \sum_{k=1}^{\infty} b_k e^{(k\pi/\ell)^2(T-t)} \sin \frac{k\pi x}{\ell}, \tag{51}$$

where

$$b_k = \frac{2\alpha}{\ell(\alpha + e^{-(k\pi/\ell)^2 T})} \int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi. \tag{52}$$

Therefore, for any $\varepsilon \in (0,1)$ we have the following estimation:

$$\begin{aligned}
&\| u(x,t) - u^{\alpha}(x,t) \|^2_{L^2(0,\ell)} \\
&\leq 4\alpha^2 \sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2(T-t)} [(\alpha + e^{-(k\pi/\ell)^2 T})^{\varepsilon} (\alpha + e^{-(k\pi/\ell)^2 T})^{1-\varepsilon}]^{-2} \\
&\cdot \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 \\
&\leq 4\alpha^2 \sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2(T-t)} [e^{2(k\pi/\ell)^2 T}]^{\varepsilon} [\alpha^{1-\varepsilon}]^{-2} \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 \\
&= 4\alpha^{2\varepsilon} \sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2((1+\varepsilon)T-t)} \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 \\
&\leq 4\alpha^{2\varepsilon} \sum_{k=1}^{\infty} e^{2(k\pi/\ell)^2(1+\varepsilon)T} \left(\int_0^{\ell} \sin \frac{k\pi\xi}{\ell} f(\xi) d\xi \right)^2 = \alpha^{2\varepsilon} M^2(\varepsilon).
\end{aligned} \tag{53}$$

Therefore, we complete the proof. □

The above two theorems are crucial to identify that the proposed regularization is workable. Although the problem we consider is ill-posed, we have assumed that the exact solution is existent in order to cast the error estimate in a manner that is typical in partial differential equation approximations.

6 Selection of regularization parameter and numerical examples

Up to this point however we not yet specify how to select the regularization parameter α. Suppose that $f(x) \in L^2(0,\ell)$ satisfying condition (47) and that $f(x)$ having a Fourier sine series expansion:

$$f(x) = \sum_{k=1}^{\infty} a_k^\star \sin\frac{k\pi x}{\ell}, \tag{54}$$

where

$$a_k^\star = \frac{2}{\ell}\int_0^\ell \sin\frac{k\pi\xi}{\ell} f(\xi)d\xi. \tag{55}$$

Substituting Eq. (54) into Eq. (39) we obtain

$$u^\alpha(x,0) = \sum_{k=1}^{\infty} \frac{e^{-(k\pi/\ell)^2T}}{\alpha + e^{-(k\pi/\ell)^2T}} a_k^\star e^{(k\pi/\ell)^2T} \sin\frac{k\pi x}{\ell}, \tag{56}$$

where we note that

$$\frac{e^{-(k\pi/\ell)^2T}}{\alpha + e^{-(k\pi/\ell)^2T}} = \frac{1}{1+\alpha e^{(k\pi/\ell)^2T}}.$$

For a better numerical solution we require that

$$\alpha e^{(k\pi/\ell)^2T} = \alpha_0 \ll 1.$$

Otherwise, the term $e^{-(k\pi/\ell)^2T}/[\alpha + e^{-(k\pi/\ell)^2T}]$ in Eq. (56) will be very small when k and/or T are large, which may lead to a large numerical error. Thus we get an approximation

$$\frac{e^{-(k\pi/\ell)^2T}}{\alpha + e^{-(k\pi/\ell)^2T}} = \frac{1}{1+\alpha_0} = 1 - \alpha_0 + \alpha_0^2 - \alpha_0^3 + \ldots.$$

When the terms with order higher than one are truncated we obtain a good approximation of $u(x,0)$ by

$$u^{\alpha_0}(x,0) = (1-\alpha_0)\sum_{k=1}^{\infty} a_k^\star e^{(k\pi/\ell)^2T} \sin\frac{k\pi x}{\ell}. \tag{57}$$

The existence of the above series is guaranteed by condition (47). The regularization parameter α_0 is a small number.

6.1 Example 1

In order to compare our numerical results with those obtained by Lesnic, Elliott and Ingham (1998), Mera, Elliott, Ingham and Lesnic (2001), Mera, Elliott and Ingham (2002) and Mera (2005), let us first consider a one-dimensional benchmark BHCP:

$$u_t = u_{xx}, \;\; 0 < x < 1, \;\; 0 < t < T, \tag{58}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{59}$$

and the final time condition

$$u(x,T) = \sin(\pi x)\exp(-\pi^2 T). \tag{60}$$

The data to be retrieved are given by

$$u_e(x,t) = \sin(\pi x)\exp(-\pi^2 t), \;\; T > t \geq 0. \tag{61}$$

For this problem we let $\ell = 1$, and substitute Eq. (60) for $f(x)$ into Eq. (38) to obtain

$$u(x,t) = \frac{\exp-(\pi^2 T)}{\alpha + \exp[-\pi^2(T-t)]}\sin(\pi x). \tag{62}$$

It is very interesting that u is identical to u_e when $\alpha = 0$. Taking $t = 0$ we recover the initial condition to be

$$u(x,0) = \frac{\exp(-\pi^2 T)}{\alpha + \exp(-\pi^2 T)}\sin(\pi x). \tag{63}$$

In Fig. 1 we show the numerical errors when compared with the exact one $\sin(\pi x)$ for different regularization parameters of $\alpha = 10^{-10}, 10^{-12}$ with a fixed $T = 1$. Upon comparing with the numerical results computed by Mera (2005) with the method of fundamental solution (MFS) together with the Tikhonov regularization technique (see Figure 5 of the above cited paper), we can say that the new method is much accurate than MFS. In Fig. 2 we just show the numerical error with a very large $T = 5$ with $\alpha = e^{-60}$, where the data $u(x,T)$ is very small about in the order of $10^{-21.5}$.

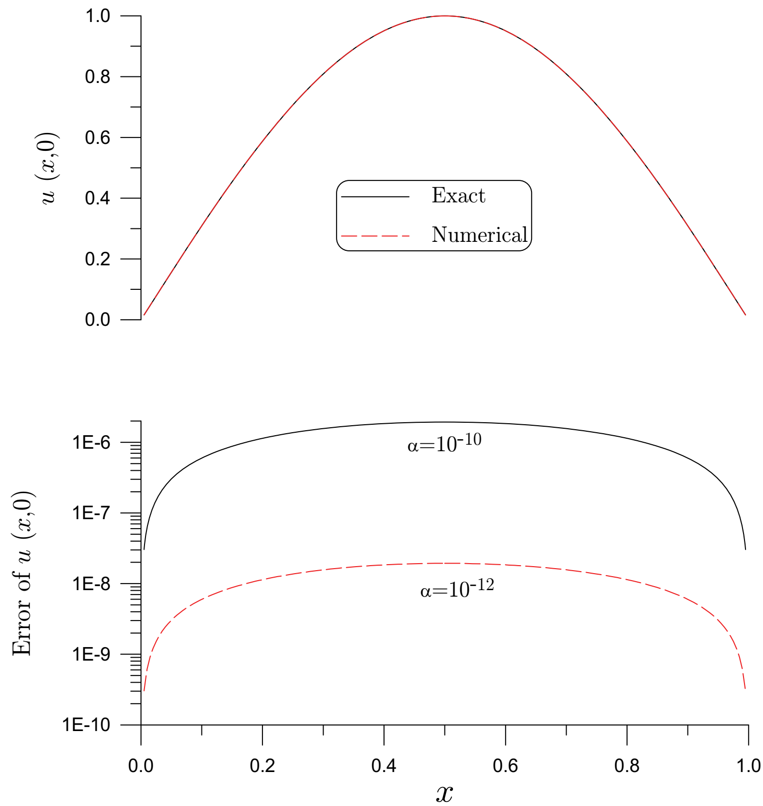

Figure 1: For Example 1 we comparing numerical and exact initial data retrieved from final data at $T = 1$, and plotting the numerical error.

6.2 Example 2

In order to compare our numerical results with those obtained by Ames, Clark, Epperson and Oppenheimer (1998), and Ames and Epperson (1997), let us consider the same problem as in Example 1 but with the final time data to be

$$f(x) = e^{-\tau\pi^2}\sin\pi x + e^{-4\tau\pi^2}\sin 4\pi x + e^{-2-9\tau\pi^2}\sin 3\pi x. \tag{64}$$

Substituting the above $f(x)$ into Eq. (55) and then into Eq. (57) we obtain a regularized initial data:

$$u^{\alpha_0}(x,0) = (1-\alpha_0)\left[e^{(T-\tau)\pi^2}\sin\pi x + e^{4(T-\tau)\pi^2}\sin 4\pi x + e^{-2+9(T-\tau)\pi^2}\sin 3\pi x\right]. \tag{65}$$

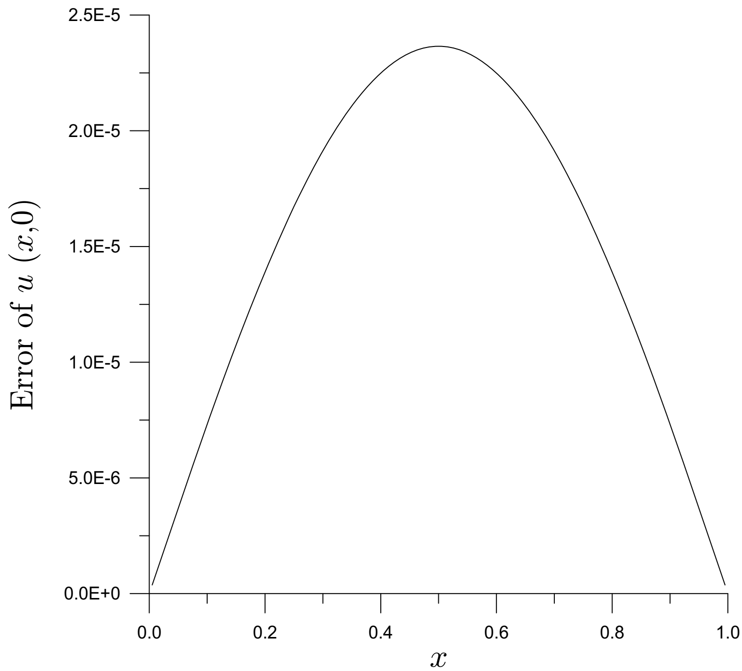

Figure 2: For Example 1 we plotting the numerical error of initial data retrieved from final data at $T = 5$.

Therefore, by Eqs. (40) and (41) we obtain a regularized solution:

$$u^{\alpha_0}(x,t) = (1-\alpha_0)\left[e^{(T-\tau-t)\pi^2}\sin\pi x + e^{4(T-\tau-t)\pi^2}\sin 4\pi x + e^{-2+9(T-\tau-t)\pi^2}\sin 3\pi x\right]. \tag{66}$$

It is different from the exact solution by a factor $1-\alpha_0$ only.

For the comparison purpose we take $\tau = T - t_0$ and $t = t_0$ to obtain

$$u^{\alpha_0}(x,t_0) = (1-\alpha_0)\left[\sin\pi x + \sin 4\pi x + e^{-2}\sin 3\pi x\right], \tag{67}$$

which is also equal to the exact one in addition the factor $1-\alpha_0$.

In Fig. 3 we compare the exact solution with the regularized solution under $\alpha_0 = 10^{-10}$. It can be seen that the numerical error is in the order of $\alpha_0 = 10^{-10}$. In Table 1 we compare our results with that calculated by Ames, Clark, Epperson and Oppenheimer (1998), denoted as ACEO, and that by Ames and Epperson (1997), denoted as AE. The parameters used in these calculations are $T = 1/8$ and $t_0 = 1/32$. In all compared data the present results are very good.

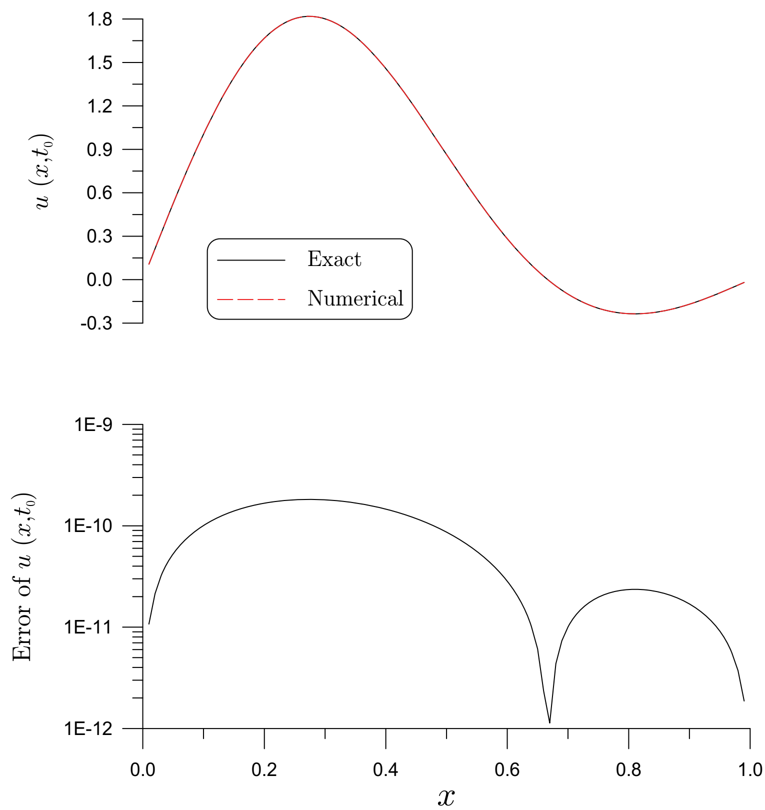

Figure 3: For Example 2 we comparing numerical and exact data at $t = t_0$ retrieved from final data at $T = 1/8$, and plotting the numerical error.

6.3 Example 3

Let us consider the one-dimensional BHCP:

$$u_t = u_{xx}, \;\; 0 < x < 1, \;\; 0 < t < T, \tag{68}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{69}$$

Table 1: Errors of Example 2; $\alpha_0 = h^2$.

h^{-1}	ACEO	AE	Present paper
8	0.500437E+00	0.685975E+00	0.281688E-01
16	0.277570E+00	0.616086E+00	0.704220E-02
32	0.130830E+00	0.444974E+00	0.176055E-02
64	0.935625E-01	0.224220E+00	0.440138E-03
128	0.770219E-01	0.115449E+00	0.110034E-03
256	0.482003E-01	0.969024E-01	0.275086E-04
512	0.193557E-01	0.943557E-01	0.687715E-05
1024	0.570397E-02	0.901759E-01	0.171929E-05
2048	0.149270E-02	0.768532E-01	0.429821E-06

and the initial condition

$$u(x,0) = \begin{cases} 2x, & 0 \le x \le 0.5, \\ 2(1-x), & 0.5 \le x \le 1. \end{cases} \tag{70}$$

The exact solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8}{\pi^2(2k+1)^2} \cos\frac{(2k+1)\pi(2x-1)}{2} \exp[-\pi^2(2k+1)^2 t]. \tag{71}$$

The backward numerical solution is subjected to the final condition at time T:

$$f(x) = u(x,T) = \sum_{k=0}^{\infty} \frac{8}{\pi^2(2k+1)^2} \cos\frac{(2k+1)\pi(2x-1)}{2} \exp[-\pi^2(2k+1)^2 T]. \tag{72}$$

The difficulty of this problem is originated from that we use a smooth final data to retrieve a non-smooth initial data. In the literature, this one-dimensional BHCP is called a triangular test [Muniz, de Campos Velho and Ramos (1999); Muniz, Ramos and de Campos Velho (2000); Chiwiacowsky and de Campos Velho (2003)].

For this problem we let $\ell = 1$ and insert Eq. (72) for $f(x)$ into Eq. (42) to obtain

$$a_k = [\alpha + e^{-(k\pi)^2 T}] \sum_{m=0}^{\infty} \frac{8(-1)^m \delta_{k,(2m+1)}}{\pi^2(2m+1)^2} e^{-[(2m+1)\pi]^2 T}. \tag{73}$$

Inserting it into Eq. (40) we obtain

$$u^\alpha(x,t) = \sum_{k=1}^{\infty} \frac{1}{[\alpha + e^{-(k\pi)^2 T}]} \sum_{m=0}^{\infty} \frac{8(-1)^m \delta_{k,(2m+1)}}{\pi^2(2m+1)^2} e^{-[(2m+1)\pi]^2 T} e^{-(k\pi)^2 t} \sin k\pi x. \tag{74}$$

Interchanging the order of summation and using the δ property we have

$$u^{\alpha}(x,t) = \sum_{m=0}^{\infty} \frac{8(-1)^m}{\pi^2(2m+1)^2} \frac{e^{-[(2m+1)\pi]^2 T}}{[\alpha + e^{-[(2m+1)\pi]^2 T}]} e^{-[(2m+1)\pi]^2 t} \sin(2m+1)\pi x. \tag{75}$$

It gives

$$u^{\alpha}(x,0) = \sum_{m=0}^{\infty} \frac{8(-1)^m}{\pi^2(2m+1)^2} \frac{e^{-[(2m+1)\pi]^2 T}}{[\alpha + e^{-[(2m+1)\pi]^2 T}]} \sin(2m+1)\pi x. \tag{76}$$

The term

$$\frac{e^{-[(2m+1)\pi]^2 T}}{[\alpha + e^{-[(2m+1)\pi]^2 T}]} = 1 - \alpha_0$$

is already derived at the beginning of this section. Hence, we have

$$u^{\alpha}(x,0) = (1-\alpha_0) \sum_{m=0}^{\infty} \frac{8(-1)^m}{\pi^2(2m+1)^2} \sin(2m+1)\pi x. \tag{77}$$

Therefore, we use this solution to compare that in Eq. (70). In practice, the datum is obtained by taking the sum of the first one thousand terms, which guarantees the convergence of the series. From Fig. 4 it can be seen that the numerical error is very small even at the coner point.

Muniz, de Campos Velho and Ramos (1999), and Muniz, Ramos and de Campos Velho (2000) have calculated this example by different regularization techniques. They have shown that the explicit inversion method does not give satisfactory results even with a small terminal time with $T = 0.008$ [Muniz, de Campos Velho and Ramos (1999)]. Muniz, Ramos and de Campos Velho (2000) have calculated the initial data with a terminal time $T = 0.01$ by the Tikhonov regularization, maximum entropy principle and truncated singular value decomposition, and good results were obtained as shown in Figures 4 and 5 of the above cited paper. However, with the current method under a very large $T = 1$, the maximum error occurring at $x = 0.5$ is only 0.00020244. The result is also better than that calculated by Liu, Chang and Chang (2006), where they used $T = 0.01$ to obtain a maximum error 0.0619522.

In the case when the input final measured data are contaminated by random noise, we are concerned with the stability of our method, which is investigated by adding the different levels of random noise on the final data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, where $R(i)$ are random numbers in $[-1,1]$. Then the given final data are multiplied by $(1+sR(i))$.

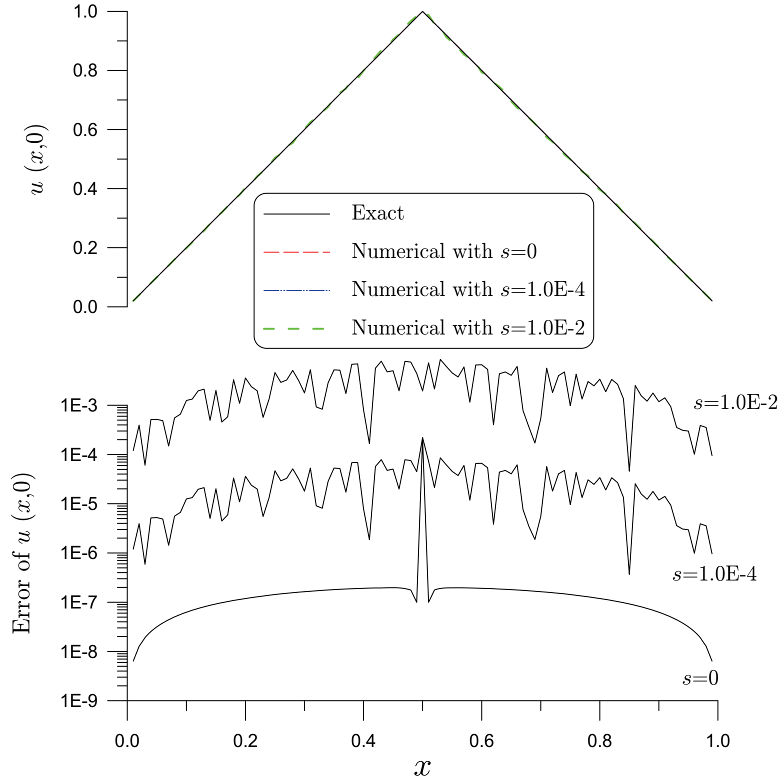

Figure 4: For Example 3 we comparing exact initial data and numerical initial data under different noise levels on the final data at $T = 1$, and plotting the numerical errors.

The numerical results with $T = 1$ were compared with the numerical result without considering random noise in Fig. 4. It can be seen that the noise levels with $s = 0.0001$ and $s = 0.01$ disturb the numerical solutions deviating from the exact solution very small. The maximum error as shown is also in the order of 10^{-3} even for a larger disturbance with $s = 0.01$.

7 Conclusions

In this paper we have transformed the 1D backward heat conduction problem into a problem to solve a second-kind Fredholm integral equation through a direct regularization technique. By using the Fourier series expansion technique and a termwise separable property of kernel function, an analytical solution for approximating the exact solution is presented. The influence of regularization parameter on the perturbed solution is clear, which led well to a better selection of the regularization parameter to avoid the inducing of a large numerical error. The uniform convergence and error estimate of the regularized solution were provided. They demonstrated that the new regularized technique is although simple but is applicable to the backward heat conduction problems. The numerical examples have shown that the new method could retrieve all initial data very excellently, even the final data is very small or noised by large disturbance, and even the initial data to be retrieved are not smooth.

References

Ames, K. A.; Cobb, S. S. (1997): Continuous dependence on modeling for related Cauchy problems of a class of evolution equations. *J. Math. Anal. Appl.*, vol. 215, pp. 15-31.

Ames, K. A.; Epperson, J. F. (1997): A kernel-based method for the approximate solution of backward parabolic problems. *SIAM J. Num. Anal.*, vol. 34, pp. 1357-1390.

Ames, K. A.; Clark, G. W.; Epperson, J. F.; Oppenheimer, S. F. (1998): A comparison of regularizations for an ill-posed problem. *Math. Comp.*, vol. 67, pp. 1451-1471.

Ames, K. A.; Payne, L. E. (1999): Continuous dependence on modeling for some well-posed perturbations of the backward heat equation. *J. Inequal. Appl.*, vol. 3, pp. 51-64.

Chang, C.-W.; Liu, C.-S.; Chang, J.-R. (2005): A group preserving scheme for inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, pp. 13-38.

Chang, C.-W.; Liu, C.-S.; Chang, J.-R. (2009): A new shooting method for quasi-boundary regularization of multi-dimensional backward heat conduction problems. *J. Chinese Inst. Engineers*, vol. 32, pp. 307-316.

Chang, J.-R.; Liu, C.-S.; Chang, C.-W. (2007): A new shooting method for quasi-boundary regularization of backward heat conduction problems. *Int. J. Heat Mass Transfer*, vol. 50, pp. 2325-2332.

Chiwiacowsky, L. D.; de Campos Velho, H. F. (2003): Different approaches for the solution of a backward heat conduction problem. *Inv. Prob. Eng.*, vol. 11, pp. 471-494.

Clark, G. W.; Oppenheimer, S. F. (1994): Quasireversibility methods for non-well-posed problems. *Elect. J. Diff. Eqns.*, vol. 1994, pp. 1-9.

Han, H.; Ingham, D. B.; Yuan, Y. (1995): The boundary element method for the solution of the backward heat conduction equation. *J. Comp. Phys.*, vol. 116, pp. 292-299.

Iijima, K. (2004): Numerical solution of backward heat conduction problems by a high order lattice-free finite difference method. *J. Chinese Inst. Engineers*, vol. 27, pp. 611-620.

Jourhmane, M.; Mera, N. S. (2002): An iterative algorithm for the backward heat conduction problem based on variable relaxation factors. *Inv. Prob. Eng.*, vol. 10, pp. 293-308.

Kirkup, S. M.; Wadsworth, M. (2002): Solution of inverse diffusion problems by operator-splitting methods. *Appl. Math. Model.*, vol. 26, pp. 1003-1018.

Lattés, R.; Lions, J. L. (1969): The Method of Quasireversibility, Applications to Partial Differential Equations. Elsevier, New York.

Lesnic, D.; Elliott, L.; Ingham, D. B. (1998): An iterative boundary element method for solving the backward heat conduction problem using an elliptic approximation. *Inv. Prob. Eng.*, vol. 6, pp. 255-279.

Liu, C.-S. (2004): Group preserving scheme for backward heat conduction problems. *Int. J. Heat Mass Transfer*, vol. 47, pp. 2567-2576.

Liu, C.-S. (2006): The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, pp. 149-163.

Liu, C.-S. (2007a): Elastic torsion bar with arbitrary cross-section using the Fredholm integral equations. *CMC: Computers, Materials & Continua*, vol. 5, pp. 31-42.

Liu, C.-S. (2007b): A meshless regularized integral equation method for Laplace equation in arbitrary interior or exterior plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 99-109.

Liu, C.-S. (2007c): A MRIEM for solving the Laplace equation in the doubly connected domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 145-161.

Liu, C.-S. (2009): Solving the inverse problems of Laplace equation to determine the Robin coefficient/cracks' position inside a disk. *CMES: Computer Modeling in Engineering & Sciences*, vol. 40, pp. 1-28.

Liu, C.-S.; Chang, C.-W.; Chang, J.-R. (2006): Past cone dynamics and backward group preserving schemes for backward heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, pp. 67-81.

Liu, C.-S.; Chang, C.-W.; Chang, J.-R. (2009): A quasi-boundary semi-analytical method for backward in time advection-dispersion equation. *CMC: Computers, Materials & Continua*, vol. 9, pp. 111-135.

Liu, C.-S.; Atluri, S. N. (2009): A Fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentiation of noisy data, and its relation to the filter theory. CMES: Computer Modeling in Engineering & Sciences, vol. 41, pp. 243-261.

Liu, C.-S.; Yeih, W.; Atluri, S. N. (2009): On solving the ill-conditioned system $\mathbf{Ax} = \mathbf{b}$: general-purpose conditioners obtained from the boundary-collocation solution of the Laplace equation, using Trefftz expansions with multiple length scales. *CMES: Computer Modeling in Engineering & Sciences*, vol. 44, pp. 281-311.

Mera, N. S. (2005): The method of fundamental solutions for the backward heat conduction problem. *Inv. Prob. Sci. Eng.*, vol. 13, pp. 65-78.

Mera, N. S.; Elliott, L.; Ingham, D. B.; Lesnic, D. (2001): An iterative boundary element method for solving the one-dimensional backward heat conduction problem. *Int. J. Heat Mass Transfer*, vol. 44, pp. 1937-1946.

Mera, N. S.; Elliott, L.; Ingham, D. B. (2002): An inversion method with decreasing regularization for the backward heat conduction problem. *Num. Heat Transfer B*, vol. 42, pp. 215-230.

Muniz, W. B.; de Campos Velho, H. F.; Ramos, F. M. (1999): A comparison of some inverse methods for estimating the initial condition of the heat equation. *J. Comp. Appl. Math.*, vol. 103, pp. 145-163.

Muniz, W. B.; Ramos, F. M.; de Campos Velho, H. F. (2000): Entropy- and Tikhonov-based regularization techniques applied to the backward heat equation. *Int. J. Comp. Math.*, vol. 40, pp. 1071-1084.

Showalter, R. E. (1983): Cauchy problem for hyper-parabolic partial differential equations. In *Trends in the Theory and Practice of Non-Linear Analysis*, Elsevier.

Showalter, R. E.; Ting, T. W. (1970): Pseudo-parabolic partial differential equations. *SIAM J. Math. Anal.*, vol. 1, pp. 1-26.

Tricomi, F. G. (1985): Integral Equations. Dover, New York.