An Improved Petrov-Galerkin Spectral Collocation Solution for Linear Stability of Circular Jet

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Abstract: A Fourier-Chebyshev Petrov-Galerkin spectral method is described for computation of temporal linear stability in a circular jet. Basis functions presented here are exponentially mapped Chebyshev functions. They satisfy the pole condition exactly at the origin, and can be used to expand vector functions efficiently by using the solenoidal condition. The mathematical formulation is presented in detail focusing on the analyticity of solenoidal vector field used for the approximation of the flow. The scheme provides spectral accuracy in the present cases studied and the numerical results are in agreement with former works.

Keywords: hydrodynamic stability; circular jet; cylindrical system singularity; coordinate transformation; spectral-Galerkin method; projection method.

1 Introduction

Jets are important in many practical applications, e.g., related to combustion, propulsion, mixing and aeroacoustic. As one of the generic flow of fluid mechanics, jets have been of scientific interest for over 100 years. The round jet results when fluid is emitted, with a given initial momentum, out of a circular orifice into a large space. At sufficiently high Reynolds numbers this jet will be turbulent. The stability properties of the flow play a fundamental role in the transition to turbulence and the formation of coherent vortex structures in a turbulent fluid (Rayleigh, 1880).

Frequently, the choice of independent variables is motivated by the symmetry of circular jet, then cylindrical coordinates are likely most appropriate. However, the choice of a particular set of independent variable might inadvertently introduce mathematically allowable but physically unrealistic terms, e.g., singularities. The

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treatment of the geometrical singularity in cylindrical and spherical coordinates has been a difficulty in the development of accurate finite difference and pseudospectral schemes for many years (Schmid and Henningson, 2000; Drazin and Reid, 2004). The use of a spectral representation is often to be preferred for the accurate solution of problems with simple geometry (Xie *et al.*, 2008a, 2008b 2009a, 2009b). Lopez *et al.* (2002) derived a regularity conditions by using the properties at the axis of the functions chosen to expand velocity and pressure along the radial direction. Pole conditions for Poisson-type equations in the physical space were derived by Huang and Sloan (1993). But the time step restriction problem that arises for advection problems due to the increased resolution near the coordinate singularity can't be avoided. One way to avoid the time step restriction is to use a Fourier filter in the azimuthal direction as used by Fornberg (1995). Priymak and Miyazakiy (1998) presented a robust numerical technique for incompressible Navier Stokes equations in cylindrical coordinates. Lin and Atluri (2000, 2001) proposed several upwinding Meshless Local Petrov-Galerkin (MLPG) schemes to solve steady convection-diffusion problems in one and two dimensions. It shows that the MLPG method is very promising to solve the convection-dominated flow and fluid mechanics problems. Meseguer and Trefethen (2003) described a Fourier-Chebyshev Petrov-Galerkin spectral method for high accuracy computation of linearized dynamics flow in finite circular pipe in the light of Chapman's analysis. Bierbrauer & Zhu (2007) present three analytical solutions, the Bounded Creeping Flow, Solenoidal and Conserved Solenoidal Solutions, which are both continuous, incompressible, retain as much of the original mathematical formulation as possible and provide a physically reasonable initial velocity field. For hydrodynamic stability problems, the linearized Navier-Stokes equations are a general eigenvalues equation, the conditioner of the matrix usually is an ill-conditioned system arising from very-high order polynomial interpolations. Trefethen *et al.* (1993) have highlighted the fact that, even when all eigenvalues of the linearized problem indicated decay, there can be transient amplification and growth in energy owing to the non-normality of the linearized operator. Liu & Atluri (2008, 2009) developed a highly accurate technique based on modified Trefftz method to deal with the ill-posed linear problems. A link between these structures is discussed by van Doorne & Westerweel (2009). Theoretical approaches to extract perturbations that are efficient in triggering turbulence are presented by Biau & Bottaro (2009) and Cohen et al. (2009), respectively.

Recently, MLPG methods mainly are used in the bounded domain for fluid mechanics problems. Shan *et al.* (2008) used local MLPG method to solve 3D incompressible viscous flows with curved boundary. Sellountos and Sequeira developed a hybrid multi-region MLPG velocity-vortices scheme for the 2D Navier

stokes equations. To improve the accuracy and stability of MLPG methods, Mohammadi (2008) and Orsini *et al.* (2008) proposed a meshless radial basis function techniques, and Sellountos & Sequeira (2009) applied the radial basis function networks to transient viscous flows. To construct a basis function set for unbounded domains, it is necessary to assume the asymptotic behavior of the approximated functions for large radius r. One way to treat this class of functions is the domain truncation method which imposes artificial boundary conditions at sufficiently large radius. The method can be made more efficient if additional mappings are used, so that standard spectral basis functions such as Chebyshev polynomial can be used. Grosch and Orszag (1977) investigated the exponential and algebraic mapping methods and found by numerical experiments that the algebraic mapping gives a better result than the exponential mapping. Boyd (1999) supported their result by examining the asymptotic behavior of the expansion coefficients of model functions by the method of steepest descent.

In the present work, an efficient spectral Petrov-Galerkin scheme for the numerical approximation of hydrodynamic stability equation in a circular jet is presented. The radial basis function and weight function has been improved based on the previous work (Xie et al. 2008a), The infinite domain is transformed into a finite unit disk domain by exponential mappings. The discrete formulation of the disk with Chebyshev spectral method proposed by Fornberg (1995) is adopted. Rational physical basis and test basis functions in a bounded domain are used for expansion in the radial direction of polar coordinates. The numerical method is validated against the results available in the literature.

2 The mathematical formulation

To derive the linearized equations of round jet, we start with the impressible dimensionless Navier-Stokes equations, these equations in cylindrical polar coordinates become:

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{1}$$

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \tag{2}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) \tag{3}$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{Re}\Delta u_z \tag{4}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}; \quad \Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (5)$$

These equations are non-dimensionalised with respect to length scale L^*, velocity scale U^*, and Reynolds number is $Re = L^* U^* / v$. The length scale and velocity scale is usually based on the jet core velocity and momentum thickness.

Our concern in this paper is the linearized problem in which only infinitesimal perturbations from the laminar flow are considered, Let

$$u_r = U_r + u'_r; \quad u_\theta = U_\theta + u'_\theta; \quad u_z = U_z + u'_z; \quad p = P + p' \qquad (6)$$

and the perturbation can be expressed as superposition of complex Fourier modes of the form:

$$\frac{u'_r}{u_r(r)} = \frac{u'_\theta}{u_\theta(r)} = \frac{u'_z}{u_z(r)} = \frac{p'}{\rho p(r)} = e^{i(n\theta + kz - \beta t)} = e^{i(n\theta + kz - kct)} \qquad (7)$$

where $u_r(r)$, $u_\theta(r)$, $u_z(r)$ and $p(r)$ are the amplitudes of the corresponding disturbances; n is the azimuthal mode of the disturbance; k is the axial wavenumber of disturbance; c (or $\beta = kc$) is the wave amplification factor. Then the linearized Navier-Stokes equations become:

$$-ikcu_r = -Dp + \frac{1}{Re}\left(D^2 u_r + \frac{1}{r}u_r - \frac{n^2+1}{r^2}u_r - k^2 u_r - \frac{2}{r^2}inu_\theta\right) - ikU_z u_r \qquad (8)$$

$$-ikcu_\theta = -\frac{in}{r}p + \frac{1}{Re}\left(D^2 u_\theta + \frac{1}{r}u_\theta - \frac{n^2+1}{r^2}u_\theta - k^2 u_\theta + \frac{2}{r^2}inu_r\right) - ikU_z u_\theta \qquad (9)$$

$$-ikcu_z = -ikp + \frac{1}{Re}\left(D^2 u_z + \frac{1}{r}u_z - \frac{n^2}{r^2}u_z - k^2 u_z\right) - u_r DU_z - ikU_z u_z \qquad (10)$$

$$\left(D + \frac{1}{r}\right)u_r + i\left(\frac{1}{r}nu_\theta + ku_z\right) = 0 \qquad (11)$$

where $D = d/dr$. The Eqs.(8)-(11) are of singular Sturm-Liouville equation and have regular singularities at $r = 0$.

3 Coordinate transform

We investigate the utility of mappings to solve the linear stability problems of round jet numerically in infinite regions. In this paper we use the formulation proposed

by Fornberg (1995). The idea is to take $x \in [-1, 1]$ instead of $x \in [0, 1]$. To expand this class of functions, we consider the exponential mapping:

$$x = \frac{1 - e^{-r/L}}{1 + e^{-r/L}}; \quad r \in (-\infty, \infty) \text{ or } r/L = \ln\frac{1 + x}{1 - x}; \ x \in (-1, 1) \tag{12}$$

where $L > 0$ is the map parameter. The exponential map gives a good resolution near $r = 0$, and it is especially useful in the treatment of geometrical singularity for cylindrical coordinates. The exponential map (x versus r) with various L is plotted in Figure 1. The larger the parameter L, the map gives a better solution as $r \to \infty$.

Then the distribution of velocity of round jet is

$$U_z = \frac{1}{(1 + r^2)^2} = \frac{1}{(1 + L^2 \ln^2(1 + x)/(1 - x))^2}; \ x \in (-1, 1) \tag{13}$$

The exponential map is especially convenient because it yield simple expressions for derivations. The derivatives with respect to r become:

$$\frac{\partial u}{\partial r} = \frac{1 - x^2}{2L} \frac{\partial u}{\partial x} \tag{14}$$

The axial component velocity profiles of the round jet and its derivation for various values L is shown in Figure 2 and 3. The larger L values, the more steeper the profiles curve near the axis. The result is that a grid is highly clustered near the origin, if the solution is smooth there. It may be wasteful in numerically computation. Thus a smaller L is needed to save the cost of computation. According to the analysis above, there is a constraint in the choice of the map parameter L (Xie & Lin, 2008b), and the problem is discussed and solved in Section 5.3.

We use Chebyshev series to represented the scalar function $u(r)$, then

$$u(x) = \sum_{n=0}^{N} a_n T_{2n}(x); \quad T_{2n}(\cos\theta) = \cos 2n\theta \tag{15}$$

The set of $T_{2n}(x)$ for fixed integer n is complete and orthogonal with respect to weight function $w = 1/\sqrt{1 - x^2}$. And the details of the application of Chebyshev series to the numerical solution of ordinary and partial differential equations are given by Boyd (1999).

Then the linearized Navier-Stokes equations become:

$$-ikcu_r = -D_* p + \frac{1}{Re}\left(D_*^2 u_r + \frac{1}{r}u_r - \frac{n^2 + 1}{r^2}u_r - k^2 u_r - \frac{2}{r^2}inu_\theta\right) - ikU_z u_r \tag{16}$$
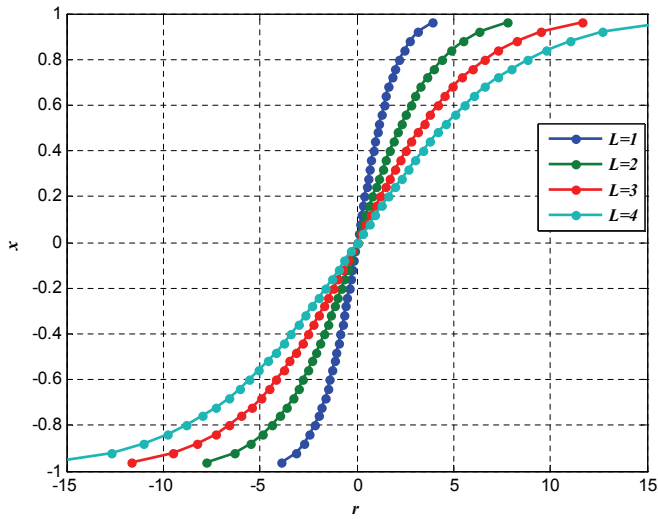
Figure 1: Variation of *r* versus *x* for the exponential map with various *L*
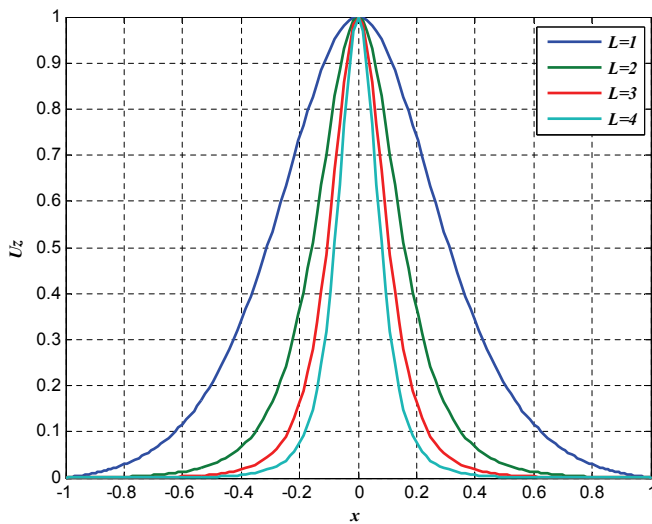

Figure 2: Distribution of axial component velocity of round jet with various values *L*
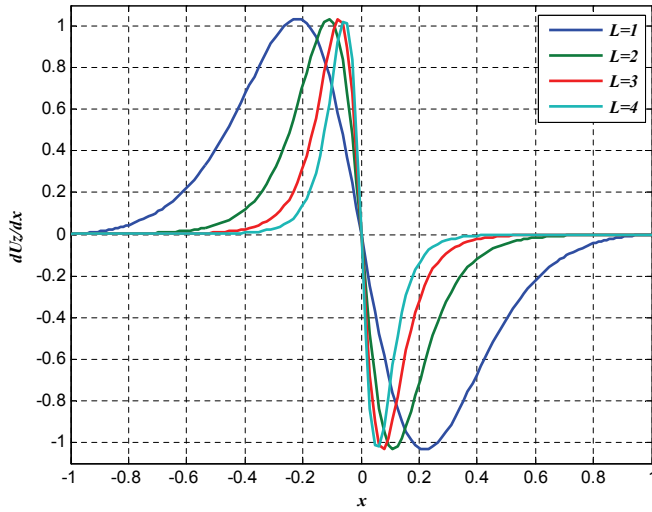
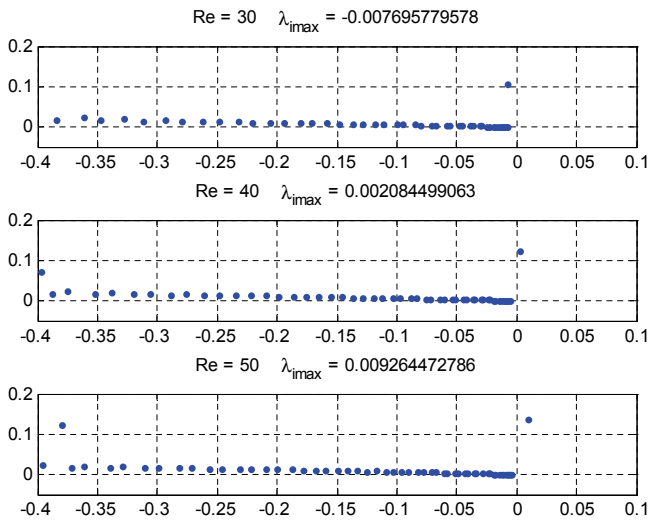Figure 3: The differential of axial velocity component of round jet with various L

Figure 4: Numerically computed eigenvalues in the complex plane for $n = 1$ and $k = 0.45$ with $Re = 30, 40, 50$

$$-ikcu_\theta = -\frac{in}{r}p + \frac{1}{Re}\left(D_*^2 u_\theta + \frac{1}{r}u_\theta - \frac{n^2+1}{r^2}u_\theta - k^2 u_\theta + \frac{2}{r^2}inu_r\right) - ikU_z u_\theta \quad (17)$$

$$-ikcu_z = -ikp + \frac{1}{Re}\left(D_*^2 u_z + \frac{1}{r}u_z - \frac{n^2}{r^2}u_z - k^2 u_z\right) - u_r D_* U_z - ikU_z u_z \quad (18)$$

$$\left(D_* + \frac{1}{r}\right)u_r + i\left(\frac{1}{r}nu_\theta + ku_z\right) = 0 \quad (19)$$

where $D_* = (1-x^2)(\partial u/\partial x)/2L$, $r = L\ln(1+x)/(1-x)$.

Then the boundary conditions for first azimuthal ($n = 1$) mode become:

$$u_r(0) + u_\theta(0) = Du_z(0) = Dp(0) = 0; \quad u_r(1) = u_\theta(1) = u_z(1) = p(1) = 0 \quad (20)$$

4 Solenoidal Petrov-Galerkin discretisation

In order to have spectral accuracy in the numerical approximation of the eigen-values problem, analyticity of the vector fields is required in the interval [0, 1]. Transformations to polar coordinates are singular at $r=0$, making necessary a special treatment of our solution functions in a neighborhood of the origin. According to the regularity analysis of Priymak & Miyazaki (1998), the solenoidal basis for the approximation of the perturbation vector field takes the form:

$$\mathbf{u} = e^{i(kz+n\theta-kct)}\sum_{m=0}^{M} a_m^{(1)}\mathbf{w}_m^{(1)}(x)\mathbf{u}_m^{(1)}(x) + a_m^{(2)}\mathbf{w}_m^{(2)}(x)\mathbf{u}_m^{(2)}(x) \quad (21)$$

where $\mathbf{u}_m$ belongs to the physical or trial space and $\mathbf{w}_m$ is a solenoidal vector field belongs to the test or projection space. Then the physical or trial basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} -ikr^{n-1}g_m(x) \\ D_*[r^n g_m(x)] \\ 0 \end{pmatrix}, \mathbf{u}_m^{(2)} = \begin{pmatrix} 0 \\ -ikr^n h_m(x) \\ inr^{n-1}h_m(x) \end{pmatrix} \quad (22)$$

In order to take the advantage of orthogonality properties of Chebyshev polynomials, the test functions should be built up suitably. In essence, the projection fields are going to have the same structure as the trial fields but the functions will be modified by the Chebyshev weight $(1-x^2)^{-1/2}$. However, the resulting matrices would be dense. Whereas they can be made to be bounded if the projection velocity fields as follows:

$$\mathbf{w}_m^{(1)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} -ikr^n g_m(x) \\ D_*[r^{n+1}g_m(x)] + r^{n+1}xg_m(x)/2L \\ 0 \end{pmatrix},$$

$$\mathbf{w}_m^{(2)} = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} 0 \\ -ikr^{n+1}h_m(x) \\ inr^n h_m(x) \end{pmatrix} \quad (23)$$

for various azimuthal mode n, except that if $k=0$, the third component of $\mathbf{w}_m^{(2)}$ is replaced by $rh_m(x)$. It should be noted that the formats of the projection velocity fields in this paper was different from the previous results of Xie and Lin (2008b), and this formats is more universal.

The Petrov-Galerkin projection scheme is carried out by substituting the spectral approximation in equations and projecting over the dual space. This procedure leads to a discretized generalized eigenvalues problem, and the coefficient $a_m^{(1,2)}$ govern the temporal behavior of the perturbation.

$$\mathbf{AX} = \lambda \mathbf{BX} \tag{24}$$

where $\lambda = -ikc$, the matrixes, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ represent as follows respectively.

$$\mathbf{A} = \begin{bmatrix} \left(\mathbf{w}_m^{(1)} \cdot \ell[\mathbf{u}_m^{(1)}] \right) & \left(\mathbf{w}_m^{(1)} \cdot \ell[\mathbf{u}_m^{(2)}] \right) \\ \left(\mathbf{w}_m^{(2)} \cdot \ell[\mathbf{u}_m^{(1)}] \right) & \left(\mathbf{w}_m^{(2)} \cdot \ell[\mathbf{u}_m^{(2)}] \right) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \left(\mathbf{w}_m^{(1)} \cdot \mathbf{u}_m^{(1)} \right) & \left(\mathbf{w}_m^{(1)} \cdot \mathbf{u}_m^{(2)} \right) \\ \left(\mathbf{w}_m^{(2)} \cdot \mathbf{u}_m^{(1)} \right) & \left(\mathbf{w}_m^{(2)} \cdot \mathbf{u}_m^{(2)} \right) \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} a_m^{(1)} \\ a_m^{(2)} \end{bmatrix}$$

where ℓ stands for the linear operator of linear stability equations

$$\ell[\cdot] = \frac{1}{Re}\Delta[\cdot] - \mathbf{u}_B \cdot \nabla[\cdot] - [\cdot] \cdot \nabla \mathbf{u}_B \tag{25}$$

where $\mathbf{u}_B$ are the basic flow velocity vector $(0, 0, U_z)$. The pressure term should be formally included in the operator ℓ, but it is cancelled when projecting it over $\mathbf{w}$, that is $(\mathbf{w}, p)=0$.

The generalized eigenvalues problem in Eq.(24) can be computed exactly by Gauss-Chebyshev-Lobatto quadrature formulas. In the present study, the temporal instability of round jet is considered. Hence, k and n is real quantity while $c = c_r + ic_i$ is generally complex. The disturbances will grow with time if $c_i > 0$ and will decay $c_i < 0$ in Eq.(7). The neutral disturbances are then characterized by $c_i = 0$.

5   Results and discussion

In stability analysis the most important eigenvalues is the one that is the most unstable or least stable. For the present framework, this corresponds to the eigenvalues

with the least imaginary part. In particular, the flow will be temporal unstable if the imaginary part of the complex amplification is positive. The results in the present section have been obtained by parametrically varying the Reynolds number and frequency for an azimuthal wave number n of 1. This mode represents the most important components of circular jet flow. Although n is necessarily an integer, k can be any real number.

5.1 *Eigenvalues*

The efficiency of Petrov-Galerkin spectral method has been discussed by Meseguer & Trefethen (2003) and Lin & Atluri (2000, 2001), and the eigenvalues are computed numerically based on Chebyshev polynomial formulas. Figure 4 shows some of the right most eigenvalues in the complex plane for $n = 1$ and $k = 0.45$ with $Re =$ 30, 37.6, 50. The distribution of the eigenvalues of circular jet is different from that of pipe flow. Although both the linearized Navier-Stokes equations are the same, an investigation of linear stability equation around the parabolic profile shows that it is linearly stable for all Reynolds number (Eckhardt, 2009). For circular jet velocity profile, it is unstable for small Reynolds number. And the critical Reynolds number is about 37.6, but the value of critical number is affected by some other parameters, such as the order of Chebyshev polynomial, M; exponential map parameter, L; and axial wavenumber, k, etc.

5.2 *The choice of the order of Chebyshev polynomial*

The wave amplifications (c_i) of round jet for M ranging from 40 to 100 are shown in Figure 5. It shows that the wave amplification of jet at $M = 80$ is almost the same trend as that of $M = 100$. So much it can be concluded that $M = 100$ is far away enough for the accuracy of wave amplifications. Hence, the order of Chebyshev polynomial is defined at $M = 100$ in the present study.

5.3 *The effect of map parameter L*

From the analysis in section 3, there is constraint in the choice of exponential map parameter L. In order to get better solution as $r \rightarrow \infty$, the larger L is needed. While for the sake of the computational cost, the smaller L is demanded. The effect of map parameter L on amplification as functions of k and Re is plotted in Figure 6 and 7. It shows that the largest amplification for various k and Re is taken place under the conditions that L is about 3. We adopted the map parameter values with $L = 3$ for the main calculation; this value represents the best compromise between the competing demands of the accuracy and the cost of computation.
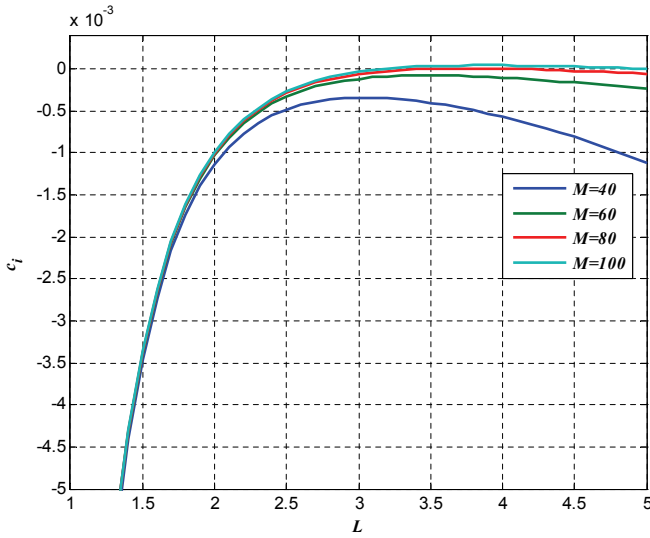
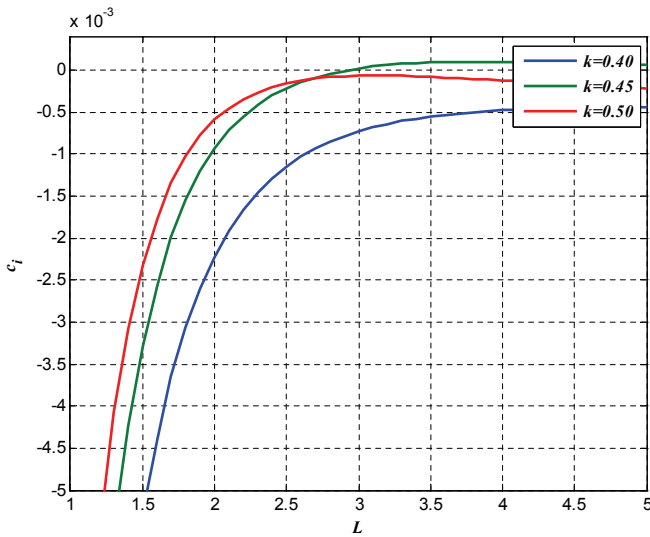Figure 5: The effect of the Chebyshev polynomial order

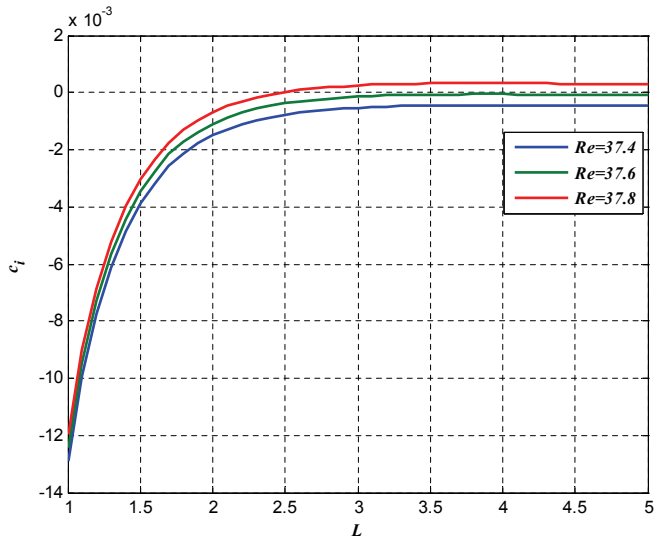Figure 6: The effect of L on the amplification factor with various k at $Re = 37.6$

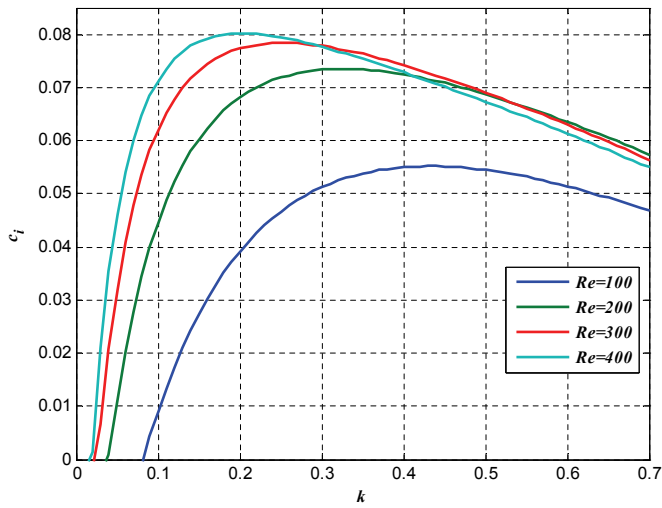Figure 7: The effect of L on amplification factor with various Re for fixed $k = 0.46$

Figure 8: Amplification factor as a function of k and Re

5.4 *Phase velocity and amplification under supercritical conditions*

The amplification factor and the phase velocity and as a function of wavenumber (k) and Reynolds number (Re) for $n=1$ mode are shown in Figure 8 and 9. The phase velocity gradually approaches the inviscid solution as Re Increases. For the $n=1$ mode the phase velocity increases monotonically with frequency. The amplification factor does not behave in such a regular manner and an unusual phenomenon occurs. With the increase in the Reynolds number, the peak value of amplification factor has also increased, but the corresponding wavenumber decreases.

5.5 *The critical Reynolds number*

To obtain the critical Reynolds number, the amplification factor for some values of Re close to the critical Reynolds number is plot in Figure 10. The critical Reynolds number is the point where the curve $c_i(k)$ becomes tangent to the $c_i=0$ line. And in k-Re plane the neutral curve ($c_i=0$ line) separates the space into two zones: one is stable and the other is unstable, which is shown in Figure 11 and 12. From the graph the critical Reynolds number is found to be $Re=37.64$, and the corresponding wave number is $k=0.469$ for $n=1$ mode, under the conditions, the amplification factor is $c_i=$ -1.981736025416046e-006, and the phase velocity is $c_r=-0.253$, the distributions of corresponding perturbation velocities are shown in Figure 13. The present result is also compared with some of the other values reported by previous researchers in Table 1.

Table 1: Comparison of critical Reynolds number for $n=1$ mode

Reference	Re	k	λ_r
Morris (1976)	37.64	0.44	0.1
Lessen & Singh (1973)	37.9	0.3989	0.08
Salgado & Sandham (2007)	37.8	0.417	0.09
Kulkarni & Agarwal (2007)	37.68	0.4505	0.104
Xie & Lin (2008b)	37.6829	0.459	0.103
present	37.64	0.469	0.119

6 Conclusion

The incompressible linear stability equation of round jet in cylindrical polar co-ordinates with Petrov-Galerkin spectral method is presented. To construct a basis function set for unbounded domains, it is necessary to assume the asymptotic behavior of the approximated functions for large r. For linear stability of round jet
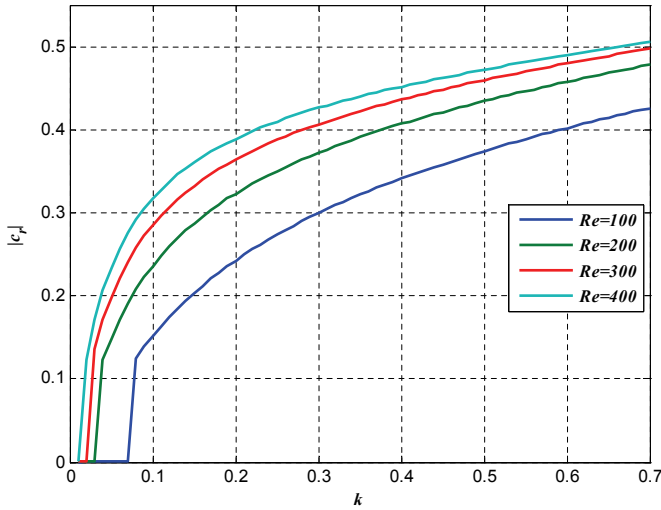
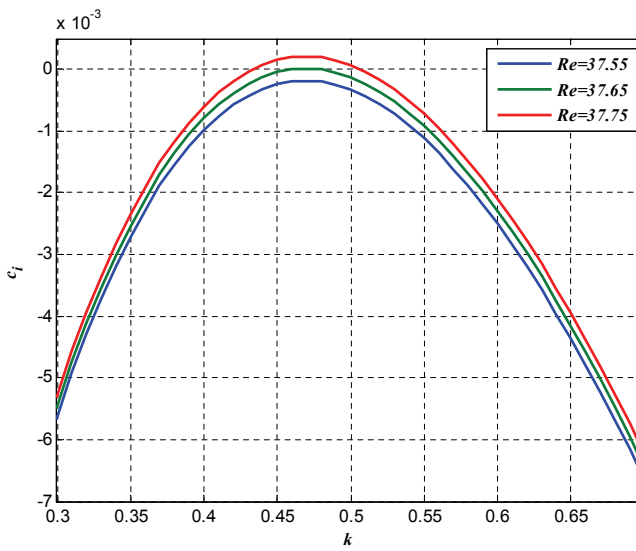Figure 9: Phase velocity as a function of *k* and *Re*

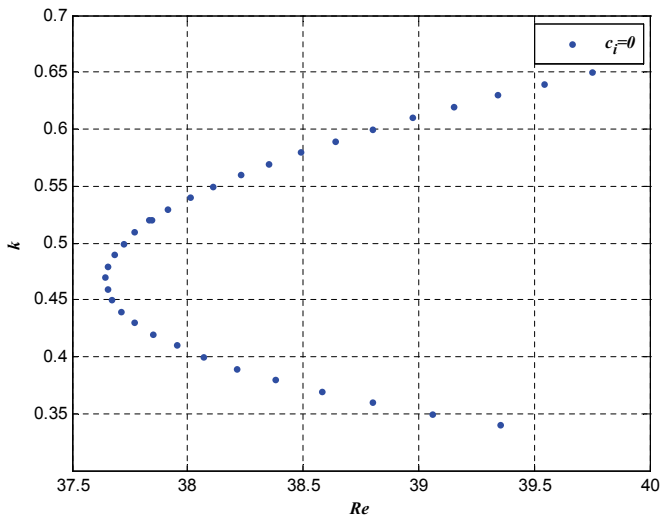Figure 10: Amplification factor as a function of *Re* and *k*

Figure 11: The neutral curve in *k-Re* plane based on numerically computed critical c_i
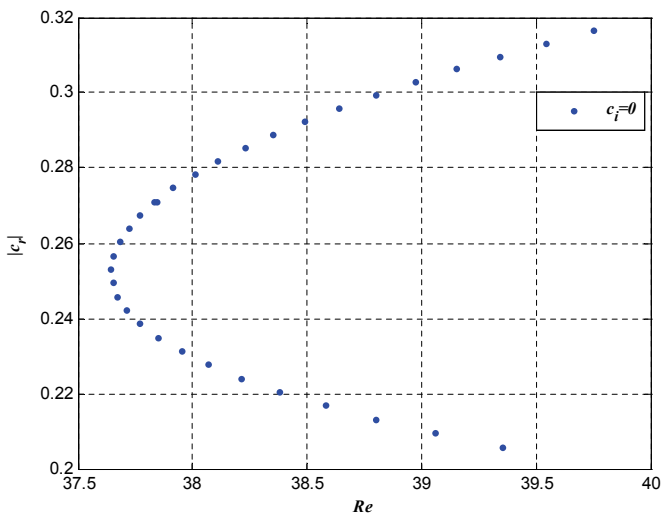

Figure 12: The neutral curve in c_r-*Re* plane based on numerically computed critical c_i
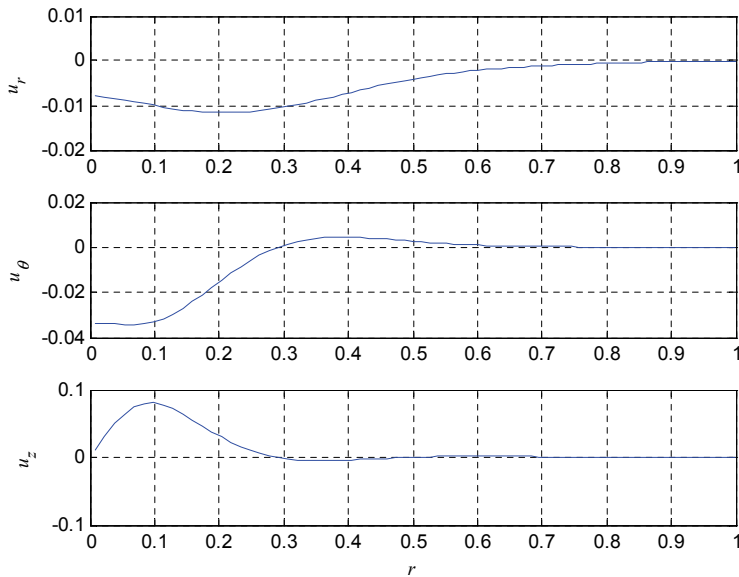
Figure 13: The distributions of the perturbation velocities under the critical conditions ($Re = 37.64$, $k = 0.469$)

the exponential mappings is favorable. Problems posed in polar coordinates can be solved efficiently by spectral methods. To weaken the coordinate singularity at $r=0$, one approach is to take $x \in [-1, 1]$ instead of $x \in [0, 1]$. The numerical simulation was performed by a Matlab code. The critical Reynolds number is also computed and shown to be in good agreement with those reported in the literature.

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