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Probabilistic Interval Response and Reliability Analysis of Structures with A Mixture of Random and Interval Properties

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Abstract: Static response and reliability of structures with a mixture of random and interval parameters under uncertain loads are investigated in this paper. Structural stiffness matrix is a random interval matrix when some structural parameters are modeled as random variables and others are considered as intervals. The structural displacement and stress responses are also random interval variables. From the static finite element governing equations, the random interval structural responses are obtained using the random interval perturbation method based on the first- and second-order perturbations. The expressions for mean value and standard deviation of random interval structural displacement and stress responses are developed by the random interval moment method. The structural reliability is not a deterministic value but an interval as the structural responses are random interval variables. The lower and upper bounds of reliability index, probability of failure and reliability of structural elements and systems are investigated using the combination of the first-order reliability method and interval approach. Four examples are used to demonstrate the validity and feasibility of the presented methods.

Keywords: random interval response; interval reliability; probabilistic interval analysis; random interval moment method; random interval perturbation method

1 Introduction

Uncertainties exist in the analysis and design of many systems. Uncertain analysis of many types of systems and structures have been addressed [Gao, Chen and Ma (2003); Ma et al. (2006); Ma, Chen and Gao (2006); Jiang and Han (2007); Jiang, Han and Liu (2007); Moens, De Munck and Vandepitte (2007); Gao, Zhang and Dai (2008); Loeven and Bijl (2008); Panda and Manohar (2008); Tian and Yang (2008)]. The properties of a real civil engineering structure are also usually

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different from those specified in design. Over the lifetime of a structure, the damaging effects associated with attacks from aggressive environmental agents such as a progressive deterioration of concrete and corrosion of steel usually lead to significant variations of system parameters. As very powerful tools, probabilistic methods have been widely used to predict the responses of structural systems with uncertainty [Elishakoff, Ren and Shinozuka (1995); Zhang et al. (1996); Beck and Melchers (2004); Val and Stewart (2005); Gao (2006); Gao and Kessissoglou (2007); Stewart and Suo (2009)]. In probabilistic methods, uncertain parameters are modeled as random variables/fields and uncertainties of loads are described by random processes/variables. These methods can provide not only the mean value but also the standard deviation and even the probability density for structural responses. Monte-Carlo simulation method [Hurtado and Barbat (1998); Radhika, Panda and Manohar (2008); Spanos and Kontsos (2008); Figiel and Kaminski (2009)], perturbation based stochastic finite element method [Elishakoff, Ren and Shinozuka (1995); Zhang et al. (1996); Kaminski and Kleiber (2000); Kaminski (2001); Kaminski (2007); Hua et al. (2008)], spectral stochastic finite element method [Verhoosel, Gutierrez and Hulshoff (2006); Chen and Soares (2008); Galal (2008); Ngah and Young (2007); Nouy (2008)] and other types of stochastic methods [Li and Chen (2004); Manjuprasad and Manohar (2007); Chen and Li (2007, 2009)] have been developed to analyse random structures. However, the probabilistic methods are only applicable when information about an uncertain parameter in the form of a preference probability function is available. The interval methods can be used when the probability function is unavailable but the range of the uncertain parameter is known. The response quantities of interest will also be intervals. In the past decade, significant progresses in interval analyses of structures with bounded parameters have been achieved. Interval static response [Qiu and Elishakoff (1998); Chen and Yang (2000); Chen, Lian and Yang (2002); Gao (2007a)], natural frequencies/eigenvalues [Chen, Lian and Yang (2003); Chen, Guo and Chen (2004); Qiu, Wang and Friswell (2005); Gao (2007b)], dynamic response [Zhang, Ding and Chen (2007); Qiu, Ma and Wang (2009); Wang and Qiu (2009)] and optimization [Maiumder and Rao (2009a, 2009b)] of structures with interval parameters have been investigated.

Structural reliability analysis plays an important role in the analysis and design of structures because the structural designer must verify, within a prescribed safety level, the serviceability and ultimate conditions. The evaluation of the failure probability taking into account the uncertainties in structural parameters and excitations is a basic problem in structural reliability analyses. The first-order reliability method (FORM) is considered to be one of the most reliable computational methods. Over the past three decades, numerous studies have contributed to the development of reliability methods based on FORM [Zhao and Ono (1999); Maincon (2000); Yang, Gang and Cheng (2006); Puatatsananon and Saouma (2006); Low and Tang (2007)]. Consequently, FORM becomes a basic method for analysis of structural reliability. Second-order reliability theory [Zhao and Ono (1999); Zhao, Ono and Kato (2002)], higher order moment method [Zhao and Lu (2007)] and response surface method [Bucher and Bourgund (1990); Guan and Melchers (2001); Gomes and Awruch (2004); Kaymaz and McMahon (2005); Gavin and Yau (2008)], Monte-Carlo simulation method [Papadrakakis, Papadopoulos and Lagaros (1996); Melchers and Ahammed (2004); Puatatsananon and Saouma (2006)] and other methods have been also used for reliability analysis. In most structural reliability analysis, uncertainties of structural parameters and loads are represented by probabilistic information. Static and dynamic responses of structures are random variables, and the structural random vibration responses are random processes. Thus, the failure probability and reliability index are deterministic values. However, structural reliability becomes an interval number having the lower and upper bounds if both random variables and interval variables are included in the structural system or the mean values and standard deviations of structural responses are intervals. Recently, a few of researchers have conducted research on probabilistic interval reliability analysis. Guo and Du (2009) investigated the reliability sensitivity analysis of a system with both random and interval variables. Qiu, Yang and Elishakoff (2008) studied the interval reliability of structural systems assuming the numerical characteristics of static stress and resistance are interval variables.

In a structural system, some structural parameters/loads can be modeled as random variables, but some of them are best considered as interval variables. In this paper, the random interval moment method and random interval perturbation method are presented to predicate the static displacement and stress responses of structures with a mixture of random and interval parameters subjected to random/interval loads. The structural probabilistic interval reliability is investigated using the combination of the FORM and interval analysis.

2 Random interval arithmetic

Let $X(R)$ be the set of all real random variables on a probability space (Ω, A, P), x^R is a random variable of X(R). R denotes the set of all real numbers. μ_x (or $\bar{x}$) and σ_x are the mean (deterministic) value and standard deviation of x^R, respectively. $y^I = [\underline{y}, \bar{y}] = \left\{t,\ \underline{y} \le t \le \bar{y} \middle| \underline{y}, \bar{y} \in R\right\}$ is an interval variable of $I(R)$ which denotes the set of all the closed real intervals. $\underline{y}$ and $\bar{y}$ are the lower and upper bounds of interval variable y^I, respectively. Interval variable y^Ican also be written as

$$y^I = y^c + \Delta y^I; \quad \Delta y^I = [-\Delta y, +\Delta y]; \quad y^c = \frac{\underline{y} + \bar{y}}{2}; \quad \Delta y = \frac{\bar{y} - \underline{y}}{2} \tag{1}$$

where y^c, Δy and Δy^I are the midpoint (deterministic) value, maximum width and uncertain interval of y^I, respectively.

A real number α is equivalent to an interval $[\alpha, \alpha]$. Such an interval is said to be degenerate [Hansen (1992)]. If we apply this concept to a random variable, x^R can be expressed as an interval $\left[x^R, x^R\right]$, which is a degenerate random interval. The real number α can also be considered as a degenerate random variable, its mean value equals to α and its variance and standard deviation are equal to zero. Therefore, interval variable $y^I = [\underline{y}, \bar{y}]$ can also be considered as a degenerate random interval. Let$+,-,*$and $\div$denote the basic operations of addition, subtraction, multiplication and division, respectively. By extending known interval arithmetic manipulations [Hansen (1992)], we define the corresponding operations between the random variable $x^R = \left[x^R, x^R\right]$ and interval variable $y^I = [\underline{y}, \bar{y}]$ as

$$x^R + y^I = y^I + x^R = [x^R + \underline{y}, x^R + \bar{y}] \tag{2}$$

$$x^R - y^I = [x^R - \bar{y}, x^R - \underline{y}] \tag{3}$$

$$y^I - x^R = [\underline{y} - x^R, \bar{y} - x^R] \tag{4}$$

$$x^R \cdot y^I = y^I \cdot x^R = [x^R \cdot \underline{y}, x^R \cdot \bar{y}] \tag{5}$$

$$x^R \div y^I = x^R \cdot \frac{1}{y^I} = x^R \cdot \left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right] = \left[\frac{x^R}{\bar{y}}, \frac{x^R}{\underline{y}}\right] \quad \left(0 \notin y^I,\ \frac{1}{\bar{y}} \le \frac{1}{\underline{y}}\right) \tag{6}$$

$$y^I \div x^R = [\underline{y}, \bar{y}] \cdot \frac{1}{x^R} = \left[\frac{\underline{y}}{x^R}, \frac{\bar{y}}{x^R}\right] \quad (0 \notin x^R) \tag{7}$$

If $0 \in y^I$, the lower and upper bounds of y^I can be restricted to finite values such as $\underline{y} \le 0 \le \bar{y}$ and $\underline{y} < \bar{y}$. Following to the interval arithmetic rules given in reference [Hansen (1992)], the corresponding expressions for $x^R \div y^I$can also be easily developed. In Eq.(6), $\frac{1}{\bar{y}} \le \frac{1}{\underline{y}}$ is assumed and the expressions can be easily derived for other cases.

If both of x^R and y^I are random variables or interval variables, the random interval arithmetic given above is also applicable. Eqs. (2) to (7) then become simple operations for random variables and interval arithmetic for interval variables. Let $\circ$ denotes any one of the basic operations, that is, $\circ \in \{+, -, *, \div\}$. We consider $z^{RI} = x^R \circ y^I$or $z^{RI} = y^I \circ x^R$ as a random interval variable because the uncertainty of z^{RI} consists of probabilistic and interval information introduced by the random variable x^R and interval variable y^I simultaneously. A method for calculating the mean value and variance of a random interval variable is presented in section 3 as the numerical characteristics of random variables are useful in most of engineering applications.

3 Random interval moment method

Here, we propose an approach called "random interval moment method" to calculate the mean value and variance of a general random interval variable. Without loss of generality, random interval variable Z^{RI} is the function of multiple random and interval variables, which are respectively represented by random vector $\vec{X}^R = (x_1^R, x_2^R, \cdots, x_n^R)$ and interval vector $\vec{Y}^I = (y_1^I, y_2^I, \cdots, y_m^I)$. The deterministic values of $\vec{X}^R$ and $\vec{Y}^I$ are $\bar{\vec{X}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ and $\vec{Y}^c = (y_1^c, y_2^c, \cdots, y_m^c)$, respectively. The Taylor series to the first-order of the random interval variable $Z^{RI} = f(\vec{X}^R, \vec{Y}^I)$ about $(\bar{\vec{X}}, \vec{Y}^c)$ is expressed as

$$\begin{aligned} Z^{RI1} =& f(\vec{X}^R, \vec{Y}^I) = f(\bar{\vec{X}}, \vec{Y}^I) + \sum_{i=1}^{n} \left\{ \left. \frac{\partial f}{\partial x_i^R} \right|_{\bar{\vec{X}}, \vec{Y}^I} \right\} \cdot (x_i^R - \bar{x}_i) + R \\ =& f(\bar{\vec{X}}, \vec{Y}^c) + \sum_{j=1}^{m} \left\{ \left. \frac{\partial f}{\partial y_j^I} \right|_{\bar{\vec{X}}, \vec{Y}^c} \right\} \cdot \Delta y_j^I \\ &+ \sum_{i=1}^{n} \left\{ \left. \frac{\partial f}{\partial x_i^R} \right|_{\bar{\vec{X}}, \vec{Y}^c} + \sum_{j=1}^{m} \left\{ \left. \frac{\partial^2 f}{\partial x_i^R \partial y_j^I} \right|_{\bar{\vec{X}}, \vec{Y}^c} \right\} \cdot \Delta y_j^I \right\} \cdot (x_i^R - \bar{x}_i) + R \end{aligned} \tag{8}$$

where R is the remainder term.

From Eq.(8), the mean value and variance of random interval variable Z^{RI1} based on the first-order Taylor expansion can be obtained as

$$\mu_{Z^{RI1}} = E(Z^{RI1}) = f(\bar{\vec{X}}, \vec{Y}^c) + \sum_{j=1}^{m} \left\{ \left. \frac{\partial f}{\partial y_j^I} \right|_{\bar{\vec{X}}, \vec{Y}^c} \right\} \Delta y_j^I \tag{9}$$

$$\begin{aligned} \sigma^2_{Z^{RI1}} =& E\left(Z^{RI1} - E(Z^{RI1})\right)^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} \left\{ \left. \frac{\partial f}{\partial x_i^R} \right|_{\bar{\vec{X}}, \vec{Y}^c} + \sum_{j=1}^{m} \left\{ \left. \frac{\partial^2 f}{\partial x_i^R \partial y_j^I} \right|_{\bar{\vec{X}}, \vec{Y}^c} \right\} \Delta y_j^I \right\} \\ & \left\{ \left. \frac{\partial f}{\partial x_k^R} \right|_{\bar{\vec{X}}, \vec{Y}^c} + \sum_{j=1}^{m} \left\{ \left. \frac{\partial^2 f}{\partial x_k^R \partial y_j^I} \right|_{\bar{\vec{X}}, \vec{Y}^c} \right\} \Delta y_j^I \right\} Cov(x_i^R, x_k^R) \end{aligned} \tag{10}$$

where $\mu_{(\bullet)}$ and $\sigma^2_{(\bullet)}$ are the mean value and variance of variable $(\bullet)$, respectively. $E(\bullet)$ is the expectation operator.

The Taylor series to the second-order of the random interval variable Z^{RI} about $(\bar{\vec{X}}, \vec{Y}^c)$ and its mean value and variance are given in Appendix A.

4 Random interval response using random interval perturbation method

The finite element equilibrium equations of a structural system in displacement format is

$$[K]\{U\} = \{f\} \tag{11}$$

where $[K]$ is the global stiffness matrix, $\{U\}$ is the unknown displacement vector and $\{f\}$ is the load vector.

Let random vector $\vec{a}^R = (a_1^R, a_2^R, \cdots, a_n^R)$ represent all random variables of the structural system whereas $\vec{b}^I = (b_1^I, b_2^I, \cdots, b_m^I)$ represent all interval variables. The structural stiffness matrix $[K]$ and load vector $\{f\}$ are functions of $\vec{a}^R$ and $\vec{b}^I$. Obviously structural displacement vector $\{U\}$ is also functions of $\vec{a}^R$ and $\vec{b}^I$. Thus, the static equilibrium Eq.(11) can be written as

$$\left[K(\vec{a}^R,\vec{b}^I)\right]\left\{U(\vec{a}^R,\vec{b}^I)\right\} = \left\{f(\vec{a}^R,\vec{b}^I)\right\} \tag{12}$$

Using the Taylor expansion, the structural stiffness matrix and load vector can be expressed as

$$\begin{aligned}\left[K(\vec{a}^R,\vec{b}^I)\right] &= \left[K(\bar{\vec{a}},\vec{b}^c)\right] + \sum_{j=1}^{m} \frac{\partial \left[K(\bar{\vec{a}},\vec{b}^c)\right]}{\partial b_j^I}\Delta b_j^I \\ &+ \sum_{i=1}^{n}\left\{\frac{\partial \left[K(\bar{\vec{a}},\vec{b}^c)\right]}{\partial a_i^R} + \sum_{j=1}^{m}\frac{\partial^2 \left[K(\bar{\vec{a}},\vec{b}^c)\right]}{\partial a_i^R \partial b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\left\{\frac{\partial^2 \left[K(\bar{\vec{a}},\vec{b}^c)\right]}{\partial a_i^R \partial a_l^R} + \sum_{j=1}^{m}\frac{\partial^3 \left[K(\bar{\vec{a}},\vec{b}^c)\right]}{\partial a_i^R \partial a_l^R \partial b_j^I}\Delta b_j^I\right\} \\ &(a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l) + R\end{aligned} \tag{13}$$

$$\begin{aligned}\left\{f(\vec{a}^R,\vec{b}^I)\right\} &= \left\{f(\bar{\vec{a}},\vec{b}^c)\right\} + \sum_{j=1}^{m} \frac{\partial \left\{f(\bar{\vec{a}},\vec{b}^c)\right\}}{\partial b_j^I}\Delta b_j^I \\ &+ \sum_{i=1}^{n}\left\{\frac{\partial \left\{f(\bar{\vec{a}},\vec{b}^c)\right\}}{\partial a_i^R} + \sum_{j=1}^{m}\frac{\partial^2 \left\{f(\bar{\vec{a}},\vec{b}^c)\right\}}{\partial a_i^R \partial b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\left\{\frac{\partial^2 \left\{f(\bar{\vec{a}},\vec{b}^c)\right\}}{\partial a_i^R \partial a_l^R} + \sum_{j=1}^{m}\frac{\partial^3 \left\{f(\bar{\vec{a}},\vec{b}^c)\right\}}{\partial a_i^R \partial a_l^R \partial b_j^I}\Delta b_j^I\right\} \\ &(a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l) + R\end{aligned} \tag{14}$$

where $\vec{\bar{a}} = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n)$ and $\vec{b}^c = (b_1^c, b_2^c, \cdots, b_m^c)$.
Neglecting the remainder term, the random interval matrix can be rewritten as

$$\left[K(\vec{a}^R, \vec{b}^I)\right] = \left[K(\vec{\bar{a}}, \vec{b}^c)\right] + \Delta_1 \left[K(\vec{a}^R, \vec{b}^I)\right] + \Delta_2 \left[K(\vec{a}^R, \vec{b}^I)\right] = K_d + \Delta_1 K + \Delta_2 K \tag{15}$$

where

$$\begin{aligned}\Delta_1 K &= \sum_{j=1}^{m} \frac{\partial \left[K(\vec{\bar{a}}, \vec{b}^c)\right]}{\partial b_j^I} \Delta b_j^I + \sum_{i=1}^{n} \left\{ \frac{\partial \left[K(\vec{\bar{a}}, \vec{b}^c)\right]}{\partial a_i^R} + \sum_{j=1}^{m} \frac{\partial^2 \left[K(\vec{\bar{a}}, \vec{b}^c)\right]}{\partial a_i^R \partial b_j^I} \Delta b_j^I \right\} \left(a_i^R - \bar{a}_i\right) \\ &= \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I + \sum_{i=1}^{n} \left\{ K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I} \Delta b_j^I \right\} \left(a_i^R - \bar{a}_i\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\Delta_2 K &= \frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} \left\{ \frac{\partial^2 \left[K(\vec{\bar{a}}, \vec{b}^c)\right]}{\partial a_i^R \partial a_l^R} + \sum_{j=1}^{m} \frac{\partial^3 \left[K(\vec{\bar{a}}, \vec{b}^c)\right]}{\partial a_i^R \partial a_l^R \partial b_j^I} \Delta b_j^I \right\} (a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} \left\{ K''_{a_i^R a_l^R} + \sum_{j=1}^{m} K'''_{a_i^R a_l^R b_j^I} \Delta b_j^I \right\} (a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l)\end{aligned} \tag{17}$$

Similarly the load vector can also be expressed as

$$\left\{ f(\vec{a}^R, \vec{b}^I) \right\} = \left\{ f(\vec{\bar{a}}, \vec{b}^c) \right\} + \Delta_1 \left\{ f(\vec{a}^R, \vec{b}^I) \right\} + \Delta_2 \left\{ f(\vec{a}^R, \vec{b}^I) \right\} = f_d + \Delta_1 f + \Delta_2 f \tag{18}$$

where

$$\begin{aligned}\Delta_1 \left\{ f(\vec{a}^R, \vec{b}^I) \right\} &= \sum_{j=1}^{m} \frac{\partial \left\{ f(\vec{\bar{a}}, \vec{b}^c) \right\}}{\partial b_j^I} \Delta b_j^I \\ &+ \sum_{i=1}^{n} \left\{ \frac{\partial \left\{ f(\vec{\bar{a}}, \vec{b}^c) \right\}}{\partial a_i^R} + \sum_{j=1}^{m} \frac{\partial^2 \left\{ f(\vec{\bar{a}}, \vec{b}^c) \right\}}{\partial a_i^R \partial b_j^I} \Delta b_j^I \right\} \left(a_i^R - \bar{a}_i\right) \\ &= \sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I + \sum_{i=1}^{n} \left\{ f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I} \Delta b_j^I \right\} \left(a_i^R - \bar{a}_i\right)\end{aligned} \tag{19}$$

$$\Delta_2 f = \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\left\{\frac{\partial^2\left\{f(\vec{\bar{a}},\vec{b}^c)\right\}}{\partial a_i^R \partial a_l^R} + \sum_{j=1}^{m}\frac{\partial^3\left\{f(\vec{\bar{a}},\vec{b}^c)\right\}}{\partial a_i^R \partial a_l^R \partial b_j^I}\Delta b_j^I\right\}(a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l)$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\left\{f''_{a_i^R a_l^R} + \sum_{j=1}^{m} f'''_{a_i^R a_l^R b_j^I}\Delta b_j^I\right\}(a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l) \quad (20)$$

Using the perturbation theory, we can get the following governing equation for static displacement response of the structure

$$(K_d + \Delta_1 K + \Delta_2 K)(U_d + \Delta_1 U + \Delta_2 U) = f_d + \Delta_1 f + \Delta_2 f \quad (21)$$

where

$$U_d = K_d^{-1} f_d \quad (22)$$

$$\begin{aligned}\Delta_1 U &= K_d^{-1}(\Delta_1 f - \Delta_1 K U_d)\\ &= K_d^{-1}\left(\Delta_1 f - \Delta_1 K K_d^{-1} f_d\right)\end{aligned} \quad (23)$$

$$\begin{aligned}\Delta_2 U &= K_d^{-1}(\Delta_2 f - \Delta_1 K \Delta_1 U - \Delta_2 K U_d)\\ &= K_d^{-1}\left(\Delta_2 f - \Delta_1 K K_d^{-1}\left(\Delta_1 f - \Delta_1 K K_d^{-1} f_d\right) - \Delta_2 K K_d^{-1} f_d\right)\end{aligned} \quad (24)$$

The random interval structural displacement based on the first-order perturbation can be obtained as

$$U^{RI1} = U_d + \Delta_1 U \quad (25)$$

Substituting Eqs.(16), (19), (22) and (23) into Eq.(25) yields

$$\begin{aligned}U^{RI1} &= K_d^{-1} f_d + K_d^{-1}\left\{\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I + \sum_{i=1}^{n}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right)\right.\\ &\left. - \left\{\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I + \sum_{i=1}^{n}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right)\right\} K_d^{-1} f_d\right\}\\ &= K_d^{-1} f_d + K_d^{-1}\left(\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right)\\ &+ \sum_{i=1}^{n}\left\{K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} - K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d\right\}\left(a_i^R - \bar{a}_i\right)\end{aligned} \quad (26)$$

Using the random interval moment method, the mean value and variance of the random interval structural displacements based on the first-order perturbation can be obtained as

$$\mu_{U^{RI1}} = K_d^{-1} f_d + K_d^{-1} \left(\sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \tag{27}$$

$$\begin{aligned}\sigma^2_{U^{RI1}} = &\sum_{i=1}^{n} \left\{ K_d^{-1} \left\{ f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I} \Delta b_j^I \right\} - K_d^{-1} \left\{ K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I} \Delta b_j^I \right\} K_d^{-1} f_d \right\}^2 \sigma^2_{a_i^R} \\ &+ \sum_{i(\neq k)=1}^{n} \sum_{k(\neq i)=1}^{n} \left\{ K_d^{-1} \left\{ f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I} \Delta b_j^I \right\} - K_d^{-1} \left\{ K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I} \Delta b_j^I \right\} K_d^{-1} f_d \right\} \\ &\cdot \left\{ K_d^{-1} \left\{ f'_{a_k^R} + \sum_{j=1}^{m} f''_{a_k^R b_j^I} \Delta b_j^I \right\} - K_d^{-1} \left\{ K'_{a_k^R} + \sum_{j=1}^{m} K''_{a_k^R b_j^I} \Delta b_j^I \right\} K_d^{-1} f_d \right\} Cov(a_i^R, a_k^R)\end{aligned} \tag{28}$$

The random interval structural displacement based on the second-order perturbation U^{RI2} and its mean value and variance are given in Appendix B.

For analysis of structures with uncertainty, the first and second-order moments (mean value, variance and covariance) of the random system parameters are much more important than higher-order statistics. Furthermore, in most of engineering problems, only the first and second-order moments of responses are of interest. In this paper, only the mean value and variance (or standard deviation) of random interval structural responses are investigated.

The lower and upper bounds of the mean value and variance of the displacements can be computed by using optimization methods. For example, the lower and upper bounds of the mean value of the displacements based on the first-order perturbation can be expressed in the following optimization form

$$\underline{\mu_{U^{RI1}}} = \min \left\{ K_d^{-1} f_d + K_d^{-1} \left(\sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \right\} \tag{29}$$

$$\overline{\mu_{U^{RI1}}} = \max \left\{ K_d^{-1} f_d + K_d^{-1} \left(\sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \right\} \tag{30}$$

Using the relationship between the node displacement and element stress, the stress response of the *ith* element in the truss structure $\left\{ \sigma_i(\vec{a}^R, \vec{b}^I) \right\}$ can be expressed as

$$\left\{ \sigma_i(\vec{a}^R, \vec{b}^I) \right\} = \left[D(\vec{a}^R, \vec{b}^I) \right] \left\{ U_i(\vec{a}^R, \vec{b}^I) \right\} \tag{31}$$

where $\left\{U_i(\vec{a}^R,\vec{b}^I)\right\}$ is the displacement of the nodal points of the *ith* element and $\left[D(\vec{a}^R,\vec{b}^I)\right]$ is the elastic matrix.

The mean value and variance of the random interval stress response can be obtained using the random interval moment method after the random interval displacements are obtained. To demonstrate how to calculate the numerical characteristic of the random interval stress response, the mathematical expressions for statistical date of structural static stress are developed using the firs-order perturbation displacements only. The Taylor series of the random interval matrix $\left[D(\vec{a}^R,\vec{b}^I)\right]$ can be expressed as

$$\begin{aligned}\left[D(\vec{a}^R,\vec{b}^I)\right] &= \left[D(\vec{\bar{a}}^R,\vec{b}^c)\right] + \sum_{j=1}^{m} \frac{\partial \left[D(\vec{\bar{a}},\vec{b}^c)\right]}{\partial b_j^I}\Delta b_j^I \\ &+ \sum_{i=1}^{n}\left\{\frac{\partial \left[D(\vec{\bar{a}},\vec{b}^c)\right]}{\partial a_i^R} + \sum_{j=1}^{m}\frac{\partial^2 \left[D(\vec{\bar{a}},\vec{b}^c)\right]}{\partial a_i^R \partial b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right) + R \\ &= D_d + \sum_{j=1}^{m} D'_{b_j^I}\Delta b_j^I + \sum_{i=1}^{n}\left\{D'_{a_i^R} + \sum_{j=1}^{m} D''_{a_i^R b_j^I}\Delta b_j^I\right\}\left(a_i^R - \bar{a}_i\right) + R \end{aligned} \quad (32)$$

Substituting Eqs.(27) and (32) into Eq.(31) and neglecting the higher order terms, we have

$$\begin{aligned}\sigma^{RI1} &= \left(D_d + \sum_{j=1}^{m} D'_{b_j^I}\Delta b_j^I\right) K_d^{-1}\left(f_d + \sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right) \\ &+ \sum_{i=1}^{n}\left\{\left(D_d + \sum_{j=1}^{m} D'_{b_j^I}\Delta b_j^I\right)\right. \\ &\cdot\left\{K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} - K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\}K_d^{-1} f_d\right\} \\ &+ \left.\left\{D'_{a_i^R} + \sum_{j=1}^{m} D''_{a_i^R b_j^I}\Delta b_j^I\right\}K_d^{-1}\left(f_d + \sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right)\right\}\left(a_i^R - \bar{a}_i\right)\end{aligned} \quad (33)$$

The mean value and variance of σ^{RI1} are obtained as

$$\mu_{\sigma^{RI1}} = \left(D_d + \sum_{j=1}^{m} D'_{b_j^I}\Delta b_j^I\right) K_d^{-1}\left(f_d + \sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right) \quad (34)$$

$$\sigma^2_{\sigma^{RI1}} = \sum_{i=1}^{n} C(a_i^R, \Delta\vec{b}^I)^2 \sigma^2_{a_i^R} + \sum_{i(\neq k)=1}^{n} \sum_{k(\neq i)=1}^{n} C(a_i^R, \Delta\vec{b}^I) C(a_k^R, \Delta\vec{b}^I) Cov(a_i^R, a_k^R) \tag{35}$$

Where

$$\begin{aligned} C(a_i^R, \Delta\vec{b}^I) = {} & \left(D_d + \sum_{j=1}^{m} D'_{b_j^I} \Delta b_j^I \right) \\ & \cdot \left\{ K_d^{-1} \left\{ f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I} \Delta b_j^I \right\} - K_d^{-1} \left\{ K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I} \Delta b_j^I \right\} K_d^{-1} f_d \right\} \\ & + \left\{ D'_{a_i^R} + \sum_{j=1}^{m} D''_{a_i^R b_j^I} \Delta b_j^I \right\} K_d^{-1} \left(f_d + \sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \end{aligned} \tag{36}$$

Furthermore, the lower and upper bounds of $\mu_{\sigma^{RI1}}$ are computed by

$$\underline{\mu_{\sigma^{RI1}}} = \min \left\{ \left(D_d + \sum_{j=1}^{m} D'_{b_j^I} \Delta b_j^I \right) K_d^{-1} \left(f_d + \sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \right\} \tag{37}$$

$$\overline{\mu_{\sigma^{RI1}}} = \max \left\{ \left(D_d + \sum_{j=1}^{m} D'_{b_j^I} \Delta b_j^I \right) K_d^{-1} \left(f_d + \sum_{j=1}^{m} f'_{b_j^I} \Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I} \Delta b_j^I K_d^{-1} f_d \right) \right\} \tag{38}$$

The lower and upper bounds of the standard deviation of the stress response are given by

$$\begin{aligned} \underline{\sigma_{\sigma^{RI1}}} = \min \Bigg\{ \Bigg\{ & \sum_{i=1}^{n} C(a_i^R, \Delta\vec{b}^I)^2 \sigma^2_{a_i^R} \\ & + \sum_{i(\neq k)=1}^{n} \sum_{k(\neq i)=1}^{n} C(a_i^R, \Delta\vec{b}^I) C(a_k^R, \Delta\vec{b}^I) Cov(a_i^R, a_k^R) \Bigg\}^{1/2} \Bigg\} \end{aligned} \tag{39}$$

$$\begin{aligned} \underline{\sigma_{\sigma^{RI1}}} = \max \Bigg\{ \Bigg\{ & \sum_{i=1}^{n} C(a_i^R, \Delta\vec{b}^I)^2 \sigma^2_{a_i^R} \\ & + \sum_{i(\neq k)=1}^{n} \sum_{k(\neq i)=1}^{n} C(a_i^R, \Delta\vec{b}^I) C(a_k^R, \Delta\vec{b}^I) Cov(a_i^R, a_k^R) \Bigg\}^{1/2} \Bigg\} \end{aligned} \tag{40}$$

5 Probabilistic interval reliability

Structural reliability analysis is to estimate the probability of exceeding the structural limit states imposed on structural components. The structural reliability problem is defined by the integral

$$P_f = \int_{g(X)\leq 0} f_X(X)dX \tag{41}$$

where P_f is the probability of failure, X is the vector of random variables, $f_X(X)$ is the joint probability density function, and $g(X)$ is the limit state function such that $g(X) \leq 0$ defines the failure domain.

The integration of Eq.(41) is highly complex since it involves multiple integrals in addition to the joint probability density functions of the random variables. There is rarely a closed-form expression to Eq.(41). The first-order reliability method is widely used to evaluate the integral in Eq.(41). This method can be divided into three steps to approximate the probability integral [Guo and Du (2009)]: (1) transforming orginal random variables X into standard normal random variables, (2) searching for the most probable point of failure, and (3) calculating the probability of failure.

5.1 Reliability analysis of elements/components

The limit state function of the *ith* element of a structure is defined as

$$g_i(W) = R_i - \sigma_i \tag{42}$$

where R_i and σ_i are the resistance (strength) and stress response of the *ith* element. Using the first-order reliability method, the element (component) reliability can be expressed as

$$P_{f_i} = \phi(-\beta_i) \tag{43}$$

where ϕ is the standard normal cumulative distribution function. β_i is the reliability index and can be calculated by

$$\beta_i = \frac{\mu_{R_i} - \mu_{\sigma_i}}{\sqrt{\sigma_{R_i}^2 + \sigma_{\sigma_i}^2}} \tag{44}$$

where μ_{R_i} and μ_{σ_i} are the mean values of R_i and σ_i, σ_{R_i} and σ_{σ_i} are standard deviations of R_i and σ_i, respectively.

The reliability index of the *ith* element β_i is an interval variable because the structural stress response σ_i is a random interval variable and μ_{σ_i} and σ_{σ_i} are interval variables, even if μ_{R_i} and σ_{R_i} and deterministic values. Let us assume that μ_{R_i} and σ_{R_i} are also interval variables, the lower and upper bounds of the interval reliability index β_i^I can be obtained as

$$\underline{\beta_i} = \frac{\underline{\mu_{R_i}} - \overline{\mu_{\sigma_i}}}{\sqrt{(\overline{\sigma_{R_i}})^2 + (\overline{\sigma_{\sigma_i}})^2}} \tag{45}$$

$$\overline{\beta_i} = \frac{\overline{\mu_{R_i}} - \underline{\mu_{\sigma_i}}}{\sqrt{(\underline{\sigma_{R_i}})^2 + (\underline{\sigma_{\sigma_i}})^2}} \tag{46}$$

The upper and lower bounds of the failure probability can be computed by

$$\overline{P_{f_i}} = \phi(-\underline{\beta_i}) = \phi(-\frac{\underline{\mu_{R_i}} - \overline{\mu_{\sigma_i}}}{\sqrt{(\overline{\sigma_{R_i}})^2 + (\overline{\sigma_{\sigma_i}})^2}}) = 1 - \phi(\frac{\underline{\mu_{R_i}} - \overline{\mu_{\sigma_i}}}{\sqrt{(\overline{\sigma_{R_i}})^2 + (\overline{\sigma_{\sigma_i}})^2}}) \tag{47}$$

$$\underline{P_{f_i}} = \phi(-\overline{\beta_i}) = \phi(-\frac{\overline{\mu_{R_i}} - \underline{\mu_{\sigma_i}}}{\sqrt{(\underline{\sigma_{R_i}})^2 + (\underline{\sigma_{\sigma_i}})^2}}) = 1 - \phi(\frac{\overline{\mu_{R_i}} - \underline{\mu_{\sigma_i}}}{\sqrt{(\underline{\sigma_{R_i}})^2 + (\underline{\sigma_{\sigma_i}})^2}}) \tag{48}$$

The midpoint and maximum width of the failure probability can be easily obtained as

$$P_{f_i}^c = \frac{\overline{P_{f_i}} + \underline{P_{f_i}}}{2}, \quad \Delta P_{f_i} = \frac{\overline{P_{f_i}} - \underline{P_{f_i}}}{2} \tag{49}$$

As $P_{r_i} = 1 - P_{f_i}$denote the reliability (probability of survival) of the *ith* element, the lower bound (worst possible value) and upper bound (best possible value) of the reliability can be expressed as

$$\underline{P_{r_i}} = 1 - \overline{P_{f_i}} = \phi(\underline{\beta_i}) = \phi(\frac{\underline{\mu_{R_i}} - \overline{\mu_{\sigma_i}}}{\sqrt{(\overline{\sigma_{R_i}})^2 + (\overline{\sigma_{\sigma_i}})^2}}) \tag{50}$$

$$\overline{P_{r_i}} = 1 - \underline{P_{f_i}} = \phi(\overline{\beta_i}) = \phi(\frac{\overline{\mu_{R_i}} - \underline{\mu_{\sigma_i}}}{\sqrt{(\underline{\sigma_{R_i}})^2 + (\underline{\sigma_{\sigma_i}})^2}}) \tag{51}$$

The midpoint and maximum width of the reliability are

$$P_{r_i}^c = \frac{\overline{P_{r_i}} + \underline{P_{r_i}}}{2}, \quad \Delta P_{r_i} = \frac{\overline{P_{r_i}} - \underline{P_{r_i}}}{2} \tag{52}$$

5.2 Reliability analysis of structures

Suppose that there are *ne* elements in the structure under consideration. If the structure is considered as a series system, the reliability of this structure is

$$P_r = 1 - P_f = \prod_{i=1}^{ne} (1 - P_{f_i}) = \prod_{i=1}^{ne} P_{r_i} \tag{53}$$

The lower and upper bound of the structural reliability (series system) can be expressed as

$$\underline{P_r} = \prod_{i=1}^{ne} \underline{P_{r_i}}, \quad \overline{P_r} = \prod_{i=1}^{ne} \overline{P_{r_i}} \tag{54}$$

If the structure is considered as a parallel system, the reliability of this structure is given by

$$P_r = 1 - P_f = 1 - \prod_{i=1}^{ne} (1 - P_{r_i}) \tag{55}$$

The lower and upper bound of the structural reliability (parallel system) can be expressed as

$$\underline{P_r} = 1 - \prod_{i=1}^{ne} (1 - \underline{P_{r_i}}), \quad \overline{P_r} = 1 - \prod_{i=1}^{ne} (1 - \overline{P_{r_i}}) \tag{56}$$

The midpoint and maximum width of the structural reliability are

$$P_r^c = \frac{\overline{P_r} + \underline{P_r}}{2}, \quad \Delta P_r = \frac{\overline{P_r} - \underline{P_r}}{2} \tag{57}$$

It should be noted that most engineering structures are hybrid parallel and series systems. Thus, all failure modes of a structure should be identified before calculating the system reliability.

6 Illustrative examples

6.1 Tension of a bar

Consider a simple example of a fixed-free bar with Young's modulus E, cross-sectional area A, length L and subjected to a tension force F at the free end. The static finite element equation is reduced as the following simple linear equation

$$[K]\{U\} = \frac{EA}{L} U = F \tag{58}$$

where U is the extension at the forced edge.

Let us now assume that cross-sectional area is an interval variable $A^I = [\underline{A}, \overline{A}] = A^c + \Delta A^I$. Young's modulus, length and applied force are considered as normal random variables and are independent of each other. The extension of the bar is now a random interval variable.

$$\frac{E^R A^I}{L^R} U^{RI} = F^R \tag{59}$$

The computational expressions for the mean value and variance of random interval extension U^{RI} based on the first- and second-order random interval perturbation method are given in Appendix C.

In this example, the values of interval and random variables are taken as $\mu_E = 7.0 \times 10^{10} N/m^2$, $\sigma_E = 1.4 \times 10^9 N/m^2$; $A^c = 5.0 \times 10^{-5} m^2$, $\Delta A = 1.0 \times 10^{-6} m^2$; $\mu_L = 1.5 \times 10^3 mm$, $\sigma_L = 30mm$; and $\mu_F = 3kN$, $\sigma_F = 60N$, respectively. The computational results of the mean value and variance of the random interval extension obtained by the fist-order random interval perturbation method (FRI) and second-order random interval perturbation method (SRI), and the combination of the algebra synthesis method and interval operations (ASM, see Appendices C and D) are given in Table 1. In order to investigate the accuracy of the random interval perturbation method, the relative error (RE) between the results (RE1=$\left|\frac{FRI-ASM}{ASM}\right|$ and RE2=$\left|\frac{SRI-ASM}{ASM}\right|$) are also given in this table.

In general, computational results obtained by the FRI and SRI are in good agreement with those computed by the ASM. The results calculated by the SRI are closer to those computed by ASM, which indicates that the accuracy of the second-order random interval perturbation method is higher. However, the second-order random interval perturbation method requires more computational work.

Table 1: Expectation and variance of bar's extension

	FRI	SRI	ASM	RE1	RE2
Lower bound of expectation (mm)	1.2599	1.2610	1.2610	0.087%	0.003%
Upper bound of expectation (mm)	1.3114	1.3124	1.3124	0.076%	0.005%
Lower bound of variance (10^{-4} mm^2)	19.3126	19.0861	19.0761	1.239%	0.005%
Upper bound of variance (10^{-4} mm^2)	20.3705	20.6592	20.6652	1.426%	0.002%

6.2 Static response of a frame

To demonstrate the efficiency of the approach presented in this paper for static analysis of complex structures, a planar frame shown in Figure 1 is used as an example. Suppose that there are no preload stresses in all elements. The deterministic values of structural parameters for all members are Young's modulus $E = 2.1 \times 10^{11} N/m^2$ and second moment of area $J = 8.0 \times 10^{-4} m^4$. A load acts on the node 18 along the negative Y-direction with the deterministic value $f = 5 \times 10^4 (N)$.

Consider that the Young's modulus are interval variables, second moment of areas and load are random variables. The values of structural parameters and load are taken as $E^I = [2.05,\ 2.16] \times 10^{11} N/m^2$, $\sigma_J = 3.2 \times 10^{-5} m^4$, $\sigma_F = 500N$. The computational results of the mean value and standard deviation of the random interval displacement of node 10 in Y-direction are given in Table 2. The mean value and standard deviation of the stress response of element 9 are given in Table 3. In the following, the first-order random interval perturbation method (FRI) is employed to calculate the structural responses. To investigate the accuracy of the FRI, results obtained by using a hybrid simulation method (HSM) are also given in Tables 2 and 3. The hybrid simulation method is the combination of direct simulations for an interval variable and Monte-Carlo simulations for random variables. In each of hybrid simulations, a value within the interval of the Young's modulus is selected. The mean and standard deviation of the structural responses can then be obtained by using 10000 Monte-Carlo simulations considering the randomness of the second moment of areas and load. Changing the value of the Young's modulus from its lower bound to upper bound with a very small increment ($0.0001 \times 10^{11} N/m^2$), large numbers of mean values and standard deviations of structural responses can be obtained. The lower and upper bounds of the mean value and standard deviation of structural responses are identified.

Table 2: Random interval displacement of node 10 in Y-direction

	FRI	HSM	Relative error
Lower bound of mean value (mm)	1.7429	1.7438	0.0516%
Upper bound of mean value (mm)	1.8278	1.8289	0.0601%
Lower bound of standard deviation ($\times 10^{-2}$mm)	7.1966	7.1902	0.0890%
Upper bound of standard deviation ($\times 10^{-2}$mm)	7.5264	7.5408	0.191%

Table 3: Random interval stress of element 9

	FRI	HSM	Relative error
Lower bound of mean value (MPa)	59.5270	59.5945	0.113%
Upper bound of mean value (MPa)	65.4737	65.5479	0.113%
Lower bound of standard deviation (MPa)	2.4616	2.4571	0.183%
Upper bound of standard deviation (MPa)	2.6924	2.7026	0.377%

From Tables 2 and 3, it can be observed that the results computed by the FRI are also in good agreement with the hybrid simulation results. The accuracy of the structural responses obtained by the FRI is acceptable although this method cannot give conservative results as the higher terms are neglected.

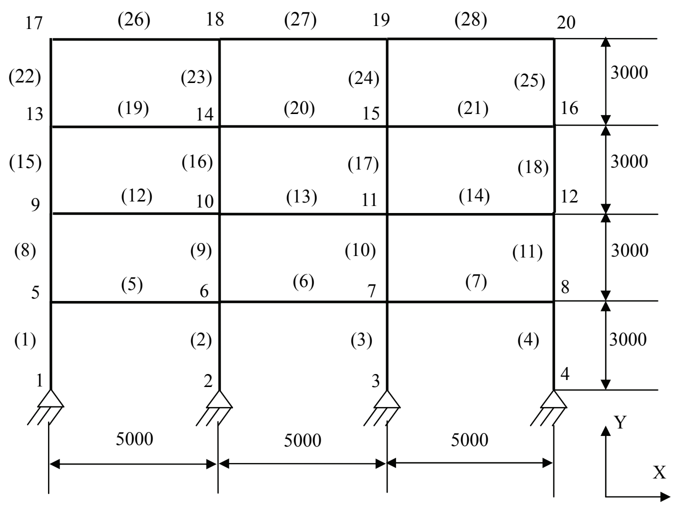


Figure 1: 28-beam structure (unit: mm).

Here, we introduce the coefficient of variation $v_{x^R}$for random variable x^R and the interval change ratio Δy_F for interval variable y^I as follows

$$v_{x^R} = \frac{\sigma_{x^R}}{\bar{x}}, \quad \Delta y_F = \frac{\Delta y}{y^c} \tag{60}$$

The dispersal degree of a random variable or an interval variable can be better reflected by the coefficient of variation or interval change ratio.

To investigate the differences between the effects of random and interval variables on structural responses, the values of coefficient of variation of random variables and interval change ratio of interval variables are varied from zero to 0.2. If *DD* denotes the dispersal degree of uncertain variables, then its value will vary from zero to 0.2. The lower and upper bounds on the mean value and standard deviation of the random interval displacement of node 14 in Y-direction are given in Figures 2 and 3, respectively.

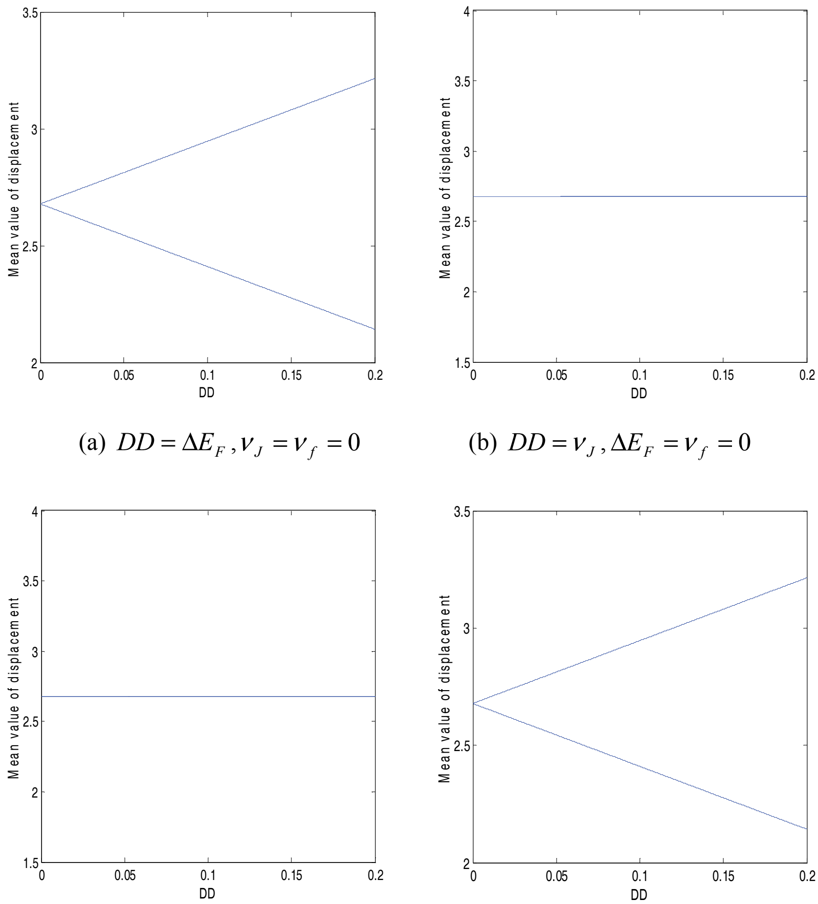
Figures 2(b) and (c) show that the random variables do not affect the mean value of the random interval structural response. In other words, the mean value of structural response is not an interval but a deterministic value if structural parameters and loads are random variables. However, the mean value of structural response is an interval if the structural has interval parameters or loads as shown in Figures 2(a) and (d). The interval width of structural response depends on the dispersal degree of the interval parameter.

Figure 3(a) shows that the standard deviation (or variance) of structural response is zero if the structure has only interval variables. Structural response is an interval variable (not a random interval variable) if all structural parameters and loads are interval variables. The standard deviation of structural response is also a deterministic value (number) not an interval variable if structural parameters and loads are random variables, and its values depend on the randomness of random structural parameters/loads as shown in Figs. 3(b) and 3(c). The standard deviation of structural response will be a random interval variable if the structural have a mixture of random and interval parameters/loads, and its interval width is dependent on both the dispersal degrees of random and interval variables as shown in Fig. 3(d).

6.3 Static response of a truss structure

In this example, the Young's modulus of all elements of the truss structure shown in Figure 4 and the load *P* are considered as a random variable and an interval variable, respectively. The cross-sectional area and length of all elements are deterministic values and$A = 8.0 \times 10^{-5} m^2$. The values of interval and random variables are taken as $\mu_E = 2.0 \times 10^{11} N/m^2$, $\sigma_E = 0.5 \times 10^{10} N/m^2$; and $P^c = 10kN$,$\Delta P = 300N$, respectively. The second-order random interval perturbation method (SRI) is employed to calculate the structural responses. The lower bound, midpoint, upper bound and interval change ratio of the mean value and standard deviation of the random interval displacement of node 12 in Y-direction are given in Table 4. The results obtained by using the hybrid simulation method (HSM) are also given in this Table.

Table 4 shows that the results calculated by the SRI are very close to the hybrid simulation results. The accuracy of the structural responses obtained by the SRI is quite good. In engineering applications, the SRI could be used to predict the structural random interval responses for important structures although this method



(a) $DD = \Delta E_F, \nu_J = \nu_f = 0$ (b) $DD = \nu_J, \Delta E_F = \nu_f = 0$

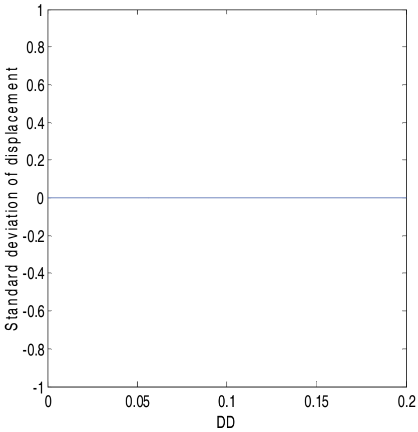
(c) $DD = \nu_f, \Delta E_F = \nu_J = 0$ (d) $DD = \Delta E_F = \nu_J = \nu_f$

Figure 2: Mean value of displacement of node 14 in Y-direction (unit: mm).

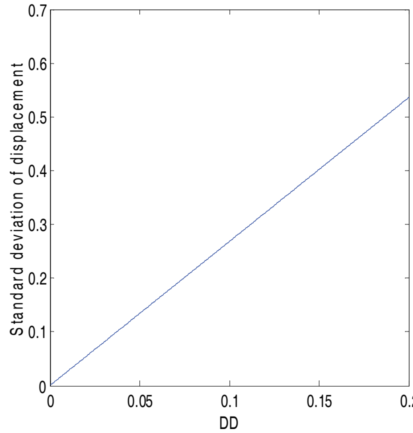
requires more computational effort.

6.4 Reliability of a cantilever truss structure

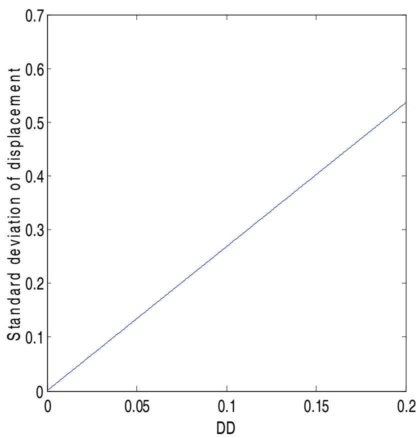
Consider a 14 bar 2D statically determinate truss structure as shown in Fig. 5. The Young's modulus and cross-sectional areas for all elements are same. Their deterministic values are $E = 2.1 \times 10^{11} N/m^2$ and $A = 6.0 \times 10^{-4} m^4$. A load acts on the node 9 along the negative Y-direction with the deterministic value $f =$



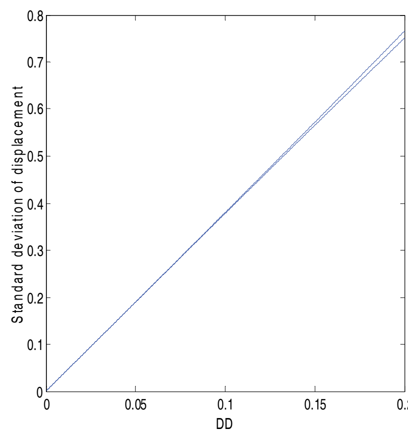
(a) $DD = \Delta A_F, \nu_E = \nu_f = 0$



(b) $DD = \nu_E, \Delta A_F = \nu_f = 0$



(c) $DD = \Delta A_F = \nu_E = \nu_f$



(d) $DD = \Delta A_F = \nu_E = \nu_f$

Figure 3: Standard deviation of displacement of node 14 in Y-direction (unit: mm).

$2.4 \times 10^4 (N)$. Young's modulus are interval variables. Cross-sectional areas and load are considered as random variables. The mean value and standard deviation of resistance (strength) R_i are given in Table 5. The first-order random interval perturbation method are employed to calculate the random interval structural displacement and stress responses. The coefficient of variation for random variables and interval factor for interval variables are all taken as 0.05 and 0.1, that

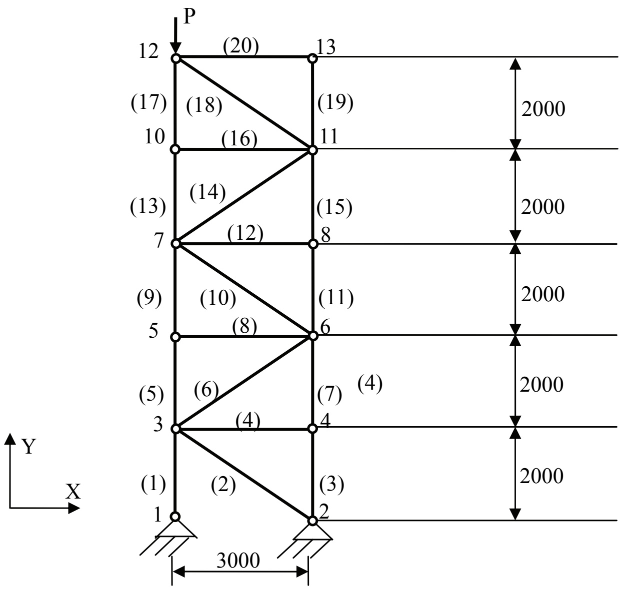


Figure 4: 20-bar planar structure (unit: mm)

Table 4: Random interval displacement of node 12 in Y-direction

	SRI	HSM	Relative error
Lower bound of mean value (mm)	6.2142	6.2141	0.0018%
Upper bound of mean value (mm)	6.5982	6.5984	0.0026%
Midpoint of mean value (mm)	6.4062	6.4062	3.1450e-6
Interval change ratio of mean value	0.0300	0.0300	0.0009%
Lower bound of standard deviation (mm)	0.1516	0.1516	0.0028%
Upper bound of standard deviation (mm)	0.1609	0.1609	0.0019%
Midpoint of standard deviation (mm)	0.1562	0.1563	3.8006e-6
Interval change ratio of standard deviation	0.0297	0.0298	0.0008%

is $DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f$=0.05 and $DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f$=0.1, respectively. The corresponding lower and upper bounds of the reliability index β_i, failure probability P_{f_i} and reliability P_{r_i} of all elements are given in Tables 6 and 7, respectively.

It can be seen that the lower bounds of reliability index, failure probability and reliability of an element in Table 6 are bigger than those of this element in Table 7, whereas the upper bounds are smaller. This denotes that the smaller dispersal degree of random and interval structural parameters, load and resistance produce smaller intervals of reliability index, failure probability and reliability of structural elements. The midpoint of the reliability of a structural element is not a deterministic value when a structural system has both random and interval variables.

Table 5: Values of mean and standard deviation of resistance (unit: MPa)

Element (i)	1	2,4,6,810-14	3,7	5,9
$\mu_{R_i}^c$	330	100	260	180
$\sigma_{R_i}^c$	50	20	40	30

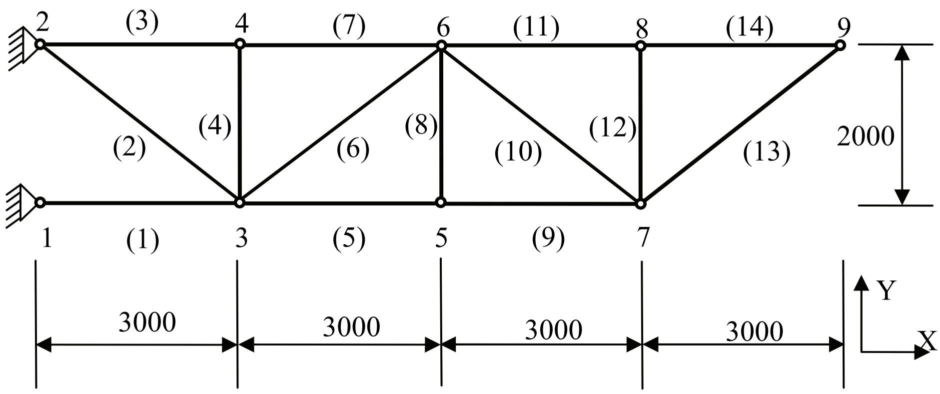
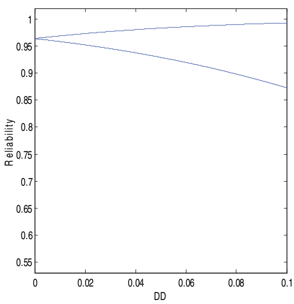


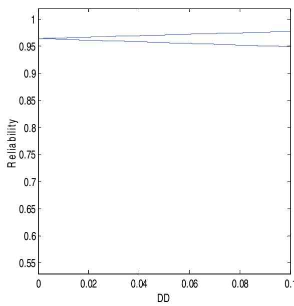
Figure 5: 14-bar cantilever truss structure (unit: mm)

The reliability of element 1 are shown in Figs. 6(a) to 6(h) while the dispersal degree of structural parameters, load and resistance are varied from zero to 0.1. The reliability is not an interval variable when a structural system has random variables as shown in Figs. 6(d) and 6(e). From Figs. 6(a) to 6(c), it can be observed that the change of the Young's modulus will produce greatest effect on the reliability of element 1. Figs. 6(a), 6(b) and 6(f) show that the structural reliability is an interval even if only resistance is an interval and structural parameters and load are deterministic values. Fig. 6(h) shows that the maximum width of the reliability is the biggest when the uncertainty of all structural parameters, load and resistance are considered simultaneously.

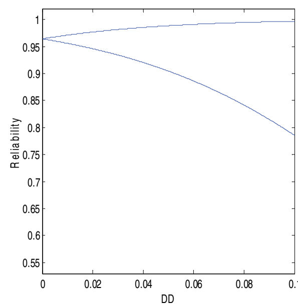
Fig. 7 shows that the intervals of reliability index, failure reliability and reliability of element 3 are all increasing as the dispersal degree of structural parameters, load and resistance increase, as we expected. To decrease the change range of structural



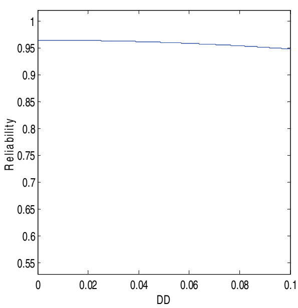
(a) $DD = \Delta\mu_{R_1 F}$



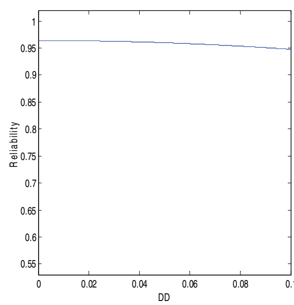
(b) $DD = \Delta\sigma_{R_1 F}$



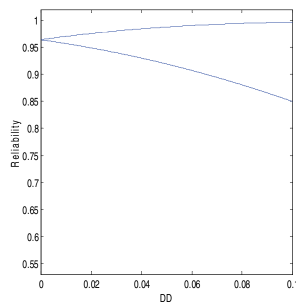
(c) $DD = \Delta E_F$



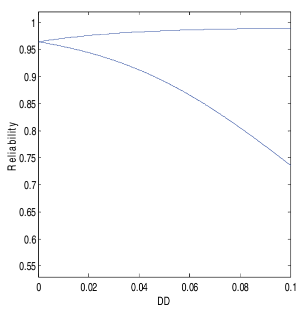
(d) $DD = \nu_f$



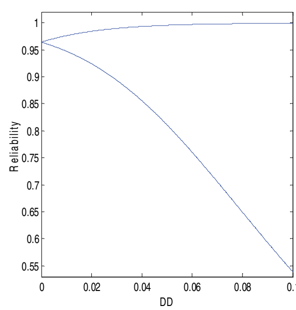
(e) $DD = \Delta\mu_{R_1 F} = \Delta\sigma_{R_1 F}$



(f) $DD = \Delta E_F = \nu_A = \nu_f$



(g) $DD = \Delta\mu_{R_1 F} = \Delta\sigma_{R_1 F} = \Delta E_F = \nu_A = \nu_f$



(h) $DD = \nu_A$

Figure 6: Reliability of element 1.

Table 6: Computational results ($DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f$=0.05)

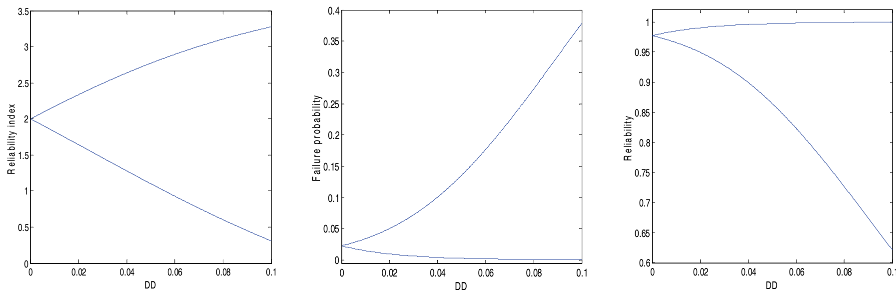
Element number	$\underline{\beta_i}$	$\overline{\beta_i}$	$\underline{P_{f_i}}$	$\overline{P_{f_i}}$	$\underline{P_{r_i}}$	$\overline{P_{r_i}}$	$P^c_{r_i}$	ΔP_{r_i}
1	0.8818	2.5889	0.0048	0.1889	0.8111	0.9952	0.9031	0.0921
2	0.7207	2.0409	0.0206	0.2355	0.7645	0.9794	0.8719	0.1075
3	1.1013	2.7716	0.0028	0.1354	0.8646	0.9972	0.9309	0.0663
4	4.5238	5.5263	1.63e-8	3.03e-6	1.0000	1.0000	1.0000	1.51e-6
5	1.1820	2.7242	0.0032	0.1186	0.8814	0.9968	0.9391	0.0577
6	0.7207	2.0409	0.0206	0.2355	0.7645	0.9794	0.8719	0.1075
7	1.1013	2.7716	0.0028	0.1354	0.8646	0.9972	0.9309	0.0663
8	4.5238	5.5263	1.63e-8	3.03e-6	1.0000	1.0000	1.0000	1.51e-6
9	1.1820	2.7242	0.0032	0.1186	0.8814	0.9968	0.9391	0.0577
10	0.7207	2.0409	0.0206	0.2355	0.7645	0.9794	0.8719	0.1075
11	1.3438	2.6178	0.0044	0.0895	0.9105	0.9956	0.9530	0.0425
12	4.5238	5.5263	1.63e-8	3.03e-6	1.0000	1.0000	1.0000	1.51e-6
13	0.7207	2.0409	0.0206	0.2355	0.7645	0.9794	0.8719	0.1075
14	1.3438	2.6178	0.0044	0.0895	0.9105	0.9956	0.9530	0.0425

Table 7: Computational results ($DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f$=0.1)

Element number	$\underline{\beta_i}$	$\overline{\beta_i}$	$\underline{P_{f_i}}$	$\overline{P_{f_i}}$	$\underline{P_{r_i}}$	$\overline{P_{r_i}}$	$P^c_{r_i}$	ΔP_{r_i}
1	0.0990	3.0891	0.0010	0.4606	0.5394	0.9990	0.7692	0.2298
2	0.1164	2.5528	0.0053	0.4537	0.5463	0.9947	0.7705	0.2242
3	0.3098	3.2750	5.28e-4	0.3784	0.6216	0.9995	0.8106	0.1889
4	4.0909	6.1111	4.94e-10	2.14e-5	1.0000	1.0000	1.0000	1.07e-5
5	0.4421	3.2407	5.96e-4	0.3292	0.6708	0.9994	0.8351	0.1643
6	0.1164	2.5528	0.0053	0.4537	0.5463	0.9947	0.7705	0.2242
7	0.3098	3.2750	5.28e-4	0.3784	0.6216	0.9995	0.8106	0.1889
8	4.0909	6.1111	4.94e-10	2.14e-5	1.0000	1.0000	1.0000	1.07e-5
9	0.4421	3.2407	5.96e-4	0.3292	0.6708	0.9994	0.8351	0.1643
10	0.1164	2.5528	0.0053	0.4537	0.5463	0.9947	0.7705	0.2242
11	0.7271	3.1349	8.59e-4	0.2336	0.7664	0.9991	0.8828	0.1164
12	4.0909	6.1111	4.94e-10	2.14e-5	1.0000	1.0000	1.0000	1.07e-5
13	0.1164	2.5528	0.0053	0.4537	0.5463	0.9947	0.7705	0.2242
14	0.7271	3.1349	8.59e-4	0.2336	0.7664	0.9991	0.8828	0.1164

reliability, the dispersal degree of system parameters and loads should be decreased greatly.

The statically determinate truss structure shown in Fig. 5 can be considered as a series system. The structural reliability is given in Table 8 when $DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f = 0.05$ and $DD = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f = 0.1$, respectively. It can be seen that the lower bound of the structural reliability, that is the worst possible value of structural reliability, is quite low as it is the product of the lower bounds of the reliability of all structural elements. For a series system,



(a) *DD = All* (b) *DD = All* (c) *DD = All*

Figure 7: Reliability index (a), failure probability (b) and reliability (c) of element 3. ($DD = ALL = \Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f$)

the reliability of all elements should be improved greatly if we want to improve the structural reliability.

Table 8: Structural reliability

	$\underline{P_r}$	$\overline{P_r}$	P_r^c	ΔP_r
$\Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f = 0.05$	0.1333	0.8968	0.5150	0.3817
$\Delta\mu_{R_1F} = \Delta\sigma_{R_1F} = \Delta E_F = \nu_A = \nu_f = 0.1$	0.0049	0.9741	0.4895	0.4846

7 Conclusions

A probabilistic interval method is proposed in this paper for static analysis of structures having both random and interval parameters/loads. The expressions for calculating the mean value and standard deviation of random interval structural responses are developed. The accuracy of the random interval perturbation method based on the first- and second- order perturbation technique is demonstrated. The interval reliability of structures is investigated using the first-order reliability method. The effects of random and interval parameters/loads on structural response are also studied. The expressions of reliability index, failure probability and reliability of structural elements, series and parallel structural systems are given in terms of intervals. From the three examples, the accuracy of the methods presented in this paper is illustrated.

Appendix A Second-order random interval moment method

The Taylor series to the second-order of the random interval variable Z^{RI} about $(\bar{\vec{X}},\vec{Y}^c)$ is developed as

$$
\begin{aligned}
Z^{RI2} &= f(\bar{\vec{X}},\vec{Y}^I) + \sum_{i=1}^{n}\left\{\left.\frac{\partial f}{\partial x_i^R}\right|_{\bar{\vec{X}},\vec{Y}^I}\right\}(x_i^R-\bar{x}_i) \\
&+ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{\left.\frac{\partial^2 f}{\partial x_k^R \partial x_l^R}\right|_{\bar{\vec{X}},\vec{Y}^I}\right\}(x_k^R-\bar{x}_k)(x_l^R-\bar{x}_l^{)} + R \\
&= f(\bar{\vec{X}},\vec{Y}^c) + \sum_{j=1}^{m}\left\{\left.\frac{\partial f}{\partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c}\right\}\Delta y_j^I \\
&+ \sum_{i=1}^{n}\left\{\left.\frac{\partial f}{\partial x_i^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{\left.\frac{\partial^2 f}{\partial x_i^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c}\right\}\Delta y_j^I\right\}(x_i^R-\bar{x}_i) \\
&+ \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{\left.\frac{\partial^2 f}{\partial x_k^R \partial x_l^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{\left.\frac{\partial^3 f}{\partial x_k^R \partial x_l^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c}\right\}\Delta y_j^I\right\} \\
&\qquad \cdot (x_k^R-\bar{x}_k)(x_l^R-\bar{x}_l) + R \qquad \text{(A1)}
\end{aligned}
$$

The mean value and variance of random interval variable Z^{RI2} can be obtained

$$
\begin{aligned}
\mu_{Z^{RI2}} &= E(Z^{RI2}) \\
&= f(\bar{\vec{X}},\vec{Y}^c) + \sum_{j=1}^{m}\{\left.\frac{\partial f}{\partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c}\}\Delta y_j^I \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{\left.\frac{\partial^2 f}{\partial x_k^R \partial x_l^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{\left.\frac{\partial^3 f}{\partial x_k^R \partial x_l^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c}\right\}\Delta y_j^I\right\} Cov(x_k^R,x_l^R) \qquad \text{(A2)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2_{Z^{RI2}} &= E\left(Z^{RI2} - E(Z^{RI2})\right)^2 \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\left\{ \left.\frac{\partial f}{\partial x_i^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^2 f}{\partial x_i^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} \\
&\cdot \left\{ \left.\frac{\partial f}{\partial x_k^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^2 f}{\partial x_k^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} Cov(x_i^R, x_k^R) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{ \left.\frac{\partial f}{\partial x_i^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^2 f}{\partial x_i^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} \\
&\cdot \left\{ \left.\frac{\partial^2 f}{\partial x_k^R \partial x_l^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^3 f}{\partial x_k^R \partial x_l^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} \\
&\cdot E\left((x_i^R - \bar{x}_i)(x_k^R - \bar{x}_k)(x_l^R - \bar{x}_l)\right) \\
&+ \frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{s=1}^{n}\left\{ \left.\frac{\partial^2 f}{\partial x_i^R \partial x_k^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^3 f}{\partial x_i^R \partial x_k^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} \\
&\cdot \left\{ \left.\frac{\partial^2 f}{\partial x_l^R \partial x_s^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^3 f}{\partial x_l^R \partial x_s^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} \\
&\cdot E\left((x_i^R - \bar{x}_i)(x_k^R - \bar{x}_k)(x_l^R - \bar{x}_l)(x_s^R - \bar{x}_s)\right) \\
&\quad - \frac{1}{4}\sum_{i=1}^{n}\sum_{l=1}^{n}\left\{\left\{ \left.\frac{\partial^2 f}{\partial x_i^R \partial x_l^R}\right|_{\bar{\vec{X}},\vec{Y}^c} + \sum_{j=1}^{m}\left\{ \left.\frac{\partial^3 f}{\partial x_i^R \partial x_l^R \partial y_j^I}\right|_{\bar{\vec{X}},\vec{Y}^c} \right\}\Delta y_j^I \right\} Cov(x_i^R, x_l^R)\right\}^2
\end{aligned} \tag{A3}
$$

Appendix B Second-order random interval perturbation method

The structural displacement based on the second-order perturbation is given by

$$U^{RI2} = U_d + \Delta_1 U + \Delta_2 U \tag{B1}$$

Substituting Eqs.(19), (20), (22), (23) to (27) into Eq.(B1) gives

$$
\begin{aligned}
U^{RI2} &= K_d^{-1} f_d \\
&+ K_d^{-1}\left(\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right) - K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I \\
&+ K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d \\
&+ \sum_{i=1}^{n}\left\{K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} - K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d\right. \\
&- K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} \\
&+ K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d \\
&- K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I \\
&\left.+ K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right\}\left(a_i^R - \bar{a}_i\right) \\
&+ \sum_{i=1}^{n}\sum_{l=1}^{n}\left\{-K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\left\{f'_{a_l^R} + \sum_{j=1}^{m} f''_{a_l^R b_j^I}\Delta b_j^I\right\}\right. \\
&+ K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d \\
&\left.+ \frac{1}{2}K_d^{-1}\left\{f''_{a_i^R a_l^R} + \sum_{j=1}^{m} f'''_{a_i^R a_l^R b_j^I}\Delta b_j^I\right\} - \frac{1}{2}K_d^{-1}\left\{K''_{a_i^R a_l^R} + \sum_{j=1}^{m} K'''_{a_i^R a_l^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d\right\} \\
&\cdot (a_i^R - \bar{a}_i)(a_l^R - \bar{a}_l) \qquad \text{(B2)}
\end{aligned}
$$

U^{RI2} in Eq.(B2) can be simply expressed as

$$
\begin{aligned}
U^{RI2} &= K_d^{-1} f_d \\
&+ K_d^{-1}\left(\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right) - K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I \\
&+ K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d \\
&\quad + \sum_{i=1}^{n} A(a_i^R,\vec{b}^I)\left(a_i^R - \bar{a}_i\right) + \sum_{i=1}^{n}\sum_{l=1}^{n} B(a_i^R, a_l^R, \vec{b}^I)\left(a_i^R - \bar{a}_i\right)\left(a_i^R - \bar{a}_i\right)
\end{aligned} \tag{B3}
$$

where

$$
\begin{aligned}
A(a_i^R,\vec{b}^I) &= K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} - K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d \\
&- K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\left\{f'_{a_i^R} + \sum_{j=1}^{m} f''_{a_i^R b_j^I}\Delta b_j^I\right\} \\
&+ K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d \\
&- K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I \\
&\quad + K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
B(a_i^R, a_l^R, \vec{b}^I) &= -K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\left\{f'_{a_l^R} + \sum_{j=1}^{m} f''_{a_l^R b_j^I}\Delta b_j^I\right\} \\
&+ K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1}\left\{K'_{a_i^R} + \sum_{j=1}^{m} K''_{a_i^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d \\
&\quad + \frac{1}{2}K_d^{-1}\left\{f''_{a_i^R a_l^R} + \sum_{j=1}^{m} f'''_{a_i^R a_l^R b_j^I}\Delta b_j^I\right\} - \frac{1}{2}K_d^{-1}\left\{K''_{a_i^R a_l^R} + \sum_{j=1}^{m} K'''_{a_i^R a_l^R b_j^I}\Delta b_j^I\right\} K_d^{-1} f_d
\end{aligned} \tag{B5}
$$

Using the random interval moment method, the mean value and variance of U^{RI2}

can be calculated by

$$\mu_{U^{RI2}} = K_d^{-1} f_d + K_d^{-1}\left(\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I - \sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d\right)$$
$$- K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} f'_{b_j^I}\Delta b_j^I + K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1}\sum_{j=1}^{m} K'_{b_j^I}\Delta b_j^I K_d^{-1} f_d$$
$$+\sum_{i=1}^{n}\sum_{l=1}^{n} B(a_i^R, a_l^R, \vec{b}^I) Cov(a_i^R, a_l^R) \quad \text{(B6)}$$

$$\sigma^2_{U^{RI2}} = \sum_{i=1}^{n}\sum_{k=1}^{n} A(a_i^R,\vec{b}^I)A(a_k^R,\vec{b}^I)Cov(a_i^R,a_k^R)$$
$$+2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A(a_i^R,\vec{b}^I)B(a_k^R,a_l^R,\vec{b}^I)\cdot E\left((a_i^R-\bar{a}_i)(a_k^R-\bar{a}_k)(a_l^R-\bar{a}_l)\right)$$
$$+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{s=1}^{n} B(a_i^R,a_k^R,\vec{b}^I)B(a_l^R,a_s^R,\vec{b}^I)$$
$$\cdot E\left((a_i^R-\bar{a}_i)(a_k^R-\bar{a}_k)(a_l^R-\bar{a}_l)(a_s^R-\bar{a}_s)\right)$$
$$-\sum_{i=1}^{n}\sum_{l=1}^{n}\left(B(a_i^R,a_l^R,\vec{b}^I)Cov(a_i^R,a_l^R)\right)^2 \quad \text{(B7)}$$

Appendix C Random interval extension of a bar

From Eq.(58), using the first-order random interval perturbation method (FRI) presented in this paper , the mean value of U^{RI1} can be obtained as

$$\mu_{U^{RI1}} = \frac{\mu_L\mu_F}{\mu_E A^c} - \frac{\mu_L}{\mu_E A^c}\frac{\mu_E}{\mu_L}\cdot\Delta A^I\cdot\frac{\mu_L\mu_F}{\mu_E A^c} = \frac{\mu_L\mu_F}{\mu_E A^c} - \frac{\mu_L\mu_F}{(A^c)^2\mu_E}\cdot\Delta A^I \quad \text{(C1)}$$

The lower and upper bounds on $\mu_{U^{RI1}}$ are

$$\underline{\mu_{U^{RI1}}} = \frac{\mu_L\mu_F}{\mu_E A^c} - \frac{\mu_L\mu_F}{(A^c)^2\mu_E}\cdot\Delta A \quad \text{(C2)}$$

$$\overline{\mu_{U^{RI1}}} = \frac{\mu_L\mu_F}{\mu_E A^c} + \frac{\mu_L\mu_F}{(A^c)^2\mu_E}\cdot\Delta A \quad \text{(C3)}$$

The variance of U^{RI1} is

$$\sigma^2_{U^{RI1}} = \left(\frac{\mu_L}{\mu_E A^c}\right)^2\left(\frac{1}{\mu_L^2}\cdot\sigma_E^2 + \frac{\mu_E^2}{\mu_L^4}\cdot\sigma_L^2\right)\left(\frac{\mu_L\mu_F}{\mu_E A^c}\right)^2\left(A^I\right)^2 + \left[\frac{\mu_L}{\mu_E A^c}\right]^2\cdot\sigma_F^2 \quad \text{(C4)}$$

Then, the lower and upper bounds on $\sigma^2_{U^{RI1}}$ are given by

$$\underline{\sigma^2_{U^{RI1}}} = \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \left(\frac{1}{\mu_L^2}\cdot\sigma_A^2 + \frac{\mu_E^2}{\mu_L^4}\cdot\sigma_L^2\right)(\underline{A})^2 \left(\frac{\mu_L\mu_F}{\mu_E A^c}\right)^2 + \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \cdot \sigma_F^2 \quad \text{(C5)}$$

$$\overline{\sigma^2_{U^{RI1}}} = \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \left(\frac{1}{\mu_L^2}\cdot\sigma_A^2 + \frac{\mu_E^2}{\mu_L^4}\cdot\sigma_L^2\right)(\overline{E})^2 \left(\frac{\mu_L\mu_F}{\mu_E A^c}\right)^2 + \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \cdot \sigma_F^2 \quad \text{(C6)}$$

From Eq.(58), using the second-order random interval perturbation method (SRI) presented in this paper , U^{RI2} can be obtained as

$$\begin{aligned} U^{RI2} = {} & \frac{\mu_L\mu_F}{\mu_E A^c} - \frac{\mu_L\mu_F}{(A^c)^2\mu_E}\cdot\Delta A^I + \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \left(\frac{\mu_E}{\mu_L}\right)^2 (\Delta A^I)(\Delta A^I)\frac{\mu_L\mu_F}{\mu_E A^c} \\ & + E1\cdot(E^R-\mu_E) + L1\cdot(L^R-\mu_L) + F1\cdot(F^R-\mu_F) \\ & + EL\cdot(E^R-\mu_E)(L^R-\mu_L) + EF\cdot(E^R-\mu_E)(F^R-\mu_F) + LF\cdot(L^R-\mu_L)(F^R-\mu_F) \\ & + E2\cdot(E^R-\mu_E)^2 + L2\cdot(L^R-\mu_L)^2 \quad \text{(C7)} \end{aligned}$$

where

$$E1 = \frac{\mu_L}{\mu_E A^c}\frac{\mu_L\mu_F}{\mu_E A^c}\left(-\frac{1}{\mu_L}A^I + 2\frac{\mu_L}{\mu_E A^c}\frac{\mu_E}{\mu_L^2}A^I\Delta A^I\right) \quad \text{(C8)}$$

$$L1 = \frac{\mu_L}{\mu_E A^c}\frac{\mu_L\mu_F}{\mu_E A^c}\left(\frac{\mu_E}{\mu_L^2}A^I - 2\frac{\mu_L}{\mu_E A^c}\frac{\mu_E^2}{\mu_L^3}A^I\Delta A^I\right) \quad \text{(C9)}$$

$$F1 = \frac{\mu_L}{\mu_E A^c}\left(1 - \frac{\mu_L}{\mu_E A^c}\frac{\mu_E}{\mu_L}\Delta A^I\right) \quad \text{(C10)}$$

$$EL = \frac{\mu_L}{\mu_E A^c}\frac{\mu_L\mu_F}{\mu_E A^c}\left(-2\frac{\mu_L}{\mu_E A^c}\frac{\mu_E}{\mu_L^3}A^I A^I - \frac{1}{\mu_L^2}A^I\right) \quad \text{(C11)}$$

$$EF = \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \frac{1}{\mu_L}A^I \quad \text{(C12)}$$

$$LF = \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \frac{\mu_E}{\mu_L^2}A^I \quad \text{(C13)}$$

$$E2 = \left(\frac{\mu_L}{\mu_E A^c}\right)^2 \frac{\mu_L\mu_F}{\mu_E A^c}\frac{1}{\mu_L^2}A^I A^I \quad \text{(C14)}$$

$$L2 = \frac{\mu_L}{\mu_E A^c}\frac{\mu_L\mu_F}{\mu_E A^c}\left(\frac{\mu_L}{\mu_E A^c}\frac{\mu_E^2}{\mu_L^4}A^I A^I - \frac{\mu_E}{\mu_L^3}A^I\right) \quad \text{(C15)}$$

The mean value and variance of U^{RI2} can be obtained as

$$\mu_{U^{RI2}} = \frac{\mu_L\mu_F}{\mu_E A^c} - \frac{\mu_L\mu_F}{(A^c)^2\mu_E}\cdot\Delta A^I + \left(\frac{\mu_L}{\mu_E A^c}\right)^2\left(\frac{\mu_E}{\mu_L}\right)^2\left(\Delta A^I\right)\left(\Delta A^I\right)\frac{\mu_L\mu_F}{\mu_E A^c} + E2\cdot\sigma_E^2 + L2\cdot\sigma_L^2 \quad \text{(C16)}$$

$$\sigma^2_{U^{RI2}} = (E1)^2\sigma_E^2 + (L1)^2\sigma_L^2 + (F1)^2\sigma_F^2 - (E2)^2(\sigma_E^2)^2 - (L2)^2(\sigma_L^2)^2 - 2\cdot E2\cdot L2\cdot\sigma_E^2\cdot\sigma_L^2 + 2\cdot E1\cdot E2\cdot\sigma_E^3 + 2\cdot L1\cdot L2\cdot\sigma_L^3 + (E2)^2\sigma_E^4 + (L2)^2\sigma_L^4 \quad \text{(C17)}$$

where $\sigma^3_{(\bullet)}$ and $\sigma^4_{(\bullet)}$ are the third-order and fourth-order moments of the random variable $(\bullet)$, respectively. Note that $\sigma^4_{(\bullet)} = 3(\sigma^2_{(\bullet)})^2$ and $\sigma^3_{(\bullet)} = 0$ for normal random variables, Eq.(C17) become as

$$\sigma^2_{U^{RI2}} = (E1)^2\sigma_E^2 + (L1)^2\sigma_L^2 + (F1)^2\sigma_F^2 + 2(E2)^2(\sigma_E^2)^2 + 2(L2)^2(\sigma_L^2)^2 - 2\cdot E2\cdot L2\cdot\sigma_E^2\cdot\sigma_L^2 \quad \text{(C18)}$$

The lower and upper bound of the mean value and variance of U^{RI2} can be computed by

$$\underline{\mu_{U^{RI2}}} = \min\left\{\mu_{U^{RI2}}\right\}, \quad \overline{\mu_{U^{RI2}}} = \max\left\{\mu_{U^{RI2}}\right\} \quad \text{(C19)}$$

$$\underline{\sigma^2_{U^{RI2}}} = \min\left\{\sigma^2_{U^{RI2}}\right\}, \quad \overline{\sigma^2_{U^{RI2}}} = \max\left\{\sigma^2_{U^{RI2}}\right\} \quad \text{(C20)}$$

The solution for the extension at the loaded end is also given by

$$U^{RI} = \frac{L^R F^R}{E^R A^I} \quad \text{(C21)}$$

From Eq.(C21) and using the algebra synthesis method (ASM) which can be found in many books (also see Appendix D) and interval operations, the expectation and variance of U^{RI} can be obtained as

$$\mu_{U^{RI}} = \frac{\mu_L\mu_F}{A^I\mu_E}\left(1 + \frac{\sigma_E^2}{\mu_E^2}\right) \quad \text{(C22)}$$

$$\sigma^2_{U^{RI}} = \frac{\mu_L^2\mu_F^2}{(A^I)^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2} + \frac{\sigma_E^2}{\mu_E^2}\right) + \frac{\mu_L^2}{(A^I)^2\mu_E^2}\left(1 + \frac{\sigma_E^2}{\mu_E^2}\right)^2\cdot\sigma_F^2 + \frac{\mu_L^2}{(A^I)^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2} + \frac{\sigma_E^2}{\mu_E^2}\right)\cdot\sigma_F^2 \quad \text{(C23)}$$

The lower and upper bounds for the expectation are

$$\underline{\mu_{U^{RI}}} = \frac{\mu_L\mu_F}{\overline{A}\mu_E}\left(1+\frac{\sigma_E^2}{\mu_E^2}\right) \tag{C24}$$

$$\overline{\mu_{U^{RI}}} = \frac{\mu_L\mu_F}{\underline{A}\,\mu_E}\left(1+\frac{\sigma_E^2}{\mu_E^2}\right) \tag{C25}$$

The lower and upper bounds on the variance are

$$\underline{\sigma^2_{U^{RI}}} = \frac{\mu_L^2\mu_F^2}{(\overline{A})^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2}+\frac{\sigma_E^2}{\mu_E^2}\right)+\frac{\mu_L^2}{(\overline{A})^2\mu_E^2}\left(1+\frac{\sigma_E^2}{\mu_E^2}\right)^2\cdot\sigma_F^2+\frac{\mu_L^2}{(\overline{A})^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2}+\frac{\sigma_E^2}{\mu_E^2}\right)\cdot\sigma_F^2 \tag{C26}$$

$$\underline{\sigma^2_{U^{RI}}} = \frac{\mu_L^2\mu_F^2}{(\underline{A})^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2}+\frac{\sigma_E^2}{\mu_E^2}\right)+\frac{\mu_L^2}{(\underline{A})^2\mu_E^2}\left(1+\frac{\sigma_E^2}{\mu_E^2}\right)^2\cdot\sigma_F^2+\frac{\mu_L^2}{(\underline{A})^2\mu_E^2}\left(\frac{\sigma_L^2}{\mu_L^2}+\frac{\sigma_E^2}{\mu_E^2}\right)\cdot\sigma_F^2 \tag{C27}$$

Appendix D Algebra synthesis method

Suppose that X and Y are normal (Gaussian) random variables, the mean value μ_Z and standard deviation $\sigma_Z$of random variable $Z = f(X,Y)$ are given in Table A1. In this table, α and β are constant, μ_X and μ_Y are the mean value of X and Yrespectively, σ_X and σ_Y are the standard deviation of X and Y respectively, and c_{XY} is the correlation coefficient of X and Y. If X and Y are not normal random variables, they should be transformed as normal random variables equivalently before use equations given in Table A1.

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Table A1: Numerical characteristics analysis using the algebra synthesis method

$Z=f(X,Y)$	mean value μ_Z	standard deviation σ_Z
$Z=\alpha$	α	0
$Z=\alpha X$	$\alpha\mu_X$	$\alpha\sigma_X$
$Z=\alpha X\pm\beta$	$\alpha X\pm\beta$	$\alpha\sigma_X$
$Z=X+Y$	$\mu_X+\mu_Y$	$(\sigma_X^2+\sigma_Y^2+2c_{XY}\sigma_X\sigma_Y)^{1/2}$
$Z=X-Y$	$\mu_X-\mu_Y$	$(\sigma_X^2+\sigma_Y^2-2c_{XY}\sigma_X\sigma_Y)^{1/2}$
$Z=XY$	$\mu_X\mu_Y+c_{XY}\sigma_X\sigma_Y$	$(\mu_X^2\sigma_Y^2+\mu_Y^2\sigma_X^2+\sigma_X^2\sigma_Y^2$ $+2c_{XY}\mu_X\mu_Y\sigma_X\sigma_Y+c_{XY}^2\sigma_X^2\sigma_Y^2)^{1/2}$
$Z=\frac{X}{Y}$	$\frac{\mu_X}{\mu_Y}[1+\frac{\sigma_Y}{\mu_Y}(\frac{\sigma_Y}{\mu_Y}-c_{XY}\frac{\sigma_X}{\mu_X})]$	$[\frac{\mu_X^2}{\mu_Y^2}(\frac{\sigma_X^2}{\mu_X^2}+\frac{\sigma_Y^2}{\mu_Y^2}-2c_{XY}\frac{\sigma_X\sigma_Y}{\mu_X\mu_Y})]^{1/2}$
$Z=X^2$	$\mu_X^2+\sigma_X^2$	$(4\mu_X^2\sigma_X^2+2\sigma_X^4)^{1/2}\approx 2\mu_X\sigma_X$
$Z=X^3$	$\mu_X^3+3\mu_X\sigma_X^2$	$(3\sigma_X^6+8\mu_X^2\sigma_X^4+5\mu_X^4\sigma_X^2)^{1/2}$ $\approx 3\mu_X^2\sigma_X$
$Z=X^n$	$\approx\mu_X^n$	$\approx\lvert n\rvert\,\mu_X^{n-1}\sigma_X$
$Z=X^{1/2}$	$(\frac{1}{2}\sqrt{(4\mu_X^2-2\sigma_X^2})^{1/2}$	$(\mu_X-\frac{1}{2}\sqrt{(4\mu_X^2-2\sigma_X^2})^{1/2}$
$Z=(X^2+Y^2)^{1/2}$	$\sqrt{(\mu_X^2+\mu_Y^2)}+\frac{\mu_X^2\sigma_Y^2+\mu_Y^2\sigma_X^2}{2\sqrt{(\mu_X^2+\mu_Y^2)^3}}$	$(\frac{\mu_X^2\sigma_Y^2+\mu_Y^2\sigma_X^2}{\mu_X^2+\mu_Y^2})^{1/2}$

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