

Pricing Options with Stochastic Volatilities by the Local Differential Quadrature Method

D. L. Young^{1,2}, C. P. Sun¹ and L. H. Shen¹

Abstract: A local differential quadrature (LDQ) method to solve the option-pricing models with stochastic volatilities is proposed. The present LDQ method is a newly developed numerical method which preserves the advantage of high-order numerical solution from the classic differential quadrature (DQ) method. The scheme also overcomes the negative effect of the ill-condition for the resultant full matrix and the sensitivity to the grid distribution. It offers a much better approach for finding the optimal order of polynomial approximation when compared to the conventional DQ method. The option-pricing problem under the stochastic volatilities is an important financial engineering topic governed by the Black-Scholes equation, a two-dimensional partial differential equation. For option-pricing problems, it would be helpful to improve the computational efficiency if we adopt the non-uniform grids to reflect the high gradient areas. Besides the requirement of non-uniform grids, a high-order solution is also easy to solve several important parameters such as “delta” and “gamma” values in the option-pricing modeling. Based on the advantages of the accuracy of the solution, the efficiency for non-uniform grids, and the appropriation for the regular-domain computation (because of its orthogonal grids), the LDQ method is proved to be very powerful to solve option-pricing problems with the stochastic volatilities. This work will consider two types of option-pricing problems under the stochastic volatilities, such as the standard options and lookback options. For standard options, we will test the effects of the final conditions, while for the lookback options we show the good capability for evaluating the exotic options. The comparisons of the numerical results for three case studies, namely the European standard call, the cash-or-nothing call and the lookback put, all indicate that the LDQ method is a very effective, stable and flexible numerical algorithm for solving the option-pricing models with stochastic volatilities.

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1 Introduction

Over the past years, several researchers tried to numerically solve the option-pricing problems. Based on the Black and Scholes equation (1973), the underlying-asset value follows the diffusion process and the option value is governed by a second-order single-variable parabolic partial differential equation (PDE). Traditional numerical methods have successfully solved this one-dimensional PDE. For example, Geske and Shastri (1985), Wu and Kwok (1997) developed a finite difference method (FDM) for the American options valuation. Besides numerical solution of the PDE, option price could also be numerically simulated by using the binomial model (Cox et al. (1979), Barraquand and Pudet (1996), Hull and White (1987), Babbs (1992) and Cheuk and Vorst (1994)) and also by the Monte-Carlo simulation (Tilley (1992), Averbukh (1997) and Longstaff and Schwartz (2001)). In recent years, the numerical solutions by using the meshfree or meshless methods for the option-pricing problem were also discussed (Hon and Mao (1999), Koc et al. (2003) and Tsai et al. (2006)). In our previous works (Sun and Young (2008)), the numerical solutions for the Black and Scholes model under constant volatility were also studied by the LDQ method.

The value of the volatility provides a crucial factor affecting the option price. However, in the Black and Scholes model, the volatility of the underlying asset is assumed to be a constant. Thus several approaches for the volatility estimation have been suggested to improve the original Black and Scholes equation, such as the jump-diffusion model (Merton (1976)), local volatility model (Coleman et al. (1999)), the nonlinear uncertain volatility model (Avellaneda et al. (1995) and Lyons (1995)) and the stochastic volatility model (Hull and White (1987), Heston (1993) and Clarke and Parrott (1999)). In this study, we compute the option value under the Heston's stochastic volatility model (1993), which is the most popular stochastic volatility model ever adopted in the literature. In the Heston's model, the volatility follows the Ornstein-Uhlenbeck process and the correlation between asset return and volatility is allowed. However these improvements have forced this one-dimensional governing equation into two dimensions. Thus the complexity for numerical approaches would also be increased. Several mesh-dependent numerical methods were proposed in the past decade for the option price in stochastic volatilities models, such as the finite volume method (FVM) (Huang et al. (2006)), the finite element method (FEM) (Forsyth et al. (1999)) and the FDM (Duffy (2006)). In this paper, we will employ the LDQ method to solve the numerical solutions for the stochastic volatility model established by Heston (1993). The LDQ method is

improved from the global differential quadrature (DQ) method which is originated by Bellman et al. (1972). Based on 1972 Bellman's work, the numerical solution by the DQ method has high accuracy with relatively fewer computational nodes than traditional numerical methods. Thus the DQ method is widely applied in many engineering applications, for example, the fluid-dynamics problems by Shu (1991 and 2000), and Shu and Richards (1990 and 1992), studies of micro-electro-mechanical systems (MEMS) by Sadeghian et al. (2007) and vibration analysis by Shina et al. (2008). However, the resultant full matrix may become easily ill-conditioned because of its large full matrix system. Furthermore, a high-order numerical approach such as the DQ method may not always be stable in applications, in particular when the initial value has discontinuity such as step function. Zong and Lam (2002) proposed the concept of the LDQ method for amending these disadvantages, which made the order of the DQ method adjustable and resulting matrix sparse, banded and no longer full. Shen et al. (2009) further extended the LDQ research to the applications in computational fluid dynamics and improvements for treating irregular geometric domains. Recently, the approximation of the weighting coefficients in the LDQ method is also an essential topic, such as Shu et al. (2005), Ding et al. (2006), Shan et al. (2008), and Ma and Qin (2008). This evolution increased flexibility of the order of approximation function and made the LDQ method more versatile and convenient to use. The localization of the DQ method also makes the large full matrix into a banded one, and thus alleviates the ill-condition for such a large linear algebraic system. For solving a banded matrix, the Bi-CG algorithm is one of the most effective matrix solvers used. Note that there are also several novel methods developed recently for solving the large systems of non-linear algebraic equations (Liu and Atluri (2008)) as well as ill-conditioned systems (Liu et al. (2009)). Moreover, because of the high-order approximation, the LDQ method also has the advantage in calculating the high-order derivatives, such as the vorticity and shear stress in the hydrodynamics or the delta " Δ " problem of the option price in this study, which is one of the most important issues for option investors and is referred to as the first-order derivative of option price with respect to stock price.

For certain option-pricing problems, the non-uniform grids are in general more efficient for the value at interesting position. For example, the adaptive mesh is more sensitive to the valuation of barrier option (Figlewski and Gao (1999), Ahn et al. (1999), and Zvan et al. (2000)); while Clarke and Parrott (1999) pointed out that the option value is especially interested for the strike price when it is equal to the underlying stock price. To the best of our knowledge, the FDM is still one of the most popular numerical skills for the option-pricing modeling. However, the mapping skills are also required if non-uniform grids are adopted. According to

the structure of the weighting function, the LDQ method can provide a much better capacity for solving the problem with non-uniform grids without additional efforts. It will also increase the efficiency of the numerical procedure.

To effectively solve the option-pricing models we require a numerical method which has to provide competitive conditions such as the advantages of the accuracy of the solution, the appropriation for the regular-domain computation, and the efficiency for non-uniform grids. The LDQ method is considered to be the best candidate among the various numerical schemes and therefore is adopted to solve the option-pricing problems in this study. The structure of this paper is organized as follows: Section 1 gives a general introduction to the numerical modeling of the pricing options with stochastic volatilities. Section 2 introduces the option-pricing models about the underlying asset with the stochastic volatilities. Section 3 demonstrates the numerical discretization by the LDQ method. Section 4 addresses the numerical experiments, and finally the capability and feasibility of the LDQ solutions for the option-pricing problems is concluded in Section 5.

2 Formulation

Three pricing options are taken as the numerical experiments by the present LDQ method. Case 1 is the European standard call; Case 2 the cash-or-nothing call and Case 3 the lookback put. Before solving these three case studies, we shall first review the stochastic volatility models, and then elaborated the governing equations, final conditions and boundary conditions for these three options.

2.1 Stochastic volatility

Suppose a stochastic differential equation (SDE) describing X is considered by the following form:

$$dX = A(X, t) dt + B(X, t) dW \quad (1)$$

where $A(X, t)$ is called the drift term, $B(X, t)$ is the noise intensity term or volatility function at time t , and dW is a Brownian motion or Wiener process. Assuming that the spot asset follows the SDE:

$$dS(t) = \mu S dt + \sqrt{y} S dz_1 \quad (2)$$

where μ is the expected rate of return on asset price S , y is the variance and z_1 is the Brownian motion. In this study, the underlying asset refers to stock and the stochastic volatility of the underlying asset, the square root of the variance y , is assumed to follow the Ornstein-Uhlenbeck process of the Heston's model as given

by Eq. (3).

$$dy = \kappa(\theta - y)dt + \xi \sqrt{y}dz_2 \tag{3}$$

where κ is the speed of reversion parameter for y , θ is the reversion level of y , ξ is the ‘volatility of volatility’, z_2 is Wiener processes with correlation parameter ρ .

2.2 Governing equations

This study assumes the underlying assets of the pricing options have stochastic volatilities followed by the Heston’s model as stated by Eq. (3). By combining Eqs. (2) and (3) and also applying the Ito’s lemma, we could obtain the partial differential equation as follows (Heston, 1993):

$$\frac{yS^2}{2}V_{SS} + \rho y\xi SV_{Sy} + \frac{\xi^2 y}{2}V_{yy} + rSV_S + (\kappa(\theta - y) - \lambda)V_y - rV + V_t = 0 \tag{4}$$

where r is the risk-free rate of interest, and λ is the market price of volatility risk. Eq. (4) is the governing equation for the Case 1 and Case 2 which are both European standard option problems.

The Case 3 is referred to as the lookback option, which is one of the exotic options with stochastic volatility, and its rule is more complex than the standard option. The payoff from the lookback options depends on the maximum or minimum asset price ever reached during the life time of the option. Generally speaking, the payoff from a lookback call is renewed as the final stock price minus the minimum stock price per observation date or the expiration date. While the put, the Case 3 in this work, is the maximum stock price minus the final stock price. Thus the lookback belongs to a path-dependent option. To obtain the governing equation of the Case 3, one may require a variable transformation (Wilmott et al. 1993). First assuming t^- and t^+ are the times just before and after an observation date, then standard no-arbitrage arguments can be used to derive the following condition:

$$V(J^+, s, y, t^+) = V(J^-, s, y, t^-) \tag{5}$$

where the new variable J denotes the maximum value of the asset price S ever achieved till the maturity. Thus the value of the variable J at observation date t_i should be changed into the following rules:

$$J^- = \begin{cases} J^+ & \text{if } \frac{J^+}{S} > 1 \\ S & \text{if } \frac{J^+}{S} < 1 \end{cases} \tag{6}$$

Then a new variable α is chosen and the process for variable transformation is as follows:

If we further define:

$$\alpha = S/J$$

$$V(J, S, y, t) = JU(\alpha, y, t) \tag{7}$$

then Eq. (4) simply becomes

$$\frac{y\alpha^2}{2}U_{\alpha\alpha} + \rho\xi y\alpha U_{\alpha y} + \frac{\xi^2 y}{2}U_{yy} + r\alpha U_{\alpha} + (\kappa(\theta - y) - \lambda)U_y - rU + U_t = 0 \tag{8}$$

Since the value of variable J varies at observation date t_i , the jump condition from Eq. (5) thus becomes:

$$U(\alpha, y, t^+) = \begin{cases} U(\alpha, y, t^-) & \text{if } \alpha < 1 \\ \alpha U(\alpha, y, t^-) & \text{if } \alpha \geq 1 \end{cases} \tag{9}$$

The following subsections 2.3 and 2.4 will demonstrate the transformation for final conditions and boundary conditions. After these transformations, this problem could be easily solved.

2.3 Final conditions

This subsection derives the final conditions of the three cases. In the following notations, we let $I_S = (0, S_{\max})$ and $I_y = (0, y_{\max})$, where the S_{\max} and y_{\max} denote the maximum of the stock price S and variance y , respectively.

(i): Case 1 is the European standard call option whose final condition is the ramp payoff given by

$$V(S, y, T) = \max(0, S - K), (S, y) \in I_S \times I_y \tag{10}$$

where K denotes the exercise price of the option and T is the maturity. Here the value of the exercise price K depends on the option contract and is less than the maximum of the stock price S . The value of the time to expiry, or maturity, T is up to the option contract.

(ii): Case 2 is the cash-or-nothing call which means the payoff of this option is set to a specified fixed price B if the final asset price is above the exercise price K ; if not, the payoff is set to zero. Thus, giving the final condition as

$$V(S, y, T) = BH(0, S - K), (S, y) \in I_S \times I_y \tag{11}$$

where B is a positive constant and H denotes the Heaviside step function. Obviously, this terminal condition shows that the option price is zero if $S < K$ and is B if $S \geq K$.

(iii): Case 3 is the lookback put whose payoff depends on the maximum value of the stock price ever achieved till the maturity. Thus the final condition for Case 3 is given by

$$V(S, y, T) = \max(J - S, 0), \quad (S, y) \in I_S \times I_y \quad (12)$$

After taking variable transformation, Eq. (12) becomes

$$U(\alpha, y, T) = \max(1 - \alpha, 0), \quad (\alpha, y) \in I_\alpha \times I_y \quad (13)$$

where $I_\alpha = (0, \alpha \max)$.

2.4 Boundary conditions

The geometric domain of the above problems contains four boundary surfaces defined by $S = 0$, $S = S \max$, $y = 0$ and $y = y \max$. The boundary conditions at the minimum ($S = 0$) and the maximum ($S = S \max$) of the asset price are shown as follows:

$$\begin{cases} V(0, y, t) = V(0, y, T) = 0 \\ V(S \max, y, t) = V(S \max, y, T) \end{cases} \quad (14)$$

To determine the boundary conditions at $y = 0$ and $y = y \max$, we treat them as Robin conditions by directly taking two particular values $y = 0$ and $y = y \max$ in Eq. (4).

The boundary condition of the Case 3 at $S = 0$ is in the Dirichlet type as:

$$V(S, y, t) = J \exp(-r(T - t)) \quad (15)$$

In addition to $y = 0$ and $y = y \max$, the boundary condition at $S = S \max$ for Case 3 can also be calculated by the simplified governing equation. After taking the variable transformation, we can transform the boundary condition from Eq. (15) to (16) as follows:

$$U(S, y, t) = \exp(-r(T - t)) \quad (16)$$

The boundary conditions at $\alpha = \alpha \max$ (the maximum value of α), $y = 0$ and $y = y \max$ are also obtained by the governing equation as mentioned above. In the following numerical procedure, the values of $\alpha \max$ and $y \max$ are set large enough to have no affect on the numerical results.

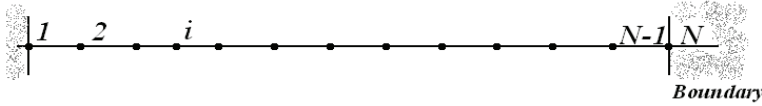


Figure 1: Illustration of the distribution for the grid nodes in the 1-D LDQ method

3 Numerical Discretization

In this section we illustrate the procedure of the application of the LDQ method to solve the option-pricing models. First we introduce the one-dimensional LDQ approximation, and then extend to higher dimension such as two-dimensional problem, and finally discretize the above governing equations by the LDQ method. We now consider there are N grid points distributed on one axis as depicted in Fig. 1. The formulation for the LDQ approximation for the first order derivative takes the following form:

$$f_x^{(1)}(x_i) = \sum_{j=1}^L a_{ij} \cdot f(x_j), \text{ for } i = 1, 2, \dots, N, L \leq N \tag{17}$$

where $f(x_j)$ represents the functional value at a grid point x_j , $f_x^{(1)}(x_i)$ indicates the first order derivative with respect to $x = x_i$, and a_{ij} are the weighting coefficients of the first order derivative. Here, L named as the number of local referenced nodes is adopted to approximate the functional value and the derivative of the grid point, in which the L could be chosen from the condition of the demand of the order or the accuracy. In this study we adopt the localized Shu's general approach (1990 and 2000) to determine the weighting coefficients a_{ij} :

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j) \cdot M^{(1)}(x_j)}, \text{ for } i \neq j \tag{18}$$

$$a_{ii} = - \sum_{j=1, j \neq i}^L a_{ij} \text{ for } i = 1, 2, \dots, N; L \leq N$$

where the $M^{(1)}(x_i)$ term is defined by:

$$M^{(1)}(x_i) = \prod_{k=1, k \neq i}^L (x_i - x_k) \tag{19}$$

For the discretization of the second order derivative, an approximation form is obtained by a similar derivation:

$$f_x^{(2)}(x_i) = \sum_{j=1}^L b_{ij} \cdot f(x_j), \text{ for } i = 1, 2, \dots, N, L \leq N \tag{20}$$

where b_{ij} is the weighting coefficients of the second order derivative $f_x^{(2)}(x_i)$, which could also be calculated by the localized Shu's approach (1990 and 2000) as follows:

$$b_{ij} = 2a_{ij} \left(a_{ii} - \frac{1}{x_i - x_j} \right), \text{ for } i \neq j$$

$$b_{ii} = - \sum_{j=1, i \neq j}^L b_{ij}, \text{ for } i = 1, 2, \dots, N; L \leq N$$
(21)

or can be obtained by using the matrix multiplication approach:

$$b_{ij} = \sum_{k=1}^N a_{ik} \cdot a_{kj}$$
(22)

The LDQ formulae for higher order derivatives can also be similarly obtained.

We can extend the higher-dimension LDQ approximation by taking the following two-dimensional case as an illustration. It is assumed that the following linear formulations are satisfied for the function $f(x, y)$ and its first order derivatives:

$$f_x^{(1)}(x_i, y_j) = \sum_{k=1}^{L_x} a_{ik}^x \cdot f(x_k, y_j), \quad L_x \leq N$$

$$f_y^{(1)}(x_i, y_j) = \sum_{k=1}^{L_y} a_{jk}^y \cdot f(x_i, y_k), \quad L_y \leq M$$
(23)

for $i = 1, 2, \dots, N; j = 1, 2, \dots, M$

where $f_x^{(1)}, f_y^{(1)}$ are the first order derivatives of the function $f(x, y)$ with respect to x and y , respectively. a_{ik}^x, a_{jk}^y are the corresponding weighting coefficients calculated by L_x local referenced nodes from N grid points on x -axis, and also L_y local referenced nodes from M grid points on y -axis. The approximation for two dimensional higher-order derivatives can be obtained analogously as one dimension, or by applying the matrix multiplication approach. We can approximate the weighting coefficients of mixed derivative term as follows:

$$a_{ij}^{xy} = \sum_{k=1}^N a_{ik}^x \cdot a_{kj}^y$$
(24)

We will first demonstrate the discretization for Eq. (4), the governing PDE of the standard European option model. By expanding the first derivative of time t by the

finite difference method, Eq. (4) becomes:

$$\begin{aligned}
 (1 - \phi) \left(\frac{yS^2}{2} V_{SS}^{n+1} + \rho y \xi S V_{Sy}^{n+1} + \frac{\xi^2 y}{2} V_{yy}^{n+1} + r S V_S^{n+1} + (\kappa(\theta - y) - \lambda) V_y^{n+1} \right. \\
 \left. - r V^{n+1} \right) + \phi \left(\frac{yS^2}{2} V_{SS}^n + \rho y \xi S V_{Sy}^n + \frac{\xi^2 y}{2} V_{yy}^n + r S V_S^n + (\kappa(\theta - y) - \lambda) V_y^n - r V^n \right) \\
 + \frac{V^{n+1} - V^n}{\Delta t} = 0 \quad (25)
 \end{aligned}$$

where ϕ is the time integrating factor. After introducing the above transposition and the discretization procedures, Eq. (25) could be rewritten as follows:

$$\begin{aligned}
 & \left[(1 - r\Delta t(\phi - 1)) [I] + \Delta t(\theta - 1) \left(\frac{1}{2} y_i S_i^2 [b_{Sij}] + \frac{1}{2} \xi^2 y_i [b_{yij}] + r S_i [a_{Sij}] \right. \right. \\
 & \left. \left. + \rho y \xi S_i [a_{Syij}] + (\kappa(\theta - y_i) - \lambda) [a_{yij}] \right) \right] \{ V^n(S_j, y_j) \} \\
 & = \left[(1 - r\Delta t\phi) [I] + \Delta t\phi \left(\frac{1}{2} y_i^2 S_i^2 [b_{Sij}] + \frac{1}{2} \xi^2 y_i [b_{vij}] + r S_i [a_{Sij}] + \rho y \xi S_i [a_{Syij}] \right. \right. \\
 & \left. \left. + (\kappa(\theta - y_i) - \lambda) [a_{yij}] \right) \right] \{ V^{n+1}(S_j, y_j) \}, \\
 & \text{for } 0 \leq \phi \leq 1, i = 1, 2, \dots, N \quad (26)
 \end{aligned}$$

The discretization for the lookback option model could also be obtained in a very similar process from Eq. (8).

4 Numerical experiments

This section will introduce the applications of the LDQ method to perform the numerical experiments. In order to show the results for solving the option-pricing models with stochastic volatilities by the LDQ method, this section will carry out three numerical experiments, the European-style standard and lookback option cases as mentioned in sections 2 and 3. For all these test problems, we choose $\phi = 0.5$ (Crank-Nicolson scheme) and $\lambda = 0$.

4.1 Case 1 A European standard call option

Case 1 is a European standard call option with the ramp-payoff final condition as Eq. (10) and the boundary conditions as Eq. (14). The parameters chosen are as follows: $S_{\max} = 100$, $y_{\max} = 1$, $T = 1$ (year), $r = 10\%$, $\rho = 0.9$, $\xi = 1$, $\mu = 0, \kappa = 0.2$ and $K = 57$.

The numerical results by the LDQ method are compared with the benchmark solutions by the FDM. The benchmark results are independent on both grid nodes and

time interval (1681 nodes and 0.001 year adopted). In this case, 143 and 441 computational nodes are adopted for the LDQ method with uniform grids. The maturity T is divided into 1000 time steps for both methods. The results are displayed in Figs. 2 and 3, where the contours drawn by dash line are the numerical results by the LDQ method, while the solid line contours by the FDM. Fig. 2 shows the V contours for the boundary conditions of the option values against time during maturity, by solving the single-variable Black-Scholes equation. In Fig. 2, the results (a) are solved by 143 nodes and (b) by 441 nodes. The LDQ method could obviously well solve the one dimensional Black-Scholes equation with the Robin boundary conditions, especially around the area $S = 50$ which is the most interesting area for option-pricing work. The governing equation Eq. (4) and the associated boundary conditions can be used to obtain the numerical results for the European standard call option price with stochastic volatility. Figure 3 shows the comparison of the results by the LDQ method with (a) 143 nodes and (b) 441 nodes with the numerical solutions by the FDM. The V contours drawn by solid line refer to the results by the FDM and the dash line by the LDQ method. The number of the number of local referenced node L (per dimension) is 4 for the numerical solution by the LDQ method in Figs. 2 and 3. Thus the order of the numerical solution by the LDQ method is much higher than the FDM.

Table 1: The results of price and delta at $S = 50$ and $y = 0.5$ for Case 1. (L is the the number of local referenced node per dimension)

	LDQ 143 nodes		LDQ 441 nodes		FDM 1681 nodes	
	Price	Delta	Price	Delta	Price	Delta
2	11.651	48.27	11.660	48.29	11.68	48.32
4	11.685	48.29	11.689	48.34		
6	11.739	48.35	11.690	48.34		
8	10.977	48.39	11.798	48.53		

Table 1 shows the comparisons between different L by the LDQ method with different set of grid nodes from the results by the FDM. Besides the option price, the numerical solutions of the delta, which is the first-order derivative of option price with respect to stock price of the option price, are also compared in Table 1. It is demonstrated that the LDQ method has a better capability to solve the high-order derivatives more efficiently with an optimal choice for the value of L . In Table 1, the results show that the optimal value of L should be 4 in both perspectives by the data in option price and delta. This is also a simple test for solving the option-pricing model by the LDQ method. In Fig. 2, it is proved again that the

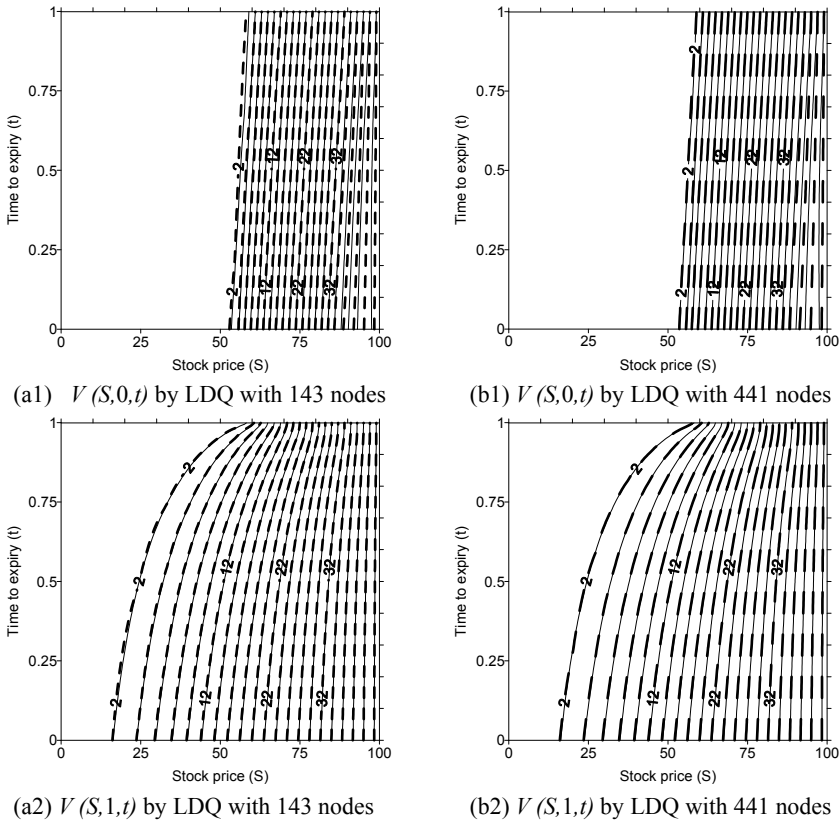


Figure 2: The boundary conditions with 143 and 441 nodes for Case 1

Black-Scholes equation by the LDQ method is stable and correct to be the boundary conditions for the two-variable option-pricing problem under a smooth final condition. Figure 3 demonstrates that the numerical solutions for this two-variable option-pricing model are convergent with increasing number of grid nodes. From Table 1, the numerical results by the LDQ method with only 143 coarse grid nodes and higher-order approach still have good numerical performance. These results verify the super capability about the high-order LDQ method to solve the standard option problems.

4.2 Case 2 A cash-or-nothing call option

Case 2 is a call option with the cash-or-nothing final condition. The formula of the final condition is shown as Eq. (11) and the boundary conditions are given by

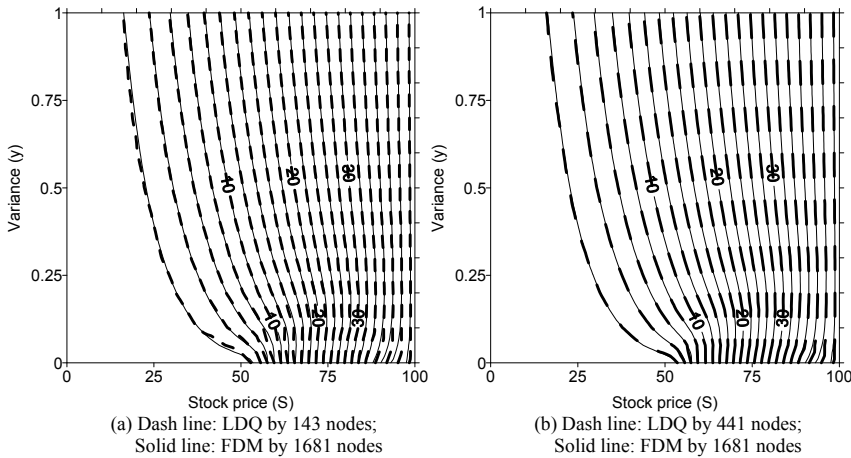


Figure 3: The LDQ solution with 143 and 441 nodes ($L = 4$) and comparison with FDM with 1681 nodes for Case 1

$V(0, y, t) = 0$ and $V(S_{\max}, y, t) = S$. The parameters for this case are as follows: $S_{\max} = 100$, $y_{\max} = 1$, $T = 1$ (year), $r = 10\%$, $\rho = 0.9$, $\xi = 1$, $\mu = 0$, $\kappa = 0.2$ and $K = 100$. As in Case 1, the LDQ solutions are also compared well with the FDM by using 1681 nodes and 1000 time steps. The results are demonstrated in Fig. 4, where the contours drawn by the dash line are the numerical results by the LDQ method, and the solid line contours by the FDM.

Based on the experience of Case 1, we first choose 441 grid nodes, $L = 4$ and divide the maturity T into 1000 time steps. However, Case 2 is unfavorable to use the high-order numerical approach because of the step-function type final condition. The numerical results with 441 nodes by $L = 4$ and 2 are depicted in Fig. 4. We obtain a better numerical solution as expected when we reduce L to 2. Table 2 demonstrates the comparison for the delta values. Although the higher-order solutions (larger L) may not be necessary to be the optimal option-price solutions because of the existence of the step-function type final condition, however they still have better performance as far as the delta value is concerned due to the advantages of derivatives calculation.

For other high-order numerical methods, the problem with discontinuous situation is not easy to solve directly. For this case, with step-function type final condition, it is obvious to observe that the better solutions of the option price for the boundary conditions are solved by the lower-order approach, especially at the $y = 0$ side in which the diffusion coefficient is zero. In other words, the LDQ method provides the flexibility for choosing the order of the numerical solution, and solves the

problem under step-function type final condition directly without additional numerical efforts. From observing the results of this case, it is convinced that the LDQ method is more convenient to solve these problems directly as comparing with other numerical methods such as the FDM. For more involved and exotic option-pricing problems, such as the payoff function of the option contract with several special conditions, the LDQ will provide a more powerful scheme for the analysis.

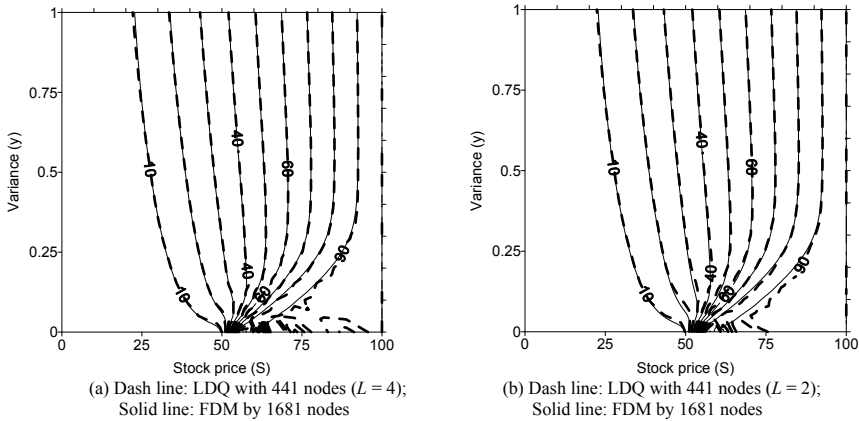


Figure 4: The LDQ solution with 441 nodes ($L = 4$ and $L = 2$) for Case 2

Table 2: The results of price and delta at $S = 50$ and $y = 0.5$ for Case 2. (L is the number of local referenced node per dimension)

	LDQ 441 nodes	FDM 441 nodes	FDM 1681 nodes	FDM 6561 nodes
L	Delta	Delta	Delta	Delta
2	121.90	121.43	122.68	122.16
4	122.03			
6	122.00			

4.3 Case 3 A lookback put option

In Case 3, a lookback option is the final test example solved by the LDQ method. By comparing to the other two cases, the lookback option problem has additional boundary condition for updating the maximum stock price J as seeing from Eqs. (6), (7) and (9). For $t = 0$, the maximum stock price J shall be equal to the stock price S . Thus the interesting option price will focus on the position $S/J = 1$. To save the computing nodes, appropriate numerical method for non-uniform grids would

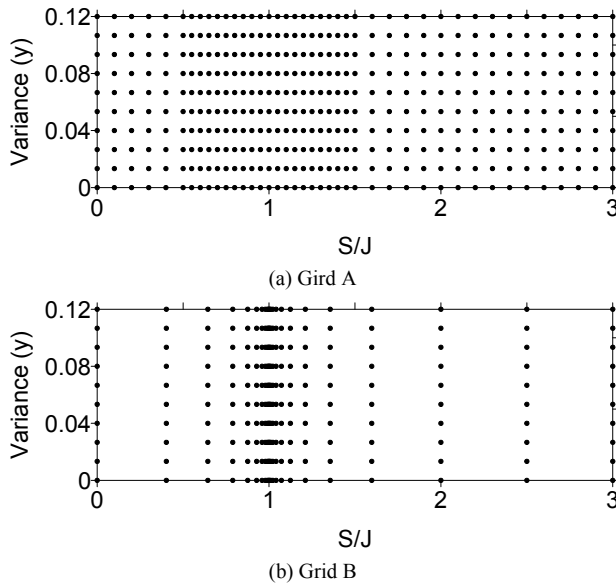


Figure 5: The distribution of non-uniform grids for Case 3

be required. This test first discusses the capacity for two different non-uniform grids A and B as shown in Fig. 5 and then the performance of the solutions by the LDQ method. It is noted that in the grid A the grid is dense between 0.5 and 1.5 and coarse in the other domain; while in the grid B the grid is dense around 1.0 and coarse in other area. This is to reflect the denser grids are used to severe gradient areas in the computational domain. In order to validate the option-pricing model for the lookback case by the LDQ method, we first give a 1-D test case with constant volatility as done by the previous research by the FEM (Forsyth et al., 1999) by setting the parameters $\kappa = \rho = \xi = 0$ and $y = 0.04$ in Eq. (8). Thus the number of variables of this model is reduced to one, only the stock price S . The following 1-D test case will be validated by the 2-D model. The other parameters are $r = 10\%$, $T = 1$ (year), $y_{\max} = 0.32$ and $\alpha_{\max} = 3$, with observation times 0.5, 1.5, 2.5... 11.5 months.

For the convenience of the comparison, the aspect ratio of the discussed domain is kept as the reference. The results by the LDQ method from grid A and B are drawn in Fig. 6. In Fig. 6(a) the contour in dash line is solved by 4625 nodes with grid A and the other one in solid line is solved by 575 nodes with grid B. In Fig. 6(b), the contours in solid line are the results by the LDQ method with 575 nodes and $L = 4$ with grid B, and the dash line is with 1750 nodes and $L = 8$ with grid B. It is

Table 3: Comparison of discrete lookback put with constant volatility by stochastic volatility model for Case 3. (Grid size is the total number of nodes in the α direction and L is the number of local referenced node per dimension)

FEM (Forsyth et al., 1999)	Wilnott et al. (1993)	LDD				
		Grid A		Grid B		
Grid size	Grid size	$At S / J=1$	Grid size (L)	$At S / J=1$	Grid size (L)	$At S / J=1$
93	0.08863	Not stated	81(4)	0.0880	23 (4)	0.0888
185	0.08883	0.089	185(4)	0.0887	37 (4)	0.0888
389	0.08885		385(4)	0.0888	70 (8)	0.0889

obvious to observe that the results by grid B have good agreement with grid A in this 1-D test though coarse non-uniform grids are used in grid B.

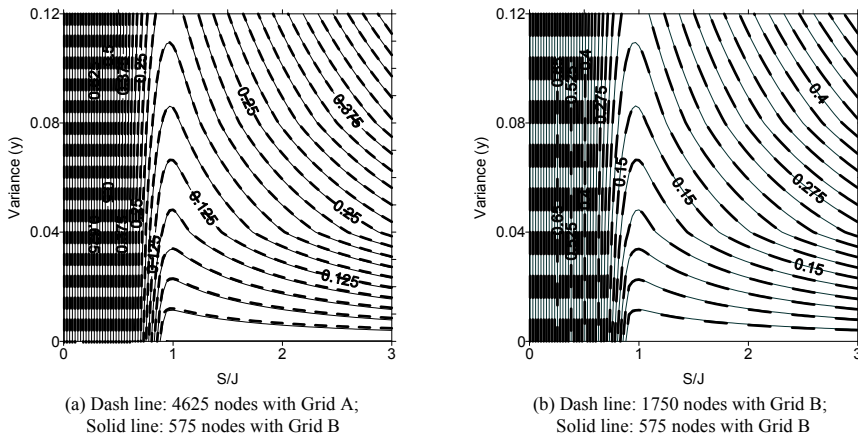


Figure 6: Lookback put with constant volatility solved by the LDQ method for Case 3

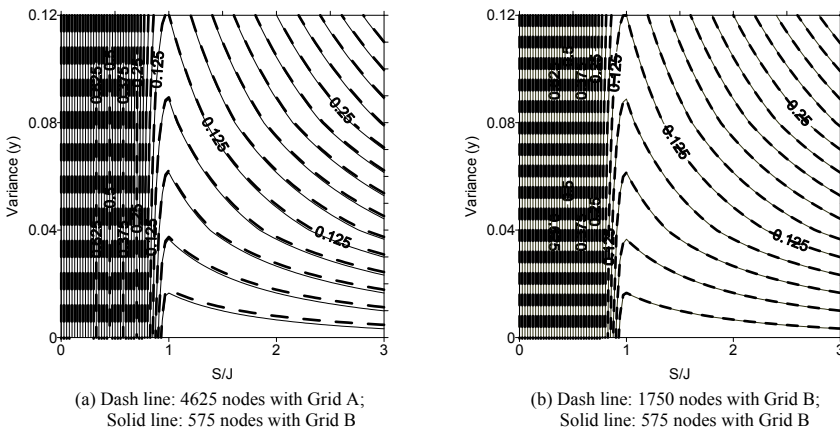


Figure 7: Lookback put with stochastic volatility solved by the LDQ method for Case 3

Table 3 shows the comparison for the results by the LDQ method at $\alpha = 1$ with those from Wilmott et al. (1993), and also from Forsyth et al. (1999) by using the FEM. From the comparison in Table 3, it is convinced that the LDQ method

provides more efficient and stable solutions for solving the option-pricing model with constant volatility. It also shows that the solution by non-uniform grid B is more efficient than grid A for the LDQ method for this 1-D test case.

After verifying the above 1-D test, we solved the following 2-D lookback option case under stochastic-volatility model by the LDQ method. The parameters are chosen as $r = 10\%$, $T = 0.5$ (year), $y_{\max} = 0.32$ and the maximum of $\alpha = 3$, with weekly observation (1/52 of a year). Different from the above 1-D numerical validation, the other parameters are set as $\kappa = 0.2$, $\rho = -0.5$, and $\xi = 0.5$. Figure 7 shows the comparison between different grids and nodes. It shows that the results by grid B also have good agreement with grid A in this 2-D case. Table 4 shows the comparison of the LDQ numerical results by two different numbers of nodes with grid B at $\alpha = 1$ and the FEM solutions from Forsyth et al. (1999). The comparison in Table 4 also reveals that the optimal order of the numerical approach, or the optimal value of L , may depend on different cases. For example the optimal value of L for 1127 grid nodes is 6 but for 3430 grid nodes it is 8. Comparing to the optimal value of L in Case 2, it is increased in Case 3. This implies that the optimal order for this numerical approach is not always needed to be reduced as in Case 2, but for some cases it is also necessary to be increased. Thus the capability of easily controlling the order of numerical approach by adjusting the value of L together with the capability of non-uniform grid distribution makes the LDQ method more flexible and powerful for the financial engineering computations.

Table 4: Comparison of discrete lookback put with stochastic volatility for Case 3. (Grid size is the total number of nodes and L is the number of local referenced node per dimension)

Grid size	FEM		LDQ		
	19551	77645	L	1127	3430
<i>At $\alpha=1$</i>	0.0541	0.0545	2	0.0536	0.0539
			4	0.0543	0.0543
			6	0.0544	0.0543
			8	0.0528	0.0544

5 Conclusions

Since Black and Scholes made remarkable contributions to the computational finance in 1973, the innovation for the option-pricing modeling in the financial market developed has been rising and flourishing during those past years. The situation for the pay-off function is various and contingent according to different option con-

tracts on the market or the portfolio of the options, such as the cash-or-nothing payoff in step-function form in Case 2. Furthermore, many types of exotic options also were developed for different demands, like the lookback option in Case 3 which desires to hedge the risk about the maximum or minimum stock price ever reached during the maturity. These changes also stimulate the developments of the research in computational financial engineering areas.

This work proposes a LDQ method to provide a numerical prediction tool for solving the option-pricing model with stochastic volatilities. The numerical results for these three option-pricing test cases demonstrate that the optimal order of numerical schemes should not always be fixed, thus the LDQ method which provides the flexibility in the approximating order is an appropriate and flexible numerical method for the application in option-pricing modeling. For certain option, such as the lookback option in Case 3, applying non-uniform grid will increase the efficiency of option pricing modeling without additional numerical difficulty. Thus the convenience and efficiency by adopting non-uniform grid to calculate the option pricing problems make the LDQ method a very competitive numerical alternative comparing to other numerical schemes. The numerical experiments of the European standard option and the lookback option valuation have shown that the LDQ method not only gives a reasonable solution, but also solves problem accurately and efficiently. Moreover, these experiments also show that the advantage for solving high-order derivatives is still maintained from the present LDQ and original DQ method. The efficiency, flexibility, and stability shown in the numerical results prove that the LDQ method is appropriate for solving option-pricing model with stochastic volatilities. It is worth investigating to apply the LDQ method to the more involved and exotic option-pricing problems with stochastic volatilities.

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References

- Ahn, D. H.; Gao, B.; Figlewski, S.** (1999): Pricing Discrete Barrier Options with the Adaptive Mesh Model. *J. Deriv.*, Vol. 6, No. 4, pp. 33–44.
- Avellaneda, M.; Levy, A.; Paras, A.** (1995): Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities. *App. Math. Fin.*, Vol. 2, pp. 73–88.
- Averbukh, V.** (1997): Pricing American Options Using Monte Carlo Simulation. Ph.D. dissertation, Cornell University, NY, USA.

- Babbs, S.** (1992): Binomial Valuation of Lookback Options. *Working paper*, Mibland Montagu, Capital Markets, London, UK.
- Barraquand, J.; Pudet, T.** (1996): Pricing of American Path-Dependent Contingent Claims. *Math. Fin.*, Vol. 6, pp. 17–51.
- Bellman, R. E.; Kashef, B. G.; Casti, J.** (1972): Differential Quadrature: A Technique for the Rapid Solution of Nonlinear Partial Differential Equations. *J. Comput. Phy.*, Vol. 10, pp. 40–52.
- Black, F.; Scholes, M.** (1973): The Pricing of Options and Corporate Liabilities. *J. Polit. Eco.*, Vol. 81, pp. 637–654.
- Cheuk, T.; Vorst, T.** (1994): *Lookback Options and the Observation Frequency: A Binomial Approach*. Erasmus University, Rotterdam, Netherland.
- Clarke, N.; Parrott, K.** (1999): Multigrid for American Option Pricing with Stochastic Volatility. *App. Math. Fin.*, Vol. 6, pp. 177–195.
- Coleman, T. F.; Li, Y.; Verma, A.** (1999): Reconstructing the Unknown Local Volatility Function. *J. Comput. Finance*, Vol. 2, pp. 77–100.
- Cox, J.; Ross, S.; Rubinstein, M.** (1979): Option Pricing: a Simplified Approach. *J. Financ. Econ.*, Vol. 7, pp. 229–263.
- Ding, H.; Shu, C.; Ye, K.S.; Xu, D.** (2006): Numerical Computation of Three-Dimensional Incompressible Viscous Flows in the Primitive Variable Form by Local Multiquadric Differential Quadrature Method. *Comput. Methods Appl. Mech. Eng.*, Vol. 195, pp. 516–533.
- Duffy, D. J.** (2006): *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*. John Wiley & Sons, Ltd, England, UK.
- Figlewski, S.; Gao, B.** (1999): The Adaptive Mesh Model: a New Approach to Efficient Option Pricing. *J. Financ. Econ.*, Vol. 53, pp. 313–351.
- Forsyth, P. A.; Vetzal, K. R.; Zvan, R.** (1999): A Finite Element Approach to the Pricing of Discrete Lookbacks with Stochastic Volatility. *App. Math. Fin.*, Vol. 6, pp. 87–106.
- Geske, R.; Shastri, K.** (1985): Valuation by Approximation: a Comparison of Option Valuation Techniques. *J. Financ. Quant. Ana.*, Vol. 20, pp. 45–771.
- Heston, S.** (1993): A Closed Form Solution for Options with Stochastic Volatility and Applications to Bond and Currency Options. *Rev. Fin. Studies*, Vol. 6, pp. 327–343.
- Hon, Y. C.; Mao, X. Z.** (1999): A Radial Basis Function Method for Solving Options Pricing Model. *J. Financ. Eng.*, Vol. 8, pp. 31–49.
- Huang, C. S.; Hung, C. H.; Wang, S.** (2006): A Fitted Finite Volume Method for

the Valuation of Options on Assets with Stochastic Volatilities. *Computing*, Vol. 77, pp. 297–320.

Hull, J.; White, A. (1987): The Pricing of Options on Assets with Stochastic Volatilities. *J. Finance*, Vol. 42, No.2, pp. 281–300.

Koc, M. B.; Boztosun, I.; Boztosun, D. (2003): On the Numerical Solution of Black-Scholes Equation. International Workshop on Meshfree Methods, July 21-23, Lisbon, Portugal.

Liu C. S.; Atluri S. N. (2008): A Novel Time Integration Method for Solving A Large System of Non-Linear Algebraic Equations. *CMES: Computer Modelling & Engineering Sciences*, Vol. 31, No. 2, pp. 71-84.

Liu C. S.; Yeih W. C.; and Atluri S. N. (2009): On Solving the Ill-Conditioned System $\mathbf{Ax}=\mathbf{b}$: General-Purpose Conditioners Obtained From the Boundary-Collocation Solution of the Laplace Equation, Using Trefftz Expansions With Multiple Length Scales. *CMES: Computer Modelling & Engineering Sciences*, Vol. 44, No. 3, pp. 281-312.

Longstaff, F. A.; Schwartz, E. S. (2001): Valuing American Options by Simulation: A Simple Least-Squares Approach. *Rev. Fin. Studies*, Vol. 14, No. 1, pp. 113—147.

Lyons, T. J. (1995): Uncertain Volatility and the Risk-free Synthesis of Derivatives. *App. Math. Fin.*, Vol. 2, pp. 117–133.

Ma, H.; Qin Q. H. (2008): An Interpolation-Based Local Differential Quadrature Method to Solve Partial Differential Equations Using Irregularly Distributed Nodes. *Commun. Numer. Meth. Engng.*, Vol. 24, pp. 573–584.

Merton, R. C. (1976): Option Pricing when Underlying Stock Returns are Discontinuous. *J. Financ. Econ.*, Vol. 3, pp. 125–144.

Merton, R. C. (1994): *Continuous-Time Finance, Revised ed.* Blackwell, Cambridge, MA, USA.

Sadeghian, H.; Rezazadeh, G.; Osterberg, P. M. (2007): Application of the Generalized Differential Quadrature Method to the Study of Pull-In Phenomena of MEMS Switches. *J. MEMS*, Vol. 16, No. 6, pp. 1334-1340.

Shan, Y. Y.; Shu, C.; Lu Z. L. (2008): Application of Local MQ-DQ Method to Solve 3D Incompressible Viscous Flows with Curved Boundary. *CMES: Computer Modelling & Engineering Sciences*, Vol. 25, No. 2, pp. 99-113.

Shen, L. H.; Young, D. L.; Lo, D. C.; Sun, C.P. (2009): Local Differential Quadrature Method for 2D Flow and Forced Convection problems in Irregular Domains. *Numer. Heat Transf., B Fundam*, Vol. 55, pp. 118–134.

Shina, Y. J.; Kwonb, K. M.; Yunc J. H. (2008): Vibration Analysis of a Circular

Arch with Variable Cross-Section Using Differential Transformation and Generalized Differential Quadrature. *J. Sound Vib.*, Vol. 309, No. 1-2, pp. 9-19.

Shu, C. (1991): Generalized Differential-Integral Quadrature and Application to the Simulation of Incompressible Viscous Flows Including Parallel Computation. Ph.D. thesis, University of Glasgow, Scotland, UK.

Shu, C. (2000): *Differential Quadrature and Its Applications in Engineering*. Springer, Berlin, Germany.

Shu, C.; Richards, B. E. (1990): High Resolution of Natural Convection in a Square Cavity by Generalized Differential Quadrature. Proc. of 3rd Conf. on Adv. in Numer. Methods in Eng.: Theory and Appl., Swansea, U.K., Vol. 7, pp. 978–985.

Shu, C.; Richards, B.E. (1992): Application of Generalized Differential Quadrature to Solve 2-Dimensional Incompressible Navier-Stokes equations. *Int. J. Numer. Methods Fluids*, Vol. 15, No. 7, pp. 791–795.

Shu, C.; Ding, H.; Yeo, K. S. (2005): Computation of Incompressible Navier-Stokes Equations by Local RBF-Based Differential Quadrature Method. *CMES: Computer Modelling & Engineering Sciences*, Vol. 7, No. 2, pp. 195–205.

Sun, C. P.; Young, D. L. (2008): Solving Option Pricing Model by Local Differential Quadrature Method. ICCES'08, March 19–22, Honolulu, Hawaii, USA.

Tilley; James, A. (1992): Valuing American Options in a Path Simulation Model. *Trans. Soc. Actuar*, Vol. 45, pp. 83–104.

Tsai, C. C.; Young, D. L.; Chiang, J. H.; Lo, D. C. (2006): The Method of Fundamental Solutions for Solving Option Pricing Model. *App. Math. Comput.*, Vol. 181, pp. 390–401.

Wilmott, P.; Dewynne, J. N.; Howison, S. D. (1993): *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, Oxford, UK.

Wu, L.; Kwok, Y. K. (1997): A Front-Fixing Finite Difference Method for the Valuation of American Options. *J. Financ. Eng.*, Vol. 6, pp. 83–97.

Zong, Z.; Lam, K. Y. (2002): A Localized Differential Quadrature (LDQ) Method and its Application to the 2D Wave Equation. *Comput. Mech.*, Vol. 29, pp. 382–391.

Zvan, R.; Vetzal, K. R.; Forsyth, P. A. (2000): PDE Methods for Pricing Barrier Options. *J. Econ. Dyn. Control*, Vol. 24, pp. 2563–2590.