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A Three-Point BVP of Time-Dependent Inverse Heat Source Problems and Solving by a TSLGSM

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Abstract: We consider an inverse problem for estimating an unknown time-dependent heat source $H(t)$ in a heat conduction equation $u_t(x,t) = u_{xx}(x,t) + H(t)$. First this inverse problem is formulated as a three-point boundary value problem (BVP) for ODEs discretized from the transformed homogeneous governing equation. To treat this three-point BVP we develop a two-stage Lie-group shooting method (TSLGSM). The novel approach is examined through numerical examples to convince that it is rather accurate and efficient; the estimation error is small even for identifying discontinuous and oscillatory heat sources under noise.

Keywords: Inverse problem, Two-stage Lie-group shooting method, Time-dependent heat source

1 Introduction

In the parabolic type diffusion problems the source terms are always not easy to detect directly. In practice, there are many researches on the inverse source identification problem to determine the source terms since 1970s. The present study aims to estimate as accurately as possible the time-varying heat source by solving an inverse heat conduction problem under an overspecified internal data of temperature. The estimation is based on a transient temperature measurement undertaken by a thermocouple on an internal point of a heat conducting rod.

Applications of inverse methods span over many heat transfer related topics. Sometimes the temperature and heat flux data on the boundary are known and one wants to determine the material properties. Those problems are often referred to as parameters' identification problems in the literature [Beck and Arnold (1997); Luce and Perez (1995)]. Most inverse problems have an inherited property of ill-posedness

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in the sense of Hadamard [Hadamard (1953); Maz'ya and Shaposhnikova (1998)]. Since the interest in these methods begun with one of the first published paper by Stolz (1960), the applications nowadays range over many scientific fields. Those fields include solid mechanics, fluid dynamics and heat transfer, to name only a few.

The parameter determination in partial differential equations using overspecified data plays a crucial role in applied mathematics and physics. These problems are widely encountered in the modeling of physical phenomena [Chao, Chen and Lin (2001); Dehghan (2005); Shamsi and Dehghan (2007); Dehghan and Tatari (2007); Huang and Shih (2007); Liu(2008a,2008b); Marin, Power, Bowtell, Sanchez, Becker, Glover and Jones (2008)]. At here, we consider an inverse problem of finding an unknown heat source $H(t)$ in a one-dimensional heat conduction equation, of which one needs to find the temperature distribution $u(x,t)$ as well as the heat source $H(t)$ that simultaneously satisfy

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + H(t)\ 0 < x < l,\ 0 < t \leq t_f \tag{1}$$

$$u(x,0) = f(x) \tag{2}$$

$$u(0,t) = F_0(t),\ u(l,t) = F_l(t) \tag{3}$$

Because the above problem has an unknown function $H(t)$, it cannot be solved directly. This point is drastically different from the direct problem, where $H(t)$ is given to find the temperature $u(x,t)$. Here, l is a length of the heat conducting rod, and $t_f$is a terminal time.

A new method will be developed to estimate the unknown heat source $H(t)$ of the above inverse problem, which is subjected to the above boundary conditions and initial condition, as well as an overspecified temperature measurement at an internal point x_m:

$$u(x_m,t) = F_m(t). \tag{4}$$

For the problem governed by Eqs. (1)-(4) there exist many studies as can be seen from the papers by Cannon and Duchateau (1998) for identifying $H(u)$, and Savateev (1995) and Borukhov and Vabishchevich (2000) for identifying $H(x,t)$ with additive or separable space and time. Many researchers sought the heat source as a function of space or time, for example, Farcas and Lesnic (2006), Ling, Yamamoto, Hon and Takeuchi (2006), and Yan, Fu and Yang (2008).

The model problem presented here used to describe a heat transfer process with a time-dependent source produces the temperature at a given point x_m in the spatial domain at time t. Thus, the purpose of solving this inverse problem can be viewed as an inverse control problem to identify the source control parameter that produces at any given time a desired temperature at a given point x_m in the spatial domain. The traditional approach approximately consists in reduction to a first-kind Volterra integral equation, and then some regularization techniques are used to solve the ill-posed problem. According to this type formulation, Maalek Ghaini (2000) has proven the existence, uniqueness and stability problems; however, no numerical procedures and examples were presented. More interestingly, Yan, Fu and Yang (2008) have transformed the above problem into a three-point boundary value problem as to be discussed below.

By considering

$$T(x,t) := u(x,t) - \rho(t) := u(x,t) - \int_0^t H(t)\, dt \tag{5}$$

we can transform the nonhomogeneous PDE in Eqs. (1)-(4) into a homogeneous one:

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2},\ 0 < x < l,\ 0 < t \leq t_f, \tag{6}$$

$$T(x,0) = f(x), \tag{7}$$

$$T(x_m,t) - T(0,t) = F_m(t) - F_0(t) =: P(t), \tag{8}$$

$$T(l,t) - T(x_m,t) = F_l(t) - F_m(t) =: Q(t). \tag{9}$$

Because the above problem has known functions of $P(t)$, $Q(t)$ and $f(x)$, it is a direct problem as an initial-boundary value problem. However, it is not a standard two-point boundary value problem; instead of, it is a three-point boundary value problem with these two boundary conditions specified at three different positions of $x = 0$, $x = x_m$ and $x = l$.

The present paper adopts a new two-stage Lie-group shooting method (TSLGSM) for the problem governed by Eqs. (6)-(9). After solving the above problem we can identify the heat source by

$$\int_0^t H(s)ds = F_0(t) - T(0,t). \tag{10}$$

However, this maybe needs a differential quadrature to calculate $H(t)$. A crucial point to well solve this problem will be the development of a stable and accurate shooting method to calculate the unknown boundary condition $T(0,t)$.

The parameter identification of $H(t)$ is one of the inverse problems for the applications in heat conduction engineering by considering thermal aging of materials. There exist many researches about inverse problems, and the followings are some of them: Marin, Power, Bowtell, Sanchez, Becker, Glover and Jones (2008); Huang and Shih (2007); Marin (2008); Noroozi, Sewell and Vinney (2006); Ling and Takeuchi (2008); Mera, Elliott and Ingham (2006); Mustata, Harris, Elliott, Lesnic and Ingham (2000); Harris, Mustata, Elliott, Ingham and Lesnic (2008). Our approach of the above inverse problem is based on the numerical method of line, which is a well-developed numerical method that transforms the partial differential equations (PDEs) into a system of ordinary differential equations (ODEs), together with the group preserving scheme (GPS) developed previously by Liu (2001) for ODEs. Recently, Liu (2006a, 2006b, 2006c) has extended the GPS technique to solve the boundary value problems (BVPs), and the numerical results reveal that the new method is rather promising to effectively calculate the two-point BVPs.

In the construction of the Lie group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of Lie group, and hence, the new shooting method has been labelled the Lie-group shooting method (LGSM). The LGSM is also shown effective on the second order general boundary value problems [Liu (2006b)], the singularly perturbed BVPs [Liu (2006c)], and the backward heat conduction problems [Chang, Liu and Chang (2007a, 2007b)]. Recently, Liu (2008b) has employed the LGSM technique to accurately solve the inverse heat conduction problems of identifying the nonhomogeneous heat conductivity functions.

On the other hand, in order to effectively solve the backward in time problems of parabolic PDEs, a past cone structure and a backward group preserving scheme have been successfully developed, such that the new one-step Lie-group numerical methods have been used to solve the backward in time Burgers equation by Liu (2006d), and the backward in time heat conduction equation by Liu, Chang and Chang (2006a).

Liu (2006e, 2006f, 2007) has used the concept of one-step GPS to develop the numerical estimation method for the unknown temperature-dependent heat conductivity and heat capacity of one-dimensional heat conduction equation. Because the Lie-group method possesses a certain advantage than other numerical methods due to its group structure, the Lie-group estimation method (LGEM) is shown to be a powerful technique to solve the inverse problems of parameters identification. In a series of papers, the Lie-group method reveals its excellent behavior on the numerical solutions of different boundary value problems, for example, Chang, Chang and Liu (2006) to treat the boundary layer equation in fluid mechanics, and Liu (2004), Liu, Chang and Chang (2006a), and Chang, Liu and Chang (2007a, 2007b) to treat the backward heat conduction equation, and Liu, Chang and Chang (2006b) to treat the Burgers equation. Under the advantage of Lie-group method, Liu (2008c) has extended it to solve the inverse Sturm-Liouville problems, and Liu (2008d) also used the LGEM technique to solve the inverse vibrational problems.

Liu (2008e) first established a TSLGSM for solving the three-point BVPs of second-order ODEs. Then Liu (2009) first applied this technique to identify a time-dependent heat source. It is interesting to note that the newly developed method of TSLGSM does not require any a priori regularization when applying it to the inverse problem of source identification, and also exhibits several advantages than other methods. It would be clear later that the new method can greatly reduce the computational time and is very easy to implement on the calculations of inverse problem of source identification. Especially, the present method of TSLGSM would provide much better computational results than others, which in turns greatly suggest us to use the TSLGSM in these calculations of inverse problems of source identification.

2 The numerical procedure

We are going to solve the present inverse problem of parameter identification through two steps. First, we solve the heat conduction problem in the spatial interval of $0 < x < l$ by subjecting it to the initial condition, and the Dirichlet boundary conditions. For this purpose, as that done by Chang, Liu and Chang (2005), Eq. (6) is transformed into the following equations:

$$\frac{\partial T(x,t)}{\partial x} = S(x,t), \tag{11}$$

$$\frac{\partial S(x,t)}{\partial x} = \frac{\partial T(x,t)}{\partial t}. \tag{12}$$

Then, by using a semi-discretization method to discretize the quantities of $T(x,t)$ and $S(x,t)$ in the time domain, we can obtain a system of ODEs for T and S with x as an independent variable. Second, the Lie-group estimation method as first developed by Liu (2006e) is thus extended and applied to the following discretized equations:

$$\frac{\partial T^i(x)}{\partial x} = S^i(x)\,,\ i = 1,2,\ldots,n, \tag{13}$$

$$\frac{\partial S^i(x)}{\partial x} = \frac{T^{i+1}(x) - T^i(x)}{\Delta t},\ i = 1,2,\ldots,n-1, \tag{14}$$

$$\frac{\partial S^n(x)}{\partial x} = \frac{T^n(x) - T^{n-1}(x)}{\Delta t}, \tag{15}$$

where $\Delta t = t_f/n$ is a uniform time increment, and $t_i = i\Delta t$ are the discretized times of which the measurement is sampling by a rate Δt. $T^i(x) = T(x, t_i)$, and $S^i(x) = S(x, t_i)$ are the discretized quantities at the nodal points of time.

The two known boundary conditions are given by

$$T^i(x_m) - T^i(0) = P^i, \ i = 1, 2, \ldots, n, \tag{16}$$

$$T^i(l) - T^i(x_m) = Q^i, \ i = 1, 2, \ldots, n, \tag{17}$$

which are obtained from Eqs. (8) and (9) by discretizations. In the above $P^i = P(t_i)$ and $Q^i = Q(t_i)$. Now, Eqs. (13)-(17) constitute a three-point BVP for the $2n$ ODEs in the x-domain.

3 Two-Stage Lie Group Shooting Method

In the followings, the two-stage Lie group shooting method developed by Liu (Liu, 2008e) for solving the three-point boundary value problem is adopted.

Let us write Eqs. (13)-(15) as in a vector form:

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \tag{18}$$

where the prime denotes the differential with respect to x, and

$$\mathbf{y} := \begin{bmatrix} \mathbf{T} \\ \mathbf{S} \end{bmatrix}, \ \mathbf{f} := \begin{bmatrix} \mathbf{S} \\ \mathbf{h}(x, \mathbf{T}) \end{bmatrix}, \tag{19}$$

in which $\mathbf{T} = \left(T^1, T^2, \ldots, T^n\right)^t$ and $\mathbf{S} = \left(S^1, S^2, \ldots, S^n\right)^t$, and the superscript t stands for the transpose. The components of $\mathbf{h}$ represent the right-hand sides of Eqs. (14) and (15). The dependence of $\mathbf{h}$ on x is due to the dependence of initial condition (7) on x.

From Eqs. (13)-(15) it follows that

$$\mathbf{T}' = \mathbf{S}, \tag{20}$$

$$\mathbf{S}' = \mathbf{h}(x, \mathbf{T}), \tag{21}$$

$$\mathbf{T}(0) = \mathbf{T}^0, \ \mathbf{T}(x_m) = \mathbf{T}^m, \tag{22}$$

$$\mathbf{S}(0) = \mathbf{S}^0, \ \mathbf{S}(x_m) = \mathbf{S}^m, \tag{23}$$

where $\mathbf{T}^0$, $\mathbf{T}^m$, $\mathbf{S}^0$ and $\mathbf{S}^m$ are unknown vectors.

By using Eq. (19) for $\mathbf{y}$ we have

$$\mathbf{y}^0 = \begin{bmatrix} \mathbf{T}^0 \\ \mathbf{S}^0 \end{bmatrix}, \ \mathbf{y}^m = \begin{bmatrix} \mathbf{T}^m \\ \mathbf{S}^m \end{bmatrix}, \tag{24}$$

According to the previous published literature (Liu, 2008e), the Lie group shooting method yields the following equations for the first interval $x \in [0, x_m]$:

$$\mathbf{T}^m = \mathbf{T}^0 + \frac{\eta_1}{\|\tilde{\mathbf{y}}_1\|}\tilde{\mathbf{S}}_1, \tag{25}$$

$$\mathbf{S}^m = \mathbf{S}^0 + \frac{\eta_1}{\|\tilde{\mathbf{y}}_1\|}\tilde{\mathbf{h}}_1, \tag{26}$$

where

$$\|\tilde{\mathbf{y}}_1\| = \sqrt{\left\|\tilde{\mathbf{T}}_1\right\|^2 + \left\|\tilde{\mathbf{S}}_1\right\|^2} = \sqrt{\|r\mathbf{T}^0 + (1-r)\mathbf{T}^m\|^2 + \|r\mathbf{S}^0 + (1-r)\mathbf{S}^m\|^2}, \tag{27}$$

$$\tilde{\mathbf{h}}_1 = \mathbf{h}\left((1-r)x_m, \tilde{\mathbf{T}}_1\right), \tag{28}$$

$$\cos\theta_1 := \frac{\left(\mathbf{T}^m - \mathbf{T}^0\right)\cdot\mathbf{T}^0 + \left(\mathbf{S}^m - \mathbf{S}^0\right)\cdot\mathbf{S}^0}{\sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2}\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}}, \tag{29}$$

$$Z_1 = \frac{(\cos\theta_1 - 1)\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}}{\cos\theta_1\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2} + \sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2} - \sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}, \tag{30}$$

$$\eta_1 = \frac{x_m\sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2}}{\ln Z_1} \tag{31}$$

For the later use $\hat{\mathbf{h}}_1$ is written explicitly as

$$\tilde{\mathbf{h}}_1 = \begin{bmatrix} \frac{\tilde{T}_1^2 - \tilde{T}_1^1}{\Delta t} \\ \vdots \\ \frac{\tilde{T}_1^n - \tilde{T}_1^{n-1}}{\Delta t} \\ \frac{\tilde{T}_1^n - \tilde{T}_1^{n-1}}{\Delta t} \end{bmatrix}, \tag{32}$$

where $\tilde{T}_1^i = rT^i(0) + (1-r)T^i(x_m),\ i = 1, \ldots, n$, and $\tilde{T}_1^0 = f((1-r)x_m)$. We should stress that $\hat{\mathbf{h}}_1$ is an unknown vector. In the above formulas, r is a weighting factor ranged from 0 to 1 and best choice of r can be made by minimizing the square norm of the target.

Similarly, we can obtain the Lie-group shooting equations in the second interval of $x \in [x_m, l]$:

$$\mathbf{T}^l = \mathbf{T}^m + \frac{\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{S}}_2, \tag{33}$$

$$\mathbf{S}^l = \mathbf{S}^m + \frac{\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{h}}_2, \tag{34}$$

where

$$\|\tilde{\mathbf{y}}_2\| = \sqrt{\|\tilde{\mathbf{T}}_2\|^2 + \|\tilde{\mathbf{S}}_2\|^2} = \sqrt{\|r\mathbf{T}^m + (1-r)\mathbf{T}^l\|^2 + \|r\mathbf{S}^m + (1-r)\mathbf{S}^l\|^2}, \tag{35}$$

$$\tilde{\mathbf{h}}_2 = \mathbf{h}\left(rx_m + (1-r)\,l, \tilde{\mathbf{T}}_2\right), \tag{36}$$

$$\cos\theta_2 := \frac{\left(\mathbf{T}^l - \mathbf{T}^m\right)\cdot\mathbf{T}^m + \left(\mathbf{S}^l - \mathbf{S}^m\right)\cdot\mathbf{S}^m}{\sqrt{\|\mathbf{T}^l - \mathbf{T}^m\|^2 + \|\mathbf{S}^l - \mathbf{S}^m\|^2}\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}, \tag{37}$$

$$Z_2 = \frac{(\cos\theta_2 - 1)\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}{\cos\theta_2\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2} + \sqrt{\|\mathbf{T}^l - \mathbf{T}^m\|^2 + \|\mathbf{S}^l - \mathbf{S}^m\|^2} - \sqrt{\|\mathbf{T}^l\|^2 + \|\mathbf{S}^l\|^2}}, \tag{38}$$

$$\eta_2 = \frac{(l - x_m)\sqrt{\|\mathbf{T}^l - \mathbf{T}^m\|^2 + \|\mathbf{S}^l - \mathbf{S}^m\|^2}}{\ln Z_2} \tag{39}$$

In the above procedures, the Lie-group shooting method is adopted twice for two intervals and it is named as the two-stage Lie group shooting method. (Liu, 2008e) The above $\mathbf{T}^0$, $\mathbf{T}^m$, $\mathbf{T}^l$, $\mathbf{S}^0$, $\mathbf{S}^m$ and $\mathbf{S}^l$ are totally six unknown vectors but the following two vector equations can be supplemented with the help of Eqs. (16) and (17):

$$\mathbf{T}^m = \mathbf{T}^0 + \mathbf{P}, \tag{40}$$

$$\mathbf{T}^l = \mathbf{T}^m + \mathbf{Q}, \tag{41}$$

where $\mathbf{P} = \left(P^1, \ldots, P^n\right)^t$ and $\mathbf{Q} = \left(Q^1, \ldots, Q^n\right)^t$.
We can evaluate these unknown vectors as follows. By using

$$\tilde{\mathbf{S}}_1 = r\mathbf{S}^0 + (1-r)\mathbf{S}^m, \quad \tilde{\mathbf{S}}_2 = r\mathbf{S}^m + (1-r)\mathbf{S}^l, \tag{42}$$

from Eqs. (25), (26), (33) and (34) we can solve

$$\mathbf{S}^0 = \frac{\|\tilde{\mathbf{y}}_1\|}{\eta_1}\left(\mathbf{T}^m - \mathbf{T}^0\right) - \frac{(1-r)\,\eta_1}{\|\tilde{\mathbf{y}}_1\|}\tilde{\mathbf{h}}_1, \tag{43}$$

$$\mathbf{S}^m = \frac{\|\tilde{\mathbf{y}}_2\|}{\eta_2}\left(\mathbf{T}^l - \mathbf{T}^m\right) - \frac{(1-r)\,\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{h}}_2, \tag{44}$$

$$\mathbf{S}^l = \frac{\|\tilde{\mathbf{y}}_2\|}{\eta_2}\left(\mathbf{T}^l - \mathbf{T}^m\right) - \frac{r\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{h}}_2, \tag{45}$$

$$\mathbf{T}^0 = \mathbf{T}^m - \frac{\eta_1}{\|\tilde{\mathbf{y}}_1\|}\left[r\mathbf{S}^0 + (1-r)\mathbf{S}^m\right], \tag{46}$$

where

$$\tilde{\mathbf{h}}_2 = \begin{bmatrix} \frac{\tilde{T}_2^2 - \tilde{T}_2^1}{\Delta t} \\ \vdots \\ \frac{\tilde{T}_2^n - \tilde{T}_2^{n-1}}{\Delta t} \\ \frac{\tilde{T}_2^n - \tilde{T}_2^{n-1}}{\Delta t} \end{bmatrix} \tag{47}$$

with $\tilde{T}_2^i = rT^i(x_m) + (1-r)T^i(l),\ i = 1, \ldots, n$ and $\tilde{T}_2^0 = f(rx_m + (1-r)l)$. Similarly, $\tilde{\mathbf{h}}_2$ is an unknown vector.

By Eqs. (40) and (41), Eqs. (43), (44) and (45) can be further refined to

$$\mathbf{S}^0 = \frac{\|\tilde{\mathbf{y}}_1\|}{\eta_1}\mathbf{P} - \frac{(1-r)\eta_1}{\|\tilde{\mathbf{y}}_1\|}\tilde{\mathbf{h}}_1, \tag{48}$$

$$\mathbf{S}^m = \frac{\|\tilde{\mathbf{y}}_2\|}{\eta_2}\mathbf{Q} - \frac{(1-r)\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{h}}_2, \tag{49}$$

$$\mathbf{S}^l = \frac{\|\tilde{\mathbf{y}}_2\|}{\eta_2}\mathbf{Q} - \frac{r\eta_2}{\|\tilde{\mathbf{y}}_2\|}\tilde{\mathbf{h}}_2. \tag{50}$$

For a specified r, starting from an initial guess of $\left(\mathbf{T}^0, \mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l\right)$, we can use Eq. (40) to calculate $\mathbf{T}^m$and Eq. (41) to calculate $\mathbf{T}^l$, then Eq. (32) to calculate $\tilde{\mathbf{h}}_1$and Eq. (47) to calculate $\tilde{\mathbf{h}}_2$, and then to generate the new $\left(\mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l, \mathbf{T}^0\right)$by Eqs. (48)-(50) and (46), until they converge according to a given stopping criterion:

$$\sqrt{\left\|\mathbf{T}_{i+1}^0 - \mathbf{T}_i^0\right\|^2 + \left\|\mathbf{S}_{i+1}^0 - \mathbf{S}_i^0\right\|^2 + \left\|\mathbf{S}_{i+1}^m - \mathbf{S}_i^m\right\|^2 + \left\|\mathbf{S}_{i+1}^l - \mathbf{S}_i^l\right\|^2} \leq \varepsilon. \tag{51}$$

Under the above new left-boundary conditions of $\mathbf{T}^0$ and $\mathbf{S}^0$, we can return to Eqs. (13)-(15) and integrate them to obtain $\mathbf{T}(x_m)$ and $\mathbf{T}(l)$. The above process can be done for all r in the interval of $r \in (0,1)$. Among these solutions we can pick up the best r, which leads to the smallest error of

$$\min_{r \in (0,1)} \sqrt{\|\mathbf{T}(x_m) - \mathbf{T}^0 - \mathbf{P}\|^2 + \|\mathbf{T}(l) - \mathbf{T}(x_m) - \mathbf{Q}\|^2}, \tag{52}$$

such that the three-point boundary conditions specified by Eqs. (16) and (17) can be fulfilled as best as possible.

When the process terminates, from Eq. (10) we can estimate the time-dependent heat source $H(t)$ at the discretized time t_i by

$$H(t_i) = \frac{F_0^{i+1} - F_0^i}{\Delta t} - \frac{T_0^{i+1} - T_0^i}{\Delta t},\ i = 1, \cdots, n-1, \tag{53}$$

$$H(t_n) = \frac{F_0^n - F_0^{n-1}}{\Delta t} - \frac{T_0^n - T_0^{n-1}}{\Delta t}. \tag{54}$$

4 Numerical examples

Now, we are ready to apply the TSLGSM on the estimations of $H(t)$ through the tests of numerical examples. We are concerned with the stability of TSLGSM by adding different levels of random noise on the measured data:

$$\hat{F}_m(t_i) = F_m(t_i) + sR(i), \tag{55}$$

where $F_m(t_i)$ is the exact data, and s specifies the level of noise.

4.1 Example 1

Let us first consider a simple inverse heat source problem with an exact solution of $H(t)$= -6t, where $T(x,t)$ is given by

$$T(x,t) = e^{-t}\sin x + 3tx^2 + \frac{1}{4}x^4. \tag{56}$$

The given data $F_0(t)$, $F_l(t)$, $f(x)$ and $F_m(t)$ can be computed from the exact solution. In this case we take l= 0.01 and $x_m = \frac{l}{2}$. In addition we consider $t_f = 1$, $\Delta t = 0.005$and $\Delta x = 0.0001$. The initial guess is given as $(\mathbf{T}^0, \mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l) = (3t^2, 3, 0.2, 0.2)\,\mathbf{I}$ with $\mathbf{I} = (1, \cdots, 1)$, where the initial time distribution of T^0 is prescribed and other time distribution functions for S^0, S^m and S^l are assumed to be constants. The noise levels are s=0.0 (no noise) and s=0.002. In the calculation we selected the stopping criterion used in Eq. (51) to be $\varepsilon = 10^{-3}$ for s=0.0 (no noise) and $\varepsilon = 5.0 \times 10^{-1}$ for s=0.002.

Before employing the numerical method of TSLGSM to calculate this example we use it to demonstrate how to pick up the best r as specified by Eq. (51). We plot the error of mis-matching the target with respect to r in Fig. 1 for the noise level s=0.0. It can be seen that there is a minimum point. Under this r the left-boundary conditions derived from the TSLGSM provide the best match to the right-boundary conditions at x_m and l. Then we can use the given $\mathbf{T}^0$ and the estimated $\mathbf{S}^0$ to calculate the whole temperature in the rod. In Fig. 2 we compare the exact H with

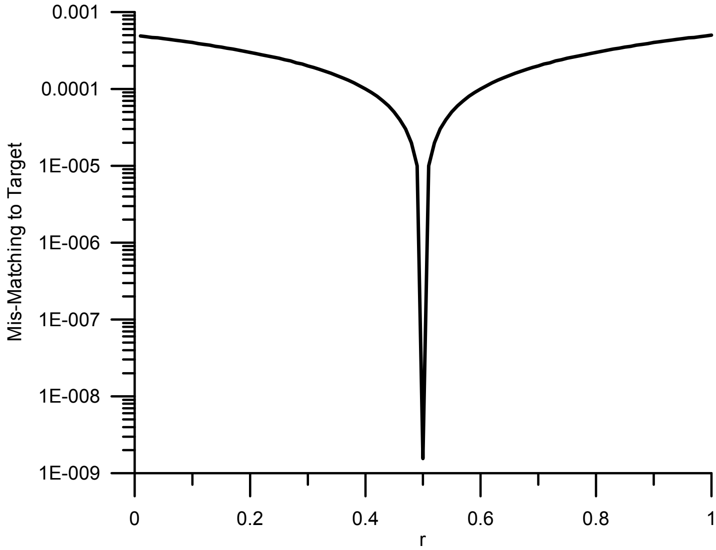

Figure 1: Demonstration of selecting the best parameter of r.

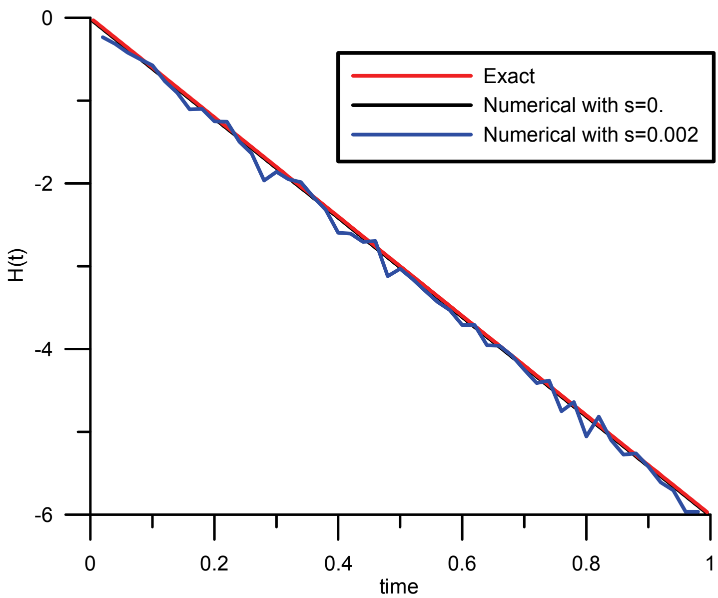

Figure 2: Comparisons between numerical results and exact solution.

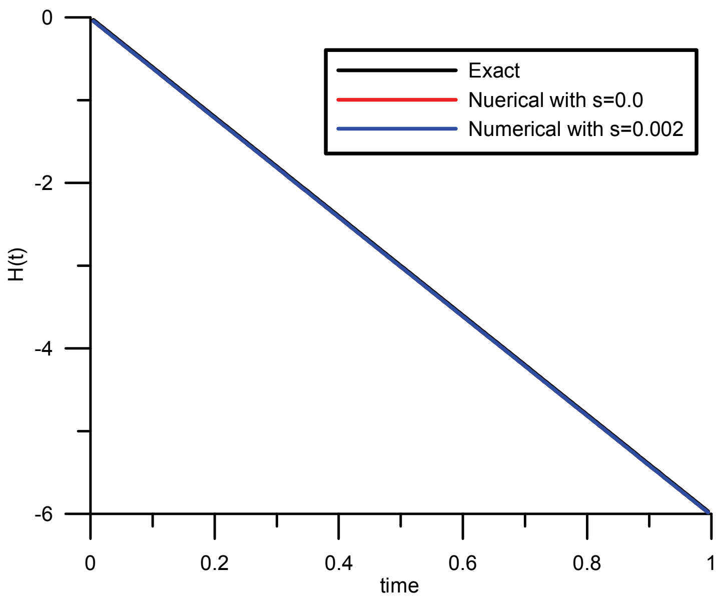

Figure 3: Inversion of heat source with l=1.0.

the numerical one with/without noise. It can be seen that inversion of the time-dependent heat source almost perfectly matches the exact one if no noise exists and is acceptable even when the noise exists.

One may wonder that the good result may not sustain when the spatial domain length l is not a small value, for example, l=1.0 (which is 100 times of the length used previously). Here we demonstrate the accuracy of solution is not affected by the domain length. Using the same designed exact solution of $H(t)$ and computed given data and other setups, the results are shown in Fig. 3. The convergent criterion selected are $\varepsilon = 100$ for s=0.0 (no noise) and s=0.002. It can be seen that with modification on the error tolerance, the results for both noise levels now perfectly match the exact solution. The way to choose a best convergence criterion will be studied in Example 3.

4.2 Example 2

In this example, we design our heat source function as a oscillatory function: $H(t) = 2\pi\cos(2\pi t)$, then $T(x,t)$ is given by

$$T(x,t) = x^2 + 2t + \sin(2\pi t). \tag{57}$$

In this case we take l= 0.01 and $x_m = \frac{l}{2}$. In addition, we consider t_f = 1, Δt =

0.005and $\Delta x = 0.0001$. The convergence criterion $\varepsilon = 10^{-3}$ for s=0.0 (no noise) and $\varepsilon = 2.0 \times 10^{-1}$ for s=0.002. The initial guess is given as $\left(\mathbf{T}^0, \mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l\right) = (2t, 0, 0.1, 0.1)\,\mathbf{I}$.

In Fig. 4, the numerical results are illustrated. It can be seen that the current approach can recover the time dependent heat source function well. For the same example, we also select a different measurement point $x_m = 0.007$ to see if the proposed method can still work well. In Fig. 5, it can be seen that the numerical results are still acceptable even though the measurement point is not at the mid-point of the domain length.

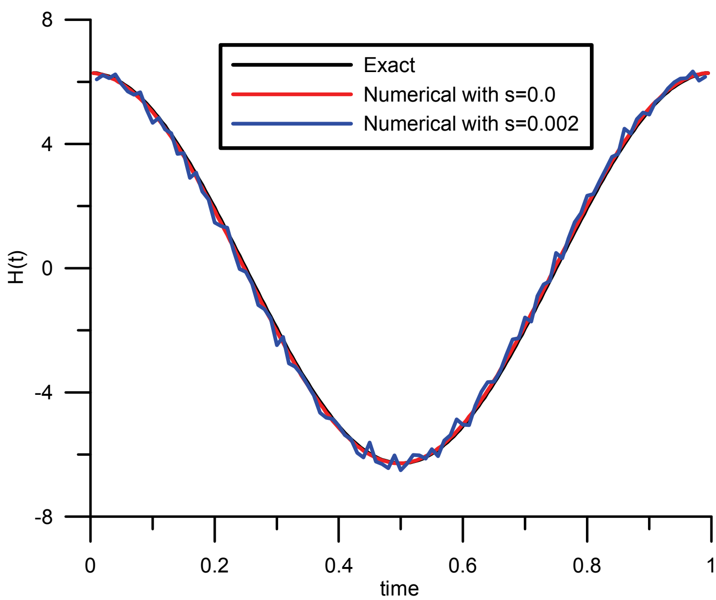

Figure 4: Inversion of an oscillatory heat source.

4.3 *Example 3*

Now let us consider the time-dependent heat source function as

$$H(t) = \begin{cases} -1 & \text{for } 0 < t < 0.25 \\ 1 & \text{for } 0.25 < t < 0.50 \\ -1 & \text{for } 0.5 < t < 0.75 \\ 1 & \text{for } 0.75 < t < 1.0 \end{cases} \tag{58}$$

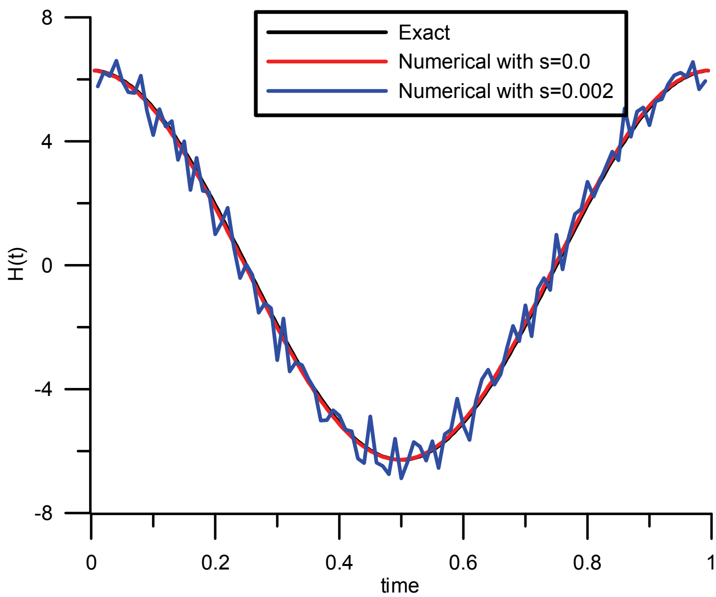

Figure 5: Inversion of the heat source with the selected measuring point not in the middle of length, $x_m = 0.007$ and l= 0.01.

which is a function with discontinuities and $T(x,t)$ is given by

$$T(x,t) = x^2 + 2t + \int_0^t H(\tau)d\tau. \tag{59}$$

Two noise levels s=0.0 and s=0.002 are given. The time increments are $\Delta t = 0.0045$ for s=0.0 and $\Delta t = 0.01$ for s=0.002, and l=0.01 and $x_m = 0.005$. The convergence criterions are $\varepsilon = 1.0 \times 10^{-7}$ for s=0.0 and $\varepsilon = 10$ for s=0.002. The initial guess is given as $\left(\mathbf{T}^0, \mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l\right) = (2t, 0, 0.1, 0.1)\,\mathbf{I}$.

The reason why we want to show this example is that in a previous published literature, Yan, Fu and Yang (2008) have calculated a similar example by using the method of fundamental solutions together with a Tikohonov regularization method. However, the result they found is not good for inversion of discontinous function. Basically, the Tikhonov's regularization method will try to smooth the solution such that it cannot be expected that a good result can be obtained when recovering a discontinous function. It can be seen from Fig. 6 that our method can perfectly recover the discontinous functions with/without noise.

In Example 1, we have mentioned that for different stopping criterion different re-

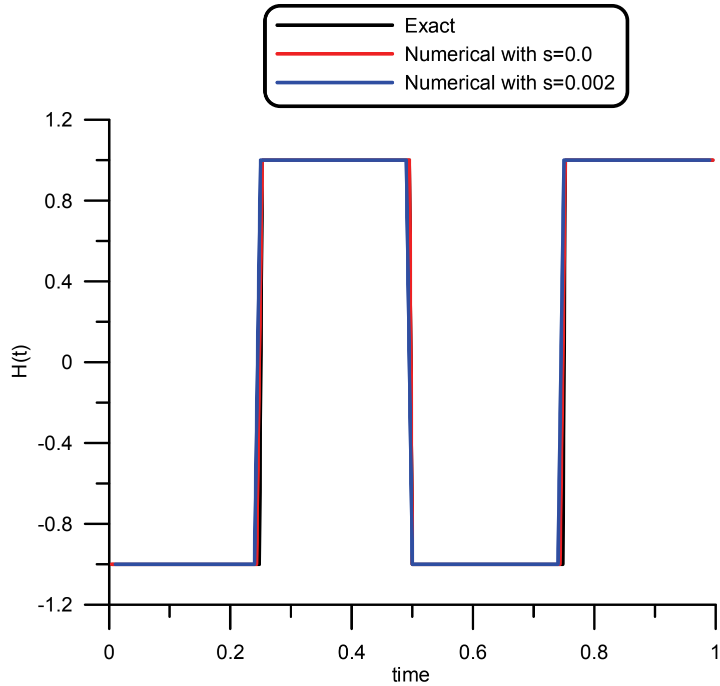

Figure 6: Inversion of a heat source with discontinuities.

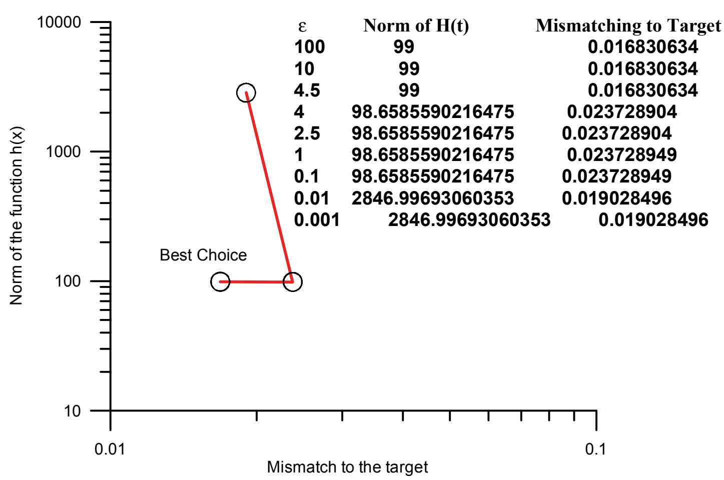

Figure 7: Demonstration of selecting the best convergence criterion ε.

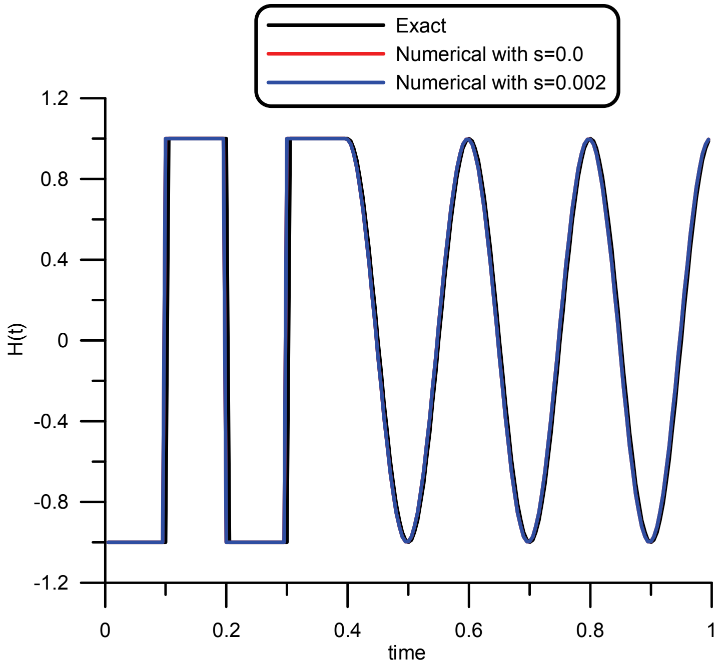

Figure 8: Inversion of a heat source with discontinuities and highly oscillatory behavior.

sults may be obtained. It needs to answer that how can one choose a best parameter. The L-curve concept developed by Hansen (1992) can be adopted here. The basic idea is to make a compromise between the norm of the unknown (which is the heat source) and the norm of mismatch of target. We hope we can find a parameter that will not yield very big value of the norm of solution, which usually happen when the problem is very ill-posed, and at the same time, have acceptable error for matching the target, which means that the solution can satisfy the equations in a reasonable range. In Fig. 7, we plot the curve between the norm of the unknown and the norm of mismatch of target for different ε for the case of s=0.002. It can be seen that after ε exceeds 4.5, the norm of unknown and the norm of mismatching keeps constant and no further change can be observed. When ε becomes smaller and smaller, the norm of unknown increases dramatically and tends to numerically unstable as expected. The optimal parameter for convergence criterion then can be determined using the abovementioned approach.

4.4 Example 4

In the last example, we examine a heat source with highly oscillating behavior and discontinuity at the same time. The heat source is designed as

$$H(t) = \begin{cases} -1 & \text{for } 0 < t < 0.1 \\ 1 & \text{for } 0.1 < t < 0.2 \\ -1 & \text{for } 0.2 < t < 0.3 \\ 1 & \text{for } 0.3 < t < 0.4 \\ \cos(10\pi(t-0.4)) & t > 0.4 \end{cases} \tag{60}$$

The corresponding $T(x,t)$ is given by Eq. (59). The time interval of measurement is $t \in [0,1]$. The noise levels s=0.0 and s=0.002 are given. The time increments are $\Delta t = 0.005$ for s=0.0 and s=0.002, and l=0.01 and $x_m = 0.005$. The convergence criterions are $\varepsilon = 1.0 \times 10^{-11}$ for s=0.0 and $\varepsilon = 10$ for s=0.002. The initial guess is given as $(\mathbf{T}^0, \mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^l) = (2t, 0, 0.1, 0.1)\mathbf{I}$.

It can be seen from Fig. 8 that the results are very good by comparing with the exact solution. It then can be concluded that the current approach is robust and very effective for computing the inverse time dependent heat source problem. In previous published literature (Liu, 2009), a similar example has been worked out. The current approach shows a better numerical accuracy than the previous approach in this example.

5 Conclusions

In order to estimate the time-dependent heat source under an extra measured temperature at an internal point, we have employed the TSLGSM to derive algebraic equations and solved them by iteration process. Numerical examples were worked out, which show that our TSLGSM is applicable even under a large noise on the measured data. Using the L-curve concept, one can determine the optimal convergence criterion. Through this study, we can conclude that the new estimation method is accurate, effective and stable. According to these facts, the present TSLGSM and one-stage LGSM can be used in practice as an accurate and effective mathematical tool to estimate the unknown time-dependent heat source.

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