 CMES, vol.44, no.3, pp.281-311, 2009

On Solving the Ill-Conditioned System $\mathbf{Ax} = \mathbf{b}$: General-Purpose Conditioners Obtained From the Boundary-Collocation Solution of the Laplace Equation, Using Trefftz Expansions With Multiple Length Scales

Chein-Shan Liu[1], **Weichung Yeih**[2] **and Satya N. Atluri**[3]

Abstract: Here we develop a general purpose pre/post conditioner $\mathbf{T}$, to solve an ill-posed system of linear equations, $\mathbf{Ax} = \mathbf{b}$. The conditioner $\mathbf{T}$ is obtained in the course of the solution of the Laplace equation, through a boundary-collocation Trefftz method, leading to: $\mathbf{Ty} = \mathbf{x}$, where $\mathbf{y}$ is the vector of coefficients in the Trefftz expansion, and $\mathbf{x}$ is the boundary data at the discrete points on a unit circle. We show that the quality of the conditioner $\mathbf{T}$ is greatly enhanced by using multiple characteristic lengths (Multiple Length Scales) in the Trefftz expansion. We further show that $\mathbf{T}$ can be multiplicatively decomposed into a dilation $\mathbf{T}_D$ and a rotation $\mathbf{T}_R$. For an odd-ordered $\mathbf{A}$, we develop four conditioners based on the solution of the Laplace equation for Dirichlet boundary conditions, while for an even-ordered $\mathbf{A}$ we develop four conditioners employing the Neumann boundary conditions. All these conditioners are well-behaved and easily invertible. Several examples involving ill-conditioned $\mathbf{A}$, such as the Hilbert matrices, those arising from the Method of Fundamental Solutions, those arising from very-high order polynomial interpolations, and those resulting from the solution of the first-kind Fredholm integral equations, are presented. The results demonstrate that the presently proposed conditioners result in very high computational efficiency and accuracy, when $\mathbf{Ax} = \mathbf{b}$ is highly ill-conditioned, and $\mathbf{b}$ is noisy.

Keywords: Ill-posed linear equations, Multi-Scale Trefftz Method (MSTM), Multi-Scale Trefftz-Collocation Laplacian Conditioner (MSTCLC), Transformation matrix, Dilation matrix, Rotation matrix

[1] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan. E-mail: liuchei.nshan@msa.hinet.net

[2] Department of Harbor & River Engineering and Computation and Simulation Center, National Taiwan Ocean University, Keelung, Taiwan

[3] Center for Aerospace Research & Education, University of California, Irvine

1 Introduction

In this paper we propose novel techniques to solve the system of linear equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

when $\mathbf{A}$ *is highly ill-conditioned.* The solution of such an ill-posed system of linear equations is an important issue for many engineering problems.

We develop a new pre- or post-conditioner $\mathbf{T}$, which is derived from a multi-scale Trefftz boundary-collocation method for solving the Laplace equation. By using this new conditioner, we redefine

$$\mathbf{x} = \mathbf{T}\mathbf{y}, \ \text{ or alternatively } \ \mathbf{x} = \mathbf{T}^{-1}\mathbf{y},$$

such that, we can solve a better-conditioned system of linear equations:

$$\mathbf{B}_i\mathbf{z}_i = \mathbf{b}_i, \ \ i = 1,2,3,4,$$

where

$$\mathbf{B}_1 = \mathbf{A}\mathbf{T}, \ \ \mathbf{B}_2 = \mathbf{A}\mathbf{T}^{-1}, \ \ \mathbf{B}_3 = \mathbf{T}\mathbf{A}, \ \ \mathbf{B}_4 = \mathbf{T}^{-1}\mathbf{A},$$

and correspondingly,

$$\mathbf{z}_1 = \mathbf{y} = \mathbf{T}^{-1}\mathbf{x}, \ \ \mathbf{z}_2 = \mathbf{y} = \mathbf{T}\mathbf{x}, \ \ \mathbf{z}_3 = \mathbf{x}, \ \ \mathbf{z}_4 = \mathbf{x},$$
$$\mathbf{b}_1 = \mathbf{b}, \ \ \mathbf{b}_2 = \mathbf{b}, \ \ \mathbf{b}_3 = \mathbf{T}\mathbf{b}, \ \ \mathbf{b}_4 = \mathbf{T}^{-1}\mathbf{b}.$$

In a practical use of linear equations in engineering problems, the data $\mathbf{b}$ are rarely given exactly; instead, noises in $\mathbf{b}$ are unavoidable due to the measurement and modeling errors. Therefore, we may encounter the problem such that the numerical solution of an ill-posed system of linear equations may deviate from the exact one to a great extent, when $\mathbf{A}$ is severely ill-conditioned and $\mathbf{b}$ is perturbed by noise.

To account for the sensitivity to noise, it is customary to use a "regularization" method to solve this sort of ill-posed problem [Kunisch and Zou (1998); Wang and Xiao (2001); Xie and Zou (2002); Resmerita (2005)], where a suitable regularization parameter is used to suppress the bias in the computed solution, by seeking a better balance of the approximation error and the propagated data error. There are several techniques developed, following the pioneering work of Tikhonov and Arsenin (1977). For a large scale system, the main choice is to use the iterative regularization algorithm, where a regularization parameter is represented by the number of iterations. The iterative method works if an early stopping criterion is used to prevent the introduction of noisy components into the approximated solutions.

The conjugate gradient method (CGM) is rather popularly employed to solve the normal equation [obtained from Eq. (1)]:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}, \tag{2}$$

where the superscript T denotes the transpose. Hanke (1992, 1995), and Hanke and Hansen (1993) have demonstrated that the performance of the CGM can be improved by the use of a right smoothing preconditioner. In the past, most researchers used the finite difference matrix as a preconditioner [Calvetti, Reichel and Shuibi (2005)].

In the last decade, a great attention has been paid to further develop the preconditioning technique, which improves the condition number of the governing matrix, for ill-posed problems. Given a suitable matrix $\mathbf{P}$, let the linear system in Eq. (1) be replaced by a preconditioned one $\mathbf{PAx} = \mathbf{Pb}$. The ideal preconditioning matrix is the inverse of $\mathbf{A}$, which evidently makes the preconditioned leading coefficient matrix to have a condition number of n. However, numerically speaking it is not possible to set up the preconditioning matrix exactly to be equal to the inverse of $\mathbf{A}$ when $\mathbf{A}$ is highly ill-conditioned. For computational reality, it is then vey important to find the approximate inverse of $\mathbf{A}$. To find a general purpose preconditioner may be very difficult and most preconditioners found in the literature are designed for specific type of applications. The following mentioned works are a few of them. For ill-posed problems, many preconditioners for regularized Tikhonov problem have been developed. For example, Jacobsen, Hasen and Saunders (2003) have developed a subspace preconditioned Least-Square QR (LSQR). Some precoditioners reformulate the original problem into a band-width matrix, such that one can use the advantages for a banded matrix. For example, the Cardinal preconditioner has been adopted to solve the PDE by the radial basis function representations [Brown, Ling, Kansa and Levesley (2005)]. For Toeplitz matrix, there exists a vast literature concerning its preconditioner, e.g., Chan (1988), and Chan and Nagy (1992). For ill-posed Toeplitz matrices with differentiable generating functions, Estatico (2009) has developed preconditioners to tackle the ill-posed behavior. An adaptive preconditioner has been developed to deal with the nonlinear system of equations [Loghin, Ruiz, Touhami (2006)]. For a general purpose preconditioner, an attempt using multigrid algorithm has been constructed [Dendy (1983)].

To solve Eq. (1), we may introduce an equivalent linear system by using a postconditioner:

$$\mathbf{A}\mathbf{M}^{-1}\mathbf{y} = \mathbf{b}, \quad \mathbf{y} = \mathbf{M}\mathbf{x}. \tag{3}$$

The challenge is to find a suitable matrix $\mathbf{M}$, such that $\mathbf{AM}^{-1}$ has a spectrum that is favorable for iterative solution with the Krylov subspace method, and whose inverse $\mathbf{M}^{-1}$ can be efficiently calculated.

In a series of papers, Erlangga and his coworkers [Erlangga, Vuik and Oosterlee (2004, 2006); Erlangga, Oosterlee and Vuik (2006); van Gijzen, Erlangga and Vuik (2007); Erlangga (2008)] extended the method of the Laplacian preconditioner first advocated by Bayliss, Goldstein and Turkel (1983) for the Helmholtz equation, to a shifted-Laplacian preconditioner for the different kind of Helmholtz equations. In the shifted-Laplacian preconditioner method, $\mathbf{M}$ is defined as arising from a discretization of

$$\mathscr{M} = -\Delta - \kappa^2(x,y)(\beta_1 - \beta_2 i), \tag{4}$$

where κ is the wave number of the Helmholtz equation. The boundary conditions were set identically to be the same as those for the original Helmholtz equation. The influence of parameters β_1 and β_2 was evaluated by Erlangga, Vuik and Oosterlee (2004).

In the above mentioned papers, the domain discretization methods, such as the FEM or the FDM, are employed to realize the discretizations of $\mathscr{M}$. The shifted-Laplacian preconditioner approach has some drawbacks in that the resulting matrix of the domain type is large-dimensional, because they act on the unknown nodal values in the domain, which may itself lead to an ill-conditioned matrix of $\mathbf{M}$, and the use of complex numbers is not suitable for the real linear system. More concisely, the shifted-Laplacian preconditioner is designed only for the Helmholtz equation, which does not guarantee its efficacy for other ill-conditioned linear systems. In the shifted-Laplacian preconditioner, the inverse of $\mathbf{M}$ should be approximated, since there does not exist an explicit form.

In this paper we propose an operator-based Laplacian conditioner, which can overcome the above drawbacks. A multi-scale Trefftz method of boundary collocation type is used to solve the Laplace equation, and thus we can explicitly derive $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$. *The new conditioners are, hopefully, universally applicable for the general purpose of reducing the ill-posedness of a linear system of equations.*

Previously, Liu and Atluri (2008) have developed a very efficient technique of a fictitious time integration method (FTIM) to solve nonlinear algebraic equations by transforming them exactly into a system of ordinary differential equations (ODEs). Then, Liu and Atluri (2009a) applied this theory to a system of linear equations, and observed that the resulting linear ODEs have a better filtering property to solve some ill-posed linear problems. Liu and Chang (2009) have further combined this FTIM technique with a non-standard group preserving scheme [Liu (2005)] to solve many linear equations of the Hilbert type. Recently, Liu and Atluri (2009b) have developed a novel interpolation technique, allowing one to use very high-order polynomials, and thus solve very accurately some ill-posed linear problems, such as the numerical differentiation of noisy data and computation of the inverse Laplace transform.

In this paper we introduce a novel and general approach to resolve the ill-posedness of highly ill-conditioned system of linear equations, by utilizing a discrete solution of the partial differential equation (PDE) of the Laplacian type. The other parts of the present paper are arranged as follows. In Section 2 we introduce a new multi-scale Trefftz method for the Laplace equation in arbitrary plane domain, and the reason for this formulation is given. This derivation naturally leads to a new set of the multi-scale T-complete functions. In Section 3 we consider a direct boundary-collocation method, to find the multi-scale transformation matrix, which can be further decomposed into a dilation matrix followed by a rotation matrix. *The use of this transformation matrix as a general-purpose pre- or post-conditioner to solve the system of equations,* $\mathbf{Ax} = \mathbf{b}$*, when* $\mathbf{A}$ *is highly ill-conditioned, is discussed.* Here, very importantly, we view the unknown vector $\mathbf{x}$ in the above linear equations to be related to a set of boundary nodal values $x_i = x(\theta_i)$ at the collocated points. Indeed, we embed the unknown vector $\mathbf{x}$ into a fictitious quantity $x(\theta)$ on a fictitious coordinate $\theta \in [0, 2\pi]$. In Section 4 we use the transformation matrix found in Section 3 as preconditioners and postconditioners of ill-posed system of linear equations. In Section 5 we solve the Neumann problem by a multi-scale Trefftz method, in order to obtain a transformation matrix for the even-dimensional system of linear equations. To test the validity of reducing the condition numbers of highly ill-posed linear systems, some examples, including the Hilbert matrices, those arising in the method of fundamental solutions (MFS), the Vandermonde matrices appearing in the polynomial interpolations, and those arising in the solution of the first-kind Fredholm integral equation, are investigated. Finally, we give some conclusions in Section 7.

2 A multiple-characteristic-length Trefftz method for solving the Laplace equation

Recently, Li, Lu, Huang and Cheng (2007) have given a fairly comprehensive comparison of the Trefftz, collocation and other boundary methods. They concluded that the collocation Trefftz method (CTM) is the simplest algorithm and provides the most accurate solution with the best numerical stability. *However, the conventional CTM may have a major drawback in that the resulting system of linear equations is extremely ill-conditioned.* In order to obtain an accurate solution of the linear equations, some special techniques, e.g., preconditioner and truncated singular value decomposition (SVD), are required.

In order to overcome these difficulties, which appear in the conventional CTM, Liu (2007a, 2007b, 2007c, 2008a) proposed a modified Trefftz method, and refined this method by taking a single characteristic length into the T-complete functions, such that, the condition number of the resulting linear equations system can be greatly reduced. Then, Liu (2008b) found that the desirable property of the modified Trefftz method can be used to modify the MFS by relating the coefficients obtained from these two methods by a linear transformation, as shown in Eq. (50) of the above cited paper.

Nevertheless, the above method, which uses a single characteristic length in the series expansion can easily result in an unstable solution, such as for degenerate scale problems and singular problems, when utilizing the high-order T-complete functions [Chen, Liu and Chang (2008)]. In this paper we first overcome this problem by a new proposal of a multiple-characteristic-length modified Trefftz method for the Laplace equation in an arbitrary plane domain.

Here we first consider a new method to solve the boundary value problem of a body in an arbitrary plane domain, posed by the Laplace equation and a Dirichlet boundary condition on a non-circular boundary:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \ \ r < \rho, \ \ 0 \le \theta \le 2\pi, \tag{5}$$

$$u(\rho, \theta) = x(\theta), \ \ 0 \le \theta \le 2\pi, \tag{6}$$

where $x(\theta)$ is a boundary function, and $r = \rho(\theta)$ is a given contour describing the boundary shape of the interior domain Ω. The contour Γ in the polar coordinates is described by $\Gamma = \{(r,\theta) | r = \rho(\theta), \ 0 \le \theta \le 2\pi\}$.

Liu (2007a, 2007b, 2007c, 2008a) has proposed a modified Trefftz method, by supposing that

$$u(r,\theta) = a_0 + \sum_{k=1}^{m} \left[a_k \left(\frac{r}{R_0}\right)^k \cos k\theta + b_k \left(\frac{r}{R_0}\right)^k \sin k\theta \right], \tag{7}$$

where

$$R_0 \ge \rho_{\max} = \max_{\theta \in [0,2\pi]} \rho(\theta) \tag{8}$$

is a constant which is greater than the characteristic length of the problem domain which is being considered. Besides, m is a positive integer chosen by the user, and $a_0, a_k, b_k, \ k = 1, \ldots, m$ are unknown coefficients.

Recently, Chen, Liu and Chang (2008) have applied the modified collocation Trefftz method (MCTM) to a discontinuous boundary value problem, a singular problem and a degenarate scale problem, of the Laplace equation, by using much higher-order terms with m larger than 100, and they showed that the MCTM is more powerful and robust against noise than other numerical methods.

A slender body has a large aspect ratio; for example, an ellipse with semiaxes a and b is slender if the aspect ratio a/b is large. Under this condition the above expansion by a single characteristic length R_0 may be ineffective and inaccurate because when we require that $a/R_0 < 1$, the power term $(b/R_0)^k$ will be very small, leading to a large round-off error in the computation of coefficients. Therefore, we replace Eq. (7) by the following expansion involving *multiple-characteristic-lengths* R_k:

$$u(r,\theta) = \frac{a_0}{R_1} + \sum_{k=1}^{m} \left[a_k \left(\frac{r}{R_{2k}} \right)^k \cos k\theta + b_k \left(\frac{r}{R_{2k+1}} \right)^k \sin k\theta \right], \tag{9}$$

where R_k is a sequence of constant numbers defined by the user.

Therefore, we have introduced a set of new multi-scale Trefftz basis functions:

$$\left\{ \frac{1}{R_1}, \left(\frac{r}{R_{2k}} \right)^k \cos k\theta, \left(\frac{r}{R_{2k+1}} \right)^k \sin k\theta, \;\; k = 1, 2, \ldots \right\}. \tag{10}$$

This set forms a new T-complete basis, and the solution of u can be expanded by these bases. The efficiency of this multi-scale Trefftz method (MSTM) to solve the Laplace equation in a slender ellipse with very large aspect ratio, is demonstrated in Section 6.1.

3 The Trefftz boundary-collocation method

By imposing condition (6) on Eq. (9) we can obtain

$$\frac{a_0}{R_1} + \sum_{k=1}^{m} \left[a_k \cos k\theta \left(\frac{\rho(\theta)}{R_{2k}} \right)^k + b_k \sin k\theta \left(\frac{\rho(\theta)}{R_{2k+1}} \right)^k \right] = x(\theta). \tag{11}$$

Here we employ the collocation method to find the coefficients a_k and b_k. Eq. (11) is imposed at a number of collocated points $(\rho(\theta_j), \theta_j)$ by choosing

$$\theta_j = \frac{2j\pi}{n}, \;\; n = 2m + 1. \tag{12}$$

When the index j runs from 1 to n we obtain a system of linear equations of the order n:

$$\mathbf{T}\mathbf{y} = \mathbf{x}, \tag{13}$$

where

$$\mathbf{y} = [a_0, a_1, b_1, \cdots, a_m, b_m]^{\mathrm{T}}$$

is the vector of unknown coefficients,

$$\mathbf{x} = [x_1, x_2, \ldots, x_{2m}, x_{2m+1}]^{\mathrm{T}}$$

is the vector of discrete boundary values with $x_j = x(\theta_j)$, and $\mathbf{T}$ is a transformation matrix, given by

$$\mathbf{T} := \begin{bmatrix} \frac{1}{R_1} & \frac{\rho_1}{R_2}\cos\theta_1 & \frac{\rho_1}{R_3}\sin\theta_1 & \ldots & \left(\frac{\rho_1}{R_{2k}}\right)^k \cos k\theta_1 & \left(\frac{\rho_1}{R_{2k+1}}\right)^k \sin k\theta_1 & \ldots & \left(\frac{\rho_1}{R_{2m}}\right)^m \cos m\theta_1 & \left(\frac{\rho_1}{R_{2m+1}}\right)^m \sin m\theta_1 \\ \frac{1}{R_1} & \frac{\rho_2}{R_2}\cos\theta_2 & \frac{\rho_2}{R_3}\sin\theta_2 & \ldots & \left(\frac{\rho_2}{R_{2k}}\right)^k \cos k\theta_2 & \left(\frac{\rho_2}{R_{2k+1}}\right)^k \sin k\theta_2 & \ldots & \left(\frac{\rho_2}{R_{2m}}\right)^m \cos m\theta_2 & \left(\frac{\rho_2}{R_{2m+1}}\right)^m \sin m\theta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{R_1} & \frac{\rho_{n-1}}{R_2}\cos\theta_{n-1} & \frac{\rho_{n-1}}{R_3}\sin\theta_{n-1} & \ldots & \left(\frac{\rho_{n-1}}{R_{2k}}\right)^k \cos k\theta_{n-1} & \left(\frac{\rho_{n-1}}{R_{2k+1}}\right)^k \sin k\theta_{n-1} & \ldots & \left(\frac{\rho_{n-1}}{R_{2m}}\right)^m \cos m\theta_{n-1} & \left(\frac{\rho_{n-1}}{R_{2m+1}}\right)^m \sin m\theta_{n-1} \\ \frac{1}{R_1} & \frac{\rho_n}{R_2}\cos\theta_n & \frac{\rho_n}{R_3}\sin\theta_n & \ldots & \left(\frac{\rho_n}{R_{2k}}\right)^k \cos k\theta_n & \left(\frac{\rho_n}{R_{2k+1}}\right)^k \sin k\theta_n & \ldots & \left(\frac{\rho_n}{R_{2m}}\right)^m \cos m\theta_n & \left(\frac{\rho_n}{R_{2m+1}}\right)^m \sin m\theta_n \end{bmatrix}, \tag{14}$$

where we use $\rho_j = \rho(\theta_j)$ for simplicity of notation.

Because our aim here is to seek a simple pre-conditioning matrix $\mathbf{T}$, which serves to transform the general variable $\mathbf{x}$ to a new variable $\mathbf{y}$ in Eq. (13), with $\mathbf{T}$ being as simple as possible, *we can conveniently assume that the problem domain is a unit disk*, i.e., $\rho_i = 1$, and thus the transformation matrix can be decomposed to

$$\mathbf{T} = \mathbf{T}_R \mathbf{T}_D, \tag{15}$$

where

$$\mathbf{T}_R = \begin{bmatrix} 1 & \cos\theta_1 & \sin\theta_1 & \cdots & \cos(m\theta_1) & \sin(m\theta_1) \\ 1 & \cos\theta_2 & \sin\theta_2 & \cdots & \cos(m\theta_2) & \sin(m\theta_2) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \cos\theta_n & \sin\theta_n & \cdots & \cos(m\theta_n) & \sin(m\theta_n) \end{bmatrix}, \tag{16}$$

$$\mathbf{T}_D = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{1}{R_{2m}}\right)^m & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \left(\frac{1}{R_{2m+1}}\right)^m \end{bmatrix}. \tag{17}$$

It is interesting to observe that $\mathbf{T}$ is the result of a transformation first by a dilation matrix $\mathbf{T}_D$, followed by a transformation by a rotation matrix $\mathbf{T}_R$. If $R_i = 1,\ i = 1,\ldots,n$, then $\mathbf{T}_D$ reduces to an identity matrix $\mathbf{I}_n$.
Furthermore, due to the orthogonal property of $\mathbf{T}_R$ we have

$$\mathbf{T}_R^{\mathrm{T}}\mathbf{T}_R = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & \frac{n}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{n}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n}{2} \end{bmatrix}. \tag{18}$$

Through some derivations we can prove that [Liu (2008b)]

$$\mathbf{T}_R^{-1} = \frac{2}{n} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 & \cdots & \cos\theta_n \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 & \cdots & \sin\theta_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cos(m\theta_1) & \cos(m\theta_2) & \cos(m\theta_3) & \cdots & \cos(m\theta_n) \\ \sin(m\theta_1) & \sin(m\theta_2) & \sin(m\theta_3) & \cdots & \sin(m\theta_n) \end{bmatrix}. \tag{19}$$

Hence, from Eq. (15) it follows that

$$\mathbf{T}^{-1} = \mathbf{T}_D^{-1}\mathbf{T}_R^{-1}, \tag{20}$$

where $\mathbf{T}_D^{-1}$ is easily calculated from Eq. (17).

Thus the conditioners $\mathbf{T}$ *and* $\mathbf{T}^{-1}$ *are easily defined in closed-form, and may be universally employed to solve an ill-conditioned system of equations,* $\mathbf{Ax} = \mathbf{b}$*, as outlined below.*

4 Presently proposed preconditioners and postconditioners to solve $\mathbf{Ax} = \mathbf{b}$, when $\mathbf{A}$ is ill-conditioned

It is known that for an ill-posed linear system

$$\mathbf{Ax} = \mathbf{b}, \tag{21}$$

it is very hard to find the solution $\mathbf{x} \in \mathbb{R}^n$ by directly inverting the system matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, when $\mathbf{A}$ is highly ill-conditioned. Therefore, some preconditioners and postconditioners have been proposed, in the past literature, to reduce the ill-posedness of the above system [Calvetti, Reichel and Shuibi (2005)]. The choice of an appropriate conditioner for ill-posed linear systems is absolutely not an easy task. To the best of our knowledge, there does not appear a systematic algorithm to find these conditioners.

In this paper we propose to use $\mathbf{T}$ in Eqs. (15), (16) and (17), and its inverse matrix $\mathbf{T}^{-1}$ of Eq. (20) as the conditioners, and the several numerical examples given below will verify that our strategy is effective.

By inserting Eq. (13) for $\mathbf{x}$ into Eq. (21) we have

$$\mathbf{By} = \mathbf{b}, \tag{22}$$

where the new system matrix is

$$\mathbf{B} := \mathbf{AT}. \tag{23}$$

We solve Eq. (22) for $\mathbf{y}$ and then find the solution of $\mathbf{x}$ by

$$\mathbf{x} = \mathbf{Ty}. \tag{24}$$

The roles of the vectors $\mathbf{x}$ and $\mathbf{y}$ in Eq. (13) can be interchanged, in as much as when one is viewed as an input, another one is viewed as an output; hence, we may also write

$$\mathbf{Tx} = \mathbf{y}, \tag{25}$$

by viewing $\mathbf{x}$ as an input and $\mathbf{y}$ as an output. Similarly, by inserting Eq. (25) for $\mathbf{x}$ into Eq. (21) we have

$$\mathbf{B}\mathbf{y} = \mathbf{b}, \tag{26}$$

where the new system matrix is

$$\mathbf{B} := \mathbf{A}\mathbf{T}^{-1}. \tag{27}$$

We solve Eq. (26) for $\mathbf{y}$ and then find the solution of $\mathbf{x}$ by

$$\mathbf{x} = \mathbf{T}^{-1}\mathbf{y}. \tag{28}$$

The above two strategies are known as the postconditioning (or right-preconditioning) techniques.

We can also give a preconditioner of Eq. (21) by $\mathbf{T}$:

$$\mathbf{B}\mathbf{x} = \mathbf{T}\mathbf{b}, \tag{29}$$

where the new system matrix is

$$\mathbf{B} := \mathbf{T}\mathbf{A}. \tag{30}$$

Then we solve Eq. (29) directly for $\mathbf{x}$. Similarly, a preconditioning of Eq. (21) by $\mathbf{T}^{-1}$ leads to

$$\mathbf{B}\mathbf{x} = \mathbf{T}^{-1}\mathbf{b}, \tag{31}$$

where the new system matrix is

$$\mathbf{B} := \mathbf{T}^{-1}\mathbf{A}. \tag{32}$$

The above two strategies are known as the preconditioning (or left-preconditioning) techniques.

For definiteness we write the above four different system matrices by

$$\mathbf{B}_1 := \mathbf{A}\mathbf{T},\ \ \mathbf{B}_2 := \mathbf{A}\mathbf{T}^{-1},\ \ \mathbf{B}_3 := \mathbf{T}\mathbf{A},\ \ \mathbf{B}_4 := \mathbf{T}^{-1}\mathbf{A}. \tag{33}$$

Correspondingly, the conditioner is called a $\mathbf{B}_i$-conditioner, if $\mathbf{B}_i$ is used as the new system matrix, instead of the ill-conditioned matrix $\mathbf{A}$.

5 The transformation matrices for even-dimensional system

In the derivations of Sections 3 and 4 we have assumed that the dimension of the linear system is $n = 2m+1$.

When we are concerned with the solution of an even-dimensional system, we can let $n = 2m$. In order to obtain the transformation matrices for such an even-dimensional system, we can simply drop out the first column of $\mathbf{T}$ in Eq. (14), where we define a set of the new angles of θ_k by

$$\theta_k = \frac{2k\pi}{n+1}, \quad n = 2m. \tag{34}$$

Consequently, while we drop out the first column of $\mathbf{T}_R$ in Eq. (16), the first row of $\mathbf{T}_R^{-1}$ in Eq. (19) is dropped out, and the first column and the first row of $\mathbf{T}_D$ in Eq. (17) is dropped out as well.

Alternatively, we can consider a Neumann boundary condition given by

$$u_n(\rho,\theta) = x(\theta), \quad 0 \leq \theta \leq 2\pi, \tag{35}$$

where n is the outward-normal direction of the boundary. Through some effort we can derive [Liu (2008c)]

$$u_n(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}\right], \tag{36}$$

where

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta)+[\rho'(\theta)]^2}}. \tag{37}$$

For the Neumann boundary condition, the general solution of u is also given by Eq. (9), but the constant term a_0/R_1 should be deleted there. By imposing condition (35) we can obtain

$$\sum_{k=1}^{m}[a_k E_k(\theta) + b_k F_k(\theta)] = x(\theta), \tag{38}$$

where

$$E_k(\theta) := \left(\frac{\rho(\theta)}{R_{2k}}\right)^k \left[\eta\frac{k}{\rho}\cos k\theta + \eta\frac{k\rho'}{\rho^2}\sin k\theta\right], \tag{39}$$

$$F_k(\theta) := \left(\frac{\rho(\theta)}{R_{2k+1}}\right)^k \left[\eta\frac{k}{\rho}\sin k\theta - \eta\frac{k\rho'}{\rho^2}\cos k\theta\right]. \tag{40}$$

When the problem domain is a unit disk, i.e., $\rho(\theta) = 1$, we have $\eta = 1$ and $\rho' = 0$, and by the collocation technique we can obtain a relation between $\mathbf{x} = [x_1, x_2, \ldots, x_{2m-1}, x_{2m}]^{\mathrm{T}}$ and $\mathbf{y} = [a_1, b_1, \cdots, a_m, b_m]^{\mathrm{T}}$ given by Eq (13), but with the following transformation matrix:

$$\mathbf{T} := \begin{bmatrix} \frac{1}{R_2}\cos\theta_1 & \frac{1}{R_3}\sin\theta_1 & \ldots & k\left(\frac{1}{R_{2k}}\right)^k \cos k\theta_1 & k\left(\frac{1}{R_{2k+1}}\right)^k \sin k\theta_1 & \ldots & m\left(\frac{1}{R_{2m}}\right)^m \cos m\theta_1 & m\left(\frac{1}{R_{2m+1}}\right)^m \sin m\theta_1 \\ \frac{1}{R_2}\cos\theta_2 & \frac{1}{R_3}\sin\theta_2 & \ldots & k\left(\frac{1}{R_{2k}}\right)^k \cos k\theta_2 & k\left(\frac{1}{R_{2k+1}}\right)^k \sin k\theta_2 & \ldots & m\left(\frac{1}{R_{2m}}\right)^m \cos m\theta_2 & m\left(\frac{1}{R_{2m+1}}\right)^m \sin m\theta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{R_2}\cos\theta_{n-1} & \frac{1}{R_3}\sin\theta_{n-1} & \ldots & k\left(\frac{1}{R_{2k}}\right)^k \cos k\theta_{n-1} & k\left(\frac{1}{R_{2k+1}}\right)^k \sin k\theta_{n-1} & \ldots & m\left(\frac{1}{R_{2m}}\right)^m \cos m\theta_{n-1} & m\left(\frac{1}{R_{2m+1}}\right)^m \sin m\theta_{n-1} \\ \frac{1}{R_2}\cos\theta_n & \frac{1}{R_3}\sin\theta_n & \ldots & k\left(\frac{1}{R_{2k}}\right)^k \cos k\theta_n & k\left(\frac{1}{R_{2k+1}}\right)^k \sin k\theta_n & \ldots & m\left(\frac{1}{R_{2m}}\right)^m \cos m\theta_n & m\left(\frac{1}{R_{2m+1}}\right)^m \sin m\theta_n \end{bmatrix}. \quad (41)$$

This transformation matrix can be decomposed into

$$\mathbf{T} = \mathbf{T}_R \mathbf{T}_D, \quad (42)$$

where

$$\mathbf{T}_R = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & \cdots & \cos(m\theta_1) & \sin(m\theta_1) \\ \cos\theta_2 & \sin\theta_2 & \cdots & \cos(m\theta_2) & \sin(m\theta_2) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \cos\theta_n & \sin\theta_n & \cdots & \cos(m\theta_n) & \sin(m\theta_n) \end{bmatrix}, \quad (43)$$

$$\mathbf{T}_D = \begin{bmatrix} \frac{1}{R_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{R_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m\left(\frac{1}{R_{2m}}\right)^m & 0 \\ 0 & 0 & 0 & \cdots & 0 & m\left(\frac{1}{R_{2m+1}}\right)^m \end{bmatrix}. \quad (44)$$

Similarly, the four different conditioners listed in Eq. (33) can employ the matrix $\mathbf{T}$ in Eq. (41), while dealing with an even-ordered system. Up to here, we have finished the theoretical developments of the new conditioners. The main ideas are viewing the unknown $\mathbf{x}$ in Eq. (21) as a set of discrete boundary values for the Laplace equation, and using a multi-scale expansion and collocation technique to find the relation between $\mathbf{x}$ and the coefficients $\mathbf{y}$ in the multi-scale Trefftz solution of the Laplace equation. Therefore, we may name this new technique as a Multi-Scale Trefftz-Collocation Laplacian Conditioner (MSTCLC). Through some calculations, we find the condition number of $\mathbf{T}$ to be in the order of n, which is very low, and this $\mathbf{T}$ is suitable to reduce the condition number of the ill-posed linear systems.

6 Numerical tests

Some well-known numerical examples which lead to the solution of equations of the type of Eq. (21) with a highly ill-conditioned $\mathbf{A}$ are investigated in this section.

A measure of the ill-posedness of Eq. (21) can be gauged by calculating the condition number of $\mathbf{A}$ [Stewart (1973)]:

$$\text{cond}(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|. \tag{45}$$

Here, $\|\mathbf{A}\|$ is the Frobenius norm of $\mathbf{A}$ defined by $\|\mathbf{A}\| := \sqrt{\sum_{i,j=1}^{n} A_{ij}^2}$, where A_{ij} is the ij-th component of $\mathbf{A}$. The Frobenius norm of a matrix is a direct extension of the Euclidean norm for a vector. For arbitrary $\varepsilon > 0$, there exists a matrix norm $\|\mathbf{A}\|$ such that $\rho(\mathbf{A}) \le \|\mathbf{A}\| \le \rho(\mathbf{A}) + \varepsilon$, where $\rho(\mathbf{A})$ is a radius of the spectrum of $\mathbf{A}$. Therefore, the condition number of $\mathbf{A}$ can be estimated by

$$\text{cond}(\mathbf{A}) = \frac{\max_{\sigma(\mathbf{A})} |\lambda|}{\min_{\sigma(\mathbf{A})} |\lambda|}, \tag{46}$$

where $\sigma(\mathbf{A})$ is the collection of the eigenvalues of $\mathbf{A}$.

Eq. (21) with the matrix $\mathbf{A}$ having a large condition number, usually implies that an arbitrarily small perturbation in the forcing function $\mathbf{b}$ may lead to an arbitrarily large perturbation of the solution vector $\mathbf{x}$. Speaking roughly, the numerical solution of Eq. (21) may lose the accuracy of k decimal points when $\text{cond}(\mathbf{A}) = 10^k$.

Without exception, the conjugate gradient method (CGM) is employed here to solve the linear equations and to find the inverse matrices. Initial guesses for $\mathbf{x}$ and for $\mathbf{y}$ are both to be zero values.

6.1 A slender ellipse

In order to validate the multi-scale Trefftz method (MSTM) proposed in Section 2, we first consider an ellipse with semiaxes a and b, and the contour in the polar coordinates is described by

$$\rho(\theta) = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}. \tag{47}$$

For comparison purpose, the exact solution of the Laplace equation is given by

$$u(x,y) = e^y\cos x = e^{r\sin\theta}\cos(r\cos\theta), \tag{48}$$

and thus the Dirichlet data on the whole boundary can be obtained by inserting Eq. (47) for r into the above equation.

We fix $a = 200b = 200$. In Fig. 1(a) we compare the exact solution with the numerical solutions, along a unit circle inside the slender ellipse, obtained by a single-characteristic-length Trefftz expansion with $R_0 = 400$ and by a multiple-characteristic-length Trefftz expansion with $R_1 = 1$, and $R_{2k} = R_{2k+1} = \rho(\theta_{2k}) + R_0$, where $R_0 = 50$. For both cases we use $m = 25$. The convergence criteria used in the CGM for the solutions of coefficients are both given by 10^{-14}. From Fig. 1(b) it can be seen that the accuracy obtained by the multiple-characteristic-length Trefftz expansion is increased by almost three orders as compared to that obtained by the single-characteristic-length Trefftz expansion.

6.2 Hilbert matrices

The problems with an ill-conditioned $\mathbf{A}$ may appear in several fields. For example, finding an $(n-1)$-order polynomial function $p(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1}$ to best match a continuous function $f(x)$ in the interval of $x \in [0,1]$:

$$\min_{\deg(p)\leq n-1}\int_0^1 |f(x) - p(x)|dx, \tag{49}$$

leads to a problem governed by Eq. (21), where $\mathbf{A}$ is the $n\times n$ Hilbert matrix defined by

$$A_{ij} = \frac{1}{i-1+j}, \tag{50}$$

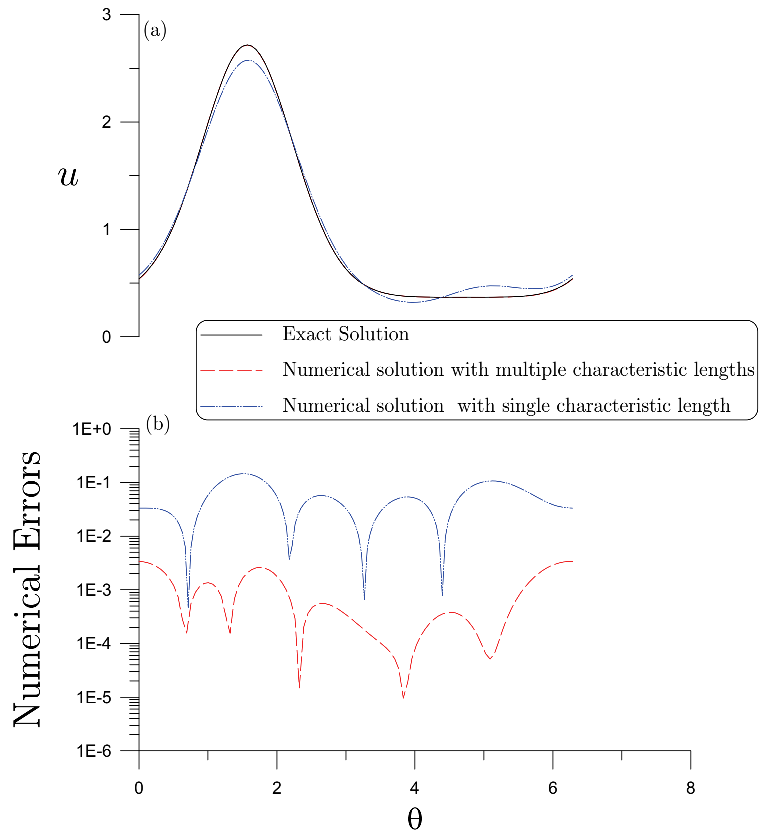

Figure 1: For a very slender ellipse with an aspect ratio 200, (a) comparison of the exact solution and numerical solutions obtained by a Single-Characteristic-Length Trefftz expansion and by a Multiple-Characteristic Length Trefftz expansion, and (b) comparison of the numerical errors.

$\mathbf{x}$ is composed of the n coefficients $a_0, a_1, \ldots, a_{n-1}$ appearing in $p(x)$, and

$$\mathbf{b} = \begin{bmatrix} \int_0^1 f(x)dx \\ \int_0^1 xf(x)dx \\ \vdots \\ \int_0^1 x^{n-1}f(x)dx \end{bmatrix} \tag{51}$$

is uniquely determined by the function $f(x)$.

The Hilbert matrix is a well-known example of a highly ill-conditioned matrix, which can be seen from the fact that $\text{cond}(\mathbf{A}) = 1.1748 \times 10^{19}$ when $n = 50$, and $\text{cond}(\mathbf{A}) = 10^{348}$ when $n = 200$. In general, $\text{cond}(\mathbf{A}) = e^{3.5n}$ when n is large.

For this case we take $\mathbf{T}_D = \mathbf{I}_n$. In Fig. 2 we plot the condition numbers of $\mathbf{B}_i$, $i = 1,2,3,4$ with respect to n. It can be seen that for those four matrices the condition numbers are controlled with a certain range of 10^5 to 10^{14}. The condition numbers of $\mathbf{B}_1$ and $\mathbf{B}_3$ are of the same orders and are smaller than that of $\mathbf{B}_2$ and $\mathbf{B}_4$.

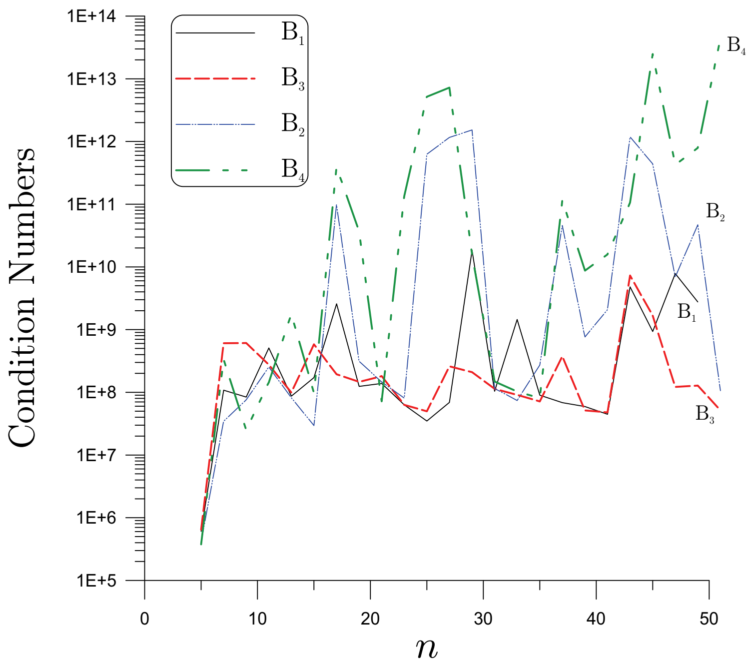

Figure 2: Comparing the condition numbers of four different conditioners with respect to the dimensions $n = 2m+1$, of the Hilbert matrices.

Encouraged by the above well-conditioning behavior of the Hilbert matrices after the MSTCLC conditioning, now, we are ready to solve a very difficult problem of a best approximation of the function e^x by an $(n-1)$-order polynomial, where we fix $n = 50$. The Neumann type $\mathbf{B}_3$-conditioner is used in this calculation, where the dilation scales are taken to be $R_{2i} = R_{2i+1} = R_0 = 10000$. We compare the exact solution e^x with the numerical solutions without noise and with a noise $\sigma = 0.01$ in Fig. 3(a). The absolute errors are also shown in Fig. 3(b). The results are rather good. The original technique using $p(x) = a_0 + a_1 x/R_0 + \ldots + a_{n-1}(x/R_0)^{n-1}$ by Liu and Atluri (2009b) for the polynomial interpolation could not be applied here to solve the best approximation by using very-high-order polynomial. Indeed, we

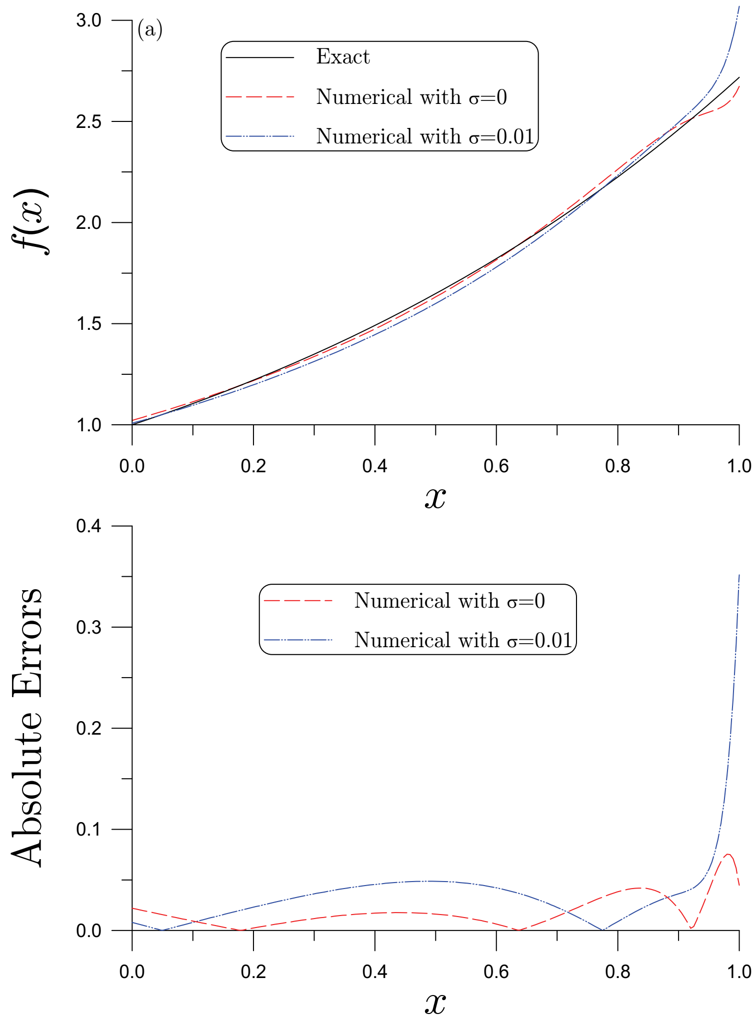

Figure 3: The best polynomial approximation by using the $\mathbf{B}_3$-conditioner: (a) comparing the exact and numerical results, and (b) the numerical errors.

have found that the technique of Liu and Atluri (2009b) leads to a huge error in the order of 10^{45}. The reason may be that in the polynomial interpolation of Liu and Atluri (2009b), the resulting Vandermonde matrix can be well-conditioned by the dilation matrix alone (see also Section 6.4 below), but in the best polynomial approximation method of Eq. (49), the resulting Hilbert matrix cannot be well-conditioned by a dilation matrix alone, but must also be accompanied by a rotation matrix.

In order to compare the numerical solutions with exact solutions we suppose that $x_1 = x_2 = \ldots = x_n = 1$, and then by Eq. (50) we have

$$b_i = \sum_{j=1}^{n} \frac{1}{i+j-1}. \tag{52}$$

Liu and Chang (2009) have calculated this problem by taking $n = 200$, and found that most algorithms failed. The numerical results for $m = 100$ ($n = 201$) calculated by the $\mathbf{B}_1$-conditioner are plotted in Fig. 4 by displaying the numerical errors. If no noise is involved, the accuracy is very good, and smaller than 0.006. When a large noise 0.001 is added on the data $\mathbf{b}$, the errors are still smaller than 0.1.

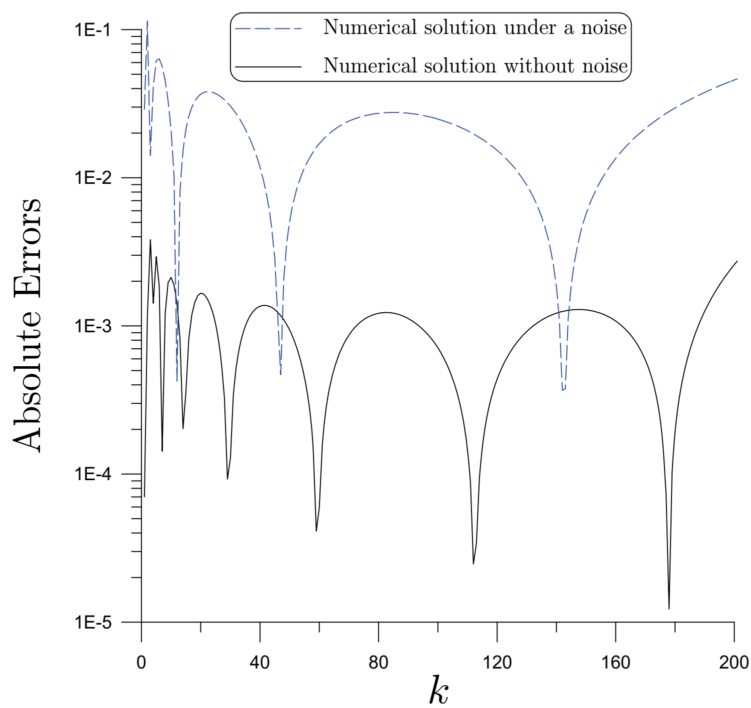

Figure 4: Numerical errors, without/with noise, for a very large $n = 201$ of the Hilbert matrix, solved by a $\mathbf{B}_1$-conditioner.

Next, we consider

$$x_i = 2\sin(p_i)\exp[p_i(1-p_i)], \quad p_i = i \times \frac{1}{n},$$

$$b_i = \sum_{j=1}^{n} \frac{1}{i+j-1} x_j + \sigma R(i) \tag{53}$$

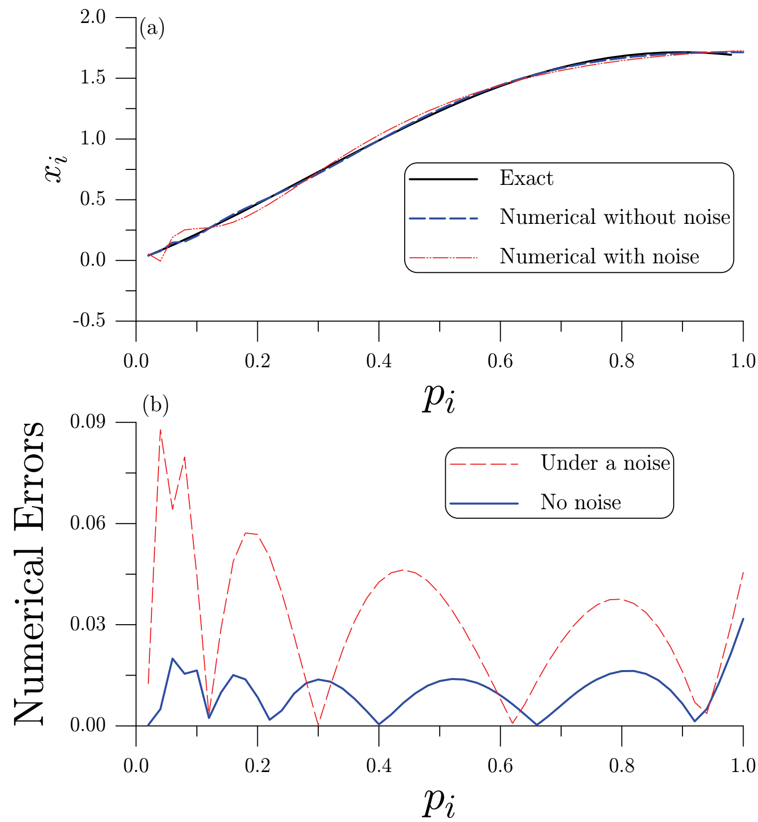

Figure 5: Solution of the Hilbert matrix with dimension $n = 100$ by using a $\mathbf{B}_3$-conditioner: (a) comparing the exact and numerical solutions without/with noise, and (b) the numerical errors.

with $m = 25$ ($n = 50$) and $0 < p_i \le 1$. This problem is more difficult than the previous one with constant $x_1 = \ldots = x_n = 1$.

We apply the technique in Section 5 to this problem by choosing $R_{2k} = R_{2k+1} = (kR_0^k)^{1/k}$, $k = 1, \ldots, m$, where $R_0 = 1.1$ is used, and using $\mathbf{T}$ as a preconditioner. When there is no noise, the numerical result is very accurate as comparing it with the exact solution in Fig. 5(a). The error as shown in Fig. 5(b) is smaller than 0.014. When the noise is imposed in a level of $\sigma = 10^{-4}$, we use the CGM with a loose convergent criterion 10^{-5}. The result is plotted in Fig. 5(a) by the dashed-dotted line, and the error as shown in Fig. 5(b) by the dashed line is smaller than 0.09. These results are better than that calculated by Liu and Chang (2009).

6.3 *The problem of ill-conditioned matrices* **A** *arising in the Method of Fundamental Solutions*

In the potential theory, it is well known that the method of fundamental solutions (MFS) can be used to solve the Laplace problems when a fundamental solution is known.

The MFS has a broad application in engineering computations. However, the MFS has a serious problem in that the resulting system of linear equations is always highly ill-conditioned, when the number of source points is increased [Golberg and Chen (1996)], or when the distances between the source points are increased [Chen, Cho and Golberg (2006)]. The convergence analysis of MFS has demonstrated that the approximation improves when the source radius tends to infinity; see. e.g., Smyrlis and Karageorghis (2004). Nevertheless, a commonly encountered problem is its poor accuracy as the source radius is increased to a large vaule in the numerical computation. The ill-conditioning of the MFS makes it very difficult to achieve very accurate approximations by the numerical solutions of the boundary value problems.

In the MFS the solution of u at the field point $z=(r\cos\theta, r\sin\theta)$ can be expressed as a linear combination of fundamental solutions $U(z,s_j)$:

$$u(z)=\sum_{j=1}^{n} c_j U(z,s_j), \quad s_j\in\Omega^c, \tag{54}$$

where $n=2m+1$ is the number of source points, c_j are the unknown coefficients, and s_j are the source points located in the complement Ω^c of Ω. For the Laplace equation we have the fundamental solutions

$$U(z,s_j)=\ln r_j, \quad r_j=|z-s_j|. \tag{55}$$

In a practical application of MFS, usually the source points are uniformly located on a circle with a radius R_0, such that after imposing the boundary condition with $u|_{\partial\Omega}=h(\theta)$ on Eq. (54) we obtain a linear equations system:

$$\mathbf{A}\mathbf{x}=\mathbf{b}, \tag{56}$$

where

$$A_{ij}=U(z_i,s_j), \quad \mathbf{x}=(c_1,\cdots,c_n)^{\mathrm{T}}, \quad \mathbf{b}=(h(\theta_1),\cdots,h(\theta_n))^{\mathrm{T}}, \tag{57}$$

and $z_i=(\rho(\theta_i)\cos\theta_i, \rho(\theta_i)\sin\theta_i)$, $s_j=(R_0\cos\theta_j, R_0\sin\theta_j)$.

In this example we consider a complex epitrochoid with the boundary shape

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)}, \tag{58}$$

$$x(\theta) = \rho\cos\theta, \quad y(\theta) = \rho\sin\theta \tag{59}$$

with $a = 4$ and $b = 1$. For comparison we also consider an exact solution, given by

$$u(x,y) = e^x \cos y. \tag{60}$$

The exact boundary data can be obtained by inserting Eqs. (58) and (59) into the above equation.

For this case by using the $\mathbf{B}_1$-conditioner with a postconditioning matrix $\mathbf{T}$, we take $R_1 = 1/(n\ln R_0)$, $R_{2k} = R_{2k+1} = 1/(2k/n)^{1/k}$, $k = 1,\ldots,m$, such that $\mathbf{T}_D$ is not an identity matrix. The condition numbers are compared in Fig. 6. It can be seen that the postconditioning technique used here can reduce the condition numbers about three orders. Numerical solutions and the exact solution along a circle with a radius 3 are compared in Fig. 7(a), and the numerical errors are shown in Fig. 7(b), from which it can be seen that the accuracy by using the postconditioning technique is raised about four orders. Liu (2008b) has proposed a modification technique of the MFS, but has not mentioned its mathematical foundation as presented here from the view point of the transformation matrix obtained from the Laplace equation.

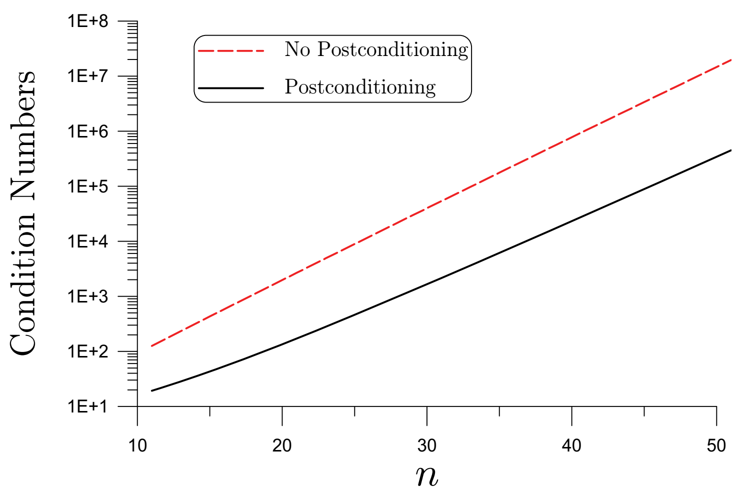

Figure 6: Comparing the condition numbers with respect to n, for the matrices arising in the MFS, without/with a $\mathbf{B}_1$-conditioner.

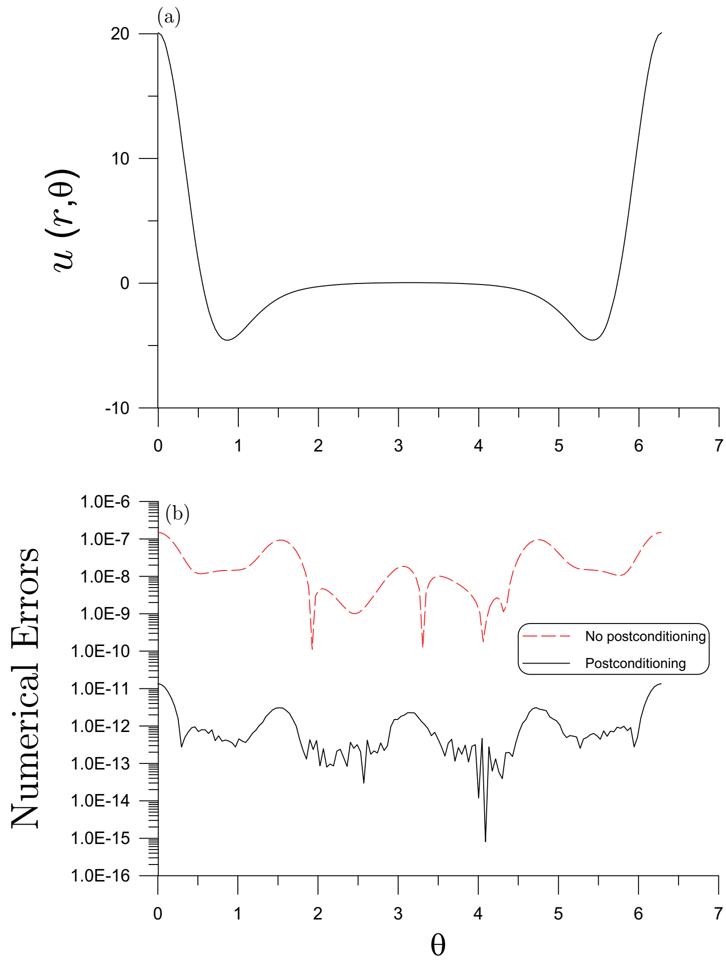

Figure 7: Solution by MFS and using a $\mathbf{B}_1$-conditioner: (a) comparing the exact and numerical solutions, and (b) the numerical errors without/with a postconditioner.

6.4 Polynomial interpolation

Polynomial interpolation is the interpolation of a given data set by a polynomial. In other words, given some data points, such as obtained by sampling of a measurement, the aim is to find a polynomial which goes exactly through these points.

Given a set of n data points (x_i, y_i) where no two x_i are the same, one is looking for a polynomial $p(x)$ of degree at most $n-1$ with the following property:

$$p(x_i) = y_i, \quad i = 1, \ldots, n, \tag{61}$$

where $x_i \in [a,b]$, and $[a,b]$ is a spatial interval of our problem domain.

The unisolvence theorem states that such a polynomial $p(x)$ exists and is unique, and can be proved by using the Vandermonde matrix. Suppose that the interpolation polynomial is in the form of

$$p(x) = \sum_{i=1}^{n} c_i x^{i-1}, \tag{62}$$

where x^i constitute a monomial basis. The statement that $p(x)$ interpolates the data points means that Eq. (61) must hold.

If we substitute Eq. (62) into Eq. (61), we obtain a system of linear equations in the coefficients c_i. The system in a matrix-vector form reads as

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-2} & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}. \tag{63}$$

We have to solve the above system for c_i to construct the interpolant $p(x)$. The matrix transpose on the left is commonly referred to as a Vandermonde matrix denoted by $\mathbf{V}$, which is known to be highly ill-conditioned.

For this case we use the $\mathbf{B}_2$-conditioner by taking

$$\mathbf{T}_D^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{r_1} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{r_{n-2}^{n-2}} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{r_{n-1}^{n-1}} \end{bmatrix}, \tag{64}$$

where $r_i = |x_{i+1} \pm r_0|$; the minus sign is used for $x_{i+1} < 0$, while the plus sign is used for $x_{i+1} \geq 0$, and r_0 is a positive constant.

In order to show the accuracy of the new postconditioning technique we consider the following interpolated function:

$$f(x) = \frac{1}{1+(x-1)^2}, \quad 0 \leq x \leq 2. \tag{65}$$

We first show the condition numbers of $\mathbf{B}_2$ with respect to n in Fig. 8(a). It is interesting that the condition numbers decrease when n increases. We take $m = 30$

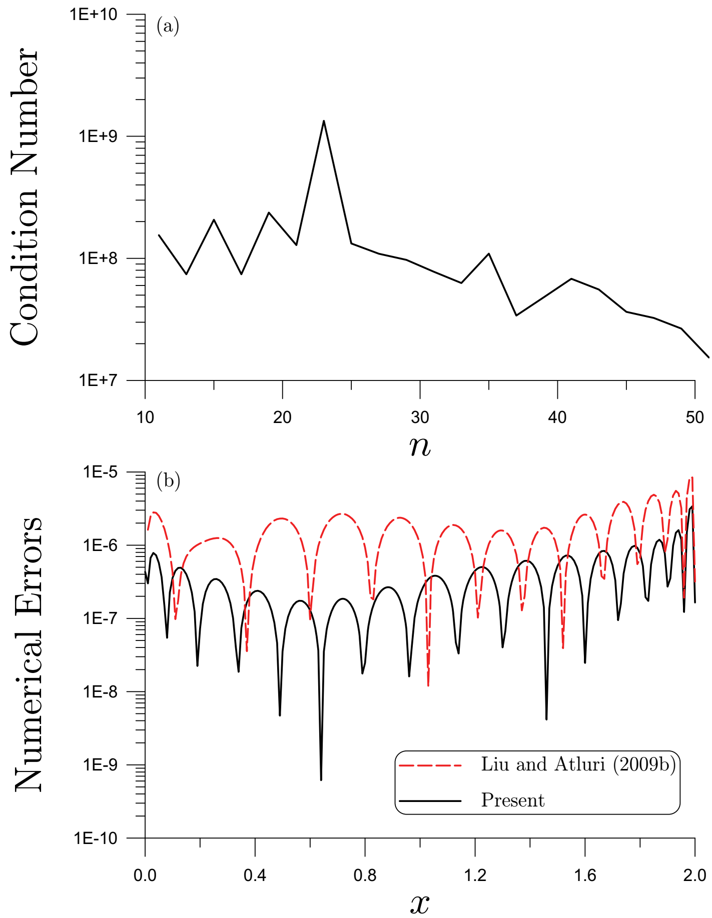

Figure 8: For an $(n-1)$-degree polynomial interpolation: (a) the condition numbers with respect to n obtained by a $\mathbf{B}_2$-conditioner, (b) the numerical errors for an example.

$(n = 61)$ and $r_0 = 1$. The absolute error is plotted in Fig. 8(b). Very accurate results are obtained, where the maximum error is about 3.4×10^{-6}.

Recently, Liu and Atluri (2009b) have proposed a novel interpolation technique of very-high-order polynomial by using

$$p(x) = \sum_{\alpha=0}^{n} \bar{a}_\alpha \left(\frac{x}{R_0}\right)^\alpha, \tag{66}$$

where R_0 is a characteristic length. Under the parameters $n = 60$ and $R_0 = 2.8$

we show the accuracy for the function in Eq. (65) in Fig. 8(b) by the dashed line, where the maximum error is about 8.4×10^{-6}. It is also very accurate, but the present MSTCLC is slightly better.

6.5 The first-kind Fredholm integral equation

To demonstrate the applications of the new conditioning theory, we further consider a highly ill-posed first-kind Fredholm integral equation:

$$\int_a^b K(s,t)x(t)dt = h(s), \ s \in [c,d]. \tag{67}$$

Let us discretize the intervals of $[a,b]$ and $[c,d]$ into $n-1$ subintervals by noting that $\Delta t = (b-a)/(n-1)$ and $\Delta s = (c-d)/(n-1)$. Let $x_j := x(t_j)$ be a numerical value of x at a grid point t_j, and let $K_{i,j} = K(s_i,t_j)$ and $h_i = h(s_i)$, where $t_j = a+(j-1)\Delta t$ and $s_i = c+(i-1)\Delta s$. Through a trapezoidal rule, Eq. (67) can be discretized into the following linear algebraic equations:

$$\frac{\Delta t}{2}K_{i,1}x_1 + \Delta t\sum_{j=2}^{n-1} K_{i,j}x_j + \frac{\Delta t}{2}K_{i,n}x_n = h_i, \ i = 1,\ldots,n. \tag{68}$$

Then, we solve

$$\int_0^\pi e^{s\cos t}x(t)dt = \frac{2}{s}\sinh s, \ s \in [0,\pi/2], \tag{69}$$

which has an exact solution $x(t) = \sin t$.

For this case we take $R_i = 10$, $i = 1,\ldots,n$, and use $m = 80$ ($n = 161$) in the $\mathbf{B}_1$-conditioner in Section 4. The numerical error is shown in Fig. 9(a), which has an accuracy in the order of 10^{-2}. This result is competive with that calculated by Liu and Atluri (2009a) by using a fictitious time integration method.

As a second example, we consider the problem of finding $x(t)$ in the following equation, solving by a $\mathbf{B}_3$-conditioner in Section 5:

$$\int_0^1 k(s,t)x(t)dt = \frac{1}{6}(s^3 - s), \ s \in [0,1], \tag{70}$$

where

$$k(s,t) = \begin{cases} s(t-1) & \text{if } s < t, \\ t(s-1) & \text{if } s \geq t, \end{cases} \tag{71}$$

and $x(t) = t$ is the exact solution.

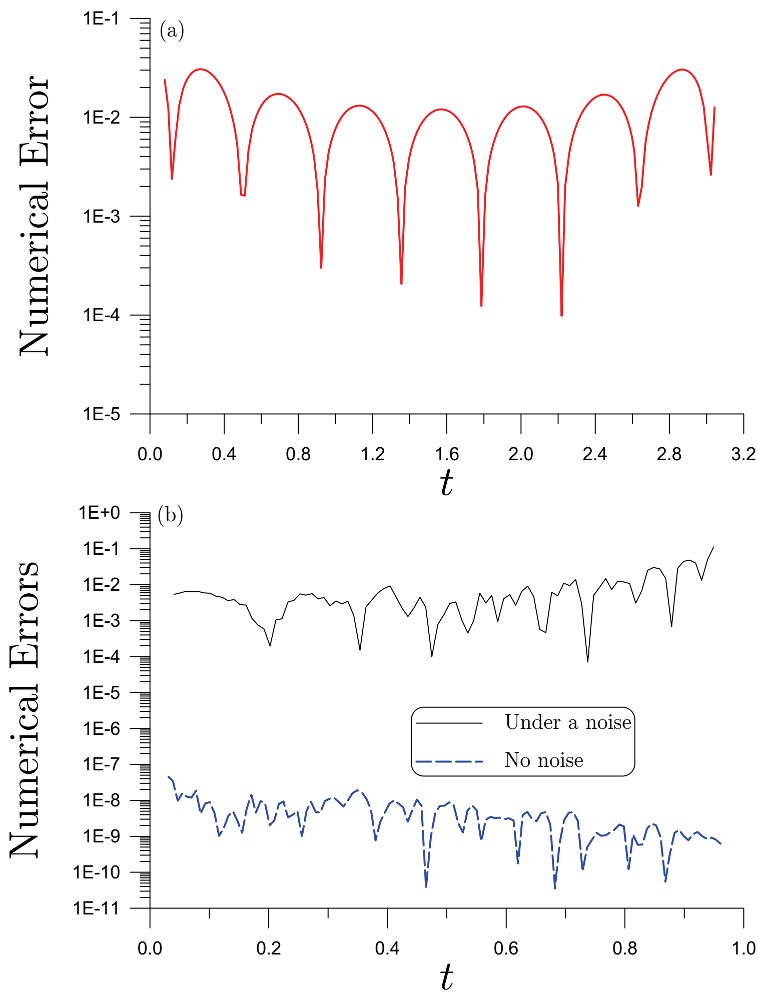

Figure 9: The numerical errors in solving the first-kind Fredholm integral equations (a) by using a $\mathbf{B}_1$-conditioner, and (b) solving by a $\mathbf{B}_3$-conditioner in Section 5.

In Fig. 9(b) we show the numerical errors under a noise $\sigma = 0.001$ by the solid line, and a dashed line for the un-noised case, where $m = 50$ ($n = 100$) were used for both cases. As compared with the results calculated by Calvetti, Reichel, Shuibi (2005) without considering noise, our result with an L_2-norm error 4.6×10^{-8} for the same case is much improved with about five orders.

7 Conclusions

Novel "Multi-Scale Trefftz-Collocation Laplacian Conditioners" (MSTCLCs) are developed in this paper, for the first time, to overcome the ill-posedness of severely ill-conditioned system of linear equations: $\mathbf{Ax} = \mathbf{b}$. We considered $\mathbf{x}$ to be a finite-dimensional set of discrete boundary values, on a unit circle, of the Laplace equation. We propose a Trefftz expansion involving multiple-characteristic-lengths to solve the Laplace equation in an arbitrary plane domain under Dirichlet or Neumann boundary conditions. By using the collocation technique, we could find the transformation matrix $\mathbf{T}$ between $\mathbf{x}$ and $\mathbf{y}$ as shown in Eq. (13). The transformation matrix $\mathbf{T}$ can be decomposed into a dilation matrix $\mathbf{T}_D$ followed by a rotation matrix $\mathbf{T}_R$. For an odd-dimensional linear system, four conditioners $\mathbf{B}_i,\ i = 1,2,3,4$ were derived in Section 4 based on the Dirichlet boundary condition, and similarly, for an even-dimensional linear system, four $\mathbf{B}_i,\ i = 1,2,3,4$ were derived in Section 5 based on the Neumann boundary condition. These conditioners have very low condition numbers and are invertible, thus greatly reducing the condition numbers of the ill-posed linear systems investigated in this paper, which include the ill-posed linear systems obtained from the Hilbert matrices, from the method of fundamental solutions, from polynomial interpolations, and from solving the first-kind Fredholm integral equations. When the new conditioners are applied, we found that the condition numbers do not increase fast or not increase at all with respect to the dimension n of the investigated systems. Therefore, we can obtain very accurate solutions of the ill-posed linear systems, even when a large noise exists in the given data. In this new theory, there exists a beautiful structure of *four dualities*: $\cos\theta$ and $\sin\theta$, dilation and rotation, Dirichlet and Neumann, and odd and even dimension. The new algorithms have better computational efficiency and accuracy, which may be applicable to many engineering problems with ill-posedness. Even for a middling ill-posed system of linear equations, the MSTCLC can increase the accuracy several orders.

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