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Numerical Study of Residual Correction Method Applied to Non-linear Heat Transfer Problem

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Abstract: This paper seeks to utilize the residual correction method in coordination with the evolutionary monotonic iteration technique to obtain upper and lower approximate solutions of non-linear heat transfer problem of the annular hyperbolic profile fins whose thermal conductivity vary with temperature. First, the monotonicity of a non-linear differential equation is reinforced by using the monotone iterative technique. Then, the cubic spline method is applied to discretize and convert the differential equation into the mathematical programming problems. Finally, based on the residual correction concept, the complicated constraint inequality equations can be transferred into the simple iterative equations. As verified by this problem, hereof the method proposed can be efficiently obtained the upper and lower approximate solutions of nonlinear heat transfer problems, and easily identify the error range between mean approximate solutions and exact solutions.

Keywords: cubic spline, monotone, residual, upper and lower approxiamte solution, mathematical programming

Nomenclature

E_p	maximum possible error
E_r	maximum actual relative error
E_{re}	maximum iteration error
$f(x)$	function
$g(x)$	boundary condition
m	dimensionless parameter,
N	number of grid points
$R(x)$	residual of a differential equation
R_i	residual correction value on a grid point

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Greek symbols

ε	dimensionless thermal conductivity
θ	dimensionless temperature
η	dimensionless radius

Superscripts

m	iteration times for residual correction
$\wedge$	initially assumed function or value
$-$	mean approximate solution
$\sim$	approximate solution
$\cup, \cap$	lower and upper approximate solution

Subscripts

i	serial number of calculation grid points

1 Introduction

Simple linear differential equation will involve into a complex nonlinear equation if radiative heat transfer on surface has to take into consideration or thermal conductivity and convection vary with the difference of temperature. That is to say that obtaining solutions of linear problems is not difficult, it is still a challenge to solve nonlinear ones, by either numerical or theoretical methods. However, there still many scholars studied on different numerical methods which included relying on Laplace heat transform to derive simple solutions of complex variable functions in relation to time, and Chiu and Chen (2002) as well as Pamuk (2005) relied on symbolic calculation to get serious solutions of approximate solutions, such as Adomian's decomposition method, and variation interaction method, derived from Khaleghi, Ganji, and Sadighi (2007). In this paper, we will develop into a more convenient and accurate way to estimate the upper and lower approximate limits of the exact solution of differential equation problems. As early as 1967, Protter has put forth the maximum which explains the relationship between the solution and residual of an equation. But according to the application of this method would be very time-consuming in calculating the inequality equations, recently Lee (2002) had successfully implemented the genetic algorithms method to confer the feasibility of using simplified equations.

In order to obtain the upper and lower solutions of the nonlinear differential equation problem, the article applies the concept of the maximum principle to establish the residual of the differential equations. At first, we will consider the nonlinear equation as follows:

$$R_{\tilde{\theta}}(x) = F(x, \tilde{u}, \tilde{u}_x, \tilde{u}_{xx}) - f(x) \text{ in } D \tag{1}$$

Boundary condition satisfies,

$$R_{\tilde{\theta}}(x) = g(x) - \tilde{\theta}(x) \text{ on } \partial D \tag{2}$$

where $R_{\tilde{\theta}}$ is known as the residual value function or residual, of the differential equation in the domain or of the surroundings of the boundary. On assumption that function $\theta(x)$ is the exact solution, Eq. (1) and Eq. (2) become the residual. The approximate functions are defined in the domain and are continuous until the second derivatives, as long as $\partial F/\partial\theta$ is continuous. And

$$\frac{\partial R}{\partial \theta} \leq 0 \text{ in } D \tag{3}$$

when

$$R_{\check{\theta}}(x) \geq R_{\theta}(x) = 0 \geq R_{\check{\theta}}(x) \text{ on } D \cup \partial D \tag{4}$$

holds, the following relations with exact solution will also hold:

$$\check{\theta}(x) \leq \theta(x) \leq \check{\theta}(x) \text{ on } D \cup \partial D \tag{5}$$

in which approximate solution $\check{\theta}(x)$ and $\check{\theta}(x)$ are called lower and upper solutions of the exact solutions $\theta(x)$, while any differential equation with such relations are considered monotonic in solutions.

In order to get the upper and lower extremes of Eq. (1) and Eq. (2), it is firstly required to probe the system that contains monotonic equations. The equation may include the nonlinear terms, so that a monotonic residual of the differential equation probably does not exist at all. Or if it does exist, the analysis would be very difficult. As far as published studies appear, no such monotonic analysis has ever been conducted. When solving the nonlinear problem, the inequality Eq. (3) may not be satisfied. In this paper we are going to apply the idea of considering the compare with the exact solutions and approximate solutions of the nonlinear heat transfer problem.

2 Residual Correction Method

The accurate and credible upper and lower approximate solution can be obtained from Eq. (3) to Eq. (5), but the mathematical programming problem of inequality equations under such constrained conditions may give much difficulty in solving the problem. There are few scholars attempting to obtain this kind of solution of mathematics programming problem. However, Lin and Chen (2003) has recently used genetic algorithm to solve this kind of inequality equation. It simplifies the inequality mathematical programming problems to be transformed into the traditional differential equations with an additional residual correction term is used in combination with iterative technique to discretize Eq. (1), and then transform these inequalities into the following equation:

$$R_i^m = F_i^{m+1}(x, u, u_x, u_{xx}) - f_i^(x) \tag{6}$$

where m stands iteration times of residual correction and the suffix of i stands the serial number of discretized calculation points. And R_i^m is the residual function of the grid point to correct the residual value and further ensure the values within adjacent subintervals ($x_{i-1} \leq x \leq x_{i+1}$) are always positive or negative. Here are complete steps to correct the residual.

Assume residual of each grid point at the first iteration time is zero: $R_i^m = 0$

Obtain F_i^{m+1}, u_i^{m+1} and its differential value within two order by the first assumption.

Residual distribution $R_i^{m+1}(x)$ used as iterative correction in the next step can be obtained.

The residual correction can be given as follows:

For getting the upper approximate solutions, each point's residual must be subtracted the minimum residual distribution:

$$R_i^{m+1} = R_i^m - \min\left(R_i^{m+1}(x)\right) \ x_{i-1} \leq x \leq x_{i+1} \tag{7}$$

For getting the lower approximate solutions, each point's residual must be subtracted the maximum residual distribution:

$$R_i^{m+1} = R_i^m - \max\left(R_i^{m+1}(x)\right) \ x_{i-1} \leq x \leq x_{i+1} \tag{8}$$

Repeat the step II to IV till the relative residual error is reached. The relative error is shown as:

$$E_{re} = \frac{\left|R_i^{m+1} - R_i^m\right|}{R_i^m} \tag{9}$$

3 The Cubic Spline

Lately, the cubic spline has been popularly used to approximate solutions to fit and fair the curves of ordinary or partial differential equations. The purpose of the cubic spline technique is to incorporate the jump condition between the interface and the second-order accurate extrapolation is applied to incorporate these jump conditions. Chawla (1975) successfully used cubic spline collocation method to solve the nonlinear transient one-dimension heat transfer problem. Rubin and Graves (1975) have used the spline technique and quasi-linearization to obtain the numerical solution of application of the partial differential equations in one space variables. In the present research, there are some scholars who have done much research into spline function about boundary problems, Johnson (2005). And Golberg and Chen (2001) used polyharmonic spline interpolants to obtain particular solutions for Helmholtz-type equations. However, in this paper, we will discuss the residual distribution within adjacent subintervals, which must be determined first before the solution of residual correction.

Since the function and derivatives obtained from traditional difference are only on grid points. They are not continuous on non-calculation points and are not applicable to this method. For this reason, the article adopts the concept of spline approximation to discretize equation. First, consider the one-dimensional second-order partial differential equation:

$$u_t = f(u, u_x, u_{xx}) \tag{10}$$

Based on the theory of the cubic spline and implicit scheme, the general linear discretize formula is:

$$u_i^{n+1} = F_i + G_i m_i^{n+1} + S_i M_i^{n+1} \tag{11}$$

where F_i, G_i, S_i are constant terms; u_i, m_i and M_i are separately indicated as the value of i, first derivative and second derivate. Based on the Wang and Kahawita (1983), the Eq. (11) can be transferred into equation only contained the second derivatives in the following form:

$$A_i M_{i-1} + B_i M_i + C_i M_{i+1} = D_i \ , \ i = 1, 2, ..., N-1 \tag{12}$$

After substituting the boundary into Eq. (12), we are going to get the following parameters relationship is obtained.

$$A_i = \frac{h_i}{6} + \frac{G_i + 2G_{i-1}}{6\Delta_i} - \frac{S_{i-1}}{h_i \Delta_i} \tag{13}$$

$$B_i = \frac{h_i + h_{i+1}}{3} - \frac{G_{i+1} + G_i}{6\Delta_{i+1}} + \frac{2G_i + G_{i-1}}{6\Delta_i} + S\left(\frac{1}{\Delta_{i+1}h_{i+1}} + \frac{1}{\Delta_i h_i}\right) \tag{14}$$

$$C_i = \frac{h_{i+1}}{6} - \frac{2G_{i+1} + G_i}{6\Delta_{i+1}} - \frac{S_{i+1}}{h_{i+1}\Delta_{i+1}} \tag{15}$$

$$D_i = \frac{F_{i+1} - F_i}{\Delta_{i+1}h_{i+1}} - \frac{F_i - F_{i-1}}{\Delta_i h_i} \tag{16}$$

$$\Delta_i = 1 - \frac{G_i - G_{i-1}}{h_i} \tag{17}$$

$$\Delta_{i+1} = 1 - \frac{G_{i+1} - G_i}{h_{i+1}} \tag{18}$$

$$\begin{bmatrix} B_0 & C_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_1 & B_1 & C_1 & \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & A_{N-1} & B_{N-1} & C_{N-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & A_N & B_N \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \\ M_N \end{bmatrix} = \begin{bmatrix} D_0 \\ D_1 \\ \vdots \\ D_{N-1} \\ D_N \end{bmatrix} \tag{19}$$

Equation (12) can be expressed in matrix form where is contained $N+1$ equations and it is in the form of tri-diagonal, like Equation (19). Then, the Thomas Algorithm can be used to figure out the second-order differential value. After obtaining second-order differential value M_i, we can use basic relation of cubic spline to get the first differential derivative m_i and function value u_i.

4 Results and Discussions

Fins or extended surfaces are widely used in augmentation of heat transfer rate from the solid surface to the ambient medium. As the technical efficiency in engineering applications gains ground, promptly, the material and boundary condition and even governing equation becomes more and more complicated. Now, in order to verify the correctness of the method mentioned above, the following example of heat transfer problem is given for validation purpose. This is the heat transfer problem of annular hyperbolic profile fin whose thermal conductivity varies with temperature. In the past few years, Mokheimer (2002) presented the performance of annular fins of different profile subject to locally variable heat transfer coefficient. Arauzo, Campo, Cortes (2005) offered an elementary analytic procedure for the approximate solution of the quasi-one-dimensional heat conduction equation or a generalized Bessel equation governing the temperature variation in annular fins of hyperbolic profile.

Annular fins with hyperbolic profile as illustrated in Fig. 1. where r_b and y_b are the radius and the thickness of the fin base; r_o and y_o are the radius and the thickness of the fin tip, respectively. The fin profile is defined that the thickness of the fin varied with the radius in the function of $y \cdot r =$constant.

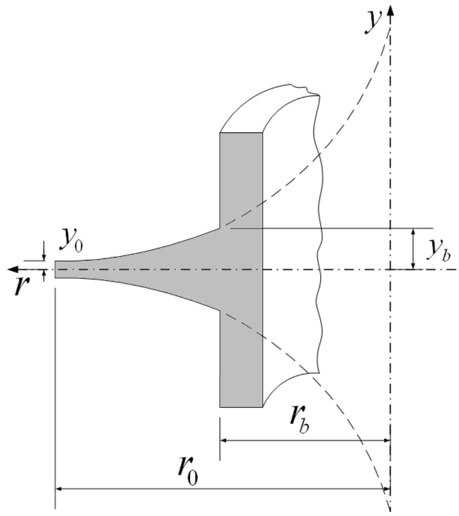

Figure 1: Sketch of the annular fin of hyperbolic profile

From the balance of heat transfer between the solid surfaces and surroundings, the dimensionless governing equation and boundary conditions can be written as follows:

$$(1+\varepsilon\theta)\frac{d^2\theta}{d\eta^2}+\varepsilon\left(\frac{d\theta}{d\eta}\right)^2-\frac{\eta}{\eta_b}m^2\theta=0 \tag{20}$$

$$\begin{aligned}&\theta=1\,,\ \eta=\eta_b\\&\tfrac{d\theta}{d\eta}=0\,,\ \eta=1\end{aligned} \tag{21}$$

where dimensionless variables are,

$$\theta=\frac{T-T_a}{T_b-T_a},\quad \eta=\frac{r}{r_o},\quad \eta_b=\frac{r_b}{r_o},\quad \varepsilon=\frac{k_b-k_a}{k_a}=\beta\left(T_b-T_a\right),\quad m=r_o\sqrt{\frac{h}{k_a y_b}} \tag{22}$$

First of all, the equation must be checked whether it satisfies the monotonicity or not. Based on Eq. (23), the equation can not be identified that totally smaller than zero. To improve this problem, we add a correction term $-\frac{\theta-\hat{\theta}}{\Delta\tau}$, which is the concept of monotone iterative method (Ladde, 1985) into Eq. (20) and shown as Eq.

(24) where the superscript $\wedge$ is an initially assumed function or a value obtained from the previous iteration time. Therefore, this paper presents a concept to reinforce the monotonicity of a differential equation, allowing the maximum principle of differential equations to be applied to a wider range of non-linear problems. $\delta\theta$ of Eq. (25) is the difference between exact and approximate solutions. Eventually, put them to Eq. (20), we can get the relationship between the residual and approximate solutions and shown as Eq. (26).

$$\frac{\partial}{\partial\theta}\left[(1+\varepsilon\theta)\,\theta''+\varepsilon\theta'^2-\frac{\eta}{\eta_b}m^2\theta\right]=\varepsilon\theta''-\frac{\eta}{\eta_b}m^2\le 0 \tag{23}$$

$$(1+\varepsilon\theta)\,\theta'''+\varepsilon\left(\theta'\right)^2-\frac{\eta}{\eta_b}m^2\theta-\frac{\theta-\hat{\theta}}{\Delta\tau}=-\frac{\theta-\hat{\theta}}{\Delta\tau} \tag{24}$$

$$\theta=\tilde{\theta}-\delta\theta \tag{25}$$

$$R_n(\eta)=\left(1+\varepsilon\tilde{\theta}\right)\delta\theta''+2\varepsilon\tilde{\theta}'\delta\theta'-\left(\frac{\eta}{\eta_b}m^2+\frac{1}{\Delta\tau}-\varepsilon\tilde{\theta}''\right)\delta\theta \tag{26}$$

Base on the maximum principle of the inequality of Eq. (3); the time step variable $\Delta\tau$ must be constrained to make the inequality of equation be satisfied.

$$\frac{\eta}{\eta_b}m^2+\frac{1}{\Delta\tau}-\varepsilon\tilde{\theta}''\ge 0 \tag{27}$$

Therefore, following summing-up of all constraint conditions, equations shall also be satisfied in the process of obtaining the lower solution:

$$R(\eta)=R_n(\eta)-\frac{\theta-\hat{\theta}}{\Delta\tau}\ge 0 \tag{28}$$

$$\theta-\hat{\theta}\ge 0 \tag{29}$$

$$R_n(\eta)\ge 0 \tag{30}$$

thus as an optimal lower solution, the approximate solution $\check{\theta}(\eta)=\max\left(\tilde{\theta}(\eta)\right)$ may be found and $\check{\theta}(\eta)\le\theta(\eta)$may be ensured. Similarly, obtaining the upper solution, the following conditions shall also be satisfied

$$R(\eta)=R_n(\eta)-\frac{\theta-\hat{\theta}}{\Delta\tau}\le 0 \tag{31}$$

$$\theta-\hat{\theta}\le 0 \tag{32}$$

$$R_n(\eta)\le 0 \tag{33}$$

An approximate solution $\check{\theta}(\eta) = \min\left(\tilde{\theta}(\eta)\right)$ which is the optimal upper solution may be obtained, and $\check{\theta}(\eta) \geq \theta(\eta)$ can be ensured.

Second, in light of this situation which bases on the concept of residual correction method to simplify the inequality mathematical programming problems of Eq. (28)-(33) into equality problems, differential equations may resemble in obtaining solutions traditionally. For this purpose, an additional residual correction term is used in combination with the iteration technique to discretize the equation and transform into the following:

$$\left(1+\varepsilon\tilde{\theta}_i^{m+1}\right)\left(\tilde{\theta''}_i\right)^{m+1}+\varepsilon\left(\tilde{\theta'}_i^2\right)^{m+1}-\frac{\eta}{\eta_b}m^2\tilde{\theta}_i^{m+1}-\frac{\tilde{\theta}_i^{m+1}-\hat{\theta}_i}{\Delta\tau}=R_i^m \tag{34}$$

Among the equation, the superscript "*m*" stands for iteration times of residual correction, while the subscript "*i*" is the serial number of discretized calculation grid points. And R_i^m serves as residual correction values at calculation grid points to correct the residual values not only at calculation grid points but within the adjacent subintervals which can be totally positive or negative. Finally, follow the complete steps for residual correction method mentioned above and constrain the relative error as follows:

$$E_{re} = \left|\frac{\theta_i^{m+1}-\theta_i^m}{\theta_i^m}\right| \tag{35}$$

Discretize the equations above based on cubic spline and use residual correction method and make relative residual error be reached 10^{-5}. It can be obtained approximate solutions and residual values. Under this condition, the computing time costs less than one second. The distribution of the residual correcting before and after respectively is shown as Fig. 2 (a) and (b), respectively. It can be understood from Fig. 2 (a) that the residual distribution of differential equation before correcting is always zero at calculation grid points and greater or smaller than zero on non-calculation grid points. Nevertheless, after residual correction, the residual distribution will not satisfy zero on the calculation grid points, but neighboring regions are all positive or negative. For instance, in order to get the lower solution, the residual at $\eta = 0.3625$ is corrected about 0.0008 and make the neighboring region $0.325<\eta<0.4$ all larger than or equal to zero. On the other hand, getting the higher solutions, the original residual distribution on the region of $0.325<\eta<0.4$ is all smaller than or equal to zero therefore it is needless to correct. After two parts of correction, the residual values in subintervals of all grid points are all positive or negative and symmetric distribution on both sides of zero as shown in Fig. 2 (b).

It is worth mentioning the small value of *m*. Because of m^2 is an extended Biot number; that is, the ratio of convection at the surface to conduction within the fin as well as the ration of the internal resistance of the fin to heat conduction to its external resistance to heat convection. The small value of m^2 indicates the inner resistance of the fin to heat conduction is small relative to resistance. The fact is that the only considered regarding the heat transfer effect is along the radius direction and can neglect the heat transfer of y direction.

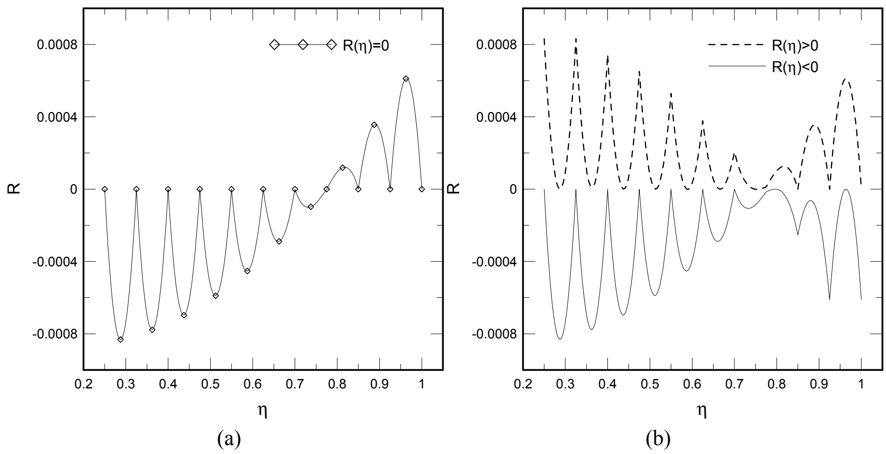

Figure 2: Residual value distribution curves before and after correction of (a) and (b), respectively; where $R_b = 0.25$, $N = 11, \varepsilon = 0.3$ and $m = 0.7$

According to the residual correction, we can obtain the distribution of mean approximate solutions and exact solutions indicated as Eq. (36) shown as Fig. 3 where ε is equal to zero. Apparently, they greatly match even when the greater value of m.

$$\theta(\eta)_{exact} = \sqrt{\frac{\eta}{\eta_b}} \left[\frac{I_{\frac{1}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\eta^{\frac{3}{2}}\right) I_{\frac{2}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\right) - I_{-\frac{1}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\eta^{\frac{3}{2}}\right) I_{-\frac{2}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\right)}{I_{\frac{1}{3}}\left(\frac{2m}{3}\eta^{\frac{3}{2}}\right) I_{\frac{2}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\right) - I_{-\frac{1}{3}}\left(\frac{2m}{3}\eta^{\frac{3}{2}}\right) I_{-\frac{2}{3}}\left(\frac{2m}{3\sqrt{\eta_b}}\right)} \right] \tag{36}$$

where I_v represents the first kind modified Bessel function with fractional order v.

If the final approximate solution is the mean value which is between upper and lower solutions, it can be plausible that the maximum error of approximate solutions at any point must be smaller than maximum possible error of mean approximate solutions (E_p) shown as Eq. (37), even though the exact solution is not identified. In this example, the maximum actual relative error (E_r) shown as Eq. (38) can be obtained between mean approximate solution and exact solution.

$$E_p = \frac{\left|\check{\theta}(\eta) - \check{\theta}(\eta)\right|}{2}, \; \eta_b \leq \eta \leq 1 \tag{37}$$

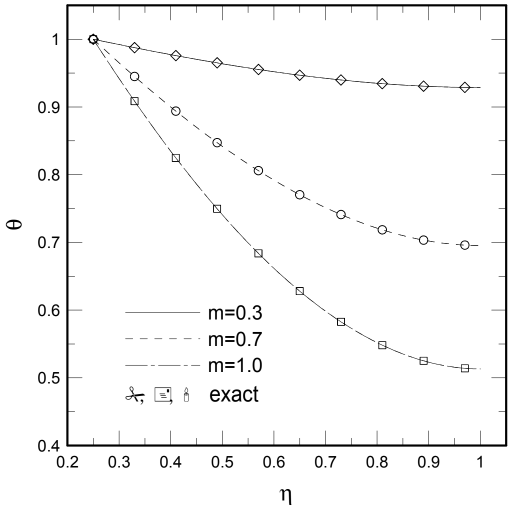

Figure 3: Distribution curves of the mean approximate and exact solutions under the different value of m when $R_b = 0.25$, $N = 11$,and $\varepsilon = 0$

$$E_r = \frac{\left|\bar{\theta}(\eta) - \theta(\eta)_{exact}\right|}{\theta(\eta)_{exact}},\ \eta_b \le \eta \le 1 \tag{38}$$

With all considered, we can realize the error magnitude is far smaller than the maximum possible error obtained shown as Table 1. Within just three grid points, the relative error between numerical and exact solution can be limited to smaller than one thousandth.

Table 1: Variation of approximate solutions and error range of this problem under different numbers of calculation grid points on $\eta = 0.5$ when $R_b = 0.25$, $\varepsilon = 0$, $m = 1$

N	lower $\check{\theta}$	mean $\bar{\theta}$	upper $\check{\theta}$	E_p	E_r
3	0.735015	0.740613	0.746211	5.5980E-03	3.3755E-04
5	0.738855	0.740942	0.743029	2.0870E-03	1.0655E-04
6	0.740313	0.740881	0.741449	5.6824E-04	2.3784E-05
11	0.740740	0.740870	0.740999	1.2964E-04	8.2637E-06
21	0.740833	0.740865	0.740896	3.1691E-05	1.5923E-06
41	0.740856	0.740864	0.740872	7.8785E-06	3.5846E-07

Owing to the fact that the distribution of residual values is symmetric, the obtained upper and lower approximate solutions are characterized by the basic symmetry on both sides of exact solution. So the mean approximate solution derived from upper and lower solution, allowing the method proposed in this paper to have this property of obtaining the optimal approximate solution with few calculating grid points. From this characteristics, we can estimate the nonlinear differential equation problem when ε is not zero shown as Fig. 4. The larger value of ε represents the thermal conductivity of the fin which is affected by temperature difference between the fin base and the surrounding fluids, so that the temperature will be rise. This tendency along with the previous conclusion provides more accurate results.

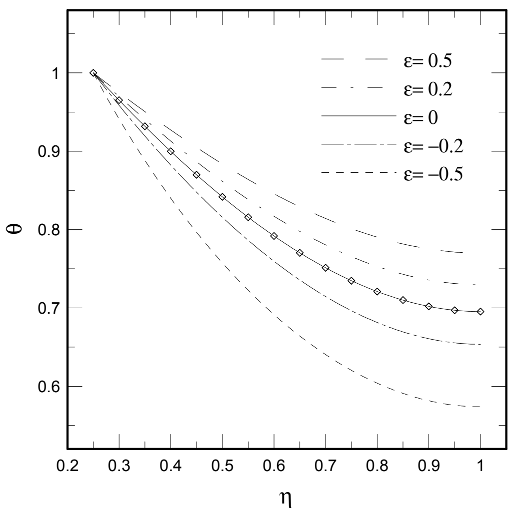

Figure 4: Distribution curves with different ε when $R_b = 0.25$, $N = 11$, and $m = 0.7$

It is more obvious in Fig. 5 that with the decrease of time steps or the increase of calculation grid points, the difference between upper and corresponding lower solution is narrowed, indicating that with the increase of discretized grid points, upper and lower solutions may effectively approximate their exact solution. The more numbers of the grid points, the less difference between upper and lower approximate solutions. For all that when η is closed to 1.0, the error is going to increase, the relative error between mean value and exact solution is still under 10^{-3} while calculating number is only 6, which shown as Fig. 6 Therefore, it can be said that on nonlinear heat transfer problem the residual correction also directly increases the accuracy of approximate solution under the increase of grid points.

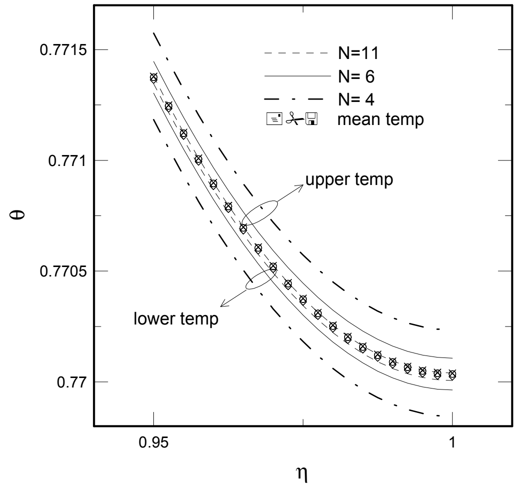

Figure 5: Distribution curves of upper and lower approximate solutions where $R_b = 0.25$, $N = 11$, $\varepsilon = 0.5$, and $m = 0.7$.

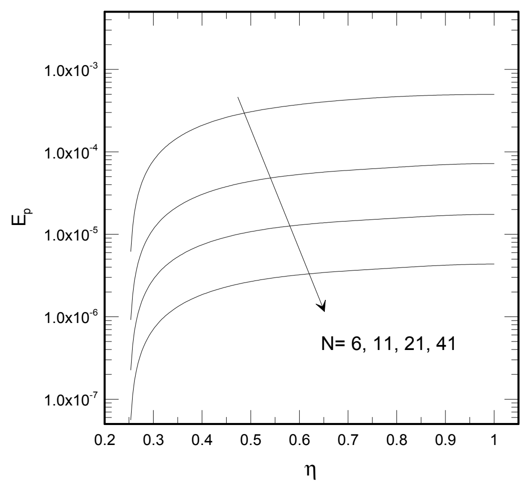

Figure 6: The maximum possible errors under different numbers of grid points when $R_b = 0.25$, $\varepsilon = 0.3$, and $m = 0.7$

5 Conclusions

The residual correction method under the concept of maximum principle can successfully obtain the upper and lower approximate solutions and correct the residual all greater or smaller than zero. The features of the correcting residual and the numerical results of upper and lower solutions are symmetric. Thus, the accuracy of the mean approximate solution can be improved. This method can solve the inequality constraint mathematical programming problems in few seconds, and obtain upper and lower approximate solutions of non-linear equation problems efficiently. From the upper and lower approximate solutions, the maximum possible error range can be easily determined when exact solution is unknown. Even if only four grid points are selected, mean approximate solutions can still be obtained for differential equation efficiently and precisely. Thus, we can conclude that it is a practical, extendable, and worthy of recommending method to study where including two dimensional equation problems. We can also do further and compare with the example of nonlinear semiconductor Poisson equation written by Wordelman, Aluru, and Ravaioli (2000) applying meshless Finite Point method and stiff ordinary differential equation using nonstandard Group-preserving scheme developed by Liu (2005).

Acknowledgement: Thanks for the subsidy of the Outlay NSC96- 2221- E006-171- MY2 given by National Science Council, the Republic of China, to help us finish this special research successfully.

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