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Numerical Solution of Nonlinear Schrodinger Equations by Collocation Method Using Radial Basis Functions

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Abstract: A mesh free method for the numerical solution of the nonlinear Schrodinger (NLS) and coupled nonlinear Schrodinger (CNLS) equation is implemented. The presented method uses a set of scattered nodes within the problem domain as well as on the boundaries of the domain along with approximating functions known as radial basis functions (RBFs). The set of scattered nodes do not form a mesh, means that no information of relationship between the nodes is needed. Error norms L_2, L_∞ are used to estimate accuracy of the method. Stability analysis of the method is given to demonstrate its practical applicability.

Keywords: RBFs; Nonlinear Schrodinger equations (NLS), Coupled nonlinear Schrodinger (CNLS) equations; Mesh free collocation method.

1 Introduction

NLS equation is a model which has application in fluid mechanics, plasma physics, nonlinear optics [Bialynicki (1979), Bullough (1979), Cowan(1986)]. Various numerical methods have been used for the solution of NLS equation [Chang (1999), Dag (1999), Karkashian (1998), Pathria (1990), Sheng (2001), YanXu, ChiWang (2005)]. The coupled nonlinear Schrodinger equation (CNLS) equations derived by [Benny and Newell (1967)] is a model for two interacting nonlinear packets in a dispersive and conservative system. CNLS equation has been solved numerically by many authors [Yan Xu and Chi Wang (2005), M. S. Ismail (2008), Ismail and Taha (2001), Ismail and Taha (2007)]

The multiquadric method was first introduced by [Hardy (1971)] to approximate two-dimensional geographical surfaces. [Kansa (1990)] derived a modified multiquadric scheme for the numerical solution of PDEs. [Micchelli (1986), Madych

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(1990), Frank and Schaback (1998)] worked on existence, uniqueness, and convergence of the RBFs approximation. The importance of shape parameter c in the MQ method was examined by [Tarwater (1985)]. [Micchelli (1986)] has shown that the system of equations obtained with this approach is always solvable for distinct interpolation points. The idea of applying the MQ technique to PDEs was proposed by [Kansa (1990)], which was extended by [Golberg (1996)] later on. [Hon and Mao (1998)] extended the use of MQ for numerical solutions of various ordinary and partial differential equations including nonlinear Burgers' equation with shock waves. Very recently a number of papers have been published on RBFs collocation methods. [Sirajul Haq et.al (2008), Marjan Uddin et.al (2009)] have used mesh free collocation method for different types of PDEs. [Nicolas ali libre et.al (2008)] have used an adaptive scheme for nearly singular PDEs, [Emdadi et. al (2008)] proposed a stable PDE solution method for large multiquadric shape parameters whereas a modified meshless control volume method was proposed by [P. Orsini, H. Power (2008)]. [G. Kosec and B. Sarler (2008)] have used local RBF collocation method for Darcy flow. [S.N. Atluri et.al (2004)] have developed a meshless finite volume method through MLPG mixed approach. [S, Chantasiriwan (2006)] has used Multiquadric Collocation method for solving Lid-driven Cavity flow problem and [C. Shu, H. Ding (2005)] have used this approach to solve the Navier-Stokes equation.

In this work, we propose a mesh free collocation method based on the radial basis functions, MQ ($\psi(r) = (r^2+c^2)^{1/2}$, where c is a shape parameter), TPS ($r^4 \log r$) and spline basis (r^5), for the numerical solution of the following types of equations.

i) Nonlinear Schrodinger equation(NLS)[Yan Xu and Chi-Wang Shu (2005)]

$$iw_t + w_{xx} + 2|w|^2 w = 0, \ -\infty < x < \infty, \ t > 0. \tag{1}$$

ii) The coupled CNLS equations [Benny and Newell (1967)] are

$$\begin{aligned} &i(w_1)_t + \frac{1}{2}(w_1)_{xx} + (|w_1|^2 + \beta|w_2|^2)w_1 = 0 \\ &i(w_2)_t + \frac{1}{2}(w_2)_{xx} + (\beta|w_1|^2 + |w_2|^2)w_2 = 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &i(w_1)_t + i\alpha(w_1)_x + \frac{1}{2}(w_1)_{xx} + (|w_1|^2 + \beta|w_2|^2)w_1 = 0 \\ &i(w_2)_t - i\alpha(w_2)_x + \frac{1}{2}(w_2)_{xx} + (\beta|w_1|^2 + |w_2|^2)w_2 = 0. \end{aligned} \tag{3}$$

In the above equations $w = v_1 + iv_2$, $w_1 = u_1 + iu_2$, $w_2 = u_3 + iu_4$ are complex valued functions of the spatial coordinate x, t stands for time variable, α and β are real parameters.

The structure of the paper is organized as follows. In Section 2, we discuss the mesh free method. In Section 3, stability analysis of the scheme is presented. In Section 4, we test the method on problems related to the NLS and CNLS equations. In Section 5, the results are concluded.

2 Description of the technique

Consider an n-dimensional ($n = 1,2,3$) time dependent boundary value problem

$$u_t + \mathscr{L}u = f(x,t),\ x \in \Omega, \quad \mathscr{B}u = g(x,t), \quad x \in \Gamma, \tag{4}$$

where $\mathscr{L}$ and $\mathscr{B}$ are derivative and boundary operators respectively. Ω and Γ stand for interior of the domain and boundary of the domain respectively.

The initail condition is given by

$$u(x,0) = u_0. \tag{5}$$

For spatial derivatives, we use the following θ-weighted scheme

$$\frac{u^{(n+1)} - u^{(n)}}{\delta t} + \theta \mathscr{L}u^{(n+1)} + (1-\theta)\mathscr{L}u^{(n)} = f(x,t^{(n+1)}), \quad 0 \le \theta \le 1. \tag{6}$$

In above equation δt is time step, $u^{(n)}$ (n is non-negative integer) is the solution at time $t^{(n)} = n\ \delta t$ and $\{x_i\}_{i=1}^{N} \in \Omega \cup \Gamma$ are collocation points.

Approximate solution of Eq. (4) in terms of RBF is given by

$$u^{(n)}(x_i) = \sum_{j=1}^{N} \psi(r_{ij})\lambda_j^{(n)},\ i = 1(1)N. \tag{7}$$

In the above eqution $\psi(r_{ij})$ represent RBFs, $r_{ij} = \|Q_i - Q_j\|$ is Euclidean norm between the points Q_i, Q_j and $\lambda_j^{(n)}$ ($j = 1(1)N$) are constants to be determined. Matrix form of the system (7) can be written as:

$$\mathbf{u^{(n)}} = \mathbf{A}\boldsymbol{\lambda}^{\mathbf{(n)}}, \tag{8}$$

where $\mathbf{A} = [\psi(r_{ij})]_{N\times N}$, $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T$.

From Eqs. (6) and (7), we can write

$$\sum_{j=1}^{N} \left(\frac{\psi(r_{ij})\lambda_j^{(n+1)} - \psi(r_{ij})\lambda_j^{(n)}}{\delta t} + \theta \mathscr{L}[\psi(r_{ij})]\lambda_j^{(n+1)} + (1-\theta)\mathscr{L}[\psi(r_{ij})]\lambda_j^{(n)} \right) = f(x_i, t^{(n+1)}), \quad i = 1(1)N_d, \tag{9}$$

$$\sum_{j=1}^{N} \mathscr{B}(\psi(r_{ij}))\lambda_j^{(n+1)} = g(x_i, t^{(n+1)}),\ i = N_d + 1(1)N. \tag{10}$$

Eqs. (9) and (10) can be written in the matrix form as

$$\mathbf{G}\lambda^{(\mathbf{n+1})} = \mathbf{H}\lambda^{(\mathbf{n})} + \mathbf{F}^{(\mathbf{n+1})}, \tag{11}$$

where

$$\mathbf{G} = \begin{bmatrix} \psi(r_{ij}) + \delta t\theta\mathscr{L}[\psi(r_{ij})] \\ \mathscr{B}\psi[(r_{ij})] \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \psi(r_{ij}) - \delta t(1-\theta)\mathscr{L}[\psi(r_{ij})] \\ 0 \end{bmatrix},$$

$$\mathbf{F}^{(\mathbf{n+1})} = \begin{bmatrix} \delta t f^{(n+1)} \\ g^{(n+1)} \end{bmatrix}.$$

Eliminating $\lambda_j^{(n)}$ ($j = 1(1)N$) from Eqs. (8) and (11) we obtaine the solution at time level $n+1$

$$\mathbf{u}^{(\mathbf{n+1})} = \mathbf{A}\mathbf{G}^{-\mathbf{1}}\mathbf{H}\mathbf{A}^{-\mathbf{1}}\mathbf{u}^{(\mathbf{n})} + \mathbf{A}\mathbf{G}^{-\mathbf{1}}\mathbf{F}^{(\mathbf{n+1})}.$$

For numerical calculation we use $\theta = 0.5$. For distinct collocation points, **A** is always invertible [Micchelli (1986)]. It was shown by [Schaback (1998)] that the matrix **G** is invertible provided the matrix **G** is symmetric.

2.1 The NLS equation

The decomposed form of the NLS Eq. (1) is given as

$$(-1)^{k-1}(v_k)_t + (v_{3-k})_{xx} + 2(v_k)^2 v_{3-k} + 2v_{3-k}^3 = 0, \quad k = 1,2. \tag{12}$$

The boundary conditions are

$$v_k(a,t) = g_{ak}(t), \quad v_k(b,t) = g_{bk}(t), \quad t > 0, \quad k = 1,2 \tag{13}$$

and initial conditions

$$v_k(x,0) = g_k(x), \quad k = 1,2 \quad \text{and} \quad a \le x \le b. \tag{14}$$

Applying scheme (6) to Eqs. (12), we get

$$(v_k)^{(n+1)} = (v_k)^{(n)} + (-1)^k \delta t \left[((v_{3-k})xx)^{(n)} + 2(v_k^2 v_{3-k})^{(n)} + 2(v_{3-k}^3)^{(n)} \right], \quad k = 1,2. \tag{15}$$

The RBF approximations of $v_k,\ k = 1,2$, for Eqs. (12) are given by

$$(v_k)^{(n)}(x) = \sum_{j=1}^{N} \xi_{kj}^{(n)} \psi(r_j), \quad k = 1,2. \tag{16}$$

Using Eqs. (16) in Eqs. (15) along with the boundary conditions (13), in matrix form we can write

$$\mathbf{A}\xi_{\mathbf{k}}{}^{(n+1)} = \mathbf{A}\xi_{\mathbf{k}}{}^{(n)} + (-1)^k \delta t \left[((\mathbf{v}_{3-k})_{xx})^{(n)} + 2\left(\mathbf{v}_k^2 \mathbf{v}_{3-k}\right)^{(n)} + 2\left(\mathbf{v}_{3-k}^3\right)^{(n)} \right] + \mathbf{g}_k^{(n+1)}, \quad k = 1,2. \tag{17}$$

where $\xi_{\mathbf{k}}{}^{(n)} = \left[\xi_{kj}^{(n)} \right]_{N\times N}^T, \quad \mathbf{g}_k^{(n+1)} = \left[\ g_{ak}^{(n+1)}, 0, 0, ..., g_{bk}^{(n+1)} \ \right]^T, \quad k = 1,2.$
In more compact form Eq. (17) can be written as

$$\xi_{\mathbf{k}}{}^{(n+1)} = \xi_{\mathbf{k}}{}^{(n)} + \mathbf{A}^{-\mathbf{1}}\mathbf{G}_k^{(n+1)}, \quad k = 1,2 \tag{18}$$

where

$$\mathbf{G}_k^{(n+1)} = \mathbf{g}_k^{(n+1)} + (-1)^k \delta t \left[((\mathbf{v}_{3-k})_{xx})^{(n)} + 2\left(\mathbf{v}_k^2 \mathbf{v}_{3-k}\right)^{(n)} + 2\left(\mathbf{v}_{3-k}^3\right)^{(n)} \right], \quad k = 1,2.$$

The solution can be obtained from the above two equations and Eq. (16) at each time level n.

2.2 CNLS equations

Decomposed form of the CNLS Eq. (3) is given by

$$(u_k)_t + \alpha(u_k)_x + \frac{(-1)^{k-1}(u_{3-k})_{xx}}{2} + (-1)^{k-1} \left[\sum_{j=1}^{2} \left(u_j^2 + \beta u_{j+2}^2 \right) \right] u_{3-k} = 0, \quad k = 1,2$$

$$(u_k)_t - \alpha(u_k)_x + \frac{(-1)^{k-1}(u_{7-k})_{xx}}{2} + (-1)^{k-1} \left[\sum_{j=1}^{2} \left(\beta u_j^2 + u_{j+2}^2 \right) \right] u_{7-k} = 0, \quad k = 3,4 \tag{19}$$

along with the boundary conditions

$$u_k(a,t) = f_{ak}(t), \quad u_k(b,t) = f_{bk}(t), \quad k = 1(1)4, \quad t > 0, \tag{20}$$

and initial conditions

$$u_k(x,0) = f_k(x), \quad k = 1(1)4, \quad a \le x \le b. \tag{21}$$

Using Eq. (6) in system of Eqs. (19), we can write

$$u_k^{(n+1)} + \frac{\alpha\delta t}{2}(u_k)_x^{(n+1)} = u_k^{(n)} - \delta t\left[\frac{\alpha}{2}(u_k)_x^{(n)} + \frac{(-1)^{k-1}(u_{3-k})_{xx}^{(n)}}{2}\right] + (-1)^k\delta t\left[\left\{\sum_{j=1}^{2}\left((u_j^2)^{(n)} + \beta(u_{j+2}^2)^{(n)}\right)\right\}(u_{3-k})^{(n)}\right], \quad k = 1,2 \tag{22}$$

$$u_k^{(n+1)} - \frac{\alpha\delta t}{2}(u_k)_x^{(n+1)} = u_k^{(n)} + \delta t\left[\frac{\alpha}{2}(u_k)_x^{(n)} + \frac{(-1)^{k}(u_{7-k})_{xx}^{(n)}}{2}\right] + (-1)^k\delta t\left[\left\{\sum_{j=1}^{2}\left(\beta(u_j^2)^{(n)} + (u_{j+2}^2)^{(n)}\right)\right\}(u_{7-k})^{(n)}\right], \quad k = 3,4. \tag{23}$$

The RBF approximations for u_k, $k = 1(1)4$ of the system (19) are given by

$$u_k^{(n)}(x) = \sum_{j=1}^{N}\lambda_{kj}^{(n)}\psi(r_j), \; k = 1(1)4. \tag{24}$$

After the use of Eq. (24), the system of Eqs. (22) and (23) along with the boundary conditions (20) can be written in matrix form as

$$\left[\mathbf{A} + \frac{\alpha\delta t}{2}\mathbf{D}_1\right]\lambda_k^{(n+1)} = \left[\mathbf{A} - \frac{\alpha\delta t}{2}\mathbf{D}_1\right]\lambda_k^{(n)} + (-1)^k\frac{\delta t}{2}(u_{3-k})_{xx}^{(n)} + (-1)^k\delta t\left[\sum_{j=1}^{2}\left((u_j^2)^{(n)} + \beta(u_{j+2}^2)^{(n)}\right)\right](u_{3-k})^{(n)} + \mathbf{f}_k^{(n+1)}, \quad k = 1,2 \tag{25}$$

$$\left[\mathbf{A} - \frac{\alpha\delta t}{2}\mathbf{D}_1\right]\lambda_k^{(n+1)} = \left[\mathbf{A} + \frac{\alpha\delta t}{2}\mathbf{D}_1\right]\lambda_k^{(n)} + (-1)^k\frac{\delta t}{2}(u_{7-k})_{xx}^{(n)} + (-1)^k\delta t\left[\sum_{j=1}^{2}\left(\beta(u_j^2)^{(n)} + (u_{j+2}^2)^{(n)}\right)\right](u_{7-k})^{(n)} + \mathbf{f}_k^{(n+1)}, \quad k = 3,4 \tag{26}$$

where
$\lambda_k^{(n)} = \left[\lambda_{kj}^{(n)}\right]_{N\times N}^T$, $\mathbf{D}_1 = [\psi'(r_{ij})]_{N\times N}$, $\mathbf{f}_k^{(n+1)} = \left[\ f_{ak}^{(n+1)}, 0, 0, ..., f_{bk}^{(n+1)}\ \right]^T$, $k = 1(1)4$.

In more compact form we can write Eqs. (25)-(26) as

$$\lambda_k^{(n+1)} = \mathbf{M}^{-1}\mathbf{N}\lambda_k^{(n)} + \mathbf{M}^{-1}\mathbf{F}_k^{(n+1)},\ k = 1,2, \tag{27}$$

$$\lambda_j^{(n+1)} = \mathbf{N}^{-1}\mathbf{M}\lambda_j^{(n)} + \mathbf{N}^{-1}\mathbf{F}_j^{(n+1)},\ j = 3,4. \tag{28}$$

where

$$\mathbf{M} = \left[\mathbf{A} + \frac{\alpha\delta t}{2}\mathbf{D}_1\right], \quad \mathbf{N} = \left[\mathbf{A} - \frac{\alpha\delta t}{2}\mathbf{D}_1\right],$$

$$\mathbf{F}_k^{(n+1)} = \mathbf{f}_k^{(n+1)} + (-1)^k\frac{\delta t}{2}(u_{3-k})_{xx}^{(n)} + (-1)^k\delta t\left[\sum_{j=1}^{2}\left((u_j^2)^{(n)} + \beta(u_{j+2}^2)^{(n)}\right)\right](u_{3-k})^{(n)}, \quad k = 1,2$$

$$\mathbf{F}_k^{(n+1)} = \mathbf{f}_k^{(n+1)} + (-1)^k\frac{\delta t}{2}(u_{7-k})_{xx}^{(n)} + (-1)^k\delta t\left[\sum_{j=1}^{2}\left(\beta(u_j^2)^{(n)} + (u_{j+2}^2)^{(n)}\right)\right](u_{7-k})^{(n)}, \quad k = 3,4.$$

Matrix forms of Eq. (24) is

$$\mathbf{u}_k^{(n)} = \mathbf{A}\lambda_k^{(n)},\ k = 1(1)4. \tag{29}$$

Using Eq. (29) in Eq. (27), we obtain

$$\mathbf{u}_j^{(n+1)} = \left[\mathbf{A}\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1}\right]\mathbf{u_j}^{(n)} + [\mathbf{A}\mathbf{M}^{-1}]\mathbf{F}_j^{(n+1)},\ j = 1,2, \tag{30}$$

$$\mathbf{u}_k^{(n+1)} = \left[\mathbf{A}\mathbf{N}^{-1}\mathbf{M}\mathbf{A}^{-1}\right]\mathbf{u_k}^{(n)} + \left[\mathbf{A}\mathbf{N}^{-1}\right]\mathbf{F}_k^{(n+1)},\ k = 3,4.$$

3 Stability Analysis

Stability of the scheme for the CNLS equations (3), using spectral norm of the amplification matrix, is discussed in this section. Let $\mathbf{u}_k$ and $\mathbf{u}_k^*$ be respectively numerical and exact solutions of Eq. (19). The error vectors are denoted and defined by $\varepsilon_k^{(n)} = \mathbf{u_k}^{(n)} - \mathbf{u_k^*}^{(n)}$, $(k = 1(1)4)$. Using Eq. (30), we arrive at the following equations

$$\begin{aligned} \varepsilon_k^{(n+1)} &= \mathbf{u}_k^{(n+1)} - \mathbf{u}_k^{*(n+1)} = \mathbf{E}_k \varepsilon_k^{(n)}, \\ \mathbf{E}_k &= \mathbf{A}\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1},\ k = 1,2;\ \mathbf{E}_k = \mathbf{A}\mathbf{N}^{-1}\mathbf{M}\mathbf{A}^{-1},\ k = 3,4. \end{aligned} \tag{31}$$

where $\mathbf{E}_k,\ k = 1(1)4$, are the amplification matrices. For the scheme to remain stable, $\varepsilon_\mathbf{k}^{(\mathbf{n})}$ must approach to zero as $n \longrightarrow \infty$ for $k = 1(1)4$ i.e $\rho(\mathbf{E_k}) \le 1$, where $\rho(\mathbf{E}_k)$ represent spectral radii of the matrices $\mathbf{E}_k$. From Eqs. (27) and (31), we can write

$$\begin{aligned} &\left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_1\right] \varepsilon_k^{(n+1)} = \left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_2\right] \varepsilon_k^{(n)},\ k = 1,2, \\ &\left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_3\right] \varepsilon_k^{(n+1)} = \left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_4\right] \varepsilon_k^{(n)},\ k = 3,4, \end{aligned} \tag{32}$$

where $\mathbf{S}_1 = \mathbf{S}_4 = \alpha \mathbf{D}_1 \mathbf{A}^{-1}, \quad \mathbf{S}_2 = \mathbf{S}_3 = -\alpha \mathbf{D}_1 \mathbf{A}^{-1}$.
For stability, maximum eigenvalues of the matrices

$$\left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_1\right]^{-1} \left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_2\right] \quad \text{and} \quad \left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_3\right]^{-1} \left[\mathbf{I} + \frac{\delta t}{2}\mathbf{S}_4\right]$$

must be less or equal to unity i.e. to write

$$\left|\frac{1 + (\delta t/2)\eta_{S_2}}{1 + (\delta t/2)\eta_{S_1}}\right| \le 1 \quad \text{and} \quad \left|\frac{1 + (\delta t/2)\eta_{S_4}}{1 + (\delta t/2)\eta_{S_3}}\right| \le 1, \tag{33}$$

where η_{S_i} stand for eigenvalues of the matrices $\mathbf{S}_i$, for $i = 1(1)4$, respectively. Above conditions will hold if $\eta_{S_1} \ge \eta_{S_2}$ and $\eta_{S_3} \ge \eta_{S_4}$. It is evident from inequality (33), that the stability of the scheme (30) depends upon the parameters δt and eigenvalues of the matrices $\mathbf{S}_i,\ i = 1(1)4$ in addition to number of collocation points N and shape parameter c. In the case of free parameter RBFs like TPS ($r^{2n} \log r$, n is positive integer) and Quintic (r^n, n is odd positive integer), stability also depends upon the same factors except shape parameter c. For an acceptable distribution of collocation points, inequality (33) must hold. In computational experiment we consider Problem 4, where behavior of minimum eigenvalues of the matrices $\mathbf{S}_i,\ i = 1,2$ versus MQ shape parameter c is shown in Fig. 6. Also in Table 4, the error norms L_∞ of the solutions are given for fixed number of collocation points N. This table confirms the fact that mesh free collocation method produce stable results when the shape c lies in the interval $(0.1, 1.7)$.

4 Numerical examples

In this section, we apply the proposed method for the numerical solution of different classes of coupled PDEs. The accuracy of the method is checked by error norms L_2, L_∞ and the invariants of the NLS and CNLS equations. These error norms and invariants are given as

$$L_2 = \|w^* - w\|_2 = \left[\delta x \sum_{j=1}^{N} (w^* - w)^2\right]^{1/2},$$

$$L_\infty = \|w^* - w\|_\infty = \mathrm{m}ax_j |w^* - w|.$$

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} |w_1|^2 \, dx, \\ I_2 &= \int_{-\infty}^{\infty} |w_2|^2 \, dx, \\ I_3 &= \int_{-\infty}^{\infty} i \sum_{j=1}^{2} (\overline{w}_j (w_j)_x - w_j (\overline{w}_j)_x) \, dx, \\ I_4 &= \int_{-\infty}^{\infty} \left(\sum_{j=1}^{2} \frac{1}{2} \left|(w_j)_x\right|^2 - \frac{1}{2} \sum_{j=1}^{2} \left|w_j\right|^4 - e|w_1|^2|w_2|^2 \right) dx, \end{aligned} \tag{34}$$

where w^* and w are used for exact and approximate solutions respectively. The tested problems are given below.

Problem 1. Consider NLS equation

$$iw_t + w_{xx} + 2|w|^2 w = 0 \tag{35}$$

with exact solution [Yan Xu and Chi-Wang Shu (2005)]

$$w(x,t) = \mathrm{sech}(x - 4t) \exp\left(2i(Cx - \frac{3}{2}t)\right). \tag{36}$$

The error norms L_∞ and L_2 of $v_k, k = 1,2$, are given in Table 1 for different values t. The numerical results are obtained by using three radial basis functions MQ, TPS ($r^4 log(r)$) and spline basis (r^5). In all these computations, we use MQ shape parameters $c_1 = c_2 = 1$, $N = 76$, $\delta x = 0.4$, $\delta t = 0.0001$ and interval $[-15, 15]$. From the table it is clear that the results of both $v_k, k = 1,2$, obtained by spline basis and MQ are better than that obtained by TPS.

Table 1: Error norms for v_1 and v_2 when $\delta x = 0.4$, $\delta t = 0.0001$ $C = 1$ and $N = 76$ in $[-15, 15]$ corresponding to Problem 1.

MQ	v_1		v_2	
Time	L_∞	L_2	L_∞	L_2
0.1	4.270E-004	5.669E-005	5.794E-004	4.253E-005
0.2	6.063E-004	2.095E-005	6.844E-004	7.365E-005
0.4	7.909E-004	6.810E-005	8.927E-004	7.910E-005
0.6	1.167E-003	2.201E-004	9.145E-004	1.985E-004
0.8	1.297E-003	9.624E-004	1.195E-003	3.404E-004
1.0	1.487E-003	1.566E-003	1.848E-003	7.511E-004
r^5	v_1		v_2	
Time	L_∞	L_2	L_∞	L_2
0.1	4.045E-004	9.547E-005	3.727E-004	7.918E-005
0.2	4.425E-004	2.255E-005	4.735E-004	1.906E-004
0.4	6.619E-004	3.254E-004	7.097E-004	2.944E-005
0.6	1.039E-003	1.692E-005	9.452E-004	4.795E-004
0.8	1.396E-003	1.336E-003	1.179E-003	8.586E-004
1.0	1.614E-003	2.469E-003	1.748E-003	8.738E-004
TPS	v_1		v_2	
Time	L_∞	L_2	L_∞	L_2
0.1	7.413E-004	1.192E-004	8.700E-004	5.056E-004
0.2	1.761E-003	1.448E-004	1.878E-003	1.503E-003
0.4	3.709E-003	2.501E-003	3.416E-003	4.331E-003
0.6	4.528E-003	8.825E-003	5.096E-003	2.768E-003
0.8	6.692E-003	1.190E-002	6.587E-003	5.059E-003
1.0	8.054E-003	6.084E-003	7.764E-003	1.342E-002

For the soliton propagation of the NLS equation (35), the initial condition is given by

$$w(x,0) = \operatorname{sech}(x - x_0)\exp\left(2i(C(x - x_0))\right). \tag{37}$$

In the above condition x_0 is center of soliton initially which is -10 here. In Fig. 1, the numerical solutions in time interval $[0,5]$ are shown. We use MQ shape parameters $c_1 = c_2 = 1$, $N = 126$, $\delta x = 0.4$ and $\delta t = 0.0001$, interval $[-25,25]$. From the figure it is clear that the single soliton is propagating as the time progresses.

Now to study the interaction of two soliton of the NLS equation (35) with initial

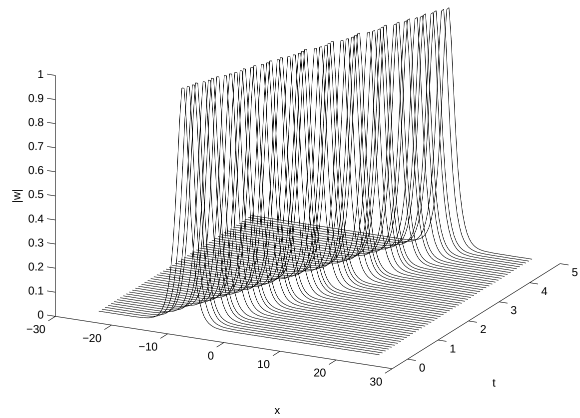

Figure 1: Propagation of the single soliton $|w|$ corresponding to Problem 1 in $[-25,25]$.

condition taken as

$$w(x,0) = \sum_{j=1}^{2} \operatorname{sech}(x-x_j)\exp\left(\frac{1}{2}iC_j(x-x_j)\right). \tag{38}$$

The above equation corresponds to sum of two solitons, one initially located at $x_1 = -10$, moving to right, while the other located at $x_2 = 10$ and is moving towards left. The problem is solved on the interval $[-25,25]$ for up to time $t = 5$ using the MQ. The parameters are $C_1 = 4$, $C_2 = -4$ and the interaction profile is shown in Fig. 2. It is noted that the two waves move toward each other, collide at time $t = 2.5$, and then move away from each other after the interaction without change of shape, and this agrees with [Yan Xu and Chi-Wang Shu (2005)]. The values of the shape parameters used here are $c_1 = c_2 = 1$, $\delta x = 0.4$ and $\delta t = 0.0001$.

Now, we consider the birth of soliton of the NLS equation (35) using Maxwellian initial condition [Yan Xu and Chi-Wang Shu (2005)]

$$w(x,0) = A\exp(-x^2). \tag{39}$$

The interval $[-45,45]$, with periodic boundary condition $w(-45,t) = w(45,t) = 0$ for up to time $t = 4$, is used to find solution of the problem by MQ. The soliton profile is shown in Fig. 3(A). We note that the standing soliton is observed and this result agrees with [Yan Xu and Chi-Wang Shu (2005)]. For computations, we use

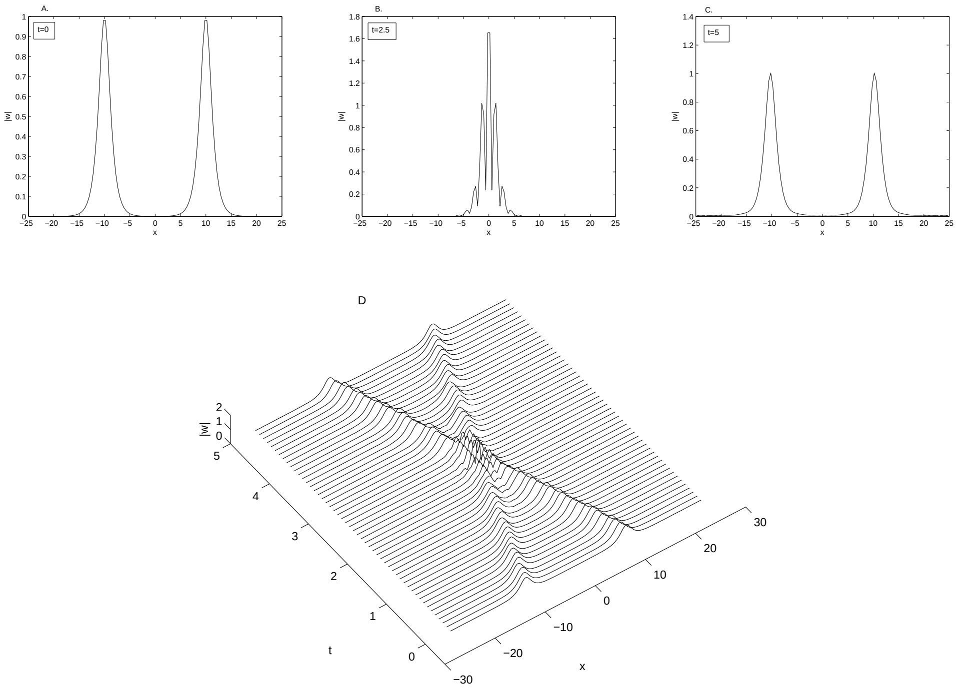

Figure 2: Interaction profile of two solitons, $C_1 = 4$, $C_2 = -4$, corresponding to Problem 1

MQ shape parameters $c_1 = c_2 = 1$, $\delta x = 0.4$ and $\delta t = 0.0001$, $A = 1.78$.
Lastly we show the birth of soliton of the equation (35) using a square well initial condition [Yan Xu and Chi-Wang Shu (2005)] given by

$$w(x,0) = A\exp\left(-x^2 + 2ix\right). \tag{40}$$

The problem is solved on the interval $[-45, 45]$, with periodic boundary condition for up to time $t = 4$, using MQ. The soliton profile is shown in Fig. 3(B). We observe a mobile soliton which is in agreement with the result of [Yan Xu and Chi Wang Shu (2005)].

Problem 2. Here we consider the bound state solution of NLS equation

$$iw_t + w_{xx} + \beta|w|^2 w = 0 \tag{41}$$

with initial condition

$$w(x,0) = \operatorname{sech}(x). \tag{42}$$

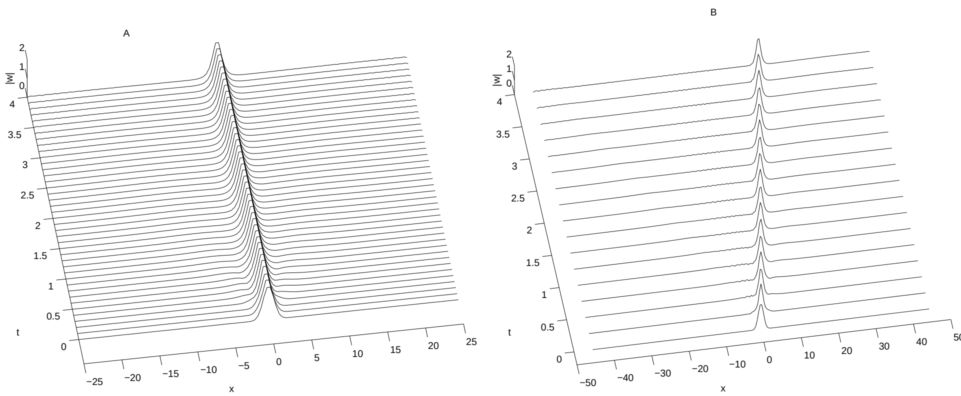

Figure 3: Birth of standing soliton Fig. 3(A) and Birth of mobile soliton Fig. 3(B) corresponding to problem 1.

It will produce a bound state of M solitons if $\beta = 2M^2$ [Yan Xu and Chi-Wang Shu (2005)]. The solution for $M = 3$ with periodic boundary condition in $[-15, 15]$ is shown in Fig. 4 where we have used MQ shape parameters $c_1 = 0.5$, $c_2 = 0.5$, time step size $\delta t = 0.0001$ and $\delta x = 0.1$. The obtained results are in good agrement with those given in [Yan Xu and Chi-Wang Shu (2005)].

Problem 3. In this problem, we consider the coupled nonlinear Schrodinger equations (CNLSE)

$$i(w_1)_t + \frac{1}{2}(w_1)_{xx} + \left(|w_1|^2 + \beta|w_2|^2\right) w_1 = 0, \quad -\infty < x < \infty \tag{43}$$

$$i(w_2)_t + \frac{1}{2}(w_2)_{xx} + \left(\beta|w_1|^2 + |w_2|^2\right) w_2 = 0, \quad -\infty < x < \infty$$

with the exact solution

$$w_1(x,t) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\left(\sqrt{2\alpha}\,(x - vt)\right)\exp\left[i\left\{vx - \left(\frac{v^2}{2} - \alpha\right)t\right\}\right] \tag{44}$$

$$w_2(x,t) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\left(\sqrt{2\alpha}\,(x - vt)\right)\exp\left[i\left\{vx - \left(\frac{v^2}{2} - \alpha\right)t\right\}\right]$$

In Table 2, we have listed the invariants, infinity error norms and comparison of the results with [M. S. Ismail (2008)]. From the comparison we found that the present method (MQ) is more accurate than Crank-Nicolson (CN) and Galerkin methods (GM). It is noted that all the conserved quantities are nearly constant. The

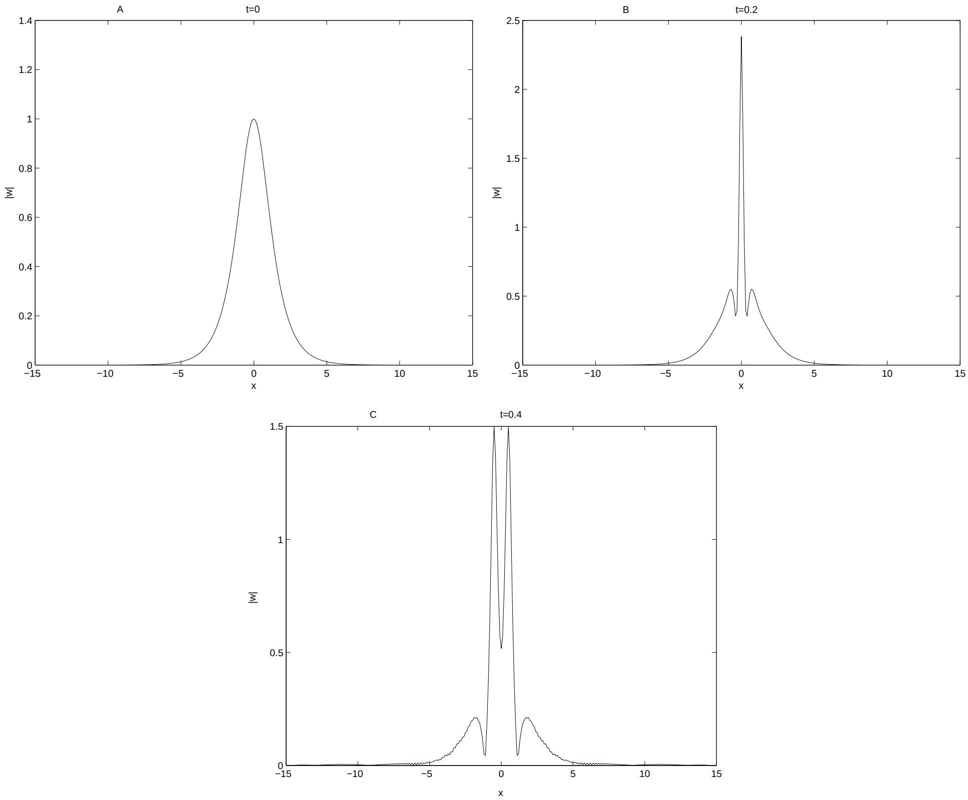

Figure 4: The bound state solution for $M = 3$ with periodic boundary condition in $[-15, 15]$ for problem 2.

parameters used are $c_1 = 0.5$, $c_2 = 0.5$, $\delta t = 0.0001$, $\delta x = 0.2$, $v = 1$, $\beta = 1, \alpha = 1$ and the interval is $[-10, 40]$.
In order to study mobility of single soliton we choose the initial condition

$$w_1(x,0) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\left(\sqrt{2\alpha}x\right)\exp(ivx) \tag{45}$$

$$w_2(x,0) = \sqrt{\frac{2\alpha}{1+\beta}}\operatorname{sech}\left(\sqrt{2\alpha}x\right)\exp(ivx)$$

Fig. 5 shows that the single soliton is moving from left to right. Parameters for computations are $c = 0.5$, $\delta t = 0.0001$, $\delta x = 0.2$, $v = 1$, $\beta = 1, \alpha = 1$ and interval

Table 2: Infinity error norms and invariants for single soliton obtained by MQ corresponding to Problem 3.

MQ						CN (2008)	GM (2008)
Time	I_1	I_2	I_3	I_4	L_∞	L_∞	L_∞
0.0001	1.414214	1.414214	-5.656853	0.471403	7.0637E-005	...	...
5	1.414862	1.414862	-5.657559	0.469809	8.4346E-004	...	...
10	1.415513	1.415513	-5.658263	0.468209	0.0027	0.037770	0.036107
15	1.416165	1.416165	-5.658965	0.466623	0.0058	...	...
20	1.417652	1.417652	-5.657945	0.480803	0.0123	0.073790	0.070403

is $[-10, 40]$.

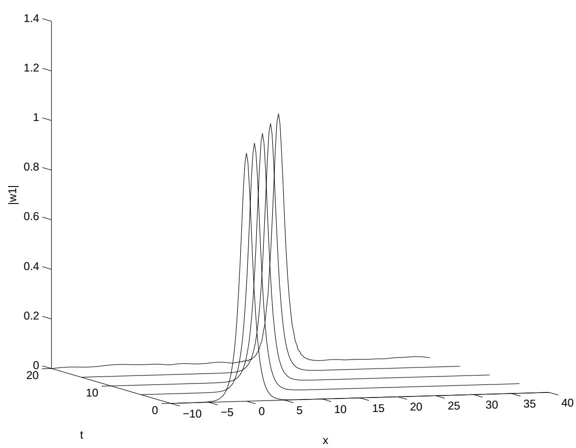

Figure 5: Propagation of the single soliton $|w_1|$ corresponding to Problem 3 in $[-10, 40]$.

Problem 4. Finally we consider the coupled nonlinear Schrodinger equations (CNLSE)

$$i(w_1)_t + i\alpha(w_1)_x + \frac{1}{2}(w_1)_{xx} + \left(|w_1|^2 + \beta|w_2|^2\right) w_1 = 0, \quad -\infty < x < \infty \tag{46}$$

$$i(w_2)_t - i\alpha(w_2)_x + \frac{1}{2}(w_2)_{xx} + \left(\beta|w_1|^2 + |w_2|^2\right) w_2 = 0, \quad -\infty < x < \infty.$$

The exact solution is given by

$$w_1(x,t) = \sqrt{\frac{2a}{1+\beta}} \operatorname{sech}(\sqrt{2a}(x-Ct)) \exp\left[i\left\{(C-\alpha)x - \left(\frac{C^2-\alpha^2}{2} - a\right)t\right\}\right] \quad (47)$$

$$w_2(x,t) = \sqrt{\frac{2a}{1+\beta}} \operatorname{sech}(\sqrt{2a}(x-Ct)) \exp\left[i\left\{(C+\alpha)x - \left(\frac{C^2-\alpha^2}{2} - a\right)t\right\}\right].$$

In Table 3 the error norms, L_2 and L_∞, of real part u_1 and imaginary part u_2 of a single soliton w_1, obtained by MQ, TPS ($r^4 log(r)$) and spline basis r^5 in the interval $[-25,25]$, are given for $\delta x = 0.2$, $\delta t = 0.0001$ $C = 1$, and $N = 251$. From the table we can say that, the results obtained by MQ and r^5 are better than those of TPS. In Table 4, L_∞ error norms for different values of MQ shape parameter are shown. Here we observe that as value of the shape parameter c increases beyond 1.7, the error norms are going to increase which gives the indication of instability. In other words we can say that $c = 1.7$ is the critical value. In Fig. 6, behavior of eigenvalues of matrices $\mathbf{S_1}$, $\mathbf{S_2}$ versus MQ shape parameter are shown corresponding to problem 4.

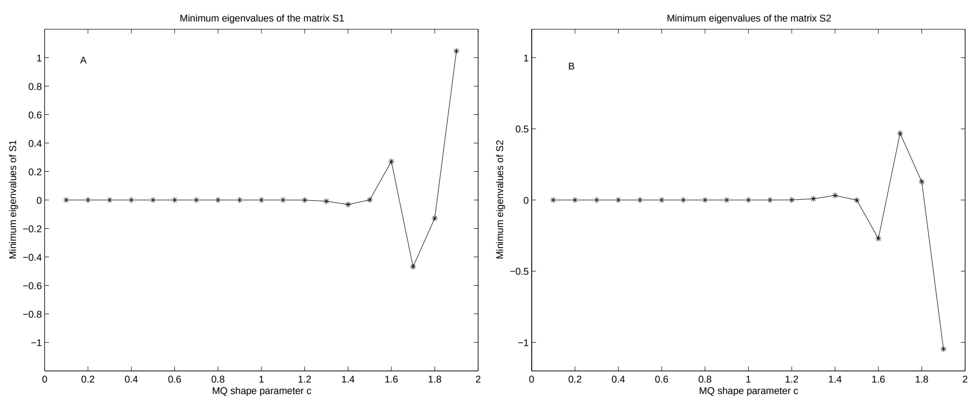

Figure 6: Stability plots, figures A,B showing minimum eigenvalues values of the matrices $\mathbf{S_1}$ and $\mathbf{S_2}$, when $t = 0.1$, $\delta t = 0.0001$ $\delta x = 0.2$, $C = 1$, $\alpha = 0.5$, $\beta = 1$ and $d = 1$ in $[-25,25]$, corresponding to problem 4.

5 Concluding remarks

In this paper, a mesh free interpolation method using radial basis functions is implemented to compute solution of NLS and CNLS equations. The performance of

Table 3: Error norms of u_1 and u_2 when $\delta x = 0.2$, $\delta t = 0.0001$ $C = 1$, and $N = 251$ in $[-25, 25]$ corresponding to Problem 4.

MQ	u_1		u_2	
Time	L_∞	L_2	L_∞	L_2
0.1	7.809E-005	1.136E-005	1.280E-004	1.014E-005
0.2	8.694E-005	1.711E-005	1.267E-004	3.910E-005
0.4	8.824E-005	5.879E-006	1.578E-004	1.388E-004
0.6	1.332E-004	2.366E-004	1.710E-004	1.482E-004
0.8	1.149E-004	2.558E-004	2.401E-004	3.791E-004
1.0	1.584E-004	4.878E-004	2.733E-004	4.473E-004
r^5	u_1		u_2	
Time	L_∞	L_2	L_∞	L_2
0.1	8.694E-005	9.814E-006	9.982E-005	5.807E-006
0.2	9.966E-005	1.031E-005	9.561E-005	2.445E-005
0.4	1.038E-004	6.756E-006	1.107E-004	7.601E-005
0.6	1.725E-004	3.028E-004	1.349E-004	2.822E-004
0.8	1.128E-004	3.920E-005	1.110E-004	1.806E-004
1.0	1.209E-004	4.491E-005	1.248E-004	2.214E-004
TPS	u_1		u_2	
Time	L_∞	L_2	L_∞	L_2
0.1	7.592E-004	2.446E-004	7.970E-004	1.371E-005
0.2	1.255E-003	3.976E-004	1.522E-003	9.199E-005
0.4	1.955E-003	7.570E-004	2.376E-003	7.819E-004
0.6	2.251E-003	1.845E-003	3.996E-003	2.331E-003
0.8	2.521E-003	3.528E-003	5.489E-003	3.099E-003
1.0	4.078E-003	6.875E-003	6.372E-003	3.471E-003

the technique is in excellent agreement with exact solution and with earlier work [YanXu and ChiWang (2005), M. S. Ismail (2008)]. Stability analysis is established. As a whole the present method produces better results with ease of implementation. The technique used in this paper provides an efficient alternative for the solution of nonlinear partial differential equations. This method is mesh free contrary to the traditional methods, like FDM and FEM. Accuracy of the technique can be increased by changing value of shape parameter, for fix number of collocation points. It is observed that time marching process reduces accuracy of the solution due to the time truncation errors. From application point of view the implementation of the method is very simple and straightforward.

Table 4: Error norms of the solutions u_1, u_2, v_1, v_2 for different values of the shape parameter c, at time $t = 0.1$, $\delta t = 0.0001$ $\delta x = 0.2$, $C = 1$, $\alpha = 0.5$, $\beta = 1$, $d = 1$ in $[-25, 25]$.

	u_1	u_2	v_1	v_2
c	L_∞	L_∞	L_∞	L_∞
0.10	1.411E-002	4.696E-002	2.790E-002	5.470E-002
0.20	9.861E-004	3.793E-003	1.955E-003	4.184E-003
0.30	5.519E-005	2.650E-004	6.827E-005	2.388E-004
0.40	7.459E-005	1.185E-004	1.150E-004	1.474E-004
0.50	7.809E-005	1.393E-004	1.280E-004	1.774E-004
0.60	7.913E-005	1.409E-004	1.292E-004	1.800E-004
0.70	7.927E-005	1.410E-004	1.293E-004	1.800E-004
0.80	7.929E-005	1.410E-004	1.293E-004	1.800E-004
0.90	7.929E-005	1.410E-004	1.293E-004	1.799E-004
1.00	7.929E-005	1.410E-004	1.293E-004	1.799E-004
1.10	7.929E-005	1.410E-004	1.293E-004	1.799E-004
1.20	7.929E-005	1.410E-004	1.293E-004	1.799E-004
1.30	7.929E-005	1.410E-004	1.293E-004	1.799E-004
1.40	7.930E-005	1.409E-004	1.293E-004	1.799E-004
1.50	7.938E-005	1.411E-004	1.293E-004	1.799E-004
1.60	7.916E-005	1.419E-004	1.292E-004	1.799E-004
1.70	7.779E-005	1.248E-004	1.305E-004	1.993E-004
1.80	1.160E+000	8.866E+001	1.289E+000	1.184E+002

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